mathematics

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Geometry of Solutions of the Quasi-Vortex Filament Equation in Euclidean 3-Space $\mathbb{E}^3$

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Abstract: This work aims at investigating the geometry of surfaces corresponding to the geometry of solutions of the vortex filament equation in Euclidean 3-space $\mathbb{E}^3$ using the quasi-frame. In particular, we discuss some geometric properties and some characterizations of parameter curves of these surfaces in $\mathbb{E}^3$.

Keywords: vortex filament equation; Hasimoto surface; quasi-frame; Euclidean space

Citation: Hamouda, E.; Moaaz, O.; Cesarano, C.; Askar S.S.; Elsharkawy, A. Geometry of Solutions of the Quasi-Vortex Filament Equation in Euclidean 3-Space $\mathbb{E}^3$. *Mathematics* **2022**, *10*, 891. https://doi.org/10.3390/math10060891

Academic Editor: Jean-Charles Pinoli

Received: 31 January 2022
Accepted: 7 March 2022
Published: 10 March 2022

1. Introduction

The aim of this article is the study of geometric flow solutions

$$\Phi_t = \Phi_s \times \Phi_{ss}, \tag{1}$$

where Φ is a curve in 3D Euclidean space parameterized by its arc length s. This equation was first considered more than a century ago by Da Rios [1,2] and rederived in 1965 by Arms and Hama [3,4] as a simplified model for dynamics of a vortex filament in an inviscid incompressible fluid. Equation (1) can be equivalently rewritten as

$$\Phi_t = \kappa \mathbf{B},$$

where κ is the curvature of Φ and $\mathbf{B}$ the binormal vector field of the Frenet frame. The other form, called the Schrödinger map onto the sphere, is obtained by differentiating (1) with respect to s and rewriting the result in terms of the tangent vector $\mathbf{T}$

$$\mathbf{T}_t = \mathbf{T} \times \mathbf{T}_{ss},$$

which is a particular case of the Landau–Lifshitz equation for ferromagnetism [5]. It is also known as the vortex filament equation (VFE), or the localized induction approximation (LIA), or the binormal Equation (BE). Some of their explicit solutions are the line, the circle, and the helix.

Many mathematicians have been interested in studying connections between geometric motions of a curve in different spaces and integrable non-linear equations (non-linear soliton equations)—in particular, a connection between the non-linear Schrodinger equation and the motion of a thin vortex filament in an incompressible fluid; see [6–10]. In soliton theory, integrable non-linear equations has an important role in the surfaces construction, in particular, Hasimoto surface that owes its name to the Japanese physicist Hidenori Hasimoto.

A quasi-Hasimoto surface is a surface whose parameter curves are equipped with the quasi-frame. It is defined in (1). This surface is generated by the vortex filament. In 2012, Abdel-All et al. [11] established Hasimoto surfaces by integrating the Gauss–Weingarten equations. In 2014, Erdogdu and Ozdemir [12] discussed Hasimoto surfaces in Minkowski 3-space $\mathbb{E}_1^3$ by using the Frenet frame. In 2015, Aydin et al. [13] studied surfaces corresponding to solutions of the local induction equation in the pseudo-Galilean space $\mathbb{G}_3^1$. In 2019, Kelleci et al. [14] discussed Hasimoto surfaces in $\mathbb{E}^3$ by using the Bishop frame. In 2021, Elzawy [15] investigated Hasimoto surfaces in the Galilean space $\mathbb{G}_3$.

The quasi-frame (or Q-frame) as one of the most efficient and effective moving frames has an important role in studying curves and surfaces in various domains (Frenet, Bishop). At all locations, the Q-frame is well-defined, it is straightforward to calculate, and its construction is unaffected by whether the curve is parameterized by arc-length or not. Among many papers dealing with this subject, we refer the reader to the following [16–18].

