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Extended Beta and Gamma Matrix Functions via 2-Parameter Mittag-Leffler Matrix Function

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Abstract: The main aim of this article is to study an extension of the Beta and Gamma matrix functions by using a two-parameter Mittag-Leffler matrix function. In particular, we investigate certain properties of these extended matrix functions such as symmetric relation, integral representations, summation relations, generating relation and functional relation.

Keywords: matrix functional calculus; Mittag-Leffler matrix function; Gamma matrix function; Beta matrix function

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1. Introduction and Preliminaries

The theory of special matrix functions has been introduced by Jó and Cortés [1,2] and they proved some basic and important properties of the Gamma and Beta matrix functions and a limit expression for the Gamma function of a matrix. After some time using these results, they also studied the hypergeometric function with matrix arguments. Motivated by the investigations of the extended Gamma, Beta, and Gauss hypergeometric matrix functions given in [1,2] many researchers [3–7] were attracted to work in the field of special functions with matrix arguments. Its study has become important both from a theoretical and applied point of view. Special matrix functions are found in the solutions for some physical problems and applications of these functions also increased in the statistics [8], Lie group theory [9] and differential equations. In mathematical physics, theoretical physics and the emerging theory of orthogonal matrix polynomials [10–12], these special matrix functions play a vital role due to their applications. Matrix extensions of some known classical special functions have become an important topic of research during recent years. To discuss our main results we require prior knowledge of some special matrix functions.

Throughout the paper, let I and O denote the Identity matrix and Zero matrix in $C^{r\times r}$, respectively, and where $C^{r\times r}$ is vector space of r-square matrices with complex entries. For a matrix $B \in C^{r\times r}$, the spectrum is denoted by $\sigma(B)$ and it is the set of all eigenvalues of the matrix B. A matrix $B \in C^{r\times r}$ is a positive stable matrix if $\Re(\mu) > 0 \;\forall\; \mu \in \sigma(B)$.

Gamma matrix function is defined as [1]:

$$\Gamma(B) = \int_0^\infty e^{-t} t^{B-I} dt, \tag{1}$$

here, B is a positive stable matrix in $C^{r\times r}$.

In addition, if $B + kI$ is invertible matrix $\forall\, k \geq 0$, then the reciprocal Gamma matrix function is defined as [1]:

$$\Gamma^{-1}(B) = B(B+I)\ldots(B+(n-1)I)\Gamma^{-1}(B+nI), n \geq 1. \tag{2}$$

Pochammer Matrix symbol is defined as [2]:

$$(B)_n = \begin{cases} I, & if\ n = 0, \\ B(B+I)...(B+(n-1)I), & if\ n \geq 1. \end{cases} \tag{3}$$

From the above definition of Pochammer Matrix symbol and Equation (2), we can observe that:

$$(B)_n = \Gamma^{-1}(B)\Gamma(B+nI),\ n \geq 1. \tag{4}$$

Beta matrix function is defined as [1]:

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}dt, \tag{5}$$

here, P and Q are positive stable matrices in $C^{r\times r}$.

In addition, if P, Q and $P + Q$ are positive stable matrices in $C^{r\times r}$ and P and Q are commutative with each other, i.e., $PQ = QP$, then

$$B(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{6}$$

In 2013, Çekim [3], studied generalizations of Euler's Beta, Gamma, hypergeometric and confluent hypergeometric matrix functions, and recurrence relations and integral representations are derived for these extended matrix functions. Some properties such as transformation formulas, recurrence relations and integral representations are obtained for extended matrix functions.

Generalised Gamma and Beta matrix functions are given as [3] :

$$\begin{aligned} \Gamma(P,r) &= \int_0^\infty t^{P-I}e^{(-t-\frac{r}{t})}dt, \\ B(P,Q;r) &= \int_0^1 t^{P-I}(1-t)^{Q-I}e^{\left(\frac{-r}{t(1-t)}\right)}dt, \end{aligned} \tag{7}$$

here, P, Q, and R are positive stable matrices in $C^{r\times r}$ and $r > 0$.

Later in 2014, Altin et al. [7] studied the matrix version of the Appell hypergeometric functions and its properties. They also derived matrix differential equations, and integral representation satisfied the F_1 matrix function.

