

Article

An Efficient Technique of Fractional-Order Physical Models Involving ρ -Laplace Transform

Nehad Ali Shah ¹, Ioannis Dassios ², Essam R. El-Zahar ^{3,4} and Jae Dong Chung ^{1,*}¹ Department of Mechanical Engineering, Sejong University, Seoul 05006, Korea; nehadali199@yahoo.com² AMPSAS, University College Dublin, D04 V1W8 Dublin, Ireland; ioannis.dassios@ucd.ie³ Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia; er.elzahar@psau.edu.sa⁴ Department of Basic Engineering Science, Faculty of Engineering, Menoufia University, Shebin El-Kom 32511, Egypt

* Correspondence: jdchung@sejong.ac.kr

Abstract: In this article, the ρ -Laplace transform is paired with a new iterative method to create a new hybrid methodology known as the new iterative transform method (NITM). This method is applied to analyse fractional-order third-order dispersive partial differential equations. The suggested technique procedure is straightforward and appealing, and it may be used to solve non-linear fractional-order partial differential equations effectively. The Caputo operator is used to express the fractional derivatives. Four numerical problems involving fractional-order third-order dispersive partial differential equations are presented with their analytical solutions. The graphs determined that their findings are in excellent agreement with the precise answers to the targeted issues. The solution to the problems at various fractional orders is achieved and found to be correct while comparing the exact solutions at integer-order problems. Although both problems are the non-linear fractional system of partial differential equations, the present technique provides its solution sophisticatedly. Including both integer and fractional order issues, solution graphs are carefully drawn. The fact that the issues' physical dynamics completely support the solutions at both fractional and integer orders is significant. Moreover, despite using very few terms of the series solution attained by the present technique, higher accuracy is observed. In light of the various and authentic features, it can be customized to solve different fractional-order non-linear systems in nature.

Keywords: third-order dispersive equations; new iterative method; ρ -Laplace transform; approximate solution; caputo's derivative



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1. Introduction

Fractional calculus (FC) has been a significant field of applied sciences for a few decades. They model actual phenomena with fractional-order integral and derivative answers better than classical derivatives. In modelling many physical phenomena, certain signification implementation can be traced, especially genetics algorithm, signal processing, visco-elasticity damping, transport schemes, electronics, biology, communication, physics, robotics, finance, and chemistry. In the area of FC, scientists are focusing on many significant contributions and discoveries [1–4]. Due to its appealing applications, FC is a significant study subject for most scholars, and the analysis of fractional-order partial differential equations (PDEs) has drawn special interest from several areas. As a result, various approaches to solving linear and non-linear fractional PDEs have been developed. The local meshless technique, for example, is used to solve fractional-order and anomalous mobile-immobile result transport producers [5–9].

Recently, fractional-order differential equations (FPDES) have been studied to model various non-linear and complex phenomena in nature. FPDES make a novel contribution towards various scientific research areas, such as a vast range of systems and processes,

memories and other fields of science and engineering. The FPDE mathematical models, either for space or time, are both more accurate and they describe many natural phenomena [10–14]. FPDEs have gained the attention of many mathematicians in the field of control theory, dielectric polarization, anomalous diffusion and other problems in science and engineering [15,16]. To describe the properties related to hereditary and the memory of different processes and materials where the past state influences the future state, FPDEs offers an outstanding tool to explain such types of systems because PDEs of integer order does not work as appropriate as fractional order. FPDEs are used to model various scientific phenomena and problems; biology, chemistry, psychology and acoustics, mathematical physics, coloured noises and continuum mechanics. There are remarkable applications of FPDEs in different branches of physics and hydrology [17–19]. The main reason FPDEs have gained popularity is that, by nature, the fractional derivative is global while the derivative of integer order is local [20–25].

Researchers have a keen interest in developing analytical and numerical methods for the solutions to FPDEs. It is always difficult to find the numerical and analytical solutions to FPDEs. However, mathematicians have developed various novel techniques to handle the solutions of FPDEs effectively. Some of these methods include the haar wavelet method [26], the homotopy analysis method [27], the Laplace transform method [28], Bernstein polynomial [29], the Adomian decomposition method [30], the Legendre base method [31], the finite difference method [32,33] variational iteration method [34], Elzaki Decomposition method [35], differential transform method [36], and the natural transform decomposition method [37].

FPDEs are implemented in many physical models in different fields of applied sciences, such as mathematical biology, fluid dynamics, chemical kinetics, fluid dynamics, linear optics, and quantum mechanics. In 1895, the Vries and Korteweg-de introduced a non-dimensionalized model identified as the KdV equation. In many fields of science and technology, such as quantum theory and particle physics, this model is used to study dispersive wave phenomena. In KdV equations, there are two significant dispersive terms: third and fifth order. Plasma physics has been described using the KdV equation of order five [38].

Fahd and Abdeljawad recently published a paper on the fractional-order ρ -Laplace transform Caputo derivative. To examine fractional-order ODEs and PDEs in the sense of Caputo fractional derivative, we proposed a new iterative approach using ρ -Laplace transform. Many fractional-order differential equations, including linear and non-linear fractional-order Zakherov–Kuznetsov equations, diffusion equations and Fokker-Planck equations, are solved using this innovative approach.

