mathematics

Review

A Survey on the Oscillation of Solutions for Fractional Difference Equations

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Abstract: In this paper, we present a systematic study concerning the developments of the oscillation results for the fractional difference equations. Essential preliminaries on discrete fractional calculus are stated prior to giving the main results. Oscillation results are presented in a subsequent order and for different types of equations. The investigation was carried out within the delta and nabla operators.

Keywords: fractional order; forward (delta) difference equation; backward (nabla) difference equation; oscillation of solutions

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1. Introduction

Over many years, the process of describing natural or real-life phenomena has been carried out using the integer-order differential equations. However, the factors involved in the phenomena are very complicated and of different natures, all of which cannot be incorporated by the ordinary differential equations. This gap in the construction of the models is covered up by arbitrary-order calculus. Fractional calculus has its origin during the same period of time as that of the classical calculus in the 17th Century. The insufficient geometrical and unsatisfactory physical interpretation of the arbitrary-order derivatives has slowed down the progress of the field. It was in the 20th Century with the development of high-speed computers and computational techniques that researchers began to understand the importance and the meaningful representation to construct and apply a certain type of nonlocal operator to real-life problems. Now, fractional calculus has turned out to be a hot topic in the fields of science and engineering. The rapid growth and inspiration of fractional calculus have been greatly due to anticipation of the memory and hereditary features that are incorporated in many phenomena by the so-called fractional differential operators [1–3]. As a result of this, the subject of fractional calculus and its widespread applications have become of great interest for the relevant audience [4]. For the same justifications that led to the investigation of the discrete analogue of integer-order differential operators, the discrete analogue of fractional differential operators, which is called fractional difference operators, has gained considerable attention, and thus, they have been significantly adopted due to their extensive applications in computations and simulations. The study of fractional difference equations was led by the pioneering works of Agarwal, Atici, Eloe, Anastassiou, Holm, Goodrich, and Peterson, who introduced a complete counterpart theory that adopts all the essential preliminaries needed to set forth similar results relevant to the qualitative theory of solutions for several types of fractional difference equations [5–11].

Every phenomenon in the world in one way or other is nonlinear in nature. Thus, the better understanding of these phenomena can be obtained from models constructed via nonlinear equations. The analytical solution of nonlinear equations is not always possible to obtain as in the case of linear equations. However, approximate solutions can be obtained for the nonlinear equations, which provide a better understanding of the behavior of the equations. In the case of nonlinear equations, without actually solving the equations, one can very well answer questions such as the existence of solutions, whether the system is stable, whether it can be controlled, whether the system is chaotic, or whether it exhibits periodicity. Thus, this direct method of analyzing the system behavior can be useful and help engineers in their research. Scientists and researchers are very much interested in the qualitative properties such as the oscillation, stability, controllability, bifurcation, chaos, and so on.

Oscillation is one of the important branches in applied mathematics and can be induced or destroyed by the introduction of nonlinearity, delay, or a stochastic term. The oscillation of differential and difference equations contributes to many realistic applications, such as torsional oscillations, the oscillation of heart beats, sinusoidal oscillation, voltage-controlled neuron models, and harmonic oscillation with damping. Not only in physical applications, oscillation theory is vital biologically in describing the synchrony in animal and plant populations due to predation and competition. Such applications have attracted the interest of many researchers who have developed systematic studies concerning the oscillation and non-oscillation of solutions of integer-order differential and difference equations; we refer the reader to the remarkable monographs [12,13]. With the explosion in the theory of fractional calculus, the oscillation of fractional-order differential equations has been under investigation in the last two decades. Grace et al. initiated this subject by studying the oscillation of fractional differential equations in [14]. Progress in this regard has continued, and several important results have been established; see for instance [15–18] and the references cited therein. In alignment with this, fractional difference equations have been the object of interested researchers in terms of the oscillation of their solutions. Several results have been reported by many researchers about the oscillation of solutions for different types of fractional difference equations.

The main aim of this work was to consolidate the recent developments in the field of the oscillation theory of discrete fractional equations and provide an insight for researchers about the future requisites in the field of the oscillations of discrete fractional calculus. The investigation in this work focused on the results for both delta- and nabla-type fractional difference equations.

2. Preliminaries

In this section, we review some notations, definitions, and well-known results of discrete fractional calculus that are widely treated throughout the remaining part of this paper. The terms and notations were adopted from different resources.

The empty sums and products were taken to be zero and one, respectively. Denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}^+$ the set of all positive real numbers. Define by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any $a, b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$.

Definition 1 ([19,20])**.** *The Euler gamma function is defined by:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using its reduction formula, the Euler gamma function can also be extended to the half-plane $\Re(z) \leq 0$ *except for* $z \in \{\ldots, -2, -1, 0\}$.

Definition 2 ([21])**.** *The generalized falling function is defined by:*

$$t^{\underline{r}} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)},$$

for those values of t and r such that the right-hand side of this equation makes sense. If $t - r + 1$ *is a nonpositive integer and* $t + 1$ *is not a nonpositive integer, then we use the convention that* $t^{\underline{r}} = 0$. *The generalized rising function is defined by:*

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)},$$

for those values of t and r so that the right-hand side of this equation is sensible. If t is a nonpositive integer, but $t + r$ *is not a nonpositive integer, then we use the convention that* $t^{\overline{r}} = 0$.

Definition 3 ([22])**.** *Let* $u : \mathbb{N}_a^b \to \mathbb{R}$ *and* $N \in \mathbb{N}_1$. *The first-order forward (delta) and backward (nabla) differences of u are defined by:*

$$\big(\Delta u\big)(t) = u(t+1) - u(t), \quad t \in \mathbb{N}_a^{b-1},$$

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1}^{b},$$

respectively. The N^{th}*-order delta and nabla differences of u are defined recursively by*

$$\big(\Delta^N u\big)(t) = \Big(\Delta\big(\Delta^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_a^{b-N},$$

and:

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}^{b},$$

respectively.

Definition 4 ([21])**.** *Let* $u : \mathbb{N}_a \to \mathbb{R}$ *and* $\nu > 0$. *Then, the* ν^{th}*-order delta fractional sum of u based at a is defined by:*

$$\big(\Delta_a^{-\nu} u\big)(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} u(s), \quad t \in \mathbb{N}_{a+\nu}.$$

Definition 5 ([21])**.** *Let* $u : \mathbb{N}_{a+1} \to \mathbb{R}$ *and* $\nu > 0$. *Then, the* ν^{th}*-order nabla fractional sum of u based at a is defined by:*

$$\big(\nabla_a^{-\nu} u\big)(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\nu-1}} u(s), \quad t \in \mathbb{N}_a.$$

Definition 6 ([21])**.** *Let* $u : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu \leq N$. *The* ν^{th}*-order Riemann–Liouville delta fractional difference of u is defined by:*

$$\big(\Delta_a^{\nu} u\big)(t) = \Big(\Delta^N\big(\Delta_a^{-(N-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N-\nu}.$$

Definition 7 ([21])**.** *Let* $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$, *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu \leq N$. *Then, the* ν^{th}*-order Riemann–Liouville nabla fractional difference of u is defined by:*

$$\big(\nabla_a^{\nu} u\big)(t) = \Big(\nabla^N\big(\nabla_a^{-(N-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 8 ([23]). *Let $u : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, and $\nu \notin \mathbb{N}$. Then, the ν^{th}-order Caputo delta fractional difference of u is defined by:*

$$(\Delta_{a*}^{\nu} u)(t) = \left(\Delta_a^{-(N-\nu)}(\Delta^N u)\right)(t), \quad t \in \mathbb{N}_{a+N-\nu},$$

where $N = [\nu] + 1$. If $\nu = N \in \mathbb{N}$, then:

$$(\Delta_{a*}^{\nu} u)(t) = (\Delta^N u)(t), \quad t \in \mathbb{N}_a.$$

Definition 9 ([24]). *Let $u : \mathbb{N}_{a-N+1} \to \mathbb{R}$ and $\nu > 0$. Then, the ν^{th}-order Caputo nabla fractional difference of u is defined by:*

$$(\nabla_{a*}^{\nu} u)(t) = \left(\nabla_a^{-(N-\nu)}(\nabla^N u)\right)(t), \quad t \in \mathbb{N}_{a+1},$$

where $N = \lceil \nu \rceil$.

3. Oscillation Results

The main results are given in this section. We carried out the presentation within delta and nabla notations.

