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Article

New Modular Fixed-Point Theorem in the Variable Exponent Spaces $\ell_{p(.)}$

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Abstract: In this work, we prove a fixed-point theorem in the variable exponent spaces $\ell_{p(.)}$, when $p^- = 1$ without further conditions. This result is new and adds more information regarding the modular structure of these spaces. To be more precise, our result concerns ρ-nonexpansive mappings defined on convex subsets of $\ell_{p(.)}$ that satisfy a specific condition which we call "condition of uniform decrease".

Keywords: electrorheological fluid; fixed point; modular vector space; Nakano; strictly convex; uniformly convex

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1. Introduction

Variable exponent spaces first appeared in a work of Orlicz in 1931 [1] (see also [2]), where he defined the following space:

$$X = \left\{ \{x_n\} \in \mathbb{R}^{\mathbb{N}}, \quad \sum_{n=0}^{\infty} |\lambda x_n|^{p(n)} < \infty, \; for\ some\ \lambda > 0 \right\}.$$

They became very important because of their use in the mathematical modeling of non-Newtonian fluids [3,4]. The typical example of such fluids are electrorheological fluids, the viscosity of which exhibits dramatic and sudden changes when exposed to an electric or magnetic field. The necessity of a clear understanding of the spaces with variable integrability is reinforced by their potential applications.

The properties of this vector space have been extensively studied in [5–7]. The norm that was commonly used to investigate the geometrical properties of X is the Minkowski functional associated to the modular unit ball and it is known as the Luxembourg norm. Whereas in the case of classical ℓ_p spaces, the natural norm is suitable for making calculations, the Luxembourg norm on X is very difficult to manipulate.

In 1950, Nakano [8] introduced for the first time the notion of modular vector space (see also [9,10]). This abstract point of view has been crucial to the development of the research on geometrical and topological properties of the variable exponent spaces $\ell_{p(.)}$.

In this work, we will introduce a class of subsets of $\ell_{p(.)}$ that have some interesting geometrical properties. This will allow us to prove a new fixed-point theorem concerning $\ell_{p(.)}$ spaces. For the study of metric fixed-point theory, we recommend the book [9].

2. Basic Notations and Terminology

For a function $p : \mathbb{N} \longrightarrow [1, +\infty)$, define the vector space

$$\ell_{p(.)} = \left\{ \{x_n\} \in \mathbb{R}^{\mathbb{N}}, \quad \sum_{n=0}^{\infty} \frac{1}{p(n)} |\lambda x_n|^{p(n)} < \infty, \ for\ some\ \lambda > 0 \right\}.$$

Nakano [8,11] introduced the concept of modular vector space.

Proposition 1 ([6,9])**.** *Consider the function $\rho : \ell_{p(.)} \longrightarrow [0, +\infty]$ defined by*

$$\rho(x) = \rho(\{x_n\}) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)}$$

then ρ satisfies the following properties

(1) $\rho(x) = 0$ *if and only if* $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$, *if* $|\alpha| = 1$,
(3) $\rho(\alpha x + (1-\alpha)y) \leq \alpha\rho(x) + (1-\alpha)\rho(y), \ \forall \alpha \in [0,1]$.

for any $x, y \in X$. The function ρ is called a convex modular.

For any subset I of $\mathbb{N}$, we consider the functional

$$\rho_I(x) = \sum_{n \in I} |x_n|^{p(n)}.$$

If $I = \emptyset$, we set $\rho_I(x) = 0$. We define on modular spaces a modular topology which is similar to the topology induced by a metric.

Definition 1. *Consider the vector space $\ell_{p(.)}$.*

(a) *We say that a sequence $\{x_n\} \subset \ell_{p(.)}$ is ρ-convergent to $x \in \ell_{p(.)}$ if and only if $\rho(x_n - x) \longrightarrow 0$. The ρ-limit is unique if it exists.*
(b) *A sequence $\{x_n\} \subset \ell_{p(.)}$ is called ρ-Cauchy if $\rho(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow +\infty$.*
(c) *A nonempty subset $C \subset \ell_{p(.)}$ is called ρ-closed if for any sequence $\{x_n\} \subset C$ which ρ-converges to x implies that $x \in C$.*
(d) *A nonempty subset $C \subset \ell_{p(.)}$ is called ρ-bounded if and only if*

$$\delta_\rho(C) = \sup\{\rho(x-y), \ \ x, y \in C\} < \infty.$$

Note that ρ satisfies the Fatou property, i.e.,

$$\rho(x-y) \leq \liminf_{n \to +\infty} \rho(x - y_n),$$

holds whenever $\{y_n\}$ ρ-converges to y, for any $x, y, y_n \in \ell_{p(.)}$. Throughout, we will use the notation $B_\rho(x, r)$ to denote the ρ-ball with radius $r \geq 0$ centered at $x \in \ell_{p(.)}$ and defined as

$$B_\rho(x, r) = \left\{ y \in \ell_{p(.)}, \ \ \rho(x-y) \leq r \right\}.$$

Note that Fatou property holds if and only if the ρ-balls are ρ-closed. That is, all ρ-balls are ρ-closed in $\ell_{p(.)}$.