The aim of this paper is to study the geometry of quasi-Hasimoto surfaces corresponding to the geometry of solutions of the quasi-vortex filament equation in 3D Euclidean space $\mathbb{E}^3$. The following is a summary of the paper's structure: We offer some basic concepts on the Q-frame along a curve and some fundamental principles about regular surfaces in $\mathbb{E}^3$, in Section 2. In Section 3, we obtain the Q-curvatures of quasi-Hasimoto surfaces in $\mathbb{E}^3$ and provide a necessary and sufficient condition for quasi-Hasimoto surfaces in $\mathbb{E}^3$ to be developable (flat) and minimal surfaces. In Section 4, we give necessary and sufficient conditions for parameter curves of quasi-Hasimoto surfaces in $\mathbb{E}^3$ to be geodesics, asymptotic lines, and principal lines. Finally, in Section 5, a summary is provided at the end of the article.

2. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ is the real vector space $\mathbb{R}^3$ with the Cartesian metric $g = dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a coordinate system of $\mathbb{E}^3$. For the vectors $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ in $\mathbb{E}^3$, we define Cartesian inner and cross products as

$$\langle P, Q \rangle = p_1 q_1 + p_2 q_2 + p_3 q_3$$

and

$$P \times Q = \begin{vmatrix} e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_2 q_3 - p_3 q_2, p_3 q_1 - p_1 q_3, p_1 q_2 - p_2 q_1).$$

Additionally, the Cartesian norm of the vector P is defined by $\|P\| = \sqrt{\langle P, P \rangle}$, where (e_1, e_2, e_3) is the natural basis of $\mathbb{R}^3$.

Assume that $\alpha(s)$ is an arc-length parameterized curve in $\mathbb{E}^3$. Along the curve $\alpha(s)$, the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is defined by

$$\begin{aligned} &\mathbf{T} = \alpha'(s) \ \text{(unit tangent vector)}, \\ &\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\| \ \text{(principal normal vector)}, \\ &\mathbf{B} = \mathbf{T} \times \mathbf{N} \ \text{(binormal vector)}. \end{aligned} \tag{2}$$

Therefore, for the curve $\alpha(s)$, the Frenet equations are

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{3}$$

where the functions κ and τ are Frenet curvatures of $\alpha(s)$ [19].

Along the curve $\alpha(s)$, the Q-frame $\{\mathbf{T}, \mathbf{N_q}, \mathbf{B_q}\}$ is defined by [20]

$$\begin{aligned} &\mathbf{T} = \alpha'(s) \text{ (unit tangent vector)}, \\ &\mathbf{N_q} = (\mathbf{T} \times \mathbf{k})/\|\mathbf{T} \times \mathbf{k}\| \text{ (quasi-normal vector)}, \\ &\mathbf{B_q} = \mathbf{T} \times \mathbf{N_q} \text{ (quasi-binormal vector)}, \end{aligned} \tag{4}$$

where $\mathbf{k}$ is called the projection vector.

Remark 1. *In our study, we have chosen* $\mathbf{k} = (0, 0, 1)$. *However, the Q-frame is singular if* $\mathbf{T}$ *and* $\mathbf{k}$ *are parallel. In this case, the projection vector can be chosen as* $\mathbf{k} = (1, 0, 0)$ *or* $\mathbf{k} = (0, 1, 0)$.

For a Cartesian angle $\phi(s)$ between the principal normal $\mathbf{N}$ and the quasi-normal $\mathbf{N_q}$, the *relation matrix* between the Q-frame and the Frenet frame is given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(s) & \sin\phi(s) \\ 0 & -\sin\phi(s) & \cos\phi(s) \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{5}$$

Thus, we have

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(s) & -\sin\phi(s) \\ 0 & \sin\phi(s) & \cos\phi(s) \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}. \tag{6}$$

Then, the quasi equations for $\alpha(s)$ are

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}'_\mathbf{q} \\ \mathbf{B}'_\mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}, \tag{7}$$

where the functions $\{\kappa_i | i = 1, 2, 3\}$ are quasi-curvatures (or Q-curvatures) of $\alpha(s)$ and are defined by

$$\kappa_1 = \kappa\cos\phi(s), \quad \kappa_2 = -\kappa\sin\phi(s), \quad \kappa_3 = \phi'(s) + \tau. \tag{8}$$