3.1. Oscillatory Behavior of Delta Fractional Difference Equations

Consider the following higher-order nonlinear delta fractional difference equations involving the Riemann–Liouville and the Caputo operators of arbitrary order:

$$\begin{cases} (\Delta^{\nu} u)(t) + f_1(t, u(t+\nu)) = r_1(t) + f_2(t, u(t+\nu)), & t \in \mathbb{N}_a, \\ (\Delta^{-(k-\nu)} u)(t)\Big|_{t=a} = u_k \in \mathbb{R}, & k = 1, 2, \cdots, N, \end{cases} \tag{1}$$

and:

$$\begin{cases} (\Delta_{*}^{\nu} u)(t) + f_1(t, u(t+\nu)) = r_1(t) + f_2(t, u(t+\nu)), & t > a \geq 0, \\ (\Delta^{k} u)(t)\Big|_{t=a} = \bar{u}_k \in \mathbb{R}, & k = 0, 1, 2, \cdots, N-1. \end{cases} \tag{2}$$

Here, $\nu > 0$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$; $f_1, f_2 : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ and $r_1 : [a, \infty) \to \mathbb{R}$ are continuous. A solution u of (1) (or (2)) is said to be oscillatory if for every natural number M, there exists $t \geq M$ such that $u(t)u(t+1) \leq 0$; otherwise, it is called non-oscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

Let $p_1, p_2 : [a, \infty) \to \mathbb{R}^+$ be continuous and β, γ be positive real numbers. We make the following assumptions:

(A1) The functions f_i satisfy the sign condition $u f_i(t, u) > 0$, $i = 1, 2$, $u \neq 0$, $t \geq a$;
(A2) $|f_1(t, u)| \geq p_1(t)|u|^{\beta}$ and $|f_2(t, u)| \leq p_2(t)|u|^{\gamma}$, $u \neq 0$, $t \geq a$;
(A3) $|f_1(t, u)| \leq p_1(t)|u|^{\beta}$ and $|f_2(t, u)| \geq p_2(t)|u|^{\gamma}$, $u \neq 0$, $t \geq a$.

In [25], Senem et al. established some oscillation theorems given in the sequel.

Theorem 1 ([25]). *Let (A1)–(A2) be satisfied with $\beta > \gamma$. If:*

$$\liminf_{t \to \infty} t^{(1-\nu)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)} [r_1(s) + G(s)] = -\infty,$$

and:

$$\limsup_{t \to \infty} t^{(1-\nu)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)} [r_1(s) - G(s)] = \infty,$$

for every sufficiently large T, where:

$$G(s) = \left(\frac{\beta}{\gamma} - 1\right)\left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}} p_1^{\frac{\gamma}{\gamma-\beta}}(s),$$

then Equation (1) is oscillatory.

Theorem 2 ([25])**.** *Let $\nu \geq 1$ and (A1)–(A3) be satisfied with $\beta < \gamma$. If*

$$\liminf_{t\to\infty} t^{(1-\nu)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) - G(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{(1-\nu)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) + G(s)] = \infty,$$

for every sufficiently large T, where G is defined as in Theorem 1, then every bounded solution of Equation (1) is oscillatory.

Theorem 3 ([25])**.** *Let (A1) and (A2) be satisfied with $\beta > \gamma$. If:*

$$\liminf_{t\to\infty} t^{(1-N)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) + G(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{(1-N)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) - G(s)] = \infty,$$

for every sufficiently large T, where G is defined as in Theorem 1, then Equation (2) is oscillatory.

Theorem 4 ([25])**.** *Let $\nu \geq 1$ and (A1)–(A3) be satisfied with $\beta < \gamma$. If:*

$$\liminf_{t\to\infty} t^{(1-N)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) - G(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{(1-N)} \sum_{s=T}^{t-\nu} (t-s-1)^{(\nu-1)}[r_1(s) + G(s)] = \infty,$$

for every sufficiently large T, where G is defined as in Theorem 1, then every bounded solution of Equation (2) is oscillatory.

Following the work in [25], Li et al. [26] investigated the oscillation of forced delta fractional difference equations with the damping term of the form:

$$\begin{cases} (1+p_3(t))(\Delta\Delta^\nu u)(t) + p_3(t)(\Delta^\nu u)(t) + f_3(t,u(t)) = g_1(t), & t \in \mathbb{N}_0, \\ (\Delta^{-(1-\nu)}u)(t)\Big|_{t=0} = u_0 \in \mathbb{R}, \end{cases} \tag{3}$$

where $0 < \nu < 1$; $p_3, g_1 : \mathbb{N}_0 \to \mathbb{R}$ and $f_3 : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ such that:

$$u f_3(t,u) > 0, \quad u \neq 0, \quad t \in \mathbb{N}_0,$$

and $p_3(t) > -1$ for $t \in \mathbb{N}_0$.

Theorem 5 ([26]). *For $t_0 \in \mathbb{N}_0$, suppose that:*

$$\liminf_{t\to\infty} \sum_{s=0}^{t-\nu} \frac{(t-s-1)^{(\nu-1)}}{V(s)} \left[M + \sum_{\xi=t_0}^{s-1} g_1(\xi)V(\xi) \right] < 0,$$

and:

$$\limsup_{t\to\infty} \sum_{s=0}^{t-\nu} \frac{(t-s-1)^{(\nu-1)}}{V(s)} \left[M + \sum_{\xi=t_0}^{s-1} g_1(\xi)V(\xi) \right] > 0,$$

where M is a constant and:

$$V(t) = \prod_{s=t_0}^{t-1} (1 + p_3(s)).$$

Then, Equation (3) is oscillatory.

Theorem 6 ([26]). *For $t_0 \in \mathbb{N}_0$, suppose that:*

$$\liminf_{t\to\infty} \sum_{s=0}^{t-1} \frac{1}{V(s)} \left[M + \sum_{\xi=t_0}^{s-1} g_1(\xi)V(\xi) \right] = -\infty,$$

and:

$$\limsup_{t\to\infty} \sum_{s=0}^{t-1} \frac{1}{V(s)} \left[M + \sum_{\xi=t_0}^{s-1} g_1(\xi)V(\xi) \right] = \infty,$$

where M is a constant and V is defined as in Theorem 5. Then, Equation (3) is oscillatory.

In this line, Seçer et al. [27] investigated the oscillation of the following nonlinear delta fractional difference equations:

$$\Delta\Big(p_4(t)\big[\Delta\big(q_1(t)((\Delta^\nu u)(t))^{\gamma_1}\big)\big]^{\gamma_2}\Big) + q_2(t) f_4\left(\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \right) = 0, \qquad (4)$$

for $t \in \mathbb{N}_{t_0+1-\nu}$. Here, $0 < \nu \le 1$, γ_1 and γ_2 are the quotients of two odd positive numbers such that $\gamma_1\gamma_2 = 1$, p_4, q_1 and q_2 are positive sequences,

$$\sum_{s=t_0}^{\infty} \left(\frac{1}{p_4^{1/\gamma_2}(s)} \right) = \infty,$$

$f_4 : \mathbb{R} \to \mathbb{R}$ is continuous, and:

$$\frac{f_4(u)}{u} \ge k, \quad k \in \mathbb{R}^+, \quad u \ne 0.$$

Theorem 7 ([27]). *If there exists a positive sequence ϕ such that:*

$$\limsup_{t\to\infty} \sum_{s=t_2}^{t-1} \left[k\phi(s)q_2(s) - \frac{q_1^{1/\gamma_1}(s)[(\Delta\phi_+)(s)]^2}{4\phi(s)\Gamma(1-\nu)\delta_1^{1/\gamma_1}(s,t_1)} \right] = \infty,$$

then Equation (4) is oscillatory. Here:

$$\delta_1(t,t_i) = \sum_{s=t_i}^{t-1} \left(\frac{1}{p_4^{1/\gamma_2}(s)} \right), \quad i = 0,1,2,3,$$

and,

$$(\Delta\phi_+)(s) = \max\{(\Delta\phi)(s), 0\}.$$

Theorem 8 ([27]). *Let ϕ be a positive sequence. Furthermore, we assume that there exists a double sequence such that:*

$$H(t,t) = 0 \text{ for } t \geq 0, \quad H(t,s) > 0 \text{ for } t > s \geq 0,$$
$$\Delta_2 H(t,s) = H(t,s+1) - H(t,s) \leq 0 \text{ for } t > s \geq 0.$$

If:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} H(t,s) \left[k\phi(s)q_2(s) - \frac{q_1^{1/\gamma_1}(s)[(\Delta\phi_+)(s)]^2}{4\phi(s)\Gamma(1-\nu)\delta_1^{1/\gamma_1}(s,t_2)} \right] = \infty.$$

Then, Equation (4) is oscillatory.

If we choose the double sequence:

$$H(t,s) = (t-s)^{\lambda}, \quad \lambda \geq 1, \quad t \geq s \geq 0,$$

we have the following corollary.

Corollary 1 ([27]). *Under the conditions of Theorem 8 and:*

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^{\lambda}} \sum_{s=t_0}^{t-1} (t-s)^{\lambda} \left[k\phi(s)q_2(s) - \frac{q_1^{1/\gamma_1}(s)[(\Delta\phi_+)(s)]^2}{4\phi(s)\Gamma(1-\nu)\delta_1^{1/\gamma_1}(s,t_2)} \right] = \infty,$$

then Equation (4) is oscillatory.

In [28], Chatzarakis et al. studied the oscillatory behavior of the delta fractional difference equation of the form:

$$\Delta\big(((\Delta^{\nu}u)(t))^{\gamma_3} + q_3(t)f_5(u(t))\big) = 0, \quad t \in \mathbb{N}_{t_0+1-\nu}, \tag{5}$$

where $0 < \nu \leq 1$; $\gamma_3 > 0$ is a quotient of odd positive integers; q_3 is a positive sequence, and $f_5 : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

$$\frac{f_5(u)}{u^n} \geq l, \quad u \neq 0, \quad l > 0, \quad n \in \mathbb{N},$$

$$\left[\frac{b}{q_3(t)}\right]^{\frac{1}{\gamma_3}} \leq -m, \quad t \geq t_0, \quad b < 0, \quad m > 0.$$

We also assume:

$$\frac{(\Delta u)(t)}{(\Delta^{\nu}u)(t+1)} \geq M_1, \quad \frac{(\Delta u)(t)}{(\Delta^{\nu}u)(t)} \geq M_2, \quad t \geq t_0,$$

for some positive constants M_1, M_2 and for all $(\Delta^{\nu}u)(t) \neq 0$ and $(\Delta^{\nu}u)(t+1) \neq 0$, and:

$$\frac{[(\Delta u)(t)]^2}{u(t)u(t+1)} \geq J_1, \quad (\Delta^2 u)(t) \geq J_2,$$

for some positive constants J_1 and J_2.