In 2016, motivated by the above work, Abdalla et al. [4] extended the Gamma and Beta matrix functions and studied some important properties.

Extended Gamma matrix function is defined as [4]:

$$\Gamma_R(B) = \int_0^\infty t^{B-I}e^{(-tI-\frac{R}{t})}dt, \tag{8}$$

here, B and R are positive stable matrices in $C^{r\times r}$.

Extended Beta matrix function is defined as [4]:

$$B(P,Q;R) = \int_0^1 t^{P-I}(1-t)^{Q-I}e^{\left(\frac{-R}{t(1-t)}\right)}dt, \tag{9}$$

here, P, Q, and R are positive stable matrices in $C^{r\times r}$.

Remark 1. *If we set $R = O$ in above Equations (8) and (9), then we obtain Gamma matrix function (1), and Beta matrix function (5), respectively:*

$$\Gamma_O(B) = \Gamma(B), \tag{10}$$

and

$$B(P, Q; O) = B(P, Q). \tag{11}$$

In the above extensions, Abdalla et al. have used an exponential matrix function as a regulator to extend Gamma matrix function and Beta matrix function. They have introduced the exponential matrix function as a kernel in definitions of Gamma matrix function and Beta matrix function.

In 2018, Garrappa et al. [5] computed Mittag-Leffler function with matrix arguments, with some applications in fractional calculus.

Two-parameter Mittag-Leffler matrix function is defined as [5]:

$$E_{(\alpha_1,\alpha_2)}(B) = \sum_{n=0}^{\infty} \frac{B^n}{\Gamma(\alpha_1 n + \alpha_2)}, \tag{12}$$

here, B is a positive stable matrix in $C^{r\times r}$ and $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Very recently, Verma and Dwivedi [13], inspired by certain extensions of the special matrix functions, introduced new extensions of Gamma and Beta matrix function by using a confluent hypergeometric matrix function.

$$\begin{aligned} \Gamma^{(R)}_{P,Q}(X) &= \int_0^\infty t^{X-I} {}_1F_1(P, Q; -tI - Rt^{-1})dt, \\ B^{(R)}_{P,Q}(X, Y) &= \int_0^1 t^{X-I}(1-t)^{Y-I} {}_1F_1\left(P, Q; \frac{-R}{t(1-t)}\right)dt, \end{aligned} \tag{13}$$

here, P, Q, and R are positive stable matrices in $C^{r\times r}$.

2. Main Results

In this section, inspired and motivated by the above certain extensions of the special matrix functions, we introduce new extensions of Gamma and Beta matrix functions by using a two-parameter Mittag-Leffler matrix function as a kernel in integral representations of these functions, and we discuss some basic properties of these extended matrix functions.

Definition 1. *Let P and R be positive stable matrices in $C^{r\times r}$, then, the new extension of Gamma function is defined as:*

$$\Gamma^{(R)}_{\alpha_1,\alpha_2}(P) = \int_0^\infty t^{P-I} E_{(\alpha_1,\alpha_2)}\left(-tI - \frac{R}{t}\right)dt, \tag{14}$$

here, $\Re(\alpha_1), \Re(\alpha_2) > 0$ and $E_{(\alpha_1,\alpha_2)}(B)$ is two-parameter Mittag-Leffler matrix function.

Definition 2. *Let P, Q and R be positive stable matrices in $C^{r\times r}$, then the new extension of Beta matrix function is defined as:*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt, \tag{15}$$

here, $\Re(\alpha_1), \Re(\alpha_2) > 0$ and $E_{(\alpha_1,\alpha_2)}(B)$ is two-parameter Mittag-Leffler matrix function.

Remark 2. *(i) If we set $\alpha_1 = \alpha_2 = 1$ in Equations (14) and (15), then we obtain the extended Gamma matrix function (8), and extended Beta matrix function (9), respectively:*

$$\Gamma^{(R)}_{1,1}(P) = \Gamma_R(P), \tag{16}$$

and

$$B^{(R)}_{(1,1)}(P,Q) = B(P,Q;R). \tag{17}$$

(ii) If we take $\alpha_1 = \alpha_2 = 1$ and $R = O$ in Equations (14) and (15), then we obtain the Gamma matrix function (1), and Beta matrix function (5), respectively:

$$\Gamma^{(O)}_{1,1}(P) = \Gamma(P), \tag{18}$$

and

$$B^{(O)}_{(1,1)}(P,Q) = B(P,Q). \tag{19}$$

In [1], it has been shown that if P and Q are positive stable matrices in $C^{r\times r}$ such that if they do not commute, then the property $B(P,Q) = B(Q,P)$ does not hold true, with a similar argument, the following result are easy to prove.

