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A Note on Exponential Stability for Numerical Solution of Neutral Stochastic Functional Differential Equations

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Abstract: This paper examines the numerical solutions of the neutral stochastic functional differential equation. This study establishes the discrete stochastic Razumikhin-type theorem to investigate the exponential stability in the mean square sense of the Euler–Maruyama numerical solution to this equation. In addition, the Borel–Cantelli lemma and the stochastic analysis theory are incorporated to discuss the almost sure exponential stability for this numerical solution of such equations.

Keywords: neutral stochastic functional differential equations; Euler–Maruyama method; discrete Razumikhin-type theorem; exponential stability; numerical solution

MSC: 60H10; 34F05; 65C20; 65C30

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1. Introduction

Some realistic dynamical systems are not only concerned with the present and past states but also the derivatives of the past states, which are mathematically characterized by neutral functional differential equations (NFDEs) in [1]. Such equations have been demonstrated to have wide applications in many fields, such as heat exchangers, chemical reaction processes, lossless transmission lines, partial element equivalent circuits, population dynamics and distributed networks in [1]. When environmental perturbation is considered, NFDEs are developed into neutral stochastic functional differential equations (NSFDEs); for more details, see [2] and the references therein. The existence, uniqueness, stability analysis, and boundedness for the solutions of NSFDEs have been investigated over the past few decades; see [3–6] and references therein. For example, in [5], by using the Picard iterative method, the existence and uniqueness of the solutions of NSFDEs have been studied under a global Lipschitz condition and a linear growth condition for the drift term and the diffusion, as well as a contractive condition for the neutral term. In [6], by establishing the stochastic version of the Razumikhin-type theorem, the exponential stability in moment for NSFDEs was investigated, and under one additional condition, the almost surely exponential stability was obtained.

Generally speaking, the explicit solutions of nonlinear stochastic differential equations (SDEs) cannot be found. Fortunately, such difficulty can be overcome with numerical solution of nonlinear SDEs. In [7], Mao and Yuan have contributed to present some classical numerical schemes for nonlinear SDEs, such as the Euler–Maruyama (EM) scheme, the stochastic θ scheme, and the Milstein scheme. More recently, in [8], Mao developed a truncated EM method for SDEs. In [9], Nguyen et al. proved the convergence of a numerical solution for hybrid SDEs with Markovian switching by using the tamed-Euler method. In [10], the convergence of Euler-type methods for nonlinear SDEs was investigated. In [11], by using the discrete Razumikhin-type technique, the stability analysis of the EM numerical solution of SFDEs was studied.

As an important type of SDEs, NSFDEs do not have explicit solutions, and we have found content with an approximation via a numerical approach. In recent years, some numerical schemes such as the EM method [12], tamed EM method [13], and stochastic θ method [14] have been proposed to discuss the convergence and the stability analysis for a numerical solution of NSFDEs. In [12], Wu and Mao discussed the convergence of the EM numerical solution for NSFDEs. In [13], Deng et al. proposed two types of explicit tamed EM schemes for NSFDEs with superlinearly growing drift and diffusion coefficients to analyze the exponential stability in moment for the numerical solution. In [14], Li and Yang studied the exponential stability in moment and the almost surely exponential stability of the EM numerical solution of NSFDEs with jumps.

The stochastic Razumikhin theorem for stochastic functional differential equations (SFDEs) was initially established in [4] and has been generalized from the continuous version to the discrete version. For example, in [11], by establishing the discrete Razumikhin-type theorem, the exponential stability in moment and the almost sure exponential stability of the EM scheme of SFDEs were investigated. In [15], the discrete stochastic version of the Razumikhin-type theorem was used to analyze the exponential stability in moment, the almost sure exponential stability for the EM scheme, and the backward EM scheme of nonlinear stochastic pantograph differential equations. Note that the continuous stochastic version of the Razumikhin-type theorem was established to investigate the exponential stability in moment for the exact solution of NSFDEs in [5], and such an excellent theorem has been widely used to discuss the stability of NSFDEs in [16]. As far as we know, there is no work on the discrete stochastic version of Razumikhin-type theorem to analyze the stability of the EM scheme for NSFDEs. In this paper, we try to close this gap.

The remainder of this paper is structured as follows. In Section 2, some necessary notations and definitions are introduced. In Section 3, some necessary assumptions and auxiliary results are presented and the discrete Razumikhin-type theorem is established. In Section 4, we give specific examples to reflect the reasonableness of the theorem.