Theorem 9 ([28]). *Assume:*

$$\sum_{s=t_0}^{\infty} q_3^{\frac{1}{\gamma_3}}(s) = \infty.$$

Furthermore, assume that there exists a positive sequence $\tilde{r}$ *such that:*

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1} H(t,s)\left[l\tilde{r}(s)q_3(s)-\frac{[(\Delta\tilde{r}_+)(s)]^2}{4\tilde{r}(s+1)M_1^{\gamma_3}}\right]=\infty,$$

where:

$$(\Delta\tilde{r}_+)(s)=\max\{(\Delta\tilde{r})(s),0\}.$$

Then, Equation (5) *is oscillatory.*

Theorem 10 ([28]). *Assume:*

$$\sum_{s=t_0}^{\infty} q_3^{\frac{1}{\gamma_3}}(s)=\infty.$$

Furthermore, assume that there exists a positive sequence $\tilde{r}$ *and a double positive sequence* $\tilde{H}(t,s)$ *such that:*

$$\tilde{H}(t,t)=0 \text{ for } t\geq t_0,\quad \tilde{H}(t,s)>0 \text{ for } t>s\geq t_0,$$
$$\Delta_2\tilde{H}(t,s)=\tilde{H}(t,s+1)-\tilde{H}(t,s)\leq 0 \text{ for } t>s\geq t_0.$$

If:

$$\limsup_{t\to\infty}\frac{1}{\tilde{H}(t,t_0)}\sum_{s=t_0}^{t-1}\left[\tilde{r}(s)q_3(s)\tilde{H}(t,s)-\frac{\tilde{h}^2(t,s)\tilde{r}(s+1)}{4l\tilde{H}(t,s)M_1^{\gamma_3}}\right]=\infty,$$

where:

$$\tilde{h}(t,s)=\Delta_2\tilde{H}(t,s)+\frac{\tilde{H}(t,s)(\Delta\tilde{r}_+)(s)}{\tilde{r}(s+1)},$$

then Equation (5) *is oscillatory.*

Theorem 11 ([28]). *Assume that there exists a positive sequence* $\tilde{r}$ *such that:*

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1} H(t,s)\left[lq_3(s)+\left(\frac{J_1}{M_2}\right)^{\gamma_3}-\frac{J_2}{M_2}-(\Delta\tilde{r}_+)(s)\right]=\infty.$$

Then, Equation (5) *is oscillatory.*

Motivated by the above works, Adiguzel [29,30] considered the oscillation behavior of the solutions of the following delta fractional difference equations:

$$\Delta\big(r_2(t)(\Delta^\nu u)(t)\big)+q_4(t)f_6\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right)=0,\quad t\in\mathbb{N}_{t_0+1-\nu}, \tag{6}$$

and:

$$\Delta\big(c_1(t)\Delta\big(c_2(t)\big(r_2(t)(\Delta^\nu u)(t)\big)\big)\big)+q_4(t)\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right)=0,\quad t\in\mathbb{N}_{t_0+1-\nu}, \tag{7}$$

where $0<\nu\leq 1$, r_2, q_4, c_1, and c_2 are positive sequences and $f_6:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $uf_6(u)>0$ for $u\neq 0$.

Theorem 12 ([29]). *Suppose that:*

$$\sum_{s=t_0}^{\infty} q_4(s)=\infty,$$

and:

$$\liminf_{t\to\infty} f_6(t)>0.$$

Then, Equation (6) is oscillatory.

Theorem 13 ([29]). *Assume that:*

$$\sum_{s=t_0}^{\infty} R(s) q_4(s) = \infty,$$

where:

$$R(t) = \sum_{s=t_0}^{t-1} \frac{1}{r_2(s)} \text{ such that } \lim_{t\to\infty} R(t) = \infty.$$

Then, every bounded solution of Equation (6) is oscillatory.

Theorem 14 ([30]). *Assume that:*

$$\sum_{s=t_0}^{\infty} \frac{1}{c_1(s)} = \sum_{s=t_0}^{\infty} \frac{1}{c_2(s)} = \sum_{s=t_0}^{\infty} \frac{1}{r_2(s)} = \infty, \tag{8}$$

and there exists a positive sequence γ such that, for all sufficiently large t,

$$\limsup_{t\to\infty} \sum_{s=t_3}^{t-1} \left[\frac{\Gamma(1-\nu)\gamma(s) q_4(s)}{\theta(s)\phi(s+1)} \sum_{\tau=t_2}^{s-1} \frac{\theta(\tau)}{r_2(\tau)} \sum_{\tau=t_2}^{s-1} \frac{\phi(\tau)}{c_2(\tau)} - \frac{c_1(s)\left[(\Delta\gamma_+)(s)\right]^2}{4\gamma(s)} \right] = \infty.$$

If there exist positive sequences β, λ such that, for all sufficiently large t,

$$\frac{\lambda(t)}{r_2(t)\sum_{s=t_1}^{t-1} \frac{1}{r_2(s)}} - (\Delta\lambda)(t) \le 0, \tag{9}$$

and:

$$\limsup_{t\to\infty} \sum_{\xi=t_2}^{t-1} \left[\frac{\beta(\xi)\lambda(\xi)}{\lambda(\xi+1)c_2(\xi)} \sum_{s=\xi}^{\infty} \left(\frac{1}{c_1(s)} \sum_{\tau=s}^{\infty} q_4(\tau) \right) - \frac{r_2(\xi)\left[(\Delta\beta_+)(\xi)\right]^2}{4\Gamma(1-\nu)\beta(\xi)} \right] = \infty, \tag{10}$$

then, Equation (7) is oscillatory. Here:

$$\phi(t) = \sum_{s=t_1}^{t-1} \frac{1}{c_1(s)}, \quad \theta(t) = \sum_{s=t_2}^{t-1} \frac{\phi(s)}{c_2(s)}, \quad \delta(t) = \sum_{s=t_3}^{t-1} \frac{\theta(s)}{r_2(s)}.$$

Further, we have:

$$(\Delta\gamma_+)(s) = \max\{0, (\Delta\gamma)(s)\}, \quad (\Delta\beta_+)(s) = \max\{0, (\Delta\beta)(s)\}.$$

Theorem 15 ([30]). *Let (8) hold. Assume that there exists a positive sequence γ such that, for all sufficiently large t,*

$$\limsup_{t\to\infty} \sum_{s=t_3}^{t-1} \left[\frac{\Gamma(1-\nu)\gamma(s) q_4(s)}{\theta(s+1)} \sum_{\tau=t_2}^{s-1} \frac{\theta(\tau)}{r_2(\tau)} - \frac{c_2(s)\theta(s+1)\left[(\Delta\gamma_+)(s)\right]^2}{4\gamma(s)\theta(s)\sum_{\tau=t_0}^{s-1}\frac{1}{c_1(\tau)}} \right] = \infty. \tag{11}$$

If there exist positive sequences β, λ such that (9) and (10) hold, then Equation (7) is oscillatory.

Theorem 16. [30] *Let* (8) *hold. Assume that there exists a positive sequence* γ *such that, for all sufficiently large* t,

$$\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left[\frac{\delta(s)\gamma(s)q_4(s)}{\delta(s+1)}-\frac{r_2(s)\phi(s)\left[(\Delta\gamma_+)(s)\right]^2}{4\gamma(s)\sum_{\tau=t_1}^{s-1}\frac{\phi(\tau)}{c_2(\tau)}\sum_{\tau=t_0}^{s-1}\frac{1}{c_1(\tau)}}\right]=\infty. \tag{12}$$

If there exist positive sequences β, λ *such that* (9) *and* (10) *hold, then Equation* (7) *is oscillatory.*

Motivated by the idea in [27], Bai et al. [31] was concerned with the oscillation of a class of nonlinear fractional difference equations with the damping term of the form:

$$\begin{aligned}\Delta\big(c_3(t)[\Delta(r_3(t)(\Delta^\nu u)(t))]^{\gamma_4}\big)+q_5(t)[\Delta(r_3(t)(\Delta^\nu u)(t))]^{\gamma_4}&\\+q_6(t)f_7\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right)&=0,\end{aligned} \tag{13}$$

for $t\in\mathbb{N}_{t_0}$, $0<\nu\le 1$, $\gamma_4\ge 1$ is a quotient of two odd positive numbers, r_3, q_5, q_6, and c_3 are positive sequences such that $c_3(t)>q_5(t)$, and $f_7:\mathbb{R}\to\mathbb{R}$ is a monotone decreasing function satisfying:

$$uf_7(u)>0,\quad \frac{f_7(u)}{u^{\gamma_4}}\ge L>0,\quad u\ne 0.$$

Theorem 17 ([31])**.** *Define:*

$$x(t)=\prod_{s=t_0}^{t-1}\frac{c_3(s)}{c_3(s)-q_5(s)}.$$

Assume:

$$\sum_{s=t_0}^{\infty}\frac{1}{(x(s)c_3(s))^{\frac{1}{\gamma_4}}}=\infty, \tag{14}$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r_3(s)}=\infty, \tag{15}$$

and:

$$\sum_{\xi=t_0}^{\infty}\frac{1}{r_3(\xi)}\sum_{\tau=\xi}^{\infty}\left[\frac{1}{c_3(\tau)x(\tau)}\sum_{s=\tau}^{\infty}x(s+1)q_6(s)\right]^{\frac{1}{\gamma_4}}=\infty. \tag{16}$$