From the above equation, we have

$$\kappa^2 = \kappa_1^2 + \kappa_2^2, \quad \phi(s) = -\arctan\left(\frac{\kappa_2}{\kappa_1}\right). \tag{9}$$

For a regular surface $\Phi = \Phi(s, t)$ in $\mathbb{E}^3$, we define the following [21]:

- The unit normal vector of Φ by

$$\mathcal{W} = \frac{\Phi_s \times \Phi_t}{\|\Phi_s \times \Phi_t\|}, \tag{10}$$

  where $\Phi_s = \frac{\partial\Phi}{\partial s}$ and $\Phi_t = \frac{\partial\Phi}{\partial t}$.
- The first fundamental form of Φ as

$$\mathbf{I} = \langle d\Phi, d\Phi \rangle = L ds^2 + 2M ds dt + N dt^2, \tag{11}$$

  where

$$L = \langle \Phi_s, \Phi_s \rangle, \quad M = \langle \Phi_s, \Phi_t \rangle, \quad N = \langle \Phi_t, \Phi_t \rangle.$$

- The second fundamental form of Φ by

$$\mathbf{II} = -\langle d\mathcal{W}, d\Phi \rangle = E ds^2 + 2F ds dt + G dt^2, \tag{12}$$

where

$$E = \langle \Phi_{ss}, \mathcal{W} \rangle, \quad F = \langle \Phi_{st}, \mathcal{W} \rangle, \quad G = \langle \Phi_{tt}, \mathcal{W} \rangle.$$

- The Gaussian curvature, mean curvature, and principal curvatures of Φ as

$$\begin{aligned} \mathcal{H} &= \frac{EG - F^2}{LN - M^2}, \\ \mathcal{M} &= \frac{LG - 2MF + NE}{2(LN - M^2)}, \\ \mathcal{B}_1 &= \mathcal{M} + \sqrt{\mathcal{M}^2 - \mathcal{H}}, \ \ \mathcal{B}_2 = \mathcal{M} - \sqrt{\mathcal{M}^2 - \mathcal{H}}, \end{aligned} \tag{13}$$

respectively.

For a regular curve $\alpha(t)$ on a surface $\Phi(s, t)$ in $\mathbb{E}^3$, the *geodesic torsion*, *geodesic curvature*, and *normal curvature* are given by

$$\mathcal{T}_g = \frac{\langle \dot{\alpha}, \mathcal{W} \times \dot{\mathcal{W}} \rangle}{\|\dot{\alpha}\|^2}, \ \ \mathcal{K}_g = \frac{\langle \ddot{\alpha}, \mathcal{W} \times \dot{\alpha} \rangle}{\|\dot{\alpha}\|^3}, \ \ \mathcal{K}_n = \frac{\langle \ddot{\alpha}, \mathcal{W} \rangle}{\|\dot{\alpha}\|^2}, \tag{14}$$

respectively.

In light of the above, we give the following definitions [22]:

Definition 1. *A regular surface $\Phi(s, t)$ is termed a developable (flat) surface if the Gaussian curvature $\mathcal{H} \equiv 0$, whereas it is termed a minimal surface if the mean curvature $\mathcal{M} \equiv 0$.*

Definition 2. *For a regular curve $\alpha(t)$ on a surface $\Phi(s, t)$, the following facts are well-known:*
- *$\alpha(t)$ is a principal line if and only if the geodesic torsion $\mathcal{T}_g \equiv 0$.*
- *$\alpha(t)$ is an asymptotic line if and only if the normal curvature $\mathcal{K}_n \equiv 0$.*
- *$\alpha(t)$ is a geodesic curve if and only if the geodesic curvature $\mathcal{K}_g \equiv 0$.*

3. Some Geometric Properties of a Quasi-Hasimoto Surface

In this section, we obtain the quasi-Gaussian curvature, quasi-mean curvature, and quasi-principal curvatures of a quasi-Hasimoto surface in $\mathbb{E}^3$ and give a necessary and sufficient condition of a quasi-Hasimoto surface in $\mathbb{E}^3$ in order for it to be a developable (flat) surface and a minimal surface.