2. Notations and Preliminaries

Unless otherwise indicated in this paper, some notations are used. Let $|\cdot|$ be the Euclidean norm in R^d. If A is a vector or matrix, A^T is the transpose of A. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. The inner product of x, y in R^d is denoted by $\langle x, y\rangle$ or x^Ty. If $x, y \in R$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. $\mathbb{M}$ represents the set of the non-negative integer numbers, namely, $\mathbb{M} = \{0, 1, \ldots, \}$, and for a positive integer M_0, $\mathbb{M}_{-M_0} = \{0, -1, -2, \ldots, -M_0\}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B^1(t), \ldots, B^m(t))^T$ be an m-dimensional Brownian motion defined on this probability space. For any $r_0 > 0$, $C := C([-r_0, 0], R^d)$ denotes the family of bounded continuous functions φ from $[-r_0, 0]$ to R^d with uniform norm $\|\varphi\| := \sup_{-r_0\leq v\leq 0} |\varphi(v)|$. Let $C^b_{\mathcal{F}_0}([-r_0, 0]; \mathbb{R}^d)$ denote the family of all $\mathcal{F}_0$-measurable and $\mathscr{C}$-valued random processes. $L^2_{\mathcal{F}_t}([-r_0, 0]; \mathbb{R}^d)$ be the family of all $\mathcal{F}_t$-measurable random processes $\varphi = \{\varphi(v) : -r_0 \leq v \leq 0\}$ such that $\mathbb{E}||\varphi||^p := \mathbb{E}\sup_{-r_0\leq v\leq 0} |\varphi(v)|^p < \infty$. If $x(t)$ is an $\mathbb{R}^d$-valued random process on $t \in [-r_0, \infty)$, we let $x_t = \{x(t + v) : -r_0 \leq v \leq 0\}$ for any $t \geq 0$.

In this paper, we shall consider the following d-dimensional NSFDEs:

$$d[x(t) - D(x_t)] = f(x_t)dt + g(x_t)dB(t),\ t \geq 0, \tag{1}$$

with initial value $x_0 = \xi \in C^b_{\mathcal{F}_0}([-r_0, 0]; R^d)$, namely, where the state $x(t) \in R^d$, $x_t = \{x(t + v) : -r_0 \leq v \leq 0\}$, the neutral term $D(\cdot) : \mathscr{C} \to R^d$, the drift term $f(\cdot, \cdot) :\ [0, +\infty) \times \mathscr{C} \to R^d$ and the diffusion term $g(\cdot, \cdot) :\ [0, +\infty) \times \mathscr{C} \to R^{d\times m}$ are some appropriate Borel measurable functions. Denote by $C^2(\mathbb{R}^d; [0, +\infty))$ the family of all nonnegative functions $V(x)$ on $\mathbb{R}^d$, which are continuously twice differentiable. For $\varphi \in \mathscr{C}$, let $\tilde{\varphi} = \varphi(0) - D(\varphi)$. Give $V \in C^2(\mathbb{R}^d; [0, +\infty))$, we define an operator $\mathcal{L}V$ from $\mathbb{R}_+ \times \mathscr{C}$ to $\mathbb{R}$ by':

$$\mathcal{L}V(\tilde{\varphi}) = V_x(\tilde{\varphi})f(t, \varphi) + \frac{1}{2}\mathrm{trace}[g^T(t, \varphi)V_{xx}(\tilde{\varphi})g(t, \varphi)]. \tag{2}$$

We often write $x(t)$ by $x(t,\xi)$ to represent the dependence on the initial data ξ.

3. Stability Analysis of the EM Numerical Solution of NSFDEs

In the following, we impose some hypotheses that are necessary.

(H1) For all $\varphi, \bar{\varphi} \in \mathscr{C}$, assume there exists a positive constant L such that:

$$|f(\varphi) - f(\bar{\varphi})|^2 \vee |g(\varphi) - g(\bar{\varphi})|^2 \leq L\|\varphi - \bar{\varphi}\|^2.$$

(H2) For all $\varphi \in \mathscr{C}$, there exists a constant $K > 0$, such that:

$$|f(\varphi)|^2 \vee |g(\varphi)|^2 \leq K(1 + \|\varphi\|^2).$$

(H3) Assume $D(\mathbf{0}) = 0$ and there exists a constant $a \in (0,1)$ such that for all $\varphi, \bar{\varphi} \in \mathscr{C}$,

$$|D(\varphi) - D(\bar{\varphi})| \leq a\|\varphi - \bar{\varphi}\|. \tag{3}$$

(H1)–(H3) are called the local Lipschitz condition, the linear growth condition, and the contractive condition, respectively. By [5] (Theorem 2.5, p. 209), these conditions guarantee the existence and uniqueness of the exact solution of NSFDEs (1). For the purpose of stability analysis, it is assumed that $f(\mathbf{0}) = 0$ and $g(\mathbf{0}) = 0$. This implies that Equation (1) admits a trivial solution $x(t,\mathbf{0}) \equiv 0$.