If:

$$\limsup_{t\to\infty}\sum_{s=T}^{t-1}\left[Lx(s)q_6(s)-\frac{\left[(\Delta x)(s)\right]^2}{4x(s)x(s+1)W(s)}\right]=\infty,$$

where T *is sufficiently large,*

$$W(t)=\left[\frac{\Gamma(1-\nu)\delta_2(t,t_1)}{r_3(t)}\right]^{\gamma_4},$$

then Equation (13) *is oscillatory or satisfies:*

$$\lim_{t\to\infty}\left[\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right]=0.$$

Here:

$$\delta_2(t,t_1)=\sum_{s=t_1}^{\infty}\frac{1}{(x(s)c_3(s))^{\frac{1}{\gamma_4}}}.$$

Theorem 18 ([31]). *Define x, W, and δ as in Theorem 17. Assume that (14)–(16) hold and there exists a positive sequence $\tilde{H}(t,s)$ such that:*

$$\tilde{H}(t,t) = 0 \text{ for } t \geq t_0, \quad \tilde{H}(t,s) > 0 \text{ for } t > s \geq t_0,$$
$$\Delta_2 \tilde{H}(t,s) = \tilde{H}(t,s+1) - \tilde{H}(t,s) \leq 0 \text{ for } t > s \geq t_0.$$

If:

$$\limsup_{t\to\infty} \frac{1}{\tilde{H}(t,t_0)} \sum_{s=t_0}^{t-1} \left[Lx(s)q_6(s)\tilde{H}(t,s) - \frac{h^2(t,s)x(s+1)}{4\tilde{H}(t,s)x(s)W(s)} \right] = \infty,$$

where:

$$h(t,s) = \Delta_2 \tilde{H}(t,s) + \frac{\tilde{H}(t,s)(\Delta x)(s)}{x(s+1)},$$

then Equation (13) is oscillatory or satisfies:

$$\lim_{t\to\infty} \left[\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \right] = 0.$$

In [32], Chatzarakis et al. studied the oscillatory behavior of the solutions of the delta fractional difference equation of the form:

$$\Delta\big(r_4(t)g_2\big((\Delta^\nu u)(t)\big)\big) + p_4(t)f_8\left(\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \right) = 0, \quad t \in \mathbb{N}_{t_0+1-\nu}, \tag{17}$$

where $0 < \nu \leq 1$, r_4, p_4, are positive sequences; g_2, $f_8 : \mathbb{R} \to \mathbb{R}$ are continuous functions with:

$$\frac{f_8(u)}{u} \geq k_1, \quad \frac{u}{g_2(u)} \geq k_2,$$

for some constants k_1, k_2 and for all $u \neq 0$. Further, we also assume that $ug_2(u) > 0$ for $u \neq 0$ and there exists a positive constant μ such that $g_2(u_1u_2) \leq \mu u_1 g_2(u_2)$ for $u_1u_2 \neq 0$.

Theorem 19 ([32]). *Assume:*

$$\sum_{s=t_1}^{\infty} g_2\left(\frac{1}{r_4(s)}\right) = \infty.$$

Furthermore, assume that there exists a positive sequence ψ such that:

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[k_1\psi(s)p_4(s) - \frac{1}{k_2}R_1(s) \right] = \infty,$$

where:

$$R_1(s) = \frac{\big[(\Delta\psi_+)(s)\big]^2 r_4(s+1)}{4\psi(s+1)\Gamma(1-\nu)}, \quad (\Delta\psi_+)(s) = \max\{(\Delta\psi)(s), 0\}.$$

Then, Equation (17) is oscillatory.

Theorem 20 ([32]). *Assume:*

$$\sum_{s=t_1}^{\infty} g_2\left(\frac{1}{r_4(s)}\right) = \infty.$$

Furthermore, assume that there exists a positive sequence ψ and a double positive sequence $\tilde{H}(t,s)$ such that:

$$\tilde{H}(t,t) = 0 \text{ for } t \geq t_0, \quad \tilde{H}(t,s) > 0 \text{ for } t > s \geq t_0,$$
$$\Delta_2 \tilde{H}(t,s) = \tilde{H}(t,s+1) - \tilde{H}(t,s) \leq 0 \text{ for } t > s \geq t_0.$$

If:

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[\psi(s)p_4(s)\tilde{H}(t,s)-\frac{\bar{h}^2(t,s)\psi(s+1)r_4(s+1)}{4k_1k_2\tilde{H}(t,s)\Gamma(1-\nu)}\right]=\infty,$$

where:

$$\bar{h}(t,s)=\Delta_2\tilde{H}(t,s)+\frac{\tilde{H}(t,s)(\Delta\psi_+)(s)}{\psi(s+1)},$$

then Equation (17) is oscillatory.

Motivated by the above-mentioned works, Alzabut at al. [33] investigated the oscillatory behavior of the nonlinear fractional difference equation with the damping term of the form:

$$\Delta\big(r_5(t)\big(\Delta^\nu u\big)(t)\big)+p_5(t)\big(\Delta^\nu u\big)(t)+q_7(t)f_9\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right)=0, \tag{18}$$

for $t\in\mathbb{N}_{t_0+1-\nu}$. Here, $0<\nu\leq 1$, p_5, q_7 are nonnegative sequences such that $1-p_5(t)>0$ for large t; $f_9:\mathbb{R}\to\mathbb{R}$ is a continuous function, and there exists a constant $k_3>0$ such that,

$$\frac{f_9(u)}{u}\geq k_3,$$

for all $u\neq 0$. Further, we also assume that $f_9(u_1)-f_9(u_2)=S(u_1,u_2)(u_1-u_2)$ for all u_1, $u_2\neq 0$, where S is a nonnegative function.

Theorem 21 ([33])**.** *Let $r_5(t)\equiv 1$ and:*

$$\sum_{t=t_0}^{\infty}\prod_{s=t_0}^{t-1}[1-p_5(s)]=\infty.$$

If there exists a positive sequence ϕ_1 such that:

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[k_3q_7(s)\phi_1(s)-\frac{\big[(\Delta\phi_1)(s)-p_5(s)\phi_1(s)\big]^2}{4\Gamma(1-\nu)\phi_1(s)}\right]=\infty,$$

then Equation (18) is oscillatory.

Theorem 22 ([33])**.** *Assume that $S(u_1,u_2)\geq\xi>0$ for all u_1, $u_2\neq 0$. If there exists a positive sequence ϕ_2 such that:*

$$\sum_{s=t_0}^{\infty}\frac{1}{r_5(s)\phi_2(s)}=\infty,\quad \sum_{s=t_0}^{\infty}q_7(s)\phi_2(s+1)=\infty,\quad r_5(t)(\Delta\phi_2)(t)\geq p_5(t)\phi_2(t+1),\quad t\geq t_0,$$

$$\sum_{s=t_0}^{\infty}\frac{\phi_2(s+1)p_5^2(s)}{r_5(s)}<\infty,\quad \sum_{s=t_0}^{\infty}\frac{r_5(s)\big[(\Delta\phi_2)(s)\big]^2}{\phi_2(s+1)}<\infty,$$

then Equation (18) is oscillatory.

Theorem 23 ([33])**.** *Let $r_5(t)\equiv 1$ and:*

$$\sum_{t=t_0}^{\infty}\prod_{s=t_0}^{t-1}[1-p_5(s)]=\infty.$$

Furthermore, assume that there exists a positive sequence ϕ_1 and a double positive sequence $\tilde{H}(t,s)$ such that:

$$\tilde{H}(t,t) = 0 \text{ for } t \geq t_0, \quad \tilde{H}(t,s) > 0 \text{ for } t > s \geq t_0,$$
$$\Delta_2\tilde{H}(t,s) = \tilde{H}(t,s+1) - \tilde{H}(t,s) \leq 0 \text{ for } t > s \geq t_0.$$

If:

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \frac{1}{\tilde{H}(t,t_0)} \left[k_3\phi_1(s)q_7(s)\tilde{H}(t,s) - \frac{\hat{h}^2(t,s)\phi_1^2(s+1)}{4\phi_1(s)\tilde{H}(t,s)\Gamma(1-\nu)} \right] = \infty, \tag{19}$$

where,

$$\hat{h}(t,s) = \Delta_2\tilde{H}(t,s) + \tilde{H}(t,s)\frac{[(\Delta\phi_1)(s) - p_5(s)\phi_1(s)]}{\phi_1(s+1)},$$

then Equation (18) is oscillatory.