2. Basic Definitions

The generalised fractional derivative, generalised fractional integral, Mittag–Leffler function, and ρ -Laplace transform are all described in this section.

Definition 1. A continuous function's generalised fractional-order integral $\delta g : [0, +\infty] \rightarrow R$ is expressed as [39]:

$$(I^{\delta, \rho} g)(\psi) = \frac{1}{\Gamma(\delta)} \int_0^\psi \left(\frac{\psi^\rho - s^\rho}{\rho} \right)^{\delta-1} \frac{g(s) ds}{s^{1-\rho}}, \quad \rho > 0, \quad \psi > 0 \quad \text{and} \quad 0 < \delta < 1.$$

Definition 2. The generalised continuous functions of a fractional-order derivative $\delta g : [0, +\infty] \rightarrow R$ is given as [39]:

$$(D^{\delta, \rho} g)(\psi) = (I^{1-\delta, \rho} g)(\psi) = \frac{1}{\Gamma(1-\delta)} \left(\frac{d}{d\psi} \right) \int_0^\psi \left(\frac{\psi^\rho - s^\rho}{\rho} \right)^{-\delta} \frac{g(s) ds}{s^{1-\rho}}, \quad \rho > 0, \quad \psi > 0 \quad \text{and} \quad 0 < \delta < 1.$$

Definition 3. The continuous function of Caputo fractional derivative of δ at $g : [0, +\infty] \rightarrow R$ is defined as [39]:

$$(D^{\delta,\rho}g)(\psi) = \frac{1}{\Gamma(1-\delta)} \left(\frac{d}{d\psi}\right) \int_0^\psi \left(\frac{\psi^\rho - s^\rho}{\rho}\right)^{-\delta} \gamma^n \frac{g(s)ds}{s^{1-\rho}}.$$

where $\rho > 0$, $\psi > 0$, $\gamma = \psi^{1-\gamma} \frac{d}{d\psi}$ and $0 < \delta < 1$.

Definition 4. The continuous function of a ρ -Laplace transform $g : [0, +\infty] \rightarrow R$ is defined as [39]:

$$L_\rho\{g(\psi)\}(s) = \int_0^\infty e^{-s\frac{\psi^\rho}{\rho}} g(\psi) \frac{d\psi}{\psi^{1-\rho}}.$$

The ρ -Laplace transform Caputo generalized fractional-order derivative of a continuous function g is given as [39]:

$$L_\rho\{D^{\delta,\rho}g(\psi)\}(s) = s^\delta L_\rho\{g(\psi)\} - \sum_{k=0}^{n-1} s^{\delta-k-1} (I^{\delta,\rho} \gamma^n g)(0).$$

Definition 5. The generalized Mittag-Leffler function is define as:

$$E_{\delta,\rho}(z) = \sum_{k=0}^{\infty} \frac{z^\delta}{\Gamma(\delta k + \rho)}.$$

3. The General Implementation of Methodology

Consider the functional equation

$$\varphi = g + G(\varphi) + N(\varphi), \quad (1)$$

where g is a source function, G is linear and N non-linear terms, respectively. Let us consider analysis of Equation (1) is given as:

$$\begin{aligned} \varphi_0 &= g, \\ \varphi_1 &= G(\varphi_0) + N(\varphi_0), \\ \varphi_{m+1} &= G(\varphi_m) + (N(\sum_{\kappa=0}^{\infty} \varphi_m(r)) - N(\sum_{\kappa=0}^{\infty} \varphi_\kappa(r))), \end{aligned} \quad (2)$$

where $m = 1, 2, 3, \dots$,

$$\varphi = \sum_{\kappa=0}^{\infty} \varphi_\kappa. \quad (3)$$

Now, linear and non-linear terms can be expressed as [30]:

$$\begin{aligned} G(\varphi) &= G(\sum_{\kappa=0}^{\infty} \varphi_\kappa) = (\sum_{\kappa=0}^{\infty} G(\varphi_\kappa)), \\ N(\varphi) &= N(\sum_{\kappa=0}^{\infty} \varphi_\kappa) = N(\varphi_0) + \sum_{\kappa=1}^{\infty} \{N(\sum_{j=0}^{\kappa} \varphi_j) - N(\sum_{j=0}^{\kappa-1} \varphi_j)\}. \end{aligned} \quad (4)$$

We obtain the solution of (1) as:

$$\varphi = g + \sum_{\kappa=0}^{\infty} G(\varphi_\kappa) + N(\varphi_0) + \sum_{\kappa=1}^{\infty} \{N(\sum_{j=0}^{\kappa} \varphi_j) - N(\sum_{j=0}^{\kappa-1} \varphi_j)\}. \quad (5)$$

Consider the fractional-order Caputo derivative equation is given as:

$$D^{\delta, \rho} \varphi(\varepsilon, \eta) + N\varphi(\varepsilon, \eta) = g(\varepsilon, \eta), \quad (6)$$

the initial condition $\varphi(\varepsilon, 0) = g(\varepsilon)$, where N is a non-linear term.