Definition 2. *Let $C \subset \ell_{p(.)}$ be a nonempty subset. A mapping $T : C \longrightarrow C$ is called ρ-Lipschitzian if there exists a constant $K \geq 0$ such that*

$$\rho(T(x) - T(y)) \leq K\, \rho(x-y), \quad \forall x, y \in C.$$

If $K = 1$, T is called ρ-nonexpansive. A point $x \in C$ is called a fixed point of T if $T(x) = x$.

The concept of modular uniform convexity was first introduced by Nakano [11], but a weaker definition of modular uniform convexity called $(UUC2)$ was introduced in [9] and seems to be more suitable to hold in $\ell_{p(.)}$ when weaker assumptions on the exponent function $p(\cdot)$ hold. The following definition is given in terms of subsets because of the subsequent results discovered in this work.

Definition 3 ([9])**.** *Consider the vector space $\ell_{p(.)}$. Let C be a nonempty subset of $\ell_{p(.)}$.*

1. *Let $r > 0$ and $\varepsilon > 0$. Define*
$$D_2(r,\varepsilon) = \left\{(x,y) \in \ell_{p(.)} \times \ell_{p(.)},\ \rho(x) \le r,\ \rho(y) \le r,\ \rho\left(\frac{x-y}{2}\right) \ge \varepsilon r\right\}.$$
If $D_2(r,\varepsilon) \cap (C \times C) \neq \emptyset$, let
$$\delta_{2,C}(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right),\ (x,y) \in D_2(r,\varepsilon) \cap (C \times C)\right\}.$$
If $D_2(r,\varepsilon) \cap (C \times C) = \emptyset$, we set $\delta_2(r,\varepsilon) = 1$. We say that ρ satisfies $(UC2)$ on C if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_{2,C}(r,\varepsilon) > 0$. When $C = \ell_{p(.)}$, we remark that for every $r > 0$, $D_2(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. In this case, we will use the notation $\delta_{2,\ell_{p(.)}} = \delta_2$.
2. *We say that ρ satisfies $(UUC2)$ on C if for every $s \ge 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$ depending on s and ε such that*
$$\delta_{2,C}(r,\varepsilon) \ge \eta_2(s,\varepsilon) > 0 \ \ for \ \ r > s.$$
3. *We say that ρ is strictly convex on C (in short (SC)), if for every $x, y \in C$ such that*
$$\rho(x) = \rho(y) \ \ and \ \ \rho\left(\frac{x+y}{2}\right) = \frac{\rho(x)+\rho(y)}{2} \ \ imply \ \ x = y.$$

In the study of the properties of $\ell_{p(.)}$ (see [12]), the following values are very important:
$$p^+ = \sup_{n\in\mathbb{N}} p(n) \ \ and \ \ p^- = \inf_{n\in\mathbb{N}} p(n).$$

In [5], the authors proved that for $\ell_{p(.)}$, with $p^- > 1$, the modular is $(UUC2)$. This modular geometrical property allows to prove the following fixed-point result:

Theorem 1. *Consider the vector space $\ell_{p(.)}$. Assume $p^- > 1$. Let C be a nonempty ρ-closed convex ρ-bounded subset of $\ell_{p(.)}$. Let $T : C \longrightarrow C$ be a ρ-nonexpansive mapping. Then T has a fixed point.*

In [13], the authors proved a similar fixed-point theorem in the case where $\{n \in \mathbb{N},\ p(n) = 1\}$ has at most one element which is an improvement from $p^- > 1$.

Before we close this section, we recall the following lemma, of a rather technical nature, which plays a crucial role when dealing with $\ell_{p(.)}$ spaces.

Lemma 1. *The following inequalities hold:*

(i) *[14]. If $p \ge 2$, then*
$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$
for any $a, b \in \mathbb{R}$.

(ii) [15]. *If* $1 < p \leq 2$, *then*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$

for any $a, b \in \mathbb{R}$ *such that* $|a| + |b| \neq 0$.

In this work, using a different approach, we obtain some fixed-point results when $p^- = 1$ without the known conditions on the function $p(\cdot)$.

3. Uniform Decrease Condition

First, we introduce an interesting class of subsets of $\ell_{p(.)}$, which will play an important part in our work. In particular, they enjoy similar modular geometric properties as $\ell_{p(.)}$ when $p^- > 1$. Before, let us introduce the following notations:

$$I_a = \Big\{n \in \mathbb{N};\ p(n) \geq a\Big\} \ \ and \ \ J_a = \mathbb{N} \setminus I_a = \Big\{n \in \mathbb{N};\ p(n) < a\Big\},$$

where $a \in [1, +\infty)$.