Proposition 1. *Let $\Phi = \Phi(s, t)$ be a solution of the vortex filament equation in $\mathbb{E}^3$. If $\Phi(s, 0)$ is an arc-length parameterized curve, then $\Phi = \Phi(s, t)$ is an arc-length parameterized curve for all t.*

Proof. It is enough to prove that $\langle \Phi_s, \Phi_s \rangle_t = 0$ for all solutions of (1).

$$\begin{aligned} \langle \Phi_s, \Phi_s \rangle_t &= 2\langle (\Phi_s)_t, \Phi_s \rangle \\ &= 2\langle (\Phi_t)_s, \Phi_s \rangle \\ &= 2\langle (\Phi_s \times \Phi_{ss})_s, \Phi_s \rangle \\ &= 2\langle \Phi_{ss} \times \Phi_{ss} + \Phi_s \times \Phi_{sss}, \Phi_s \rangle \\ &= 2\langle \Phi_s \times \Phi_{sss}, \Phi_s \rangle \\ &= 0. \end{aligned}$$

The proof is complete. □

Theorem 1. *Let $\Phi = \Phi(s, t)$ be a quasi-Hasimoto surface in E^3 such that $\Phi = \Phi(s, t)$ is an arc-length parameterized quasi-curve for all t. Then, the evolution equations for the Q-frame $\{\mathbf{T}, \mathbf{N_q}, \mathbf{B_q}\}$:*

- *with respect to s can be expressed as*

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}; \tag{15}$$

- *with respect to t can be expressed as*

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}_t = \begin{bmatrix} 0 & \rho & \sigma \\ -\rho & 0 & \omega \\ -\sigma & -\omega & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}, \tag{16}$$

where the functions ρ, σ and ω are

$$\begin{aligned} \rho &= -\kappa_{2s} - \kappa_1\kappa_3, \\ \sigma &= \kappa_{1s} - \kappa_2\kappa_3, \\ \omega &= \frac{1}{\kappa^2}\Big[(\kappa_{1ss} - 2\kappa_{2s}\kappa_3 - \kappa_{2t})\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3 + \kappa_{1t})\kappa_2 - \kappa^2\kappa_3^2\Big], \end{aligned} \tag{17}$$

respectively. Here, the functions $\{\kappa_i | i = 1, 2, 3\}$ are Q-curvatures of $\Phi = \Phi(s, t)$ for all t and $\kappa^2 = \kappa_1^2 + \kappa_2^2$.

Proof. Firstly, It is evident that the evolution equations for the quasi frame $\{\mathbf{T}, \mathbf{N_q}, \mathbf{B_q}\}$ with respect to s are given directly from the quasi-equations given in (7).

Secondly, It is known that the general time evolution equations for the Q-frame $\{\mathbf{T}, \mathbf{N_q}, \mathbf{B_q}\}$ take the form [9,11]

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix}_t = \begin{bmatrix} 0 & \rho & \sigma \\ -\rho & 0 & \omega \\ -\sigma & -\omega & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N_q} \\ \mathbf{B_q} \end{bmatrix},$$

where ρ, σ and ω are smooth functions of s and t. Our goal is to find the functions $\{\rho, \sigma, \omega\}$ in terms of the Q-curvatures $\{\kappa_i | i = 1, 2, 3\}$ of $\Phi = \Phi(s, t)$ for all t. By applying the compatibility conditions $\mathbf{T}_{ts} = \mathbf{T}_{st}$ and $(\mathbf{N_q})_{ts} = (\mathbf{N_q})_{st}$, we get

$$\begin{aligned} \rho_s &= \kappa_{1t} - \omega\kappa_2 + \sigma\kappa_3, \\ \sigma_s &= \kappa_{2t} + \omega\kappa_1 - \rho\kappa_3, \\ \omega_s &= \kappa_{3t} - \sigma\kappa_1 + \rho\kappa_2. \end{aligned} \tag{18}$$