Now applying ρ -Laplace transform of Equation (6), we get:

$$L_{\rho}(D^{\delta, \rho} \varphi(\varepsilon, \eta) + N\varphi(\varepsilon, \eta)) = L_{\rho}(g(\varepsilon, \eta)); \quad (7)$$

applying ρ -Laplace transform, we achieve:

$$s^{\delta} \varphi(\varepsilon, \eta) - s^{\delta-1} \varphi(\varepsilon, 0) = L_{\rho}(g(\varepsilon, \eta)) - L_{\rho}(N\varphi(\varepsilon, \eta)). \quad (8)$$

Simplify (8), we get:

$$\varphi(\varepsilon, \eta) = \frac{\varphi_0(\varepsilon, 0)}{s} + s^{-\delta} L_{\rho}(g(\varepsilon, \eta)) - s^{-\delta} L_{\rho}(N\varphi(\varepsilon, \eta)). \quad (9)$$

Using inverse Laplace transform of (10), we have:

$$\varphi(\varepsilon, \eta) = \varphi_0(\varepsilon, 0) + L_{\rho}^{-1}(s^{-\delta} L_{\rho}(g(\varepsilon, \eta))) - L_{\rho}^{-1}(s^{-\delta} L_{\rho}(N\varphi(\varepsilon, \eta))). \quad (10)$$

Applying new iterative method, we get

$$\begin{aligned} g &= \varphi(\varepsilon, 0), \\ L(\varphi) &= L_{\rho}^{-1}(s^{-\delta} L_{\rho}(g(\varepsilon, \eta))), \\ N(\varphi) &= -L_{\rho}^{-1}(s^{-\delta} L_{\rho}(N\varphi(\varepsilon, \eta))). \end{aligned} \quad (11)$$

The recurrence relation is defined as:

$$\begin{aligned} \varphi_0 &= g, \\ \varphi_1 &= L(\varphi_0) + N(\varphi_0), \\ \varphi_{m+1} &= L(\varphi_m) + (N(\sum_{\kappa=0}^{\infty} \varphi_{\kappa}(r)) - N(\sum_{\kappa=0}^{m-1} \varphi_{\kappa}(r))), \end{aligned} \quad (12)$$

where $m = 1, 2, 3, \dots$, Hence, the solution of Equation (6) is define as,

$$\sum_{\kappa=0}^{\infty} \varphi_{\kappa} = g + L(\sum_{\kappa=0}^{\infty} \varphi_{\kappa}) + N(\sum_{\kappa=0}^{\infty} \varphi_{\kappa}). \quad (13)$$

4. Convergence of NITM

Now, we discuss the condition for convergence of NITM.

Theorem 1. If N is $C^{(\infty)}$ in a neighborhood of u_0 and [40]

$$\|N^{(n)}(u_0)\| = \text{Sup} \left\{ N^{(n)}(u_0)(h_1, \dots, h_n) : \|h_i\| \leq 1, 1 \leq i \leq n \right\} \leq L$$

for any n and for some real $L > 0$ and $\|u_i\| \leq M < 1/e, i = 1, 2, \dots$, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$\|G_n\| \leq LM^n e^{n-1}(e-1), \quad n = 1, 2, \dots$$

Proof. In view of (13),

$$\|G_n\| \leq LM^n \sum_{i_n=1}^{\infty} \sum_{n-1}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\prod_{j=1}^n \frac{1}{i_j!} \right) = LM^n e^{n-1} (e-1).$$

Thus, the series $\sum_{n=1}^{\infty} \|G_n\|$ is dominated by the convergent series $LM(e-1) \sum_{n=1}^{\infty} (Me)^{n-1}$, where $M < 1/e$. Hence, $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, due to the comparison test. $\square$

As it is difficult to show boundedness of $\|u_i\|$, for all i , a more useful result is proved in the following theorem, where conditions on $N^{(k)}(u_0)$ are given which are sufficient to guarantee convergence of the series.

Theorem 2. If N is $C^{(\infty)}$ and $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}$, for all n , then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent [40].

Proof. Consider the recurrence relation

$$\xi_n = \xi_0 \exp(\xi_{n-1}), \quad n = 1, 2, 3, \dots,$$

where $\xi_0 = M$. Define $\eta_n = \xi_n - \xi_{n-1}$, $n = 1, 2, 3, \dots$. Using (13), (11), and the hypothesis of Theorem 2, we observe that

$$\|G_n\| \leq \eta_n, \quad n = 1, 2, 3, \dots$$

Let

$$\sigma_n = \sum_{i=1}^n \eta_i = \xi_n - \xi_0.$$

Note that $\xi_0 = e^{-1} > 0$, $\xi_1 = \xi_0 \exp(\xi_0) > \xi_0$, and $\xi_2 = \xi_0 \exp(\xi_1) > \xi_0 \exp(\xi_0) = \xi_1$. In general, $\xi_n > \xi_{n-1} > 0$. Hence, $\sum \eta_n$ is a series of positive real numbers. Note that:

$$\begin{aligned} 0 < \xi_0 &= M = e^{-1} < 1 \\ 0 < \xi_1 &= \xi_0 \exp(\xi_0) < \xi_0 e^1 = e^{-1} e^1 = 1 \\ 0 < \xi_2 &= \xi_0 \exp(\xi_1) < \xi_0 e^1 = 1 \end{aligned}$$