Definition 4. *Consider the vector space* $\ell_{p(.)}$. *A nonempty subset* C *of* $\ell_{p(.)}$ *is said to satisfy the uniform decrease condition (in short* (UD)*) if for any* $\alpha > 0$, *there exists* $a > 1$ *such that*

$$\sup_{x \in C} \ \rho_{J_a}(x) \leq \alpha.$$

Obviously the condition (UD) passes from a set to its subsets. Moreover, if $p(\cdot)$ is identically equal to 1, then the only (UD) subset is $C = \{0\}$. Since this case is not interesting, we will assume throughout that $p(\cdot)$ is not identically equal to 1. Moreover, if $p^- > 1$, then any nonempty subset of $\ell_{p(.)}$ satisfies the condition (UD). Indeed, let C be a nonempty subset of $\ell_{p(.)}$ and $\alpha > 0$. Let $a \in (1, p^-)$. Then $J_a = \emptyset$ which implies

$$\sup_{x \in C} \rho_{J_a}(x) = 0 \leq \alpha.$$

Therefore, the condition (UD) is interesting to study only when $p^- = 1$ and $p(\cdot)$ is not identically equal to 1, which will be the case throughout.

Example 1. *Consider the function* $p(\cdot)$ *defined by*

$$p(n) = 1 + \frac{1}{n+1}, \ \ n \in \mathbb{N}.$$

Consider the subset

$$C = \left\{x \in \ell_{p(.)};\ |x_n| \leq \frac{1}{(n+1)^2},\ n \in \mathbb{N}\right\}.$$

C is nonempty, convex and ρ*-closed. Let us show that it satisfies the condition* (UD). *Indeed, fix* $\alpha > 0$. *Let* $N \geq 1$ *be such that* $\sum\limits_{k \geq N} \dfrac{1}{(k+1)^2} \leq \alpha$. *Set* $a = 1 + \frac{1}{N}$. *We have*

$$\begin{aligned}\rho_{J_a}(x) &= \sum_{n\in J_a}\frac{|x_n|^{p(n)}}{p(n)}\\ &\leq \sum_{n\geq N}\frac{|x_n|^{p(n)}}{p(n)}\\ &\leq \sum_{n\geq N}\frac{1}{p(n)}\frac{1}{(n+1)^{2p(n)}}\\ &\leq \sum_{n\geq N}\frac{1}{(n+1)^2}\\ &\leq \alpha,\end{aligned}$$

for all $x \in C$, which proves our claim that C is (UD).

Before we give a characterization of subsets which satisfy the condition (UD), we need to introduce a new class of subsets of $\ell_{p(.)}$.

Definition 5. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0,+\infty) \to (1,2]$ be a nondecreasing function. Define the set C_f to be*

$$C_f = \Big\{x \in \ell_{p(.)};\ \ \rho_{J_{f(\alpha)}}(x) \leq \alpha,\ \ for\ all\ \ \alpha > 0\Big\}.$$

Note that C_f is never empty since $0 \in C_f$. Some of the basic properties of C_f are given in the following lemma.

Lemma 2. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0,+\infty) \to (1,2]$ be a non-decreasing function. Then the following properties hold:*

1. *C_f is convex.*
2. *C_f is symmetrical, i.e., $-z \in C_f$ whenever $z \in C_f$.*
3. *The Fatou property implies easily that C_f is ρ-closed as a subset of $\ell_{p(.)}$ which in turn implies that C_f is ρ-complete.*

Proposition 2. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. A subset C of $\ell_{p(.)}$ satisfies the condition (UD) if and only if there exists $f : (0,+\infty) \to (1,2]$ non-decreasing such that $C \subset C_f$.*

Proof. First, we prove that C_f satisfies the condition (UD). Fix $\alpha > 0$. If we take $a = f(\alpha)$, we obtain

$$\sup_{x\in C_f} \rho_{J_a}(x) \leq \alpha,$$

which proves our claim. Clearly, any subset C of C_f will also satisfy the condition (UD). Conversely, let C be a nonempty subset of $\ell_{p(.)}$ which satisfies the condition (UD). For any $\alpha > 0$, there exists $a > 1$ such that $\sup_{x\in C} \rho_{J_a}(x) \leq \alpha$. Set

$$[\alpha] = \Big\{a > 1;\ \sup_{x\in C} \rho_{J_a}(x) \leq \alpha\Big\}.$$

Define

$$f(\alpha) = \begin{cases} 2 & \text{if } [\alpha] \subset [2,+\infty),\\ \sup\Big([\alpha]\cap(1,2]\Big) & \text{if } [\alpha]\cap(1,2] \neq \emptyset.\end{cases}$$

Clearly, f is well defined and $f(\alpha) \in (1,2]$, for all $\alpha > 0$. Let α and β be such that $0 < \alpha \leq \beta$. We claim that $f(\alpha) \leq f(\beta)$. Indeed, it is easy to see that $[\alpha] \subset [\beta]$. If $[\alpha]\cap(1,2] \neq \emptyset$, then we have $[\beta]\cap(1,2] \neq \emptyset$ which easily implies $f(\alpha) \leq f(\beta)$. Otherwise, assume $[\alpha] \subset [2,+\infty)$.