If we set $\phi_1(t) = 1$ for all $t \geq t_0$ and,

$$H(t,s) = (t-s)^\lambda, \quad \lambda \geq 1, \quad t \geq s \geq t_0,$$

we have the following corollary.

Corollary 2 ([33]). *If the condition (19) in Theorem 23 is replaced by:*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \frac{1}{(t-t_0)^\lambda} \left[k_3q_7(s)(t-s)^\lambda - \frac{\left[\lambda(t-s-1)^{\lambda-1} + p_5(s)(t-s)^\lambda\right]^2}{4(t-s)^\lambda\Gamma(1-\nu)} \right] = \infty,$$

then Equation (18) is oscillatory.

Selvam et al. [34,35] examined the new oscillation criteria for forced delta fractional nonlinear difference equations of the form:

$$\Delta\big(r_6(t)\phi_3(u(t))(\Delta^\nu u)(t)\big) + q_8(t)f_{10}\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right) = g_2(t), \tag{20}$$

and:

$$\Delta\big(r_6(t)(\Delta^\nu u)(t)\big) + p_6(t)(\Delta^\nu u)(t) + q_8(t)f_{11}\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right) = 0, \tag{21}$$

for $t \geq t_0 > 0$. Here, $0 < \nu \leq 1$; $p_6, q_8, g_2 : [t_0,\infty) \to \mathbb{R}$ are continuous functions such that $p_6(t) < 0$, and $q_8(t) \geq 0$; $r_6 : [t_0,\infty) \to \mathbb{R}^+$ is a continuously differentiable function such that $r_6(t) \leq \lambda_1$ for some $\lambda_1 > 0$; $0 < \phi_3(u(t)) < m_1$ for some positive constant m_1 and for all $u \neq 0$:

$$f_{10}\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right) \geq 0 \text{ such that } \frac{f_{10}\left(\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right)}{\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)} \geq k_4,$$

for some positive constant k_4 and,

$$\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s) \neq 0, \quad t \geq t_0;$$

$f_{11} : \mathbb{R} \to \mathbb{R}$ is a continuous function with $u f_{11}(u) > 0$ for $u \neq 0$, and there exists a constant μ_1 such that,

$$\frac{f_{11}(u)}{u} \geq \mu_1, \quad u \neq 0.$$

Theorem 24 ([34]). *Assume that for any $L_1 \geq t_0$, there exists α_1, β_1, α_2, β_2 such that $L_1 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2$ satisfying:*

$$g_2(t) \begin{cases} \leq 0, & t \in [\alpha_1, \beta_1], \\ \geq 0, & t \in [\alpha_2, \beta_2]. \end{cases}$$

If there exists a positive function $\rho \in C^\nu[[t_0, \infty), \mathbb{R}^+]$ such that:

$$\lim_{t\to\infty} \frac{\Gamma(1-\nu)}{m_1 \lambda_1} \sum_{s=t_0}^{t-1} \frac{1}{\rho(s)} = \infty,$$

and:

$$\lim_{t\to\infty} \sum_{s=t_0}^{t-1} \left[k_4 \rho(s) q_8(s) - \frac{(\Delta^2 \rho)(s) m_1 \lambda_1}{4\rho(s)\Gamma(1-\nu)} \right] = \infty,$$

then Equation (20) is oscillatory.

Theorem 25 ([34]). *Assume that for any $L_1 \geq t_0$, there exists α_1, β_1, α_2, β_2 such that $L_1 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2$ satisfying:*

$$g_2(t) \begin{cases} \leq 0, & t \in [\alpha_1, \beta_1], \\ \geq 0, & t \in [\alpha_2, \beta_2]. \end{cases}$$

If there exists a positive function $\rho \in C^\nu[[t_0, \infty), \mathbb{R}^+]$ and a double positive sequence $\tilde{H}(t,s)$ such that:

$$\tilde{H}(t,t) = 0 \text{ for } t \geq t_0, \quad \tilde{H}(t,s) > 0 \text{ for } t > s \geq t_0,$$
$$\Delta_2 \tilde{H}(t,s) = \tilde{H}(t,s+1) - \tilde{H}(t,s) \leq 0 \text{ for } t > s \geq t_0.$$

If:

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t-1} \frac{1}{\tilde{H}(t,t_0)} \left[k_4 \rho(s) q_8(s) \tilde{H}(t,s) - \frac{h_2^2(t,s)\rho(s) m_1 \lambda_1}{4\tilde{H}(t,s)\Gamma(1-\nu)} \right] = \infty,$$

where,

$$h_2(t,s) = \Delta_2 \tilde{H}(t,s) + \tilde{H}(t,s) \frac{(\Delta\rho)(s)}{\rho(s)},$$

then Equation (20) is oscillatory.

Theorem 26 ([35]). *Assume there exists a positive function $\rho_1(t)$, $t \geq t_0$, such that:*

$$\lim_{t\to\infty} \left(\frac{\Gamma(1-\nu)}{\lambda_1} \right)^{\frac{1}{2}} \sum_{s=t_0}^{t-1} \frac{1}{\rho_1(s)} = \infty,$$

and:

$$\lim_{t\to\infty} \left\{ \left(\frac{1}{16\Gamma(1-\nu)\lambda_1} \right)^{\frac{1}{2}} \sum_{s=t_0}^{t-1} \left[\frac{p_6^2(s)\rho_1(s)}{\lambda_1} + \frac{\lambda_1 (\Delta^2 \rho_1)(s)}{\rho_1(s)} \right.\right.$$
$$\left.\left. - 2p_6(s)(\Delta\rho_1)(s) - 4\mu_1 \Gamma(1-\nu)\rho_1(s) q_8(s) \right] + \left(\frac{1}{4\Gamma(1-\nu)} \right)^{\frac{1}{2}} (\Delta\rho_1)(s) \right\} = \infty.$$

Then, Equation (21) is oscillatory.

Theorem 27 ([35]). *Assume there exists a positive function $\rho_1(t)$, $t \geq t_0$, and a double positive sequence $\tilde{H}(t,s)$ such that:*

$$\tilde{H}(t,t) = 0 \text{ for } t \geq t_0, \quad \tilde{H}(t,s) > 0 \text{ for } t > s \geq t_0,$$
$$\Delta_2 \tilde{H}(t,s) = \tilde{H}(t,s+1) - \tilde{H}(t,s) \leq 0 \text{ for } t > s \geq t_0.$$

If:

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t-1} \frac{1}{\tilde{H}(t,t_0)} \left[\mu_1 \rho_1(s) q_8(s) \tilde{H}(t,s) - \frac{h_3^2(t,s)\rho_1(s)\lambda_1}{4\tilde{H}(t,s)\Gamma(1-\nu)} \right] = \infty,$$

where,

$$h_3(t,s) = \Delta_2 \tilde{H}(t,s) + \tilde{H}(t,s) \left[\frac{(\Delta\rho_1)(s)}{\rho(s)} - \frac{p_6(s)}{\lambda_1} \right],$$

then Equation (21) is oscillatory.

Chatzarakis et al. [36] examined the oscillatory behavior for a class of nonlinear delta fractional difference equations with the damping term of the form:

$$\Delta\big(c_4(t)[\Delta(r_7(t)g_3((\Delta^\nu u)(t)))]^{\gamma_5}\big) + q_9(t)[\Delta(r_7(t)g_3((\Delta^\nu u)(t)))]^{\gamma_5} \\ + f_{12}\left(t, \sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s)\right) = 0, \quad t \in \mathbb{N}_{t_0}, \tag{22}$$

where $0 < \nu \leq 1$; c_4, r_7, $q_9 : [t_0,\infty) \to \mathbb{R}^+$ are continuous sequences with $c_4(t) > q_9(t)$; $\gamma_5 \geq 1$ is a quotient of two odd positive integers; for the continuous function $f_{12} : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$, there exists a continuously differentiable function $q_{10} : [t_0,\infty) \to \mathbb{R}^+$ such that,

$$\frac{f_{12}\left(t, \sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s)\right)}{\left[\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s)\right]^{\gamma_5}} \geq q_{10}(t),$$

for:

$$\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \neq 0, \quad u \neq 0, \quad t \geq t_0.$$

Furthermore, g_3 is an increasing function, for which there exists a constant l_1 such that,

$$\frac{u}{g_3(u)} \geq l_1 > 0, \quad u g_3(u) \neq 0.$$

$g_3^{-1} : \mathbb{R} \to \mathbb{R}$ is a continuous function with:

$$u g_3^{-1}(u) > 0, \quad u \neq 0,$$

and for that function, there exists a positive constant l_2 such that,

$$g^{-1}(u_1 u_2) \leq l_2 u_1 g_3^{-1}(u_2), \quad u_1 u_2 \neq 0.$$

Theorem 28 ([36]). *Define:*

$$y(t) = \prod_{s=t_0}^{t-1} \frac{c_4(s)}{c_4(s) - q_9(s)}.$$

Assume that u is an eventually positive solution of Equation (22) and:

$$\lim_{t\to\infty} \sum_{s=t_0}^{t-1} \frac{1}{(y(s)c_4(s))^{\frac{1}{\gamma_5}}} = \infty, \tag{23}$$

$$\lim_{t\to\infty}\sum_{s=t_0}^{t-1} g_3^{-1}\left(\frac{1}{r_7(s)}\right)=\infty, \tag{24}$$

and:

$$\lim_{t\to\infty}\sum_{\xi=t_0}^{t-1} g_3^{-1}\left(\frac{1}{r_7(\xi)}\sum_{\tau=\xi}^{\infty}\left[\frac{1}{c_4(\tau)y(\tau)}\sum_{s=\tau}^{\infty}y(s+1)q_{10}(s)\right]^{\frac{1}{\gamma_5}}\right)=\infty, \tag{25}$$

then there exists a sufficiently large $T\in\mathbb{N}_{t_0}$ such that,

$$[\Delta(r_7(t)g_3((\Delta^\nu u)(t)))]>0,\quad t\in[T,\infty),$$

and one of the following two conditions holds: (i) $(\Delta^\nu u)(t)>0$ on $[T,\infty)$ or (ii) $(\Delta^\nu u)(t)<0$ on $[T,\infty)$ and,

$$\lim_{t\to\infty}\left[\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right]=0.$$