In general, $0 < \xi_n < 1$. Hence, $\sigma_n = \xi_n - \xi_0 < 1$. This implies that $\{\sigma_n\}_{n=1}^{\infty}$ is bounded above by 1, and hence convergent. Therefore, $\sum G_n$ is absolutely convergent by the comparison test. $\square$

5. Application to Kdv Equations

Example 1. Consider the fractional dispersive KdV equation is given as [41]:

$$D_{\eta}^{\delta, \rho} \varphi(\varepsilon, \eta) + 2 \frac{\partial \varphi(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \varepsilon^3} = 0, \quad \eta > 0, \quad 0 < \delta \leq 1, \quad (14)$$

with $\varphi(x, 0) = \sin \varepsilon$. Applying ρ -Laplace transform of Equation (14), we have:

$$\varphi(\varepsilon, s) = \frac{L_{\rho}(\varphi_{\eta}) + s^{\delta-1} \varphi(\varepsilon, 0)}{s^{\delta}}, \quad (15)$$

from which,

$$\varphi(\varepsilon, \eta) = \frac{\sin \varepsilon}{s} - \frac{L_{\rho} \left\{ 2 \frac{\partial \varphi(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \varepsilon^3} \right\}}{s^{\delta}}. \quad (16)$$

Using inverse ρ -Laplace transform of Equation (16), we get:

$$\varphi(\varepsilon, \eta) = \sin \varepsilon - L_{\rho}^{-1} \left[\frac{L_{\rho} \left\{ 2 \frac{\partial \varphi(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \varepsilon^3} \right\}}{s^{\delta}} \right]. \quad (17)$$

Implementing the proposed technique, we get:

$$\begin{aligned} f &= \varphi_0 = \sin \varepsilon, \\ \varphi_1(\varepsilon, \eta) &= -L_{\rho}^{-1} \left[\frac{L_{\rho} \left\{ 2 \frac{\partial \varphi_0(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi_0(\varepsilon, \eta)}{\partial \varepsilon^3} \right\}}{s^{\delta}} \right] = -\frac{\cos \varepsilon}{\Gamma(1+\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{\delta}, \\ \varphi_2(\varepsilon, \eta) &= -L_{\rho}^{-1} \left[\frac{L_{\rho} \left\{ 2 \frac{\partial \varphi_1(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi_1(\varepsilon, \eta)}{\partial \varepsilon^3} \right\}}{s^{\delta}} \right] = -\frac{\sin \varepsilon}{\Gamma(1+2\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{2\delta}, \\ \varphi_3(\varepsilon, \eta) &= -L_{\rho}^{-1} \left[\frac{L_{\rho} \left\{ 2 \frac{\partial \varphi_2(\varepsilon, \eta)}{\partial \varepsilon} + \frac{\partial^3 \varphi_2(\varepsilon, \eta)}{\partial \varepsilon^3} \right\}}{s^{\delta}} \right] = \frac{\cos \varepsilon}{\Gamma(1+3\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{3\delta}, \\ &\vdots \end{aligned} \quad (18)$$

Hence, the approximate solution of series form of Equation (14) is given as:

$$\begin{aligned} \varphi(\varepsilon, \eta) &= \varphi_0(\varepsilon, \eta) + \varphi_1(\varepsilon, \eta) + \varphi_2(\varepsilon, \eta) + \varphi_3(\varepsilon, \eta) + \dots \\ \varphi(\varepsilon, \eta) &= \sin \varepsilon - \frac{\cos \varepsilon}{\Gamma(1+\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{\delta} - \frac{\sin \varepsilon}{\Gamma(1+2\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{2\delta} + \frac{\cos \varepsilon}{\Gamma(1+3\delta)} \left(\frac{\eta^{\rho}}{\rho} \right)^{3\delta} + \dots \end{aligned} \quad (19)$$

The series form solution is given by:

$$\begin{aligned} \varphi(\varepsilon, \eta) &= \sin \varepsilon \left(1 - \frac{\left(\frac{\eta^{\rho}}{\rho} \right)^{2\delta}}{\Gamma(1+2\delta)} + \frac{\left(\frac{\eta^{\rho}}{\rho} \right)^{4\delta}}{\Gamma(1+4\delta)} - \dots \right) \\ &\quad - \cos \varepsilon \left(\frac{\left(\frac{\eta^{\rho}}{\rho} \right)^{\delta}}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^{\rho}}{\rho} \right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^{\rho}}{\rho} \right)^{5\delta}}{\Gamma(1+5\delta)} - \dots \right), \end{aligned} \quad (20)$$

when $\delta = 1$, then NITM solution is:

$$\varphi(\varepsilon, \eta) = \sin(\varepsilon - \eta). \quad (21)$$

Example 2. Consider the following fractional dispersive KdV equation is given as [41]:

$$D_{\eta}^{\delta, \rho} \varphi(\varepsilon, \xi, \eta) + \frac{\partial^3 \varphi(\varepsilon, \xi, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi(\varepsilon, \xi, \eta)}{\partial \xi^3} = 0, \quad \eta > 0, \quad 0 < \delta \leq 1, \quad (22)$$