We suppose that the time evolution equation for the quasi-curve $\Phi = \Phi(s, t)$ can be expressed by the form

$$\Phi_t = \lambda\mathbf{T} + \mu\mathbf{N_q} + \nu\mathbf{B_q}. \tag{19}$$

By applying the compatibility condition $\Phi_{ts} = \Phi_{st}$, we get the following equations:

$$\begin{aligned} 0 &= \lambda_s - \mu\kappa_1 - \nu\kappa_2, \\ \rho &= \mu_s + \lambda\kappa_1 - \nu\kappa_3, \\ \sigma &= \nu_s + \lambda\kappa_2 + \mu\kappa_3. \end{aligned} \tag{20}$$

From the first and second equation of (18), we obtain

$$\omega = \frac{1}{\kappa^2}[(\rho\kappa_3 + \sigma_s - \kappa_{2t})\kappa_1 + (\sigma\kappa_3 - \rho_s + \kappa_{1t})\kappa_2]. \tag{21}$$

We know that the time evolution equation for the quasi-curve $\Phi = \Phi(s, t)$ is given by

$$\begin{aligned} \Phi_t &= \Phi_s \times \Phi_{ss} \\ &= \mathbf{T} \times \left(\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right) \\ &= -\kappa_2 \mathbf{N_q} + \kappa_1 \mathbf{B_q}. \end{aligned} \tag{22}$$

From (19) and (22), we find

$$\{\lambda, \mu, \nu\} = \{0, -\kappa_2, \kappa_1\}.$$

By substituting from the above equation in (20) and (21), we get the functions ρ, σ, and ω given in (17). The proof is complete. □

Theorem 2. *Let $\Phi = \Phi(s, t)$ be a quasi-Hasimoto surface in $\mathbb{E}^3$ such that $\Phi = \Phi(s, t)$ is an arc-length parameterized quasi-curve for all t. Then, the quasi-Gaussian curvature, quasi-mean curvature, and quasi-principal curvatures of the surface $\Phi = \Phi(s, t)$ are given by*

$$\begin{aligned} \mathcal{H}_{\mathbf{q}} &= \frac{-1}{\kappa^4}\left[(\kappa_{1s}\kappa_2 - \kappa_{2s}\kappa_1)^2 + (\kappa_{1ss}\kappa_1 + \kappa_{2ss}\kappa_2)\kappa^2\right], \\ \mathcal{M}_{\mathbf{q}} &= \frac{1}{2\kappa^3}\left[(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2\right], \\ \mathcal{B}_{1\mathbf{q}} &= \frac{1}{2\kappa^3}\left[(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2\right] \\ &\quad + \frac{1}{2\kappa^3}\Bigg[\left((\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2\right)^2 \\ &\quad + 4\left((\kappa_{1s}\kappa_2 - \kappa_{2s}\kappa_1)^2 + (\kappa_{1ss}\kappa_1 + \kappa_{2ss}\kappa_2)\kappa^2\right)\kappa^2\Bigg]^{1/2}, \\ \mathcal{B}_{2\mathbf{q}} &= \frac{1}{2\kappa^3}\left[(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2\right] \\ &\quad - \frac{1}{2\kappa^3}\Bigg[\left((\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2\right)^2 \\ &\quad + 4\left((\kappa_{1s}\kappa_2 - \kappa_{2s}\kappa_1)^2 + (\kappa_{1ss}\kappa_1 + \kappa_{2ss}\kappa_2)\kappa^2\right)\kappa^2\Bigg]^{1/2}, \end{aligned} \tag{23}$$

respectively. Here, the functions $\{\kappa_i | i = 1, 2, 3\}$ are Q-curvatures of $\Phi = \Phi(s, t)$ for all t and $\kappa^2 = \kappa_1^2 + \kappa_2^2$.