Theorem 29 ([36]). *Assume that u is an eventually positive solution of Equation (22) such that,*

$$[\Delta(r_7(t)g_3((\Delta^\nu u)(t)))]>0,\quad (\Delta^\nu u)(t)>0,\quad t\in[t_1,\infty),$$

where t_1 is sufficiently large and $t_1\geq t_0$. Then:

$$\Delta\left[\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right] \geq \frac{l_1\Gamma(1-\nu)(y(t)c_4(t))^{\frac{1}{\gamma_5}}[\Delta(r_7(t)g_3((\Delta^\nu u)(t)))]}{r_7(t)}\sum_{s=t_1}^{t-1}\frac{1}{(y(s)c_4(s))^{\frac{1}{\gamma_5}}}.$$

Theorem 30 ([36]). *Assume that (23)–(25) hold. If:*

$$\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left[q_{10}(s)-\frac{q_9^2(s)}{4c_4^2(s)R_2(s)y(s)}\right]=\infty,$$

where t_2 is sufficiently large,

$$R_2(t)=\left[\frac{l_1\Gamma(1-\nu)}{r_7(t)}\sum_{s=t_1}^{t-1}\frac{1}{(y(s)c_4(s))^{\frac{1}{\gamma_5}}}\right]^{\gamma_5},$$

then Equation (22) is oscillatory or satisfies:

$$\lim_{t\to\infty}\left[\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right]=0.$$

Theorem 31 ([36]). *Assume that (23)–(25) hold. If:*

$$\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left[q_{10}(s)y(s)-\frac{[(\Delta y)(s)]^2}{4R_2(s)y(s)y(s+1)}\right]=\infty,$$

where t_2 is sufficiently large, then Equation (22) is oscillatory or satisfies:

$$\lim_{t\to\infty}\left[\sum_{s=t_0}^{t-1+\nu}(t-s-1)^{(-\nu)}u(s)\right]=0.$$

Theorem 32 ([36]). *Assume that* (23)–(25) *hold. Furthermore, we assume that there exists a double sequence such that:*

$$H(t,t) = 0 \text{ for } t \geq 0, \quad H(t,s) > 0 \text{ for } t > s \geq 0,$$
$$\Delta_2 H(t,s) = H(t,s+1) - H(t,s) \leq 0 \text{ for } t > s \geq 0.$$

If:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left[H(t,s) q_{10}(s) - \frac{h_4^2(t,s)}{4H(t,s)R_2(s)y(s)} \right] = \infty,$$

where,

$$h_4(t,s) = \Delta_2 H(t,s) - H(t,s)\frac{q_9(s)}{c_4(s)},$$

then Equation (22) *is oscillatory or satisfies:*

$$\lim_{t\to\infty} \left[\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \right] = 0.$$

Theorem 33 ([36]). *Assume that* (23)–(25) *hold. Furthermore, we assume that there exists a double sequence such that:*

$$H(t,t) = 0 \text{ for } t \geq 0, \quad H(t,s) > 0 \text{ for } t > s \geq 0,$$
$$\Delta_2 H(t,s) = H(t,s+1) - H(t,s) \leq 0 \text{ for } t > s \geq 0.$$

If:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left[H(t,s) q_{10}(s) y(s) - \frac{h_5^2(t,s)}{4H(t,s)R_2(s)} \right] = \infty,$$

where,

$$h_5(t,s) = \Delta_2 H(t,s) + H(t,s)\frac{(\Delta y)(s)}{y(s+1)},$$

then Equation (22) *is oscillatory or satisfies:*

$$\lim_{t\to\infty} \left[\sum_{s=t_0}^{t-1+\nu} (t-s-1)^{(-\nu)} u(s) \right] = 0.$$

Grace et al. [37] investigated the non-oscillatory solutions of the delta fractional difference equations of the following form:

$$\begin{cases} (\Delta_*^\nu v)(t) = e(t+\nu) + f(t+\nu, u(t+\nu)), & t \in \mathbb{N}_{1-\nu}, \\ v(0) = c_0, \end{cases} \tag{26}$$

where $0 < \nu \leq 1$; $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $uf(t,u) > 0$ for $u \neq 0$, and e is a positive sequence. Grace at al. carried out the investigation for the following particular cases of (26):

$$v(t) = \Delta\Big(r(t)\big|(\Delta u)(t)\big|^{\delta-1}(\Delta u)(t)\Big), \quad \delta \geq 1, \tag{27}$$
$$v(t) = (\Delta u)(t), \tag{28}$$
$$v(t) = u(t). \tag{29}$$

where r is a positive sequence.

Theorem 34 ([37]). *Consider* (26) *with* (27). *Assume that the function f satisfies:*

$$uf(t,u) \le t^{(\gamma-1)}h(t)|u|^{\beta+1}, \quad u \neq 0,$$

for some function $h : (t_1, \infty) \to \mathbb{R}^+$ and real numbers $\gamma > 0$ and $0 < \beta < \delta$. For the sake of simplification, define:

$$R(t) = \sum_{s=1}^{t-1} r^{-1/\delta}(s),$$

and,

$$g_1(t) = \sum_{s=t_1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)}(s+\nu)^{(\gamma-1)}m^{\beta/(\beta-\delta)}(s+\nu)h^{\delta/(\delta-\beta)}(s+\nu),$$

where $t_1 \in \mathbb{N}_1$ and m is a positive sequence. Let q be a conjugate number of $p > 1$, $p(\nu-1)+1 > 0$, and $\gamma = 2 - \nu - \frac{1}{p}$. Suppose that for any positive integer t_1, we have:

$$\sum_{s=t_1-\nu}^{\infty} (s+\nu)^q R^{q\delta}(s+\nu)m^q(s+\nu) < \infty, \quad \limsup_{t\to\infty} \frac{1}{t}\sum_{s=t_1}^{t-1} g_1(s) < \infty,$$

$$\liminf_{t\to\infty} \frac{1}{t}\sum_{\tau=t_1}^{t-1}\sum_{s=1-\nu}^{\tau-\nu} (\tau-s-1)^{(\nu-1)}e(s+\nu) > -\infty,$$

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=t_1}^{t-1}\sum_{s=1-\nu}^{\tau-\nu} (\tau-s-1)^{(\nu-1)}e(s+\nu) < \infty.$$

Then, every non-oscillatory solution u satisfies:

$$|u(t)| = O\Big(t^{1/\delta}R(t)\Big), \quad t \to \infty.$$

Theorem 35 ([37]). *Consider* (26) *with* (28). *Assume that the function f satisfies:*

$$uf(t,u) \le t^{(\gamma-1)}h(t)|u|^{\lambda+1}, \quad u \neq 0,$$

for some function $h : (t_1, \infty) \to \mathbb{R}^+$ and real numbers $\gamma > 0$ and $0 < \lambda < 1$. For the sake of simplification, define:

$$g_2(t) = \sum_{s=t_1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)}(s+\nu)^{(\gamma-1)}m^{\lambda/(\lambda-1)}(s+\nu)h^{1/(1-\lambda)}(s+\nu),$$

where $t_1 \in \mathbb{N}_1$ and m is a positive sequence. Let q be a conjugate number of $p > 1$, $p(\nu-1)+1 > 0$, and $\gamma = 2 - \nu - \frac{1}{p}$. Suppose that for any positive integer t_1, we have:

$$\sum_{s=t_1-\nu}^{\infty} (s+\nu)^q m^q(s+\nu) < \infty, \quad \limsup_{t\to\infty} g_2(t) < \infty,$$

$$\liminf_{t\to\infty} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)}e(s+\nu) > -\infty, \quad \limsup_{t\to\infty} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)}e(s+\nu) < \infty.$$

Then, every non-oscillatory solution u satisfies:

$$|u(t)| = O(t), \quad t \to \infty.$$

Theorem 36 ([37]). *Consider* (26) *with* (29). *Let q be a conjugate number of $p > 1$, $p(\nu-1)+1 > 0$ and $\gamma = 2-\nu-\frac{1}{p}$. Suppose that for any positive integer t_1, we have:*

$$\sum_{s=t_1-\nu}^{\infty} m^q(s+\nu) < \infty, \quad \limsup_{t\to\infty} g_2(t) < \infty,$$

$$\liminf_{t\to\infty} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} e(s+\nu) > -\infty, \quad \limsup_{t\to\infty} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} e(s+\nu) < \infty.$$

Then, every non-oscillatory solution u is bounded.