with $\varphi(\varepsilon, \xi, 0) = \cos(\varepsilon + \xi)$. Applying ρ -Laplace transform of Equation (22), we get:

$$\varphi(\varepsilon, \xi, s) = \frac{L_{\rho}(\varphi_{\eta}) + s^{\delta-1} \varphi(\varepsilon, \xi, 0)}{s^{\delta}}, \quad (23)$$

from which,

$$\varphi(\varepsilon, \zeta, \eta) = \frac{\cos(\varepsilon + \zeta)}{s} - \frac{L_\rho \left\{ \frac{\partial^3 \varphi(\varepsilon, \zeta, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi(\varepsilon, \zeta, \eta)}{\partial \zeta^3} \right\}}{s^\delta}. \quad (24)$$

Applying inverse ρ -Laplace transform of (24), we have:

$$\varphi(\varepsilon, \zeta, \eta) = \cos(\varepsilon + \zeta) - L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi(\varepsilon, \zeta, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi(\varepsilon, \zeta, \eta)}{\partial \zeta^3} \right\}}{s^\delta} \right]. \quad (25)$$

Using the proposed technique, we get:

$$\begin{aligned} f &= \varphi_0(\varepsilon, \zeta, \eta) = \cos(\varepsilon + \zeta), \\ \varphi_1(\varepsilon, \zeta, \eta) &= -L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi_0(\varepsilon, \zeta, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi_0(\varepsilon, \zeta, \eta)}{\partial \zeta^3} \right\}}{s^\delta} \right] = -\frac{2 \sin(\varepsilon + \zeta)}{\Gamma(1 + \delta)} \left(\frac{\eta^\rho}{\rho} \right)^\delta, \\ \varphi_2(\varepsilon, \zeta, \eta) &= -L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi_1(\varepsilon, \zeta, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi_1(\varepsilon, \zeta, \eta)}{\partial \zeta^3} \right\}}{s^\delta} \right] = -\frac{4 \cos(\varepsilon + \zeta)}{\Gamma(1 + 2\delta)} \left(\frac{\eta^\rho}{\rho} \right)^{2\delta}, \\ \varphi_3(\varepsilon, \zeta, \eta) &= -L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi_2(\varepsilon, \zeta, \eta)}{\partial \varepsilon^3} + \frac{\partial^3 \varphi_2(\varepsilon, \zeta, \eta)}{\partial \zeta^3} \right\}}{s^\delta} \right] = \frac{8 \sin(\varepsilon + \zeta)}{\Gamma(1 + 3\delta)} \left(\frac{\eta^\rho}{\rho} \right)^{3\delta}, \\ &\vdots \end{aligned} \quad (26)$$

Hence, the approximate solution of series form of Equation (22) is given as:

$$\begin{aligned} \varphi(\varepsilon, \zeta, \eta) &= \varphi_0(\varepsilon, \zeta, \eta) + \varphi_1(\varepsilon, \zeta, \eta) + \varphi_2(\varepsilon, \zeta, \eta) + \varphi_3(\varepsilon, \zeta, \eta) + \dots \\ \varphi(\varepsilon, \eta) &= \cos(\varepsilon + \zeta) - 2 \frac{\sin(\varepsilon + \zeta)}{\Gamma(1 + \delta)} \left(\frac{\eta^\rho}{\rho} \right)^\delta - 4 \frac{\cos(\varepsilon + \zeta)}{\Gamma(1 + 2\delta)} \left(\frac{\eta^\rho}{\rho} \right)^{2\delta} + 8 \frac{\sin(\varepsilon + \zeta)}{\Gamma(1 + 3\delta)} \left(\frac{\eta^\rho}{\rho} \right)^{3\delta} + \dots \end{aligned} \quad (27)$$

The series form solution is given by:

$$\begin{aligned} \varphi(\varepsilon, \zeta, \eta) &= \cos(\varepsilon + \zeta) \left(1 - 4 \frac{\left(\frac{\eta^\rho}{\rho} \right)^{2\delta}}{\Gamma(1 + 2\delta)} + 16 \frac{\left(\frac{\eta^\rho}{\rho} \right)^{4\delta}}{\Gamma(1 + 4\delta)} - \dots \right) \\ &\quad - \sin(\varepsilon + \zeta) \left(\frac{\left(\frac{\eta^\rho}{\rho} \right)^\delta}{\Gamma(1 + \delta)} - 8 \frac{\left(\frac{\eta^\rho}{\rho} \right)^{3\delta}}{\Gamma(1 + 3\delta)} + 32 \frac{\left(\frac{\eta^\rho}{\rho} \right)^{5\delta}}{\Gamma(1 + 5\delta)} - \dots \right), \end{aligned} \quad (28)$$

when $\delta = 1$, then NITM solution is:

$$\varphi(\varepsilon, \zeta, \eta) = \cos(\varepsilon + \zeta + 2\eta). \quad (29)$$