3.1. The EM Numerical Solutions of NSFDEs

The EM numerical solutions of SFDEs and NSFDEs were first introduced by Mao [4] and Hu et. al. [17], respectively. Let us recall the EM numerical scheme for NSFDE (1). Let the step size h be a fraction of the delay r_0, that is, $h = r_0/M_0$ for some integer M_0. After that, by using the EM method, (1) has the following approximation:

$$\begin{cases} \overline{Y}(nh) = \xi(nh), & -M_0 \leq n \leq 0 \\ \overline{Y}((n+1)h) - D(\overline{Y}_{nh}) = \overline{Y}(nh) - D(\overline{Y}_{(n-1)h}) + f(\overline{Y}_{nh})h + g(\overline{Y}_{nh})\Delta B_n, & n \geq 0, \end{cases} \tag{4}$$

where $\Delta B_n = B((n+1)h) - B(nh)$ is the Brownian motion increment and $\overline{Y}_{nh} = \{\overline{Y}_{nh}(v) : -r_0 \leq v \leq 0\}$ is a $\mathscr{C}$-valued random variable defined by piecewise linear interpolation:

$$\overline{Y}_{nh}(v) = \overline{Y}((n+l)h) + \frac{v - lh}{h}[\overline{Y}(n+l+1)h - \overline{Y}(n+l)h] \tag{5}$$

for $lh \leq v \leq (l+1)h$, $l = -M_0, -M_0+1, \ldots, -1$. In order for $\overline{Y}_{-h}$ to be well defined, we set $\overline{Y}(-(M_0+1)h) = \xi(-M_0 h)$. Equation (5) can be written as:

$$\overline{Y}_{nh}(v) = \frac{h - (v - lh)}{h}\overline{Y}((n+l)h) + \frac{v - lh}{h}\overline{Y}((n+l+1)h), \tag{6}$$

which yields that:

$$\|\overline{Y}_{-h}\| \leq \|\overline{Y}_0\| \text{ and } \|\overline{Y}_{nh}\| = \max_{-M_0 \leq l \leq 0} |\overline{Y}((n+l)h)|, \ \forall n \geq 0. \tag{7}$$

In order to use continuous-time approximation in the future, we now introduce the $\mathscr{C}$-valued step process:

$$\overline{Y}_t = \sum_{n=0}^{\infty} \overline{Y}_{nh} I_{[nh,(n+1)h)}(t), \tag{8}$$

and define the continuous EM approximate solution as follows:

(i) for $-r_0 \leq t \leq 0$, $Y(t) = \xi(t)$;

(ii) for $t \in [nh,(n+1)h]$, $n \geq 0$,

$$
\begin{aligned}
Y(t) = \xi(0) + D\Big(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})\Big) - D(\overline{Y}_{-h}) \\
+ \int_0^t f(\overline{Y}_s)ds + \int_0^t g(\overline{Y}_s)dB(s).
\end{aligned} \tag{9}
$$

Clearly, (9) can also be written as:

$$
\begin{aligned}
Y(t) = \overline{Y}(nh) + D\Big(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})\Big) - D(\overline{Y}_{(n-1)h}) \\
+ \int_{nh}^t f(\overline{Y}_s)ds + \int_{nh}^t g(\overline{Y}_s)dB(s).
\end{aligned} \tag{10}
$$

Especially, one can see that $Y(nh) = \overline{Y}(nh)$, which illustrates the coincidence of the continuous and discrete EM approximate solutions at the grid points. Because we know that $Y(t)$ is not computable, not only just its h-increments, but it also requires knowledge of the entire Brownian path. However, $Y(nh) = \overline{Y}(nh)$, thus the error bound for $Y(t)$ will automatically imply the error bound for $\overline{Y}(nh)$. Then, it is quite obvious that:

$$
\|\overline{Y}_t\| \le \sup_{-r_0 \le s \le t} |Y(s)| \text{ and } \|\overline{Y}_{nh}|| \le \|Y_{nh}\|, \forall n \ge 0. \tag{11}
$$

These properties will frequently be applied to later proofs, without more explanation. The following definition is needed for the stability of the EM scheme.