Theorem 1. *For the positive stable matrices P, Q and R such that $PQ = QP$, the new extended Beta matrix function satisfies the symmetric relation*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = B^{(R)}_{(\alpha_1,\alpha_2)}(Q,P), \tag{20}$$

provided $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. From the definition of new extended Beta matrix function, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right) dt. \tag{21}$$

Then, putting $t = 1-u$ in above equation, we obtain:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \int_0^1 u^{Q-I}(1-u)^{P-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{u(1-u)}\right) du. \tag{22}$$

Then, from Equation (15), we obtain our desired result.

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = B^{(R)}_{(\alpha_1,\alpha_2)}(Q,P). \tag{23}$$

Hence, the proof of Theorem 1 is completed. □

Theorem 2. *For the positive stable matrices P, Q and R, the new extended Beta matrix function satisfies the functional relation*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q+I) + B^{(R)}_{(\alpha_1,\alpha_2)}(P+I,Q), \tag{24}$$

provided $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. To prove our result, consider the right-hand side of the Equation (24), and using the definition of the new extended Beta function, we obtain:

$$\begin{aligned} &B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q+I) + B^{(R)}_{(\alpha_1,\alpha_2)}(P+I,Q) = \\ &\int_0^1 t^{P-I}(1-t)^{Q} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right) dt + \int_0^1 t^{P}(1-t)^{Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right) dt. \end{aligned} \tag{25}$$

Then, after re-arrangements of terms, we have:

$$\begin{aligned} &B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q+I) + B^{(R)}_{(\alpha_1,\alpha_2)}(P+I,Q) = \\ &\qquad \int_0^1 \left[t^{-I} + (1-t)^{-I}\right] t^{P}(1-t)^{Q} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right) dt. \end{aligned} \tag{26}$$

On some calculations, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q+I)+B^{(R)}_{(\alpha_1,\alpha_2)}(P+I,Q)=\int_0^1 t^{P-I}(1-t)^{Q-I}E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{27}$$

Then, from the definition of the new extended Beta matrix function (15), we obtain our desired result.

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q+I)+B^{(R)}_{(\alpha_1,\alpha_2)}(P+I,Q). \tag{28}$$

Hence, the proof of Theorem 2 is completed. □

Theorem 3. *For the positive stable matrices P, Q and R, the new extended Beta matrix function satisfies the summation relation*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\sum_{n=0}^{\infty}B^{(R)}_{(\alpha_1,\alpha_2)}(P+nI,Q+I) \tag{29}$$

provided $\Re(\alpha_1),\Re(\alpha_2)>0$.

Proof. From the definition of the new extended Beta matrix function, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q-I}E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{30}$$

Then, after re-arrangement, we obtain:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q}(1-t)^{-I}E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{31}$$

Then, using matrix identity $(1-t)^{-I}=\sum_{n=0}^{\infty}t^{nI}$ in above equation, we obtain:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q}\left(\sum_{n=0}^{\infty}t^{nI}\right)E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{32}$$

On changing the order of integration and summation and re-arranging the terms, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\sum_{n=0}^{\infty}\int_0^1 t^{P+nI-I}(1-t)^{Q}E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{33}$$

Then, from definition of the new extended Beta matrix function (15), we obtain our desired result.

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q)=\sum_{n=0}^{\infty}B^{(R)}_{(r_1,r_2)}(P+nI,Q+I). \tag{34}$$

Hence, the proof of Theorem 3 is completed. □

Theorem 4. *For the positive stable matrices P, Q, R, and I-Q, the new extended Beta matrix function satisfies the another summation relation*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,I-Q)=\sum_{n=0}^{\infty}\frac{(Q)_n}{n!}B^{(R)}_{(\alpha_1,\alpha_2)}(P+nI,I) \tag{35}$$

provided $\Re(\alpha_1),\Re(\alpha_2)>0$.