Example 3. Consider the fractional dispersive KdV equation is given as [41]:

$$D_\eta^{\delta, \rho} \varphi(\varepsilon, \eta) + \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \zeta^3} = -\sin \pi \varepsilon \sin \eta - \pi^3 \cos \pi \varepsilon \cos \eta, \quad \eta > 0, \quad 0 < \delta \leq 1, \quad (30)$$

with $\varphi(\varepsilon, 0) = \sin \pi \varepsilon$. Applying ρ -Laplace transform of Equation (30), we have:

$$\varphi(\varepsilon, s) = \frac{L_\rho(\varphi_\eta) + s^{\delta-1} \varphi(\varepsilon, 0)}{s^\delta}, \quad (31)$$

from which,

$$\varphi(\varepsilon, \eta) = \frac{\sin \pi \varepsilon}{s} + \frac{\sin \pi \varepsilon \sin \eta + \pi^3 \cos \pi \varepsilon \cos \eta}{s^\delta} - \frac{L_\rho \left\{ \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \xi^3} \right\}}{s^\delta}. \quad (32)$$

Using inverse ρ -Laplace transform of (32), we get:

$$\begin{aligned} \varphi(\varepsilon, \eta) = & \sin \pi \varepsilon - \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{4\delta}}{\Gamma(1+4\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} \right) \\ & - \pi^3 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right) - L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi(\varepsilon, \eta)}{\partial \xi^3} \right\}}{s^\delta} \right]. \end{aligned} \quad (33)$$

Implementing the proposed technique, we get:

$$\begin{aligned} f = \varphi_0 = & \sin \pi \varepsilon - \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{4\delta}}{\Gamma(1+4\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} \right) \\ & - \pi^3 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right), \\ \varphi_1(\varepsilon, \eta) = & -L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi_0(\varepsilon, \eta)}{\partial \xi^3} \right\}}{s^\delta} \right] = \pi^3 \cos \pi \varepsilon \frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \pi^3 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} \right. \\ & \left. - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{11\delta}}{\Gamma(1+11\delta)} \right) - \pi^6 \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{4\delta}}{\Gamma(1+4\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} \right), \\ \varphi_2(\varepsilon, \eta) = & -L_\rho^{-1} \left[\frac{L_\rho \left\{ \frac{\partial^3 \varphi_1(\varepsilon, \eta)}{\partial \xi^3} \right\}}{s^\delta} \right] = -\pi^6 \sin \pi \varepsilon \frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} + \pi^6 \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+4\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} \right. \\ & \left. - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{12\delta}}{\Gamma(1+12\delta)} \right) + \pi^9 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{11\delta}}{\Gamma(1+11\delta)} \right). \\ & \vdots \end{aligned} \quad (34)$$

Hence, the approximate solution of Equation (30) is given as:

$$\varphi(\varepsilon, \eta) = \varphi_0(\varepsilon, \eta) + \varphi_1(\varepsilon, \eta) + \varphi_2(\varepsilon, \eta) + \varphi_3(\varepsilon, \eta) + \dots$$

$$\begin{aligned} \varphi(\varepsilon, \eta) = & \sin \pi \varepsilon - \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{4\delta}}{\Gamma(1+4\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} \right) \\ & - \pi^3 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right) + \pi^3 \cos \pi \varepsilon \frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} \\ & - \pi^3 \cos \pi^3 \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{11\delta}}{\Gamma(1+11\delta)} \right) \\ & - \pi^6 \sin \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{4\delta}}{\Gamma(1+4\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} \right) \\ & - \pi^6 \sin \pi \varepsilon \frac{\left(\frac{\eta^\rho}{\rho}\right)^{2\delta}}{\Gamma(1+2\delta)} + \pi^6 \sin \pi^3 \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+4\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{6\delta}}{\Gamma(1+6\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{8\delta}}{\Gamma(1+8\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{10\delta}}{\Gamma(1+10\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{12\delta}}{\Gamma(1+12\delta)} \right) \\ & + \pi^9 \cos \pi \varepsilon \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{11\delta}}{\Gamma(1+11\delta)} \right) + \dots \end{aligned} \quad (35)$$

when $\delta = 1$, then NITM solution is

$$\varphi(\varepsilon, \eta) = \sin \pi \varepsilon \cos \eta. \quad (36)$$

Example 4. Consider the fractional three dimensional nonhomogeneous dispersive KdV equation [41]:

$$\begin{aligned} D_{\eta}^{\delta, \rho} \varphi(\varepsilon, \xi, \Im, \eta) + \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \xi^3} + \frac{1}{8} \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \xi^3} + \frac{1}{27} \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \Im^3} = & -\sin(\varepsilon + 2\xi + 3\Im) \cos \eta \\ & + \sin(\varepsilon + 2\xi + 3\Im) \cos \eta, \quad 0 < \delta \leq 1, \end{aligned} \quad (37)$$

with $\varphi(\varepsilon, \xi, \Im, 0) = 0$. Applying ρ -Laplace transform of Equation (37), we get:

$$\varphi(\varepsilon, \xi, \Im, s) = \frac{L_{\rho}(\varphi_{\eta}) + s^{\delta-1} \varphi(\varepsilon, \xi, \Im, 0)}{s^{\delta}}, \quad (38)$$

from which,

$$\begin{aligned} \varphi(\varepsilon, \xi, \Im, \eta) = & \frac{0}{s} - \frac{\sin(\varepsilon + 2\xi + 3\Im) \cos \eta - \sin(\varepsilon + 2\xi + 3\Im) \cos \eta}{s^{\delta}} \\ & - \frac{L_{\rho} \left\{ \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \xi^3} + \frac{1}{8} \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \xi^3} + \frac{1}{27} \frac{\partial^3 \varphi(\varepsilon, \xi, \Im, \eta)}{\partial \Im^3} \right\}}{s^{\delta}}. \end{aligned} \quad (39)$$