Definition 1. *Given step size $h > 0$ and any bounded initial sequence $\{\xi(nh)\}_{n \in \mathbb{M}_{-M_0}}$ if*

$$
\lim_{n\to\infty} \sup \frac{1}{nh} \log \mathbb{E}|\overline{Y}(nh)|^2 < 0,
$$

then the EM scheme is said to be exponentially stable in the mean square, and if

$$
\lim_{n\to\infty} \sup \frac{1}{nh} \log |\overline{Y}(nh)| < 0, \;\; a.s.
$$

then the EM scheme is said to be is exponentially almost surely stable.

3.2. Stability of Numerical Solutions of NSFDEs

In this subsection, we establish the discrete version of the Razumikhin-type theorem on exponential stability of the EM scheme (4).

Lemma 1. *Let Assumption (H3) hold. Then for $t \in [nh, (n+1)h]$ and $n \in \mathbb{M} \cup \mathbb{M}_{-M_0}$. It holds that:*

$$
\left| D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})) \right|^2 \le a^2\Big(\|\overline{Y}_{nh}\|^2 \vee \|\overline{Y}_{(n-1)h}\|^2\Big),
$$

and

$$
\left| \overline{Y}_{nh} - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})) \right|^2 \le (1+a)^2 \sup_{-r_0 \le s \le 0} |Y(s)|^2.
$$

Since the proof of the Lemma is standard, we omit it here.

Lemma 2. *Let n^* be any positive integer and $0 < \lambda_h < r_0^{-1} \log(\frac{1}{3a^2})$. If:*

$$
e^{\lambda_h nh} E\left| \overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})) \right|^2 \le (1+a)^2 E\|\xi(s)\|^2 \tag{12}
$$

holds for $t \in [nh, (n+1)h)$, all $0 \le n \le n^*$ and $n \in \mathbb{M} \cup \mathbb{M}_{-M_0}$, then:

$$
e^{\lambda_h nh} E|\overline{Y}(nh)|^2 \le \frac{(1+a)^2 + 1}{(1 - \sqrt{3}ae^{\frac{\lambda_h r_0}{2}})^2} E\|\xi(s)\|^2, \qquad \forall 0 \le n \le n^*.
$$

Proof. Since $0 < \lambda_h < r_0^{-1}\log(\frac{1}{3a^2})$, we can choose an ε such that $3a^2 e^{\lambda_h r_0} < \varepsilon < 1$. For $0 \le n \le n^*$, $n \in \mathbb{M}$, noting that:

$$\begin{aligned}
&\left|\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right|^2 \\
&\ge |\overline{Y}(nh)|^2 - 2\Big(|\overline{Y}(nh)||D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))|\Big) \\
&+ \left|D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right|^2 \\
&\ge (1-\varepsilon)|\overline{Y}(nh)|^2 - (\varepsilon^{-1}-1)\left|D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right|^2,
\end{aligned} \tag{13}$$

we have:

$$\begin{aligned}
&|\overline{Y}(nh)|^2 \\
&\le \frac{1}{1-\varepsilon}\left|\overline{Y}_{nh} - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right|^2 + \frac{a^2}{\varepsilon}\Big(\|\overline{Y}_{nh}\|^2 \vee \|\overline{Y}_{(n-1)h}\|^2\Big).
\end{aligned} \tag{14}$$

By the condition (12), we then derive that for all $0 \le n \le n^*$:

$$\begin{aligned}
&E\sup_{0\le n\le n^*} e^{\lambda_h nh}|\overline{Y}(nh)|^2 \\
&\le \frac{1}{1-\varepsilon}E\sup_{0\le n\le n^*}\left[e^{\lambda_h nh}\left|\overline{Y}_{nh} - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right|^2\right] \\
&+ \frac{a^2}{\varepsilon}E\sup_{0\le n\le n^*}\left[e^{\lambda_h nh}\Big(\|\overline{Y}_{nh}\|^2 \vee \|\overline{Y}_{(n-1)h}\|^2\Big)\right] \\
&\le \frac{(1+a)^2}{1-\varepsilon}\sup_{-r_0\le s\le 0} E\|\xi\|^2 \\
&+ E\sup_{0\le n\le n^*}\left[\frac{a^2}{\varepsilon}\left(\sup_{-r_0\le v\le 0} e^{\lambda_h nh}|\overline{Y}(nh+v)|^2 \vee \sup_{-r_0\le v\le 0} e^{\lambda_h nh}|\overline{Y}((n-1)h+v)|^2\right)\right] \\
&\le \frac{(1+a)^2}{1-\varepsilon}\sup_{-r_0\le s\le 0} E\|\xi\|^2 + E\sup_{0\le n\le n^*}\left[\frac{a^2}{\varepsilon}\left(\sup_{-r_0\le v\le 0} e^{\lambda_h((nh+v)-v)}|\overline{Y}(nh+v)|^2\right.\right. \\
&\qquad\qquad \left.\left.\vee \sup_{-r_0\le v\le 0} e^{\lambda_h((n-1)h+v+h-v)}|\overline{Y}((n-1)h+v)|^2\right)\right] \\
&\le \frac{(1+a)^2}{1-\varepsilon}\sup_{-r_0\le s\le 0} E\|\xi\|^2 + E\sup_{0\le n\le n^*}\left[\frac{a^2}{\varepsilon}e^{\lambda_h r_0}\left(\sup_{-r_0\le v\le 0} e^{\lambda_h(nh+v)}|\overline{Y}(nh+v)|^2\right.\right. \\
&\qquad\qquad \left.\left.\vee \sup_{-r_0\le v\le 0} e^{\lambda_h((n-1)h+v)}e^{\lambda_h h}|\overline{Y}((n-1)h+v)|^2\right)\right] \\
&\le \frac{(1+a)^2}{1-\varepsilon}\sup_{-r_0\le s\le 0} E\|\xi\|^2 \\
&+ \frac{a^2}{\varepsilon}e^{\lambda_h r_0}E\left(\sup_{-r_0\le n\le n^*} e^{\lambda_h nh}|\overline{Y}(nh)|^2\right) + \frac{a^2 e^{\lambda_h h}}{\varepsilon}e^{\lambda_h r_0}E\left(\sup_{-r_0\le n\le n^*} e^{\lambda_h nh}|\overline{Y}(nh)|^2\right) \\
&\le \left[\frac{(1+a)^2}{1-\varepsilon} + \frac{a^2}{\varepsilon}e^{\lambda_h r_0}(1+e^{\lambda_h h})\right]\sup_{-r_0\le s\le 0} E\|\xi(s)\|^2 \\
&\qquad + \frac{a^2}{\varepsilon}e^{\lambda_h r_0}(1+e^{\lambda_h h})E\left(\sup_{0\le n\le n^*} e^{\lambda_h nh}|\overline{Y}(nh)|^2\right).
\end{aligned}$$

Moreover, this holds for all $-M_0 \le n \le 0$ as well. Therefore,

$$
\begin{aligned}
&E \sup_{-M_0 \le n \le n^*} e^{\lambda_h nh} |\overline{Y}(nh)|^2 \\
&\le \frac{(1+a)^2}{1-\varepsilon} \sup_{-r_0 \le s \le 0} E\|\xi(s)\|^2 \\
&+ \frac{a^2}{\varepsilon} e^{\lambda_h r_0}(1 + e^{\lambda_h h}) E \sup_{-M_0 \le n \le n^*} e^{\lambda_h nh} |\overline{Y}(nh)|^2.
\end{aligned}
\tag{15}
$$

Choosing h to be sufficiently small such that $e^{\lambda_h nh} < 2$ and noting $1 > 3a^2 e^{\lambda_h r_0}/\varepsilon$, we obtain that:

$$
\begin{aligned}
&\sup_{-M_0 \le n \le n^*} e^{\lambda_h nh} E|\overline{Y}(nh)|^2 \\
&\le E \sup_{-M_0 \le n \le n^*} e^{\lambda_h nh} |\overline{Y}(nh)|^2 \\
&\le \frac{\varepsilon(1+a)^2}{(1-\varepsilon)(\varepsilon - a^2 e^{\lambda_h r_0}(1 + e^{\lambda_h h}))} \sup_{-r_0 \le s \le 0} E|Y(s)|^2 \\
&\le \frac{\varepsilon(1+a)^2}{(1-\varepsilon)(\varepsilon - 3a^2 e^{\lambda_h r_0})} \sup_{-r_0 \le s \le 0} E|Y(s)|^2.
\end{aligned}
\tag{16}
$$

Finally, the required assertion follows by taking $\varepsilon = \sqrt{3} a e^{\frac{\lambda_h r_0}{2}}$. □

Theorem 1. *Let Assumptions* $(H1)$–$(H3)$ *hold. Fix* $h > 0$*. Let* ζ_h*,* p_h*,* a*,* γ *all be positive constants,* $q_h > 1$*,* $h\zeta_h < 1$*,* $a \in (0,1)$ *and* $0 < \gamma < r_0^{-1} \log(\frac{1}{3a^2})$*. For* $t \in [nh, (n+1)h]$*,* $n \in \mathbb{M}$*, assume that there exists a function* $V_h : R^d \to R_+$ *such that* $V(x) \le c_h |x|^2$ *and the following conditions hold:*