Let $a \in [\alpha]$. We have $a \geq 2$ and $a \in [\beta]$. By definition of the sets J, we have $J_2 \subset J_a$. Since $\rho_{J_2}(x) \leq J_a(x)$, for all $x \in \ell_{p(.)}$, we obtain

$$\sup_{x \in C} \rho_{J_2}(x) \leq \sup_{x \in C} \rho_{J_a}(x) \leq \beta,$$

i.e., $2 \in [\beta]$. This fact, will force $f(\beta) = 2$. In all cases, we have $f(\alpha) \leq f(\beta)$. In other words, the function $f : (0, +\infty) \to (1, 2]$ is non-decreasing. Finally, let us show that $C \subset \ell_g$, where $g(\alpha) = (1 + f(\alpha))/2$, for all $\alpha > 0$. Since $1 < f(\alpha)$, then we have $1 < g(\alpha) < f(\alpha)$, for all $\alpha > 0$. If $[\alpha] \subset [2, +\infty)$, pick $a \in [\alpha]$. Then $g(\alpha) = 3/2 < a$ which implies $J_{g(\alpha)} \subset J_a$. Hence

$$\rho_{J_{g(\alpha)}}(x) \leq \rho_{J_a}(x), \quad for\ all \quad x \in C,$$

which implies $\sup_{x \in C} \rho_{J_{g(\alpha)}}(x) \leq \sup_{x \in C} \rho_{J_a}(x) \leq \alpha$. Otherwise, assume $[\alpha] \cap (1, 2] \neq \emptyset$, then $f(\alpha) = \sup\Big([\alpha] \cap (1, 2]\Big)$. Since $g(\alpha) < f(\alpha)$, there exists $a \in [\alpha]$ such that $g(\alpha) < a \leq f(\alpha)$. Similar argument will show that

$$\sup_{x \in C} \rho_{J_{g(\alpha)}}(x) \leq \sup_{x \in C} \rho_{J_a}(x) \leq \alpha.$$

In both cases, we showed that $\sup_{x \in C} \rho_{J_{g(\alpha)}}(x) \leq \alpha$, for all $\alpha > 0$, i.e., $C \subset C_g$ as claimed. □

Proposition 2 allows us to focus on the subsets C_f instead of subsets which satisfy the condition (UD). The next result is amazing and surprising since it tells us that the subsets C_f enjoy nice modular geometric properties despite the fact that $p^- = 1$.

Theorem 2. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0, +\infty) \to (1, 2]$ be a non-decreasing function. Then, ρ is $(UUC2)$ on C_f.*

Proof. Let $r > 0$ and $\varepsilon > 0$. Let $x, y \in C_f$ such that $\rho(x) \leq r$, $\rho(y) \leq r$ and $\rho\left(\frac{x-y}{2}\right) \geq r\varepsilon$. Since ρ is convex, we have

$$r\varepsilon \leq \rho\left(\frac{x-y}{2}\right) \leq \frac{\rho(x) + \rho(y)}{2} \leq r,$$

which implies $\varepsilon \leq 1$. Set $\alpha = \frac{r\varepsilon}{2}$. The properties of C_f imply $\frac{x-y}{2} \in C_f$. So

$$\rho_{J_{f(\alpha)}}\left(\frac{x-y}{2}\right) \leq \alpha,$$

which implies

$$\rho_{I_{f(\alpha)}}\left(\frac{x-y}{2}\right) = \rho\left(\frac{x-y}{2}\right) - \rho_{J_{f(\alpha)}}\left(\frac{x-y}{2}\right) \geq r\varepsilon - \alpha = \frac{r\varepsilon}{2}.$$

Next, set

$$K = I_{f(\alpha)} \cap \{n, p(n) \geq 2\} \ and \ L = I_{f(\alpha)} \cap \{n, p(n) < 2\}.$$

Since $I_{f(\alpha)} = K \cup L$, we obtain $\rho_{I_{f(\alpha)}}(z) = \rho_K(z) + \rho_L(z)$, for all $z \in C_f$. From our assumptions, we have

$$\rho_K\left(\frac{x-y}{2}\right) \geq \frac{r\varepsilon}{4} \ \ or \ \ \rho_L\left(\frac{x-y}{2}\right) \geq \frac{r\varepsilon}{4}.$$

Assume first that

$$\rho_K\left(\frac{x-y}{2}\right) \geq \frac{r\varepsilon}{4}.$$

Using Lemma 1, we obtain

$$\rho_K\left(\frac{x+y}{2}\right) + \rho_K\left(\frac{x-y}{2}\right) \leq \frac{\rho_K(x)+\rho_K(y)}{2},$$

which implies

$$\rho_K\left(\frac{x+y}{2}\right) \leq \frac{\rho_K(x)+\rho_K(y)}{2} - \frac{r\varepsilon}{4}.$$

Using the convexity of the modular, we have

$$\rho_{L\cup J_{f(\alpha)}}\left(\frac{x+y}{2}\right) \leq \frac{\rho_{L\cup J_{f(\alpha)}}(x)+\rho_{L\cup J_{f(\alpha)}}(y)}{2},$$

which implies

$$\rho\left(\frac{x+y}{2}\right) \leq \frac{\rho(x)+\rho(y)}{2} - \frac{\varepsilon r}{4} \leq r\Big(1-\frac{\varepsilon}{4}\Big).$$

For the second case, assume

$$\rho_L\left(\frac{x-y}{2}\right) \geq \frac{\varepsilon r}{4}.$$

Set

$$c = \frac{\varepsilon}{8},\ \ L_1 = \Big\{n,\ |x_n-y_n| \leq c\,\Big(|x_n|+|y_n|\Big)\Big\}\ \ and\ \ L_2 = L\backslash L_1.$$