Using inverse ρ -Laplace transform of (39), we get

$$\begin{aligned} \varphi(\varepsilon, \zeta, \Im, \eta) = & \sin(\varepsilon + 2\zeta + 3\Im) \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right) \\ & - \frac{L_\rho \left\{ \frac{\partial^3 \varphi(\varepsilon, \zeta, \Im, \eta)}{\partial \zeta^3} + \frac{1}{8} \frac{\partial^3 \varphi(\varepsilon, \zeta, \Im, \eta)}{\partial \zeta^3} + \frac{1}{27} \frac{\partial^3 \varphi(\varepsilon, \zeta, \Im, \eta)}{\partial \Im^3} \right\}}{s^\delta}. \end{aligned} \quad (40)$$

Implementing the proposed technique, we get

$$\begin{aligned} f = \varphi_0(\varepsilon, \zeta, \Im, \eta) = & \sin(\varepsilon + 2\zeta + 3\Im) \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right), \\ \varphi_1(\varepsilon, \zeta, \Im, \eta) = & - \frac{L_\rho \left\{ \frac{\partial^3 \varphi_0(\varepsilon, \zeta, \Im, \eta)}{\partial \zeta^3} + \frac{1}{8} \frac{\partial^3 \varphi_0(\varepsilon, \zeta, \Im, \eta)}{\partial \zeta^3} + \frac{1}{27} \frac{\partial^3 \varphi_0(\varepsilon, \zeta, \Im, \eta)}{\partial \Im^3} \right\}}{s^\delta} = 0. \\ & \vdots \end{aligned} \quad (41)$$

Hence, the approximate solution of Equation (37) is given as:

$$\begin{aligned} \varphi(\varepsilon, \zeta, \Im, \eta) = & \varphi_0(\varepsilon, \zeta, \Im, \eta) + \varphi_1(\varepsilon, \zeta, \Im, \eta) + \varphi_2(\varepsilon, \zeta, \Im, \eta) + \varphi_3(\varepsilon, \zeta, \Im, \eta) + \dots \\ \varphi(\varepsilon, \zeta, \Im, \eta) = & \sin(\varepsilon + 2\zeta + 3\Im) \left(\frac{\left(\frac{\eta^\rho}{\rho}\right)^\delta}{\Gamma(1+\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{3\delta}}{\Gamma(1+3\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{5\delta}}{\Gamma(1+5\delta)} - \frac{\left(\frac{\eta^\rho}{\rho}\right)^{7\delta}}{\Gamma(1+7\delta)} + \frac{\left(\frac{\eta^\rho}{\rho}\right)^{9\delta}}{\Gamma(1+9\delta)} \right) + \dots \end{aligned} \quad (42)$$

when $\delta = 1$, then NITM solution is

$$\varphi(\varepsilon, \zeta, \eta) = \sin(\varepsilon + 2\zeta + 3\Im) \sin \eta. \quad (43)$$

6. Graphical Discussion

The aim this article to investigate an approximate result of the third-order fractional dispersive PDEs, implemented the analytical technique. The new iterative transformation method is applied to analysis of the given problems. The validity show of the suggested method, the result to some illustrative equations are suggested. In Figure 1, (a) the exact result graph of Example 1 at $\delta = 1$ and (b) the approximate result graph of $\delta = 1$. In Figure 2, (a) the the NITM result of Example 1 at different fractional-order of $\delta = 1$ with respective to ε and (b) the different fractional-order solution at $\delta = 1, 0.8, 0.6$ and 0.4 with respect to η . It is analyzed that the NITM and actual result is in close contact with the exact solution of the propose problem. Also in Figure 3, (a) the exact and (b) NITM solutions plot of Example 2, In Figure 4, (a) and (b) show that are different fractional-order $\gamma = 0.8$ and 0.6 . In Figure 5, (a) the the NITM result of Example 2 at different fractional-order of $\delta = 1$ with respective to ε and (b) the analytical solution of different fractional-order of $\delta = 1, 0.8, 0.6$ and 0.4 with respect to η . A similar graphical analysis and discussion can be made for the solutions of Examples 3 and 4 shown in Figures 6–9. It has been demonstrated that the suggested methods are similarly accurate It is analyzed whether fractional-order problems, like fractional-order observation, converge to an integer-order result. In Table 1 show that absolute error of different fractional order represent close contact with each other. The convergence of fractional-order solutions to integral-order approaches is observed in the same way.