(1) *for all* $l \in \mathbb{M}_{-M_0}$*,*

$$
\begin{aligned}
&EV_h\Big(\overline{Y}(n+l)h - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\Big) \\
&\le q_h EV_h\Big(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\Big)
\end{aligned}
$$

implies that:

$$
\begin{aligned}
&EV_h\Big(\overline{Y}(n+1)h - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\Big) \\
&\le (1-\zeta_h) EV_h\Big(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\Big).
\end{aligned}
$$

(2) *for some* $l \in \mathbb{M}_{-M_0} - \{0\}$*,*

$$
\begin{aligned}
&EV_h\Big(\overline{Y}(n+l)h - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\Big) \\
&> e^{\lambda_h h} EV_h\Big(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\Big)
\end{aligned}
$$

implies that:

$$
\begin{aligned}
&EV_h\Big(\overline{Y}(n+1)h - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\Big) \\
&\le \frac{1}{q_h} \max_{l \in M_{-M_0}} \Big\{ EV_h\Big(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\Big)\Big\},
\end{aligned}
$$

where $\lambda_h = [\frac{\log q_h}{((M_0+1)h)}] \wedge [\frac{\log(1-\zeta_h h)^{-1}}{h}]$*.*

Then, for any bounded initial data ξ*, there exists* $q > (1-\sqrt{3}a)^{-2}$ *such that:*

$$
E|\overline{Y}(nh)|^2 \le q(1+a)^2 e^{-\lambda_h nh} E||\xi||^2, \tag{17}
$$

for all $n \geq 0$*, namely, the sequence* $\{\overline{Y}(nh)\}_{n\geq 1}$ *is exponentially stable in terms of the mean square.*

Proof. For any $n \in \mathbb{M}$, define the sequence:

$$u_n = \max_{l \in \mathbb{M}_{-M_0}} \left\{ e^{\lambda_h (n+l)h} EV_h \left(\overline{Y}(n+l)h - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h})) \right) \right\}$$

and:

$$\bar{l} = \bar{l}(n) = \max\{l \in \mathbb{M}_{-M_0} : u_l = u_n\}.$$

This implies:

$$u_n = e^{\lambda_h (n+\bar{l})h} EV_h \left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h})) \right).$$

It will be shown that $u_{n+1} \leq u_n$.
When $\bar{l} \leq -1$, for any $l \in \mathbb{M}_{-M_0} - \{0\}$,

$$\begin{aligned} & e^{\lambda_h (n+l+1)h} EV_h \left(\overline{Y}((n+l+1)h) - D(\overline{Y}_{(n+l)h} + \frac{t-(n+l+1)h}{h}(\overline{Y}_{(n+l+1)h} - \overline{Y}_{(n+l)h})) \right) \\ & \leq e^{\lambda_h (n+\bar{l})h} EV_h \left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h})) \right), \end{aligned} \tag{18}$$

which implies that:

$$\begin{aligned} & \max_{l \in \mathbb{M}_{-M_0} - \{0\}} \{u_{l,n}\} \\ & \leq e^{\lambda_h (n+\bar{l})h} EV_h \left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h})) \right) \\ & = u_n, \end{aligned} \tag{19}$$

where:

$$u_{l.n} = e^{\lambda_h (n+l+1)h} EV_h \left(\overline{Y}((n+l+1)h) - D(\overline{Y}_{(n+l)h} + \tfrac{t-(n+l+1)h}{h}(\overline{Y}_{(n+l+1)h} - \overline{Y}_{(n+l)h})) \right).$$

Additionally, we can show that,

$$\begin{aligned} & e^{\lambda_h (n+1)h} EV_h \left(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh})) \right) \\ & \leq e^{\lambda_h (n+\bar{l})h} EV_h \left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h})) \right) \\ & = u_n. \end{aligned} \tag{20}$$

In fact, by the definition of $\bar{l}$,

$$\begin{aligned} u_n &= e^{\lambda_h (n+\bar{l})h} EV_h \left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h})) \right) \\ & > e^{\lambda_h nh} EV_h \left(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h})) \right). \end{aligned}$$

This, together with $\bar{l} \leq -1$, yields that:

$$
\begin{aligned}
&EV_h\left(\overline{Y}((n+\bar{l})h) - D(\overline{Y}_{(n+\bar{l}-1)h} + \frac{t-(n+\bar{l})h}{h}(\overline{Y}_{(n+\bar{l})h} - \overline{Y}_{(n+\bar{l}-1)h}))\right) \\
&> e^{-\lambda_h \bar{l} h} EV_h\left(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right) \\
&\geq e^{\lambda_h h} EV_h\left(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right).
\end{aligned}
$$