Since $c<1$, we obtain

$$\rho_{L_1}\left(\frac{x-y}{2}\right) \leq \sum_{n\in L_1}\frac{c^{p(n)}}{p(n)}\left(\frac{|x_n|+|y_n|}{2}\right)^{p(n)} \leq \frac{c}{2}\sum_{n\in L_1}\frac{|x_n|^{p(n)}+|y_n|^{p(n)}}{p(n)}.$$

Hence

$$\rho_{L_1}\left(\frac{x-y}{2}\right) \leq \frac{c}{2}\Big(\rho_{L_1}(x)+\rho_{L_1}(y)\Big) \leq \frac{c}{2}\Big(\rho(x)+\rho(y)\Big) \leq \frac{c}{2}\,r.$$

Our assumption on $\rho_L\left(\dfrac{x-y}{2}\right)$ implies

$$\rho_{L_2}\left(\frac{x-y}{2}\right) = \rho_L\left(\frac{x-y}{2}\right) - \rho_{L_1}\left(\frac{x-y}{2}\right) \geq r\frac{\varepsilon}{4} - \frac{c}{2}r \geq r\frac{\varepsilon}{8}.$$

For any $n\in L_2$, we have

$$f\Big(\frac{r\varepsilon}{2}\Big) - 1 = f(\alpha)-1 \leq p(n)-1 \leq p(n)(p(n)-1)$$

$$c \leq c^{2-p(n)} \leq \left(\frac{|x_n-y_n|}{|x_n|+|y_n|}\right)^{2-p(n)}.$$

Using Lemma 1, we obtain

$$\left|\frac{x_n+y_n}{2}\right|^{p(n)} + \frac{(f(\alpha)-1)}{2}c\left|\frac{x_n-y_n}{2}\right|^{p(n)} \leq \frac{1}{2}\Big(|x_n|^{p(n)}+|y_n|^{p(n)}\Big),$$

for any $n\in L_2$. Hence

$$\rho_{L_2}\left(\frac{x+y}{2}\right) \leq \frac{\rho_{L_2}(x)+\rho_{L_2}(y)}{2} - \frac{r(f(\alpha)-1)\varepsilon^2}{128},$$

which implies

$$\rho\left(\frac{x+y}{2}\right) \leq r\left(1-\frac{(f(\alpha)-1)\varepsilon^2}{128}\right).$$

Both cases imply that ρ is $(UC2)$ on C_f with

$$\delta_{2,C_f}(r,\varepsilon) \geq \min\left(\frac{\varepsilon}{4}, \frac{\left(f\left(\frac{r\varepsilon}{2}\right)-1\right)\varepsilon^2}{128}\right) > 0,$$

since $f(a) > 1$, for any $a > 0$. Since $f(\cdot)$ is nondecreasing, we may set

$$\eta_2(r,\varepsilon) = \min\left(\frac{\varepsilon}{4}, \frac{\left(f\left(\frac{r\varepsilon}{2}\right)-1\right)\varepsilon^2}{128}\right)$$

to see that in fact ρ is $(UUC2)$ on C_f which completes the proof of Theorem 2. □

The following lemma will be useful:

Lemma 3. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0,+\infty) \to (1,2]$ be a non-decreasing function. Set $g(\alpha) = f\left(\frac{\alpha}{4}\right)$, for $\alpha > 0$. We have*

$$C_f + C_f = \left\{x + y;\ \ x, y \in C_f\right\} \subset C_g.$$

Proof. Let $x, y \in C_f$. For any $n \in J_{g(\alpha)} = \left\{n;\ p(n) \leq f\left(\frac{\alpha}{4}\right)\right\}$, we have

$$\left|\frac{x_n + y_n}{2}\right|^{p(n)} \leq \frac{1}{2}\left(|x_n|^{p(n)} + |y_n|^{p(n)}\right),$$

which implies

$$\frac{1}{p(n)}|x_n + y_n|^{p(n)} \leq \frac{2^{p(n)-1}}{p(n)}\left(|x_n|^{p(n)} + |y_n|^{p(n)}\right).$$

Hence

$$\begin{aligned}\rho_{J_{f(\frac{\alpha}{4})}}(x+y) &\leq 2^{f(\frac{\alpha}{4})-1}\left(\rho_{J_{f(\frac{\alpha}{4})}}(x) + \rho_{J_{f(\frac{\alpha}{4})}}(y)\right)\\ &\leq 2\left(\frac{\alpha}{4} + \frac{\alpha}{4}\right)\\ &= \alpha.\end{aligned}$$

Therefore $\rho_{J_{g(\alpha)}}(x+y) \leq \alpha$, that is $x + y \in C_g$, which completes the proof of Lemma 3. □

In the next section, we will prove a fixed-point theorem for modular nonexpansive mappings.

4. Application

As an application to Theorem 2, we will prove a fixed-point result for modular nonexpansive mappings. The classical ingredients will be needed. First, we prove the proximinality of ρ-closed convex subsets which satisfies the condition (UD).