3.2. Oscillatory Behavior of Nabla Fractional Difference Equations

Let $\nu > 0$, and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu < N$. Take $p, q_i, r : \mathbb{N}_{a+N-1} \to \mathbb{R}$, $i = 1,2,\cdots n$; $f_1, f_2 : \mathbb{N}_{a+N-1} \times \mathbb{R} \to \mathbb{R}$; $f : \mathbb{R} \to \mathbb{R}$; $p_1, p_2 : \mathbb{N}_{a+N-1} \to \mathbb{R}^+$; $w, h : \mathbb{N}_a \to \mathbb{R}$; $q, g : \mathbb{N}_{a+1} \to \mathbb{R}$; $x, z : \mathbb{N}_1 \to \mathbb{R}$; $w_2 : \mathbb{N}_1 \to \mathbb{R}^+$; y is a positive function defined on $\mathbb{N}_1$; β, γ are positive real numbers; λ_i $(1 \le i \le n)$ are the ratios of odd positive integers with $\lambda_1 > \cdots > \lambda_l > 1 > \lambda_{l+1} > \cdots > \lambda_n$.

We make the following assumptions:

(H1). *The functions f_i satisfy the sign condition $uf_i(t,u) > 0$, $i = 1,2$, $u \neq 0$, $t \in \mathbb{N}_{a+N-1}$;*

(H2). $|f_1(t,u(t)| \ge p_1(t)|u|^\beta$ and $|f_2(t,u(t)| \le p_2(t)|u|^\gamma$, $u \neq 0$, $t \in \mathbb{N}_{a+N-1}$;

(H3). $|f_1(t,u(t)| \le p_1(t)|u|^\beta$ and $|f_2(t,u(t)| \ge p_2(t)|u|^\gamma$, $u \neq 0$, $t \in \mathbb{N}_{a+N-1}$;

(H4). $\frac{f(t)}{t} > 0$ *for all $t \neq 0$ and $x(t) < 1$ for all $t \in \mathbb{N}_1$;*

(H5). *$uf(u) > 0$ for $u \neq 0$ and $q(t) \ge 0$ for all $t \in \mathbb{N}_{a+1}$;*

(H6). *$uf(u) > 0$ for $u \neq 0$ and $w(t) \ge 0$ for all $t \in \mathbb{N}_a$.*

Alzabut et al. [38] initiated the study of the oscillation of solutions of nabla fractional difference equations. In [38], the authors established several oscillation criteria for the following nonlinear nabla fractional difference equations involving the Riemann–Liouville and Caputo operators of arbitrary order.

$$\begin{cases} (\nabla^{\nu}_{a+N-2}u)(t) + f_1(t,u(t)) = r(t) + f_2(t,u(t)), & t \in \mathbb{N}_{a+N-1}, \\ (\nabla^{-(1-\nu)}_{a+N-2}u)(t)\Big|_{t=a+N-1} = u(a+N-1) = c, & c \in \mathbb{R}, \end{cases} \tag{30}$$

and:

$$\begin{cases} (\nabla^{\nu}_{a+N-1*}u)(t) + f_1(t,u(t)) = r(t) + f_2(t,u(t)), & t \in \mathbb{N}_{a+N-1}, \\ (\nabla^k u(a+N-1) = b_k, \quad b_k \in \mathbb{R}, & k = 0,1,2,\cdots,N-1. \end{cases} \tag{31}$$

A solution u of (30) (or (31)) is said to be oscillatory if for every natural number M, there exists $t \ge M$ such that $u(t)u(t+1) \le 0$; otherwise, it is called non-oscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

Theorem 37 ([38]). *Let $f_2 = 0$ and Condition (H1) hold. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} r(s) = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} r(s) = \infty,$$

then Equation (30) is oscillatory.

Theorem 38 ([38])**.** *Let Conditions (H1) and (H2) hold with $\beta > 1$ and $\gamma = 1$. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + H_\beta(s)\right] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + H_\beta(s)\right] = \infty,$$

where,

$$H_\beta(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s),$$

then Equation (30) is oscillatory.

Theorem 39 ([38])**.** *Let Conditions (H1) and (H2) hold with $\beta = 1$ and $\gamma < 1$. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\gamma(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\gamma(s)] = \infty,$$

where,

$$H_\gamma(s) = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s),$$

then Equation (30) is oscillatory.

Theorem 40 ([38])**.** *Let Conditions (H1) and (H2) hold with $\beta > 1$ and $\gamma < 1$. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + H_{\beta,\gamma}(s)\right] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + H_{\beta,\gamma}(s)\right] = \infty,$$

where,

$$H_{\beta,\gamma}(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s)\xi^{\frac{\beta}{\beta-1}}(s) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s),$$

with $\xi : \mathbb{N}_{a+N-1} \to \mathbb{R}^+$, then Equation (30) is oscillatory.

Theorem 41 ([38])**.** *Let $f_2 = 0$ and Condition (H1) hold. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} r(s) = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} r(s) = \infty,$$

then Equation (31) *is oscillatory.*

Theorem 42 ([38]). *Let Conditions (H1) and (H2) hold with $\beta > 1$ and $\gamma = 1$. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\beta(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\beta(s)] = \infty,$$

where,

$$H_\beta(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s),$$

then Equation (31) *is oscillatory.*

Theorem 43 ([38]). *Let Conditions (H1) and (H2) hold with $\beta = 1$ and $\gamma < 1$. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\gamma(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_\gamma(s)] = \infty,$$

where,

$$H_\gamma(s) = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s),$$

then Equation (31) *is oscillatory.*

Theorem 44 ([38]). *Let Conditions (H1) and (H2) hold with $\beta > 1$ and $\gamma < 1$. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_{\beta,\gamma}(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N-1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H_{\beta,\gamma}(s)] = \infty,$$

where,

$$H_{\beta,\gamma}(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(s)\xi^{\frac{\beta}{\beta-1}}(s) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}\xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s),$$

with $\xi : \mathbb{N}_{a+N-1} \to \mathbb{R}^+$, then Equation (31) *is oscillatory.*

Following the work in [38], Abdalla et al. [39] established new oscillation criteria for (30) and (31) using the fractional Volterra sum equations and Young's inequalities. The authors in [39] observed that the cases $\beta > \gamma > 1$ and $\gamma > \beta > 1$ were not considered for (30) in [38]. The purpose of the paper [39] was to cover this gap and establish new oscillation criteria that improve the results in [38].

Theorem 45 ([39]). *Let Condition (H2) hold with $\beta > \gamma > 0$. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}} [r(s) + H(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)-H(s)] = \infty,$$

for sufficiently large T, where,

$$H(s) = \left(\frac{\beta}{\gamma}-1\right)\left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}} p_1^{\frac{\gamma}{\gamma-\beta}}(s),$$

then Equation (30) is oscillatory.

Theorem 46 ([39]). *Let $\nu \geq 1$ and Condition (H3) hold with $\gamma > \beta > 0$. If:*

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)-H(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)+H(s)] = \infty,$$

for sufficiently large T, where H is defined in Theorem 45, then every bounded solution of Equation (30) is oscillatory.

Theorem 47 ([39]). *Let Condition (H2) hold with $\beta > \gamma > 0$. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)+H(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)-H(s)] = \infty,$$

for sufficiently large T, where H is defined in Theorem 45, then Equation (31) is oscillatory.

Theorem 48 ([39]). *Let $\nu \geq 1$ and Condition (H3) hold with $\gamma > \beta > 0$. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)-H(s)] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=T+1}^{t} (t-s+1)^{\overline{\nu-1}}[r(s)+H(s)] = \infty,$$

for sufficiently large T, where H is defined in Theorem 45, then every bounded solution of Equation (31) is oscillatory.

In alignment with the above works, Abdalla et al. [40] investigated the oscillation of solutions for nabla fractional difference equations with mixed nonlinearities of the forms:

$$\begin{cases} (\nabla^{\nu}_{a+N-2}u)(t) - p(t)u(t) + \sum_{i=1}^{n} q_i(t)|u(t)|^{\lambda_i-1} = r(t), & t \in \mathbb{N}_{a+N}, \\ (\nabla^{-(N-\nu)}_{a+N-2}u)(t)\Big|_{t=a+N-1} = u(a+N-1) = c, & c \in \mathbb{R}, \end{cases} \tag{32}$$

and:

$$\begin{cases} (\nabla^{\nu}_{a+N-1*}u)(t) - p(t)u(t) + \sum_{i=1}^{n} q_i(t)|u(t)|^{\lambda_i - 1} = r(t), \quad t \in \mathbb{N}_{a+N-1}, \\ (\nabla^k u(a+N-1) = b_k, \quad b_k \in \mathbb{R}, \quad k = 0,1,2,\cdots,N-1. \end{cases} \tag{33}$$

Theorem 49 ([40]). *Let:*

$$p(t) > 0 \text{ and } q_i(t) \begin{cases} \geq 0, & 1 \leq i \leq l; \\ \leq 0, & l+1 \leq i \leq n. \end{cases} \tag{34}$$

If for some constant $K > 0$, we have:

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = -\infty, \tag{35}$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = \infty, \tag{36}$$

then Equation (32) is oscillatory.