By the definition of λ_h and condition in (2), we obtain:

$$
\begin{aligned}
&\max_{l \in \mathbb{M}_{-M_0}} EV_h\left(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\right) \\
&\geq q_h EV_h\left(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\right) \\
&\geq e^{\lambda_h (M_0+1)h} EV_h\left(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\right).
\end{aligned}
$$

Immediately, one can see that:

$$
\begin{aligned}
&e^{\lambda_h (n+1)h} EV_h\left(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\right) \\
&\leq e^{\lambda_h (n+1)h} e^{-(M_0+1)\lambda_h h} \max_{l \in \mathbb{M}_{-M_0}} \Big\{ EV_h(\overline{Y}((n+l)h) \\
&\qquad - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h})) \Big\} \\
&\leq \max_{l \in \mathbb{M}_{-M_0}} \left\{ e^{\lambda_h (n-M_0)h} EV_h\left(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\right) \right\} \\
&\leq \max_{l \in \mathbb{M}_{-M_0}} \left\{ e^{\lambda_h (n+l)h} EV_h\left(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\right) \right\} \\
&= u_n.
\end{aligned}
$$

Hence, (20) holds. This, together with (19), yields that:

$$
u_{n+1} \leq u_n, \ \text{when } \bar{l} \leq -1. \tag{21}
$$

If $\bar{l} = 0$, by the definition of $\bar{l}$, then for any $l \in \mathbb{M}_{-M_0}$,

$$
\begin{aligned}
&e^{\lambda_h (n+l)h} EV_h\left(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\right) \\
&\leq e^{\lambda_h nh} EV_h\left(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\right).
\end{aligned}
$$

Hence, by the definition of λ_h,

$$\begin{aligned}
&EV_h\bigg(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\bigg)\\
&\le e^{-\lambda_h lh} EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg)\\
&\le e^{\lambda_h M_0 h} EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg)\\
&\le e^{\lambda_h (M_0+1) h} EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg)\\
&\le q_h EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg).
\end{aligned} \tag{22}$$

The condition (1) gives:

$$\begin{aligned}
&EV_h\bigg(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\bigg)\\
&\le (1-\zeta_h h) EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg).
\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned}
&e^{\lambda_h nh} EV_h\bigg(\overline{Y}(nh) - D(\overline{Y}_{(n-1)h} + \frac{t-nh}{h}(\overline{Y}_{nh} - \overline{Y}_{(n-1)h}))\bigg)\\
&\ge e^{\lambda_h nh}(1-\zeta_h h)^{-1} EV_h\bigg(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\bigg)\\
&= e^{\lambda_h nh} e^{\frac{h log(1-\zeta_h h)^{-1}}{h}} EV_h\bigg(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\bigg)\\
&\ge e^{\lambda_h (n+1)h} EV_h\bigg(\overline{Y}((n+1)h) - D(\overline{Y}_{nh} + \frac{t-(n+1)h}{h}(\overline{Y}_{(n+1)h} - \overline{Y}_{nh}))\bigg).
\end{aligned}$$

Thus $u_{n+1} \le u_n$ also holds for $\bar{l} = 0$. Therefore, combining with (21) yields $u_{n+1} \le u_n$ for all $n \in \mathbb{M}$. This yields:

$$u_0 \ge e^{\lambda_h (n+l)h} EV_h\bigg(\overline{Y}((n+l)h) - D(\overline{Y}_{(n+l-1)h} + \frac{t-(n+l)h}{h}(\overline{Y}_{(n+l)h} - \overline{Y}_{(n+l-1)h}))\bigg),$$

By the definitions of $V_h(Y_n)$ and Lemma (1), for all $l \in \mathbb{M}_{-M_0}$,

$$\begin{aligned}
u_0 &= \max_{l\in\mathbb{M}_{-M_0}}\left\{e^{\lambda_h lh} EV_h\bigg(\overline{Y}(lh) - D(\overline{Y}_{(l-1)h} + \frac{t-lh}{h}(\overline{Y}_{lh} - \overline{Y}_{(l-1)h}))\bigg)\right\}\\
&\le \max_{l\in\mathbb{M}_{-M_0}}\left\{E\bigg|\overline{Y}(lh) - D(\overline{Y}_{(l-1)h} + \frac{t-lh}{h}(\overline{Y}_{lh} - \overline{Y}_{(l-1)h}))\bigg|^2\right\}\\
&\le (1+a)^2 \sup_{-r_0\le s\le 0} E|\xi(s)|^2.
\end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned}
e^{\lambda_h lh} E|\overline{Y}(lh)|^2 &\le \frac{(1+a)^2}{(1-\sqrt{3}ae^{\frac{\lambda_h r_0}{2}})^2} \sup_{-r_0\le s\le 0} E|\xi(s)|^2\\
&\le \frac{(1+a)^2}{(1-\sqrt{3}ae^{\frac{\lambda_h r_0}{2}})^2} E||\text{¸}||^2.
\end{aligned}$$