Proof. The first partial derivatives set $\{\Phi_s, \Phi_t\}$ of the surface $\Phi(s, t)$ is given by

$$\begin{aligned} &\Phi_s = \mathbf{T}, \\ &\Phi_t = -\kappa_2 \mathbf{N_q} + \kappa_1 \mathbf{B_q}. \end{aligned} \tag{24}$$

From (10), the unit normal vector of $\Phi(s, t)$ is

$$\mathcal{W} = \frac{-1}{\kappa}\left[\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right]. \tag{25}$$

From (11), the coefficients of the first fundamental form of $\Phi(s, t)$ are

$$\begin{cases} L = 1, \\ M = 0, \\ N = \kappa^2. \end{cases} \tag{26}$$

The second partial derivatives set $\{\Phi_{ss}, \Phi_{st}, \Phi_{tt}\}$ of the surface $\Phi(s,t)$ is given by

$$\begin{aligned}
\Phi_{ss} &= \kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}, \\
\Phi_{st} &= \rho \mathbf{N_q} + \sigma \mathbf{B_q}, \\
\Phi_{tt} &= (\rho\kappa_2 - \sigma\kappa_1)\mathbf{T} - (\kappa_{2t} + \omega\kappa_1)\mathbf{N_q} + (\kappa_{1t} - \omega\kappa_2)\mathbf{B_q},
\end{aligned} \tag{27}$$

From (12), the coefficients of the second fundamental form of $\Phi(s,t)$ are

$$\begin{cases}
E = -\kappa, \\
F = \frac{-1}{\kappa}[\rho\kappa_1 + \sigma\kappa_2], \\
G = \frac{-1}{\kappa}\left[\kappa_{1t}\kappa_2 - \kappa_{2t}\kappa_1 - \omega\kappa^2\right],
\end{cases} \tag{28}$$

where the functions ρ, σ, ω are given in (17). Thus, from (13), the quasi-Gaussian curvature, quasi-mean curvature, and quasi-principal curvatures can be expressed as in (23), respectively. The proof is complete. □

Corollary 1. *The quasi-Hasimoto surface $\Phi = \Phi(s,t)$ in E^3 is*
- *a developable (flat) surface if and only if*

$$(\kappa_{1s}\kappa_2 - \kappa_{2s}\kappa_1)^2 + (\kappa_{1ss}\kappa_1 + \kappa_{2ss}\kappa_2)\kappa^2 = 0;$$

- *a minimal surface if and only if*

$$(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \left(\kappa^2 + \kappa_3^2\right)\kappa^2 = 0,$$

where the functions $\{\kappa_i | i = 1,2,3\}$ are Q-curvatures of $\Phi = \Phi(s,t)$ for all t and $\kappa^2 = \kappa_1^2 + \kappa_2^2$.

Proof. The proof follows directly from Theorem (2). □

4. Some Characterizations of Parameter Curves of a Quasi-Hasimoto Surface

In this section, we give necessary and sufficient conditions for parameter curves of a quasi-Hasimoto surface in $\mathbb{E}^3$ to be geodesics, asymptotic lines, and principal lines. Here, we define the following:

Definition 3. *The family of all s parameter curves and the family of all tparameter curves of a quasi-Hasimoto surface in $\mathbb{E}^3$ are denoted by Ω and Γ, respectively.*

Theorem 3. *Let $\Phi = \Phi(s,t)$ be a quasi-Hasimoto surface in $\mathbb{E}^3$ such that $\Phi = \Phi(s,t)$ is an arc-length parameterized quasi-curve for all t. Then, the following statements are satisfied:*
- *Every curve that belongs to Ω is a geodesic.*
- *Every curve that belongs to Γ is a geodesic if and only if*

$$(\kappa_{1s} - \kappa_2\kappa_3)\kappa_1 + (\kappa_{2s} + \kappa_1\kappa_3)\kappa_2 = 0,$$

where the functions $\{\kappa_i | i = 1,2,3\}$ are Q-curvatures of $\Phi = \Phi(s,t)$ for all t.

Proof. Firstly, from (24), (25) and (27), we have

$$\begin{aligned}
\Phi_s &= \mathbf{T}, \\
\mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right], \\
\Phi_{ss} &= \kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}.
\end{aligned}$$

Thus, from (14), we find that the quasi-geodesic curvature equals zero. This means that every curve that belongs to Ω is a geodesic.