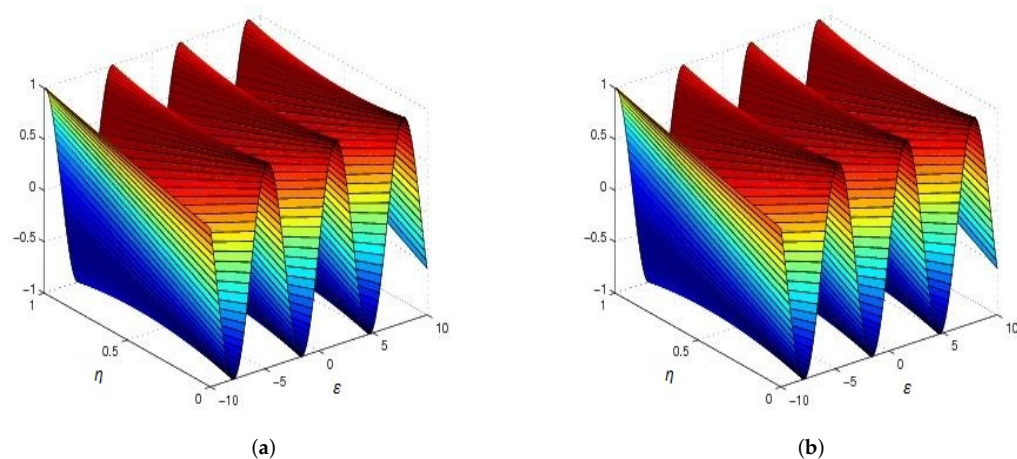


Figure 1. (a) The actual result plot of Example 3.1. (b) The approximate result plot of Example 1.

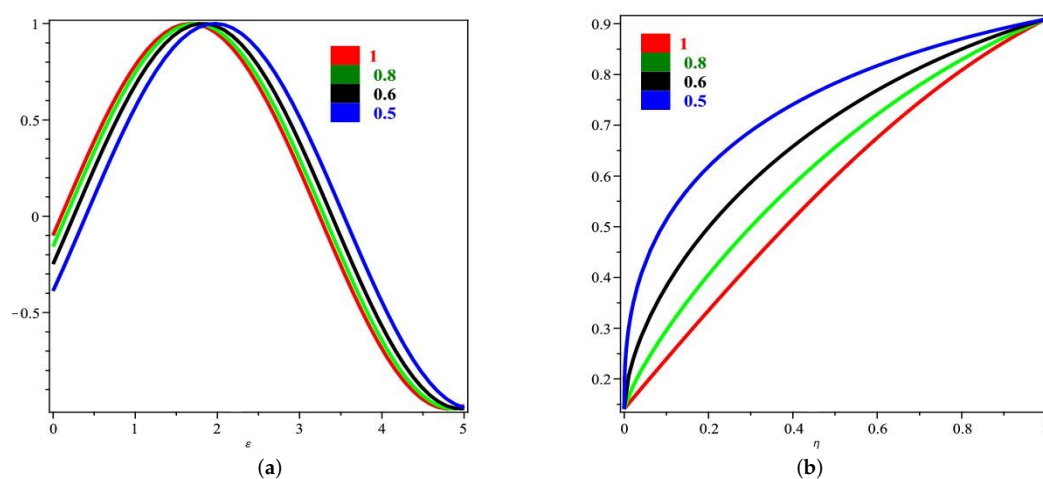


Figure 2. (a) The NITM solution graph of different fractional-order δ of Example 1. (b) The NITM solution graph of different fractional-order δ of Example 1.

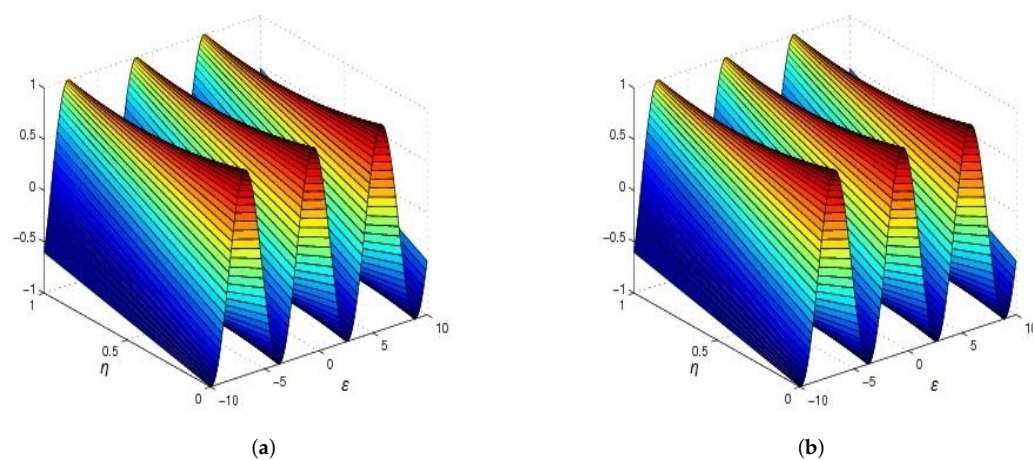


Figure 3. (a) The actual result graph of Example 2. (b) The approximate result plot of Example 2.