Proof. From the definition of the new extended Beta matrix function, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, I-Q) = \int_0^1 t^{P-I}(1-t)^{-Q} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{36}$$

Then, using matrix identity $(1-t)^{-Q} = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} t^{nI}$ in above equation, we obtain:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, I-Q) = \int_0^1 t^{P-I}\left(\sum_{n=0}^{\infty} \frac{(Q)_n}{n!} t^{nI}\right) E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{37}$$

On interchanging the order of integration and summation and re-arranging the terms, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, I-Q) = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} \int_0^1 t^{P+nI-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right)dt. \tag{38}$$

Then, from definition of the new extended Beta matrix function (15), we obtain our desired result.

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, I-Q) = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} B^{(R)}_{(\alpha_1,\alpha_2)}(P+nI, I). \tag{39}$$

Hence, the proof of Theorem 4 is completed. □

Theorem 5. *For the positive stable matrices P, Q and R, the new extended Beta matrix function satisfies the another integral representation as follows:*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, Q) = 2\int_0^{\frac{\pi}{2}} (\cos(x))^{2P-I}(\sin(x))^{2Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{\cos^2(x)\sin^2(x)}\right)dx, \tag{40}$$

provided $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. On substituting $t = \cos^2(x)$ and using the definition of the new extended Beta matrix function (15), we obtain our desired result of Theorem 5. □

Theorem 6. *For the positive stable matrices P, Q and R, the new extended Beta matrix function satisfies the another integral representation as follows:*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, Q) = 2\int_0^{\frac{\pi}{2}} (\sin(x))^{2P-I}(\cos(x))^{2Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{\cos^2(x)\sin^2(x)}\right)dx, \tag{41}$$

provided $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. On taking $t = \sin^2(x)$ and using the definition of new extended Beta matrix function (15), we obtain our desired result of Theorem 6. □

Theorem 7. *For the positive stable matrices P, Q and R, the new extended Beta matrix function satisfies the another integral representation as follows:*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P, Q) = \int_0^{\infty} \frac{u^{P-I}}{(1+u)^{P+Q}} E_{(\alpha_1,\alpha_2)}\left(-2R - R\left(u + \frac{1}{u}\right)\right)du, \tag{42}$$

provided $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. On letting $t = \frac{u}{1+u}$ and using the definition of the new extended Beta matrix function (15), we obtain our desired result of Theorem 7. □

Theorem 8. *Generating relation for the new extended Beta matrix function*

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \sum_{n=0}^{\infty} \frac{(-R)^n}{\Gamma(\alpha_1 n + \alpha_2)} B(P - nI, Q - nI), \tag{43}$$

provided, P, Q and R are positive stable matrices and $\Re(\alpha_1), \Re(\alpha_2) > 0$.

Proof. From the definition of the new extended Beta matrix function, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} E_{(\alpha_1,\alpha_2)}\left(\frac{-R}{t(1-t)}\right) dt. \tag{44}$$

Then, using the definition of two-parameter Mittag-Leffler function (12) in the above equation, we obtain:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} \left(\sum_{n=0}^{\infty} \frac{(-R)^n}{t^n(1-t)^n \Gamma(\alpha_1 n + \alpha_2)}\right) dt. \tag{45}$$

On interchanging the order of integration and summation, we have:

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \sum_{n=0}^{\infty} \frac{(-R)^n}{\Gamma(\alpha_1 n + \alpha_1)} \int_0^1 t^{P-nI-I}(1-t)^{Q-nI-I} dt. \tag{46}$$

After using the definition of the new extended Beta matrix function (15), we obtain our desired result.

$$B^{(R)}_{(\alpha_1,\alpha_2)}(P,Q) = \sum_{n=0}^{\infty} \frac{(-R)^n}{\Gamma(\alpha_1 n + \alpha_2)} B(P - nI, Q - nI). \tag{47}$$

Hence, the proof of Theorem 8 is completed. □

3. Concluding Remarks

We conclude our analysis by remarking that the results presented in this paper are new and potentially useful. First, we have introduced new extensions of Gamma and Beta matrix functions by using the two-parameter Mittag-Leffler matrix function. Then, we investigated some important properties of these extended matrix functions. The results presented in this paper find an interesting application in the evaluation of certain infinite integrals whose specialised forms arise frequently in a number of applied problems.

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