Proposition 3. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0,+\infty) \to (1,2]$ non-decreasing. Any nonempty ρ-closed convex subset C of C_f is proximinal, i.e., for any $x \in C_f$ such that*

$$d_\rho(x, C) = \inf\left\{\rho(x - y);\ y \in C\right\} < \infty,$$

there exists a unique $c \in C$ such that $d_\rho(x, C) = \rho(x - c)$.

Proof. Without loss of generality, we assume that $x \notin C$. Since C is ρ-closed we have, $R = d_\rho(x, C) > 0$. For any $n \geq 1$, there exists $y_n \in C$ such that $\rho(x - y_n) < R(1 + 1/n)$. We claim that $\{y_n/2\}$ is ρ-Cauchy. Assume not. Then there exists a subsequence $\{y_{\phi(n)}\}$ of $\{y_n\}$ and $\varepsilon_0 > 0$ such that

$$\rho\left(\frac{y_{\phi(n)} - y_{\phi(m)}}{2}\right) \geq \varepsilon_0,$$

for any $n > m \geq 1$. According to Lemma 3, $\{x - y_{\phi(n)}\}$ is in C_g, where $g(\alpha) = f(\alpha/4)$, for any $\alpha > 0$. Fix $n > m \geq 1$. We have

$$\max\left\{\rho\left(x - y_{\phi(n)}\right), \rho\left(x - y_{\phi(m)}\right)\right\} \leq R\left(1 + \frac{1}{\phi(m)}\right).$$

Since

$$\varepsilon_0 = R\left(1 + \frac{1}{\phi(m)}\right)\frac{\varepsilon_0}{R\left(1 + \frac{1}{\phi(m)}\right)} \geq R\left(1 + \frac{1}{\phi(m)}\right)\varepsilon_1,$$

with $\varepsilon_1 = \dfrac{\varepsilon_0}{2R}$, and using Theorem 2, we obtain

$$\begin{aligned}
\rho\left(x - \frac{y_{\phi(n)} + y_{\phi(m)}}{2}\right) &\leq R(1 + 1/\phi(m))\left(1 - \delta_{2,C_g}\left(R\left(1 + \frac{1}{\phi(m)}\right), \varepsilon_1\right)\right) \\
&\leq R(1 + 1/\phi(m))\Big(1 - \eta_2(R, \varepsilon_1)\Big),
\end{aligned}$$

where

$$\eta_2(R, \varepsilon_1) = \min\left(\frac{\varepsilon_1}{4}, \frac{\left(g\left(\frac{R\varepsilon_1}{2}\right) - 1\right)\varepsilon_1^2}{128}\right).$$

Since $y_{\phi(n)}$ and $y_{\phi(m)}$ are in C and C is convex, we obtain

$$R = d_\rho(x, C) \leq \rho\left(x - \frac{y_{\phi(n)} + y_{\phi(m)}}{2}\right) \leq R(1 + 1/\phi(m))\Big(1 - \eta_2(R, \varepsilon_1)\Big).$$

If we let $m \to +\infty$, we obtain

$$R \leq R(\Big(1 - \eta_2(R, \varepsilon_1)\Big) < R.$$

This contradiction implies that $\{y_n/2\}$ is ρ-Cauchy. Since $\ell_{p(\cdot)}$ is ρ-complete, there exists $y \in \ell_{p(\cdot)}$ such that $\{y_n/2\}$ ρ-converges to y. Since C is convex and ρ-closed, we conclude that $2y \in C$. Using the Fatou property, we have

$$\begin{aligned}
R = d_\rho(x, C) &\leq \rho(x - 2y) \\
&\leq \liminf_{m \to +\infty} \rho\left(x - \left(y + \frac{y_m}{2}\right)\right) \\
&\leq \liminf_{m \to +\infty} \liminf_{n \to +\infty} \rho\left(x - \frac{y_n + y_m}{2}\right) \\
&\leq \liminf_{m \to +\infty} \liminf_{n \to +\infty} \frac{\rho(x - y_n) + \rho(x - y_m)}{2} = R = d_\rho(x, C).
\end{aligned}$$

If we set $c = 2y$, we obtain $d(x, C) = \rho(x - c)$. The uniqueness of the point c comes from the fact that ρ is strictly convex on C_g since it is $(UUC2)$. $\square$

The next result discusses an intersection property known as the property (R) [9]. Recall that a nonempty ρ-closed convex subset C of $\ell_{p(\cdot)}$ is said to satisfy the property (R)

if for any decreasing sequence of nonempty ρ-closed ρ-bounded convex subsets of C have a nonempty intersection.