Corollary 3 ([40]). *Let $l = n$ in (32), then $\lambda_1 > \lambda_2 > \cdots \lambda_n > 1$. Suppose $p(t) > 0$, $q_i(t) \geq 0$, $i = 1,2,\cdots n$. If (35) and (36) hold for some constant $K_1 > 0$, then Equation (32) is oscillatory.*

Corollary 4 ([40]). *Let $l = 0$ in (32), then $1 > \lambda_1 > \lambda_2 > \cdots \lambda_n$. Suppose $p(t) < 0$, $q_i(t) \leq 0$, $i = 1,2,\cdots n$. If (35) and (36) hold for some constant $K_2 > 0$, then Equation (32) is oscillatory.*

Corollary 5 ([40]). *Let:*

$$p(t) \equiv 0 \text{ and } q_i(t) \begin{cases} \geq 0, & 1 \leq i \leq l; \\ \leq 0, & l+1 \leq i \leq n. \end{cases} \tag{37}$$

If there exists a positive function v on $\mathbb{N}_{a+N-1}$ such that for some constant $K_3 > 0$, we have:

$$\liminf_{t\to\infty} t^{1-\nu} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} v^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-\nu} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} v^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = \infty,$$

then Equation (32) is oscillatory.

Theorem 50 ([40]). *Assume that Condition (34) holds. If:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = -\infty, \tag{38}$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} p^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = \infty, \tag{39}$$

for some constant $K > 0$, then Equation (33) is oscillatory.

Corollary 6 ([40]). *Suppose $p(t) > 0$, $q_i(t) \geq 0$, $i = 1,2,\cdots n$. If (38) and (39) hold for some constant $K_1 > 0$, then Equation (33) is oscillatory.*

Corollary 7 ([40]). *Suppose $p(t) < 0$, $q_i(t) \leq 0$, $i = 1, 2, \cdots n$. If (38) and (39) hold for some constant $K_2 > 0$, then Equation (33) is oscillatory.*

Corollary 8 ([40]). *Let (37) hold. If there exists a positive function v on $\mathbb{N}_{a+N-1}$ such that for some constant $K_3 > 0$, we have:*

$$\liminf_{t\to\infty} t^{1-N} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} v^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = -\infty,$$

and:

$$\limsup_{t\to\infty} t^{1-N} \sum_{s=a+N}^{t} (t-s+1)^{\overline{\nu-1}} \left[r(s) + K \sum_{i=1}^{n} v^{\frac{\lambda_i}{\lambda_i - 1}}(s) |q_i(s)|^{\frac{1}{1-\lambda_i}} \right] = \infty,$$

then Equation (32) is oscillatory.

Following the above trend, in [41], Alzabut et al. considered the following forced and damped nabla fractional difference equation:

$$\begin{cases} (1-x(t))\big(\nabla\nabla_0^{\nu} u\big)(t) + x(t)\big(\nabla_0^{\nu} u\big)(t) + w_2(t) f(u(t)) = z(t), & t \in \mathbb{N}_1, \\ \big(\nabla_0^{-(1-\nu)} u\big)(t)\Big|_{t=1} = u(1) = c, \quad c \in \mathbb{R}, \end{cases} \tag{40}$$

where $0 < \nu < 1$, and established sufficient conditions for the oscillation of the solutions of Equation (40).

Theorem 51 ([41]). *Let Assumption (H5) and the following conditions hold:*

$$\liminf_{t\to\infty} \sum_{s=1}^{t} \frac{(t-s+1)^{\overline{\nu-1}}}{P(s)} \left[A + \sum_{\tau=t_0+1}^{s} z(\tau) P(\tau) \right] < 0,$$

and:

$$\limsup_{t\to\infty} \sum_{s=1}^{t} \frac{(t-s+1)^{\overline{\nu-1}}}{P(s)} \left[A + \sum_{\tau=t_0+1}^{s} z(\tau) P(\tau) \right] > 0,$$

where A is a constant and,

$$P(t) = \prod_{s=t_0}^{t} \left(\frac{1}{1-x(s)} \right), \quad t_0 \in \mathbb{N}_1.$$

Then, Equation (40) is oscillatory.

Theorem 52 ([41]). *Let Assumption (H5) and the following conditions hold:*

$$\liminf_{t\to\infty} \sum_{s=1}^{t} \frac{1}{P(s)} \left[A + \sum_{\tau=t_0+1}^{s} z(\tau) P(\tau) \right] = -\infty,$$

and:

$$\limsup_{t\to\infty} \sum_{s=1}^{t} \frac{(t-s+1)^{\overline{\nu-1}}}{P(s)} \left[A + \sum_{\tau=t_0+1}^{s} z(\tau) P(\tau) \right] = \infty,$$

where A is a constant and,

$$P(t) = \prod_{s=t_0}^{t} \left(\frac{1}{1-x(s)} \right), \quad t_0 \in \mathbb{N}_1.$$

Then, Equation (40) is oscillatory.

Motivated by the paper [38], the authors [42] investigated the oscillation of a nonlinear fractional nabla difference system of the form:

$$\begin{cases} (\nabla_a^\nu u)(t) + q(t)f(u(t)) = g(t), & t \in \mathbb{N}_{a+1}, \\ (\nabla_a^{-(1-\nu)} u)(t)\Big|_{t=a} = u(a) = c, & c \in \mathbb{R}, \end{cases} \tag{41}$$

where $0 < \nu < 1$, and obtained some sufficient conditions for oscillation.

Theorem 53 ([42])**.** *Let Condition (H5) hold. If:*

$$\liminf_{t\to\infty} (t-a)^{1-\nu} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\nu-1}} g(s) = -\infty,$$

and:

$$\limsup_{t\to\infty} (t-a)^{1-\nu} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\nu-1}} g(s) = \infty,$$

then Equation (41) *is oscillatory.*

Theorem 54 ([42])**.** *Let Condition (H5) hold. Assume that there exists $t_0 \in \mathbb{N}_{a+1}$ such that:*

$$\liminf_{t\to\infty} \sum_{s=t_0+1}^{t} g(s) = -\infty,$$

and:

$$\limsup_{t\to\infty} \sum_{s=t_0+1}^{t} g(s) = \infty,$$

then Equation (41) *is oscillatory.*

In [43], the authors investigated the oscillation of fractional nabla difference equations of the form:

$$\begin{cases} (\nabla\nabla_a^\nu u)(t) + w(t)f(u(t)) = h(t), & t \in \mathbb{N}_{a}, \\ (\nabla_a^{-(1-\nu)} u)(t)\Big|_{t=a} = u(a) = c, & c \in \mathbb{R}, \end{cases} \tag{42}$$

where $0 < \nu < 1$.

Theorem 55 ([43])**.** *Let Condition (H6) hold. If the inequality:*

$$(\nabla\nabla_a^\nu u)(t) \le h(t), \quad t \in \mathbb{N}_a,$$

has no eventually positive solutions and the inequality:

$$(\nabla\nabla_a^\nu u)(t) \ge h(t), \quad t \in \mathbb{N}_a,$$

has no eventually negative solutions, then every solution u of Equation (42) *is oscillatory.*

Theorem 56 ([43])**.** *Let condition (H6) be hold. Assume that u is a solution of* (42) *and there exists $t_0 \in \mathbb{N}_a$ such that $(\nabla_a^\nu u)(t)\Big|_{t=t_0} = C$ exists. If:*

$$\liminf_{t\to\infty} (t-a)^{1-\nu} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\nu-1}} \left[C + \sum_{\tau=t_0+1}^{s} h(\tau) \right] = -\infty,$$

and:

$$\limsup_{t\to\infty}(t-a)^{1-\nu}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\nu-1}}\left[C+\sum_{\tau=t_0+1}^{s}h(\tau)\right]=\infty,$$

then, Equation (42) is oscillatory.

Theorem 57 ([42]). *Let condition (H6) be hold. Assume that u is a solution of (42) and there exists $t_0 \in \mathbb{N}_a$ such that $\left(\nabla_a^{\nu}u\right)(t)\Big|_{t=t_0} = C$ exists. If:*

$$\liminf_{t\to\infty}\sum_{s=t_0+1}^{t}\left(1-\frac{s-1}{t}\right)h(s)=-\infty,$$

and:

$$\limsup_{t\to\infty}\sum_{s=t_0+1}^{t}\left(1-\frac{s-1}{t}\right)h(s)=\infty,$$

then, Equation (42) is oscillatory.

4. Conclusions

The oscillation of difference equations has been a considerable topic due to its widespread applications in science and engineering. For this purpose, many researchers have contributed to this topic by studying several types of equations. With the rise of fractional calculus, the oscillation of fractional difference equations has become the object of an extensive investigation, and consequently, distinguishable results have been elaborated during the recent years.

In this paper, we presented a scientific platform that provided a comprehensive survey on the recent developments for the oscillation results of fractional difference equations. Different types of equations were investigated and presented by using both the nabla and delta operators. We believe that the results presented in this paper will provide a cornerstone literature for the relevant audience that is interested in the investigation of oscillation theory. The theoretical presentation in this paper is promising in the sense that it can be used to develop results for the oscillation of solutions for other types of equations such as the functional dynamic equations and fuzzy dynamic equations.

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