Secondly, From (24), (25) and (27), we have

$$\begin{aligned}
\Phi_t &= -\kappa_2 \mathbf{N_q} + \kappa_1 \mathbf{B_q}, \\
\mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right], \\
\Phi_{tt} &= (\rho\kappa_2 - \sigma\kappa_1)\mathbf{T} - (\kappa_{2t} + \omega\kappa_1)\mathbf{N_q} + (\kappa_{1t} - \omega\kappa_2)\mathbf{B_q},
\end{aligned}$$

where the functions ρ, σ, ω are given in (17). Thus, from (14), we obtain the quasi-geodesic curvature as

$$\mathcal{K}_g = \frac{(\kappa_{1s} - \kappa_2\kappa_3)\kappa_1 + (\kappa_{2s} + \kappa_1\kappa_3)\kappa_2}{\kappa^2}.$$

This means that every curve that belongs to Γ is a geodesic if and only if

$$(\kappa_{1s} - \kappa_2\kappa_3)\kappa_1 + (\kappa_{2s} + \kappa_1\kappa_3)\kappa_2 = 0.$$

The proof is complete. □

Theorem 4. *Let $\Phi = \Phi(s, t)$ be a quasi-Hasimoto surface in $\mathbb{E}^3$ such that $\Phi = \Phi(s, t)$ is an arc-length parameterized quasi-curve for all t. Then, the following statements are satisfied:*

- *Every curve that belongs to Ω is an asymptotic line if and only if*

$$\kappa = 0.$$

- *Every curve that belongs to Γ is an asymptotic line if and only if*

$$(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \kappa^2\kappa_3^2 = 0,$$

where the functions $\{\kappa_i | i = 1, 2, 3\}$ are Q-curvatures of $\Phi = \Phi(s, t)$ for all t and $\kappa^2 = \kappa_1^2 + \kappa_2^2$.

Proof. Firstly, from (24), (25) and (27), we have

$$\begin{aligned}
\Phi_s &= \mathbf{T}, \\
\mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right], \\
\Phi_{ss} &= \kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}.
\end{aligned}$$

Thus, from (14), we get the quasi-normal curvature as

$$\mathcal{K}_n = -\kappa.$$

This means that every curve that belongs to Ω is an asymptotic line if and only if $\kappa = 0$.

Secondly, from (24), (25), and (27), we have

$$\begin{aligned}
\Phi_t &= -\kappa_2 \mathbf{N_q} + \kappa_1 \mathbf{B_q}, \\
\mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1 \mathbf{N_q} + \kappa_2 \mathbf{B_q}\right], \\
\Phi_{tt} &= (\rho\kappa_2 - \sigma\kappa_1)\mathbf{T} - (\kappa_{2t} + \omega\kappa_1)\mathbf{N_q} + (\kappa_{1t} - \omega\kappa_2)\mathbf{B_q},
\end{aligned}$$

where the functions ρ, σ, and ω are given in (17). Thus, from (14), we obtain the quasi-normal curvature as

$$\mathcal{K}_n = \frac{(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \kappa^2\kappa_3^2}{\kappa^3}.$$

This means that every curve that belongs to Γ is an asymptotic line if and only if

$$(\kappa_{1ss} - 2\kappa_{2s}\kappa_3)\kappa_1 + (\kappa_{2ss} + 2\kappa_{1s}\kappa_3)\kappa_2 - \kappa^2\kappa_3^2 = 0.$$

The proof is complete. □

Corollary 2. *If every curve that belongs to Ω is an asymptotic line, then every curve that belongs to Γ is also an asymptotic line.*

Proof. The proof follows directly from Theorem 4. □

Theorem 5. *Let $\Phi = \Phi(s, t)$ be a quasi-Hasimoto surface in $\mathbb{E}^3$ such that $\Phi = \Phi(s, t)$ is an arc-length parameterized quasi-curve for all t. Then, every curve that belongs to Ω and Γ is a principal line if and only if*

$$\kappa_1\kappa_{2s} - \kappa_{1s}\kappa_2 + \kappa_3\kappa^2 = 0,$$

where the functions $\{\kappa_i | i = 1, 2, 3\}$ are Q-curvatures of $\Phi = \Phi(s, t)$ for all t and $\kappa^2 = \kappa_1^2 + \kappa_2^2$.