Proposition 4. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ not identically equal to 1. Let $f : (0, +\infty) \to (1, 2]$ be a non-decreasing function. Then C_f satisfies the property (R).*

Proof. Let $\{C_n\}$ be a decreasing sequence of nonempty ρ-closed ρ-bounded convex subsets of C_f. Let $x \in C_1$. We have

$$d_\rho(x, C_n) = \inf \{\rho(x - x_n); x_n \in C_n\} \leq \sup\{\rho(x - y), \ \ x, y \in C_1\} = \delta_\rho(C_1) < \infty.$$

Since $\{C_n\}$ is decreasing, the sequence $\{d_\rho(x, C_n)\}$ is increasing bounded above by $\delta_\rho(C_1)$. Set $R = \lim\limits_{n \to +\infty} d_\rho(x, C_n) = \sup\limits_n d_\rho(x, C_n)$. If $R = 0$, then $x \in C_n$ for any $n \geq 1$, which will imply $\bigcap\limits_{n \geq 1} C_n \neq \emptyset$. Otherwise, assume $R > 0$. Using Proposition 3, there exists $c_n \in C_n$ such that $d_\rho(x, C_n) = \rho(x - c_n)$, for any $n \geq 1$. Similar argument as the one used in the proof of Proposition 3 will show that $\{c_n/2\}$ is ρ-Cauchy and converges to $c \in \ell_{p(\cdot)}$. Since $\{C_n\}$ is a decreasing sequence of ρ-closed subsets, we conclude that $2c \in \bigcap\limits_{n \geq 1} C_n$. Again this will show that $\bigcap\limits_{n \geq 1} C_n \neq \emptyset$ which completes the proof of Proposition 4. Moreover, using Fatou property, we note that

$$\rho(x - 2c) \leq \liminf_{m \to +\infty} \liminf_{n \to +\infty} \rho\left(x - \frac{c_n + c_m}{2}\right),$$

which will imply

$$d_\rho\left(x, \bigcap_{n \geq 1} C_n\right) = \lim_{n \to +\infty} d_\rho(x, C_n).$$

□

Remark 1. *Let us note that under the assumptions of Proposition 4, the conclusion still holds when we consider any family $\{C_\alpha\}_{\alpha \in \Gamma}$ of nonempty, convex, ρ-closed subsets of C, where $(\Gamma, \prec)$ is upward directed, such that there exists $x \in C$ which satisfies $\sup\limits_{\alpha \in \Gamma} d_\rho(x, C_\alpha) < \infty$. Indeed, set $d = \sup\limits_{\alpha \in \Gamma} d_\rho(x, C_\alpha)$. Without loss of generality, we may assume $d > 0$. For any $n \geq 1$, there exists $\alpha_n \in \Gamma$ such that*

$$d\left(1 - \frac{1}{n}\right) < d_\rho\ (x, C_{\alpha_n}) \leq d.$$

Since $(\Gamma, \prec)$ is upward directed, we may assume $\alpha_n \prec \alpha_{n+1}$ which implies $C_{\alpha_{n+1}} \subset C_{\alpha_n}$. Proposition 4 implies $C_0 = \bigcap\limits_{n \geq 1} C_{\alpha_n} \neq \emptyset$. Clearly C_0 is ρ-closed and using the last noted point in the proof of Proposition 4, we obtain

$$d_\rho(x, C_0) = \lim_{n \to +\infty} d_\rho(x, C_{\alpha_n}) = \sup_{n \geq 1} d_\rho(x, C_{\alpha_n}) = d.$$

Let $c_0 \in C_0$ such that $d_\rho(x, C_0) = \rho(x - c_0)$. We claim that $c_0 \in C_\alpha$, for any $\alpha \in \Gamma$. Indeed, fix $\alpha \in \Gamma$. If for some $n \geq 1$ we have $\alpha \prec \alpha_n$, then obviously we have $c_0 \in C_{\alpha_n} \subset C_\alpha$. Therefore let us assume that $\alpha \not\prec \alpha_n$, for any $n \geq 1$. Since Γ is upward directed, there exists $\beta_n \in \Gamma$ such that $\alpha_n \prec \beta_n$ and $\alpha \prec \beta_n$, for any $n \geq 1$. We can also assume that $\beta_n \prec \beta_{n+1}$ for any $n \geq 1$. Again we have $C_1 = \bigcap\limits_{n \geq 1} C_{\beta_n} \neq \emptyset$. Since $C_{\beta_n} \subset C_{\alpha_n}$, for any $n \geq 1$, we obtain $C_1 \subset C_0$. Moreover we have

$$d = d_\rho(x, C_0) \leq d_\rho(x, C_1) = \sup_{n \geq 1} d_\rho(x, C_{\beta_n}) \leq d.$$

Hence, $d_\rho(x, C_1) = d$ which implies the existence of a unique point $c_1 \in C_1$ such that $d_\rho(x, C_1) = \rho(x - c_1) = d$. Since ρ is (SC) on C_f, we obtain $c_0 = c_1$. In particular, we have $c_0 \in C_{\beta_n}$, for any $n \geq 1$. Since $\alpha \prec \beta_n$, we conclude that $C_{\beta_n} \subset C_\alpha$, for any $n \geq 1$, which implies $c_0 \in C_\alpha$. Since α was taking arbitrary in Γ, we obtain $c_0 \in \bigcap_{\alpha \in \Gamma} C_\alpha$, which implies $\bigcap_{\alpha \in \Gamma} C_\alpha \neq \emptyset$ as claimed.

The next result is necessary to obtain the fixed-point theorem sought for ρ-nonexpansive mappings.