Therefore,

$$E|\overline{Y}(nh)|^2 \le \frac{(1+a)^2}{(1-\sqrt{3}ae^{\frac{\lambda_h r_0}{2}})^2} e^{-\lambda_h nh} E||\xi||^2.$$

Taking $q > (1 - \sqrt{3}a)^{-2}$, we then have:

$$E|\overline{Y}(nh)|^2 \le \frac{(1+a)^2}{(1 - \sqrt{3}ae^{\frac{\lambda_h r_0}{2}})^2} e^{-\lambda_h nh} E||¸||^2$$
$$\le q(1+a)^2 e^{-\lambda_h nh} E||¸||^2,$$

as required. □

The following theorem reveals that the EM scheme for (4) is exponentially almost surely stable.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then we have:*

$$\lim_{n\to\infty} \sup \frac{1}{nh} \log|\overline{Y}(nh)| \le -\frac{\lambda_h}{2}.$$

Since the proof is similar to that of [11], we omit it here.

4. Example

We now illustrate the theoretical results in Theorem 1 by the following example.

Example 1. *Let us think about the neutral stochastic delay differential equation as follows:*

$$d[y(t) - 0.1y(t-1)] = -y(t)dt + 0.5y(t-1)dB(t), t \ge 0, \quad (24)$$

with initial value $y_0 = \xi(v) = 1, v \in [-\tau, 0], \tau = 1$.

Denote $f(y_t, t) = -y(t)$ and $g(y_t, t) = 0.5y(t-1)$, so assumptions (H1–H2) hold. In addition, $u(y_t) = 0.1y(t-1)$, so assumptions (H3) holds. We plot the EM method of Equation (24) for step size $h = 0.001$. It can be seen that with the increase in n, y_n gradually stabilizes, the equation has unique global solution, and the solution for the Euler–Maruyama method is stable.

This clearly shows the efficiency of our numerical method. Owing to the convergence of Euler–Maruyama method, Figure 1 illustrates that the numerical solution has stability properties.

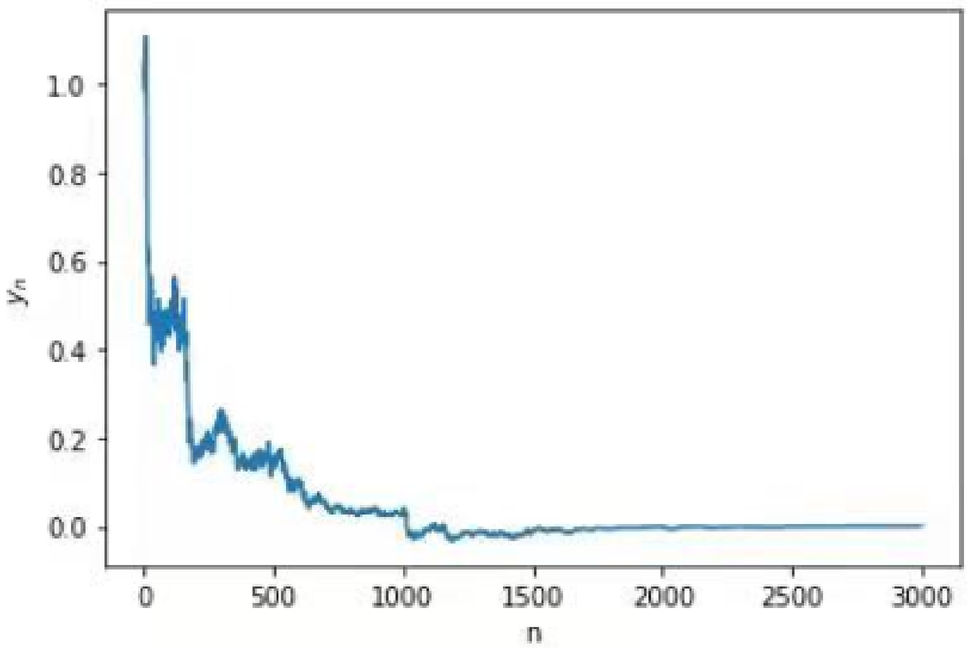

Figure 1. The simulation of the EM numerical solution of Equation (24).

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