Proof. Firstly, from (24) and (25), we have

$$\begin{aligned} \Phi_s &= \mathbf{T}, \\ \mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1\mathbf{N_q} + \kappa_2\mathbf{B_q}\right]. \end{aligned}$$

By using (15), we find

$$\mathcal{W}_s = \frac{1}{\kappa^2}\Big[\kappa^3\mathbf{T} + (\kappa_1\kappa_s - \kappa_{1s}\kappa + \kappa_2\kappa_3\kappa)\mathbf{N_q} + (\kappa_2\kappa_s - \kappa_{2s}\kappa - \kappa_1\kappa_3\kappa)\mathbf{B_q}\Big].$$

Thus, from (14), we get the quasi-geodesic torsion as

$$\mathcal{T}_g = \frac{1}{\kappa^2}\Big[\kappa_1\kappa_{2s} - \kappa_{1s}\kappa_2 + \kappa_3\kappa^2\Big].$$

This means that every curve that belongs to Ω is a principal line if and only if

$$\kappa_1\kappa_{2s} - \kappa_{1s}\kappa_2 + \kappa_3\kappa^2 = 0.$$

Secondly, from (24) and (25), we have

$$\begin{aligned} \Phi_t &= -\kappa_2\mathbf{N_q} + \kappa_1\mathbf{B_q}, \\ \mathcal{W} &= \frac{-1}{\kappa}\left[\kappa_1\mathbf{N_q} + \kappa_2\mathbf{B_q}\right]. \end{aligned}$$

By using (16), we find

$$\mathcal{W}_t = \frac{1}{\kappa^2}\left[(\rho\kappa_1\kappa + \sigma\kappa_2\kappa)\mathbf{T} + (\kappa_1\kappa_t - \kappa_{1t}\kappa + \omega\kappa_2\kappa)\mathbf{N_q} + (\kappa_2\kappa_t - \kappa_{2t}\kappa - \omega\kappa_1\kappa)\mathbf{B_q}\right],$$

where the functions ρ, σ, and ω are given in (17). Thus, from (14), we obtain the quasi-geodesic torsion as

$$\mathcal{T}_g = \frac{1}{\kappa^2}\Big[\kappa_1\kappa_{2s} - \kappa_{1s}\kappa_2 + \kappa_3\kappa^2\Big].$$

This means that every curve that belongs to Γ is a principal line if and only if

$$\kappa_1\kappa_{2s} - \kappa_{1s}\kappa_2 + \kappa_3\kappa^2 = 0.$$

The proof is complete. □

Corollary 3. *If every curve that belongs to Ω is an asymptotic line, then every curve that belongs to Ω and Γ is also a principal line.*

Proof. The proof follows directly from Theorems 4 and 5. □

5. Conclusions

In this article, by employing the Q-frame, the geometry of Hasimoto surfaces corresponding to the geometry of solutions of the vortex filament equation in Euclidean 3-space $\mathbb{E}^3$ was investigated. In detail, the Q-curvatures of a quasi-Hasimoto surface in $\mathbb{E}^3$ were obtained, and the necessary and sufficient condition of a quasi-Hasimoto surface in $\mathbb{E}^3$ in order for it to be a developable (flat) surface and a minimal surface was presented. The necessary and sufficient conditions for parameter curves of a quasi-Hasimoto surface in $\mathbb{E}^3$ to be geodesics, asymptotic lines, and principal lines were given. This surface is a new contribution to the field, and it may be useful for some specific applications in theoretical physics and fluid dynamics.

Author Contributions: Conceptualization, E.H. and O.M.; methodology, O.M. and A.A.; formal analysis, C.C. and A.A.; investigation, C.C. and S.A.; writing—original draft preparation, E.H. and S.A.; writing—review and editing, O.M, A.A. and C.C.; All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Supporting Project number (RSP-2022/167), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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