Proposition 5. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ are not identically equal to 1. Let $f : (0, +\infty) \to (1, 2]$ be a nondecreasing function. Then C_f has the ρ-normal structure property, i.e., for any nonempty ρ-closed convex ρ-bounded subset C of ℓ_f not reduced to one point, there exists $x \in C$ such that*

$$\sup_{y \in C} \rho(x - y) < \delta_\rho(C).$$

Proof. Let C be a ρ-closed convex ρ-bounded subset C of C_f not reduced to one point. Since C is not reduced to one point, we have $\delta_\rho(C) > 0$. Let $x, y \in C$ such that $x \neq y$. Set

$$\varepsilon_0 = \frac{1}{\delta_\rho(C)}\, \rho\left(\frac{x - y}{2}\right) > 0.$$

Fix $c \in C$. Using Lemma 3, we have $x - c$ and $y - c$ are in $C_f - C_f \subset C_g$, where $g(\alpha) = f(\alpha/4)$, for any $\alpha > 0$. So far we have

$$\max\Big\{\rho(x - c), \rho(y - c)\Big\} \leq \delta_\rho(C) \;\; \textit{and} \;\; \rho\left(\frac{x - y}{2}\right) \geq \delta_\rho(C)\, \varepsilon_0.$$

Theorem 2 implies

$$\rho\left(c - \frac{x + y}{2}\right) \leq \delta_\rho(C)\, \Big(1 - \delta_{2, C_g}(R, \varepsilon_0)\Big).$$

Since c was taken arbitrary in C, we conclude that

$$\sup_{c \in C}\, \rho\left(c - \frac{x + y}{2}\right) \leq \delta_\rho(C)\, \Big(1 - \delta_{2, C_g}\big(\delta_\rho(C), \varepsilon_0\big)\Big) < \delta_\rho(C) > 0.$$

Therefore the proof of Proposition 5 is complete. □

Putting all this together, we are ready to prove the main fixed-point result of our work.

Theorem 3. *Consider the vector space $\ell_{p(.)}$ such that $p^- = 1$ and $p(\cdot)$ are not identically equal to 1. Let C be a nonempty ρ-closed convex ρ-bounded subset of $\ell_{p(.)}$, which satisfies the condition (UD). Any ρ-nonexpansive mapping $T : C \to C$ has a fixed point.*

Proof. Since C satisfies the condition (UD), Proposition 2 secures the existence of a nondecreasing function $f : (0, +\infty) \to (1, 2]$ such that C is a subset of C_f. The conclusion is trivial if C is reduced to one point. Therefore, we will assume that C is not reduced to one point, i.e., $\delta_\rho(C) > 0$. Consider the family

$$\mathcal{F} = \{K \subset C,\; K \neq \emptyset,\; \rho - \textit{closed convex and}\; T(K) \subset K\}$$

The family $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. Since C is bounded, we use Remark 1 to be able to use Zorn's lemma and conclude that $\mathcal{F}$ contains a minimal element K_0. Let us show that K_0 is reduced to one point. Assume not, i.e., K_0 contains more than one point. Set $co(T(K_0))$ to be the intersection of all ρ-closed convex subset of C containing $T(K_0)$. Hence $co(T(K_0)) \subset K_0$ since $K_0 \in \mathcal{F}$. Moreover, we have

$$T(co(T(K_0))) \subset T(K_0) \subset co(T(K_0)),$$

which implies that $co(T(K_0)) \in \mathcal{F}$. K_0 being a minimal element of $\mathcal{F}$ we deduce that $K_0 = co(T(K_0))$. Using Proposition 5, we deduce the existence of $x_0 \in K_0$ such that

$$r_0 = \sup_{y \in K_0} \rho(x_0 - y) < \delta_\rho(K_0).$$

Define the subset $K = \left\{ x \in K_0,\ \sup_{y \in K_0} \rho(x - y) \le r_0 \right\}$. K is not empty since $x_0 \in K$. Note that we have $K = \bigcap_{y \in K_0} B_\rho(y, r_0) \cap K_0$. Using the properties of modular balls, K is a ρ-closed and convex subset of K_0. Next, we prove that $T(K) \subset K$. Indeed, let $x \in K$. Since T is ρ-nonexpansive, we have

$$\rho(T(x) - T(y)) \le \rho(x - y) \le r_0,$$

for all $y \in K_0$. So we have $T(y) \in B_\rho(T(x), r_0) \cap K_0$, which implies $T(K_0) \subset B_\rho(T(x), r_0)$. Since $K_0 = co(T(K_0))$, we conclude that $K_0 \subset B_\rho(T(x), r_0)$, which implies

$$\rho(T(x) - y) \le r_0,$$

for all $y \in K_0$. Hence $T(x) \in K$. Since x was taken as arbitrary in K, we obtain $T(K) \subset K$. The minimality of K_0 will force $K = K_0$. Hence

$$r_0 < \delta_\rho(K_0) = \delta_\rho(K) \le r_0.$$

This is a contradiction. Therefore, K_0 is reduced to one point and it is a fixed point of T because $T(K_0) \subset K_0$. □

Remark 2. *In Theorem 3, the condition (UD) can be replaced by the following condition which is slightly more general:*

there exists $x_0 \in \ell_{p(.)}$ such that $x_0 + C$ satisfies the condition (UD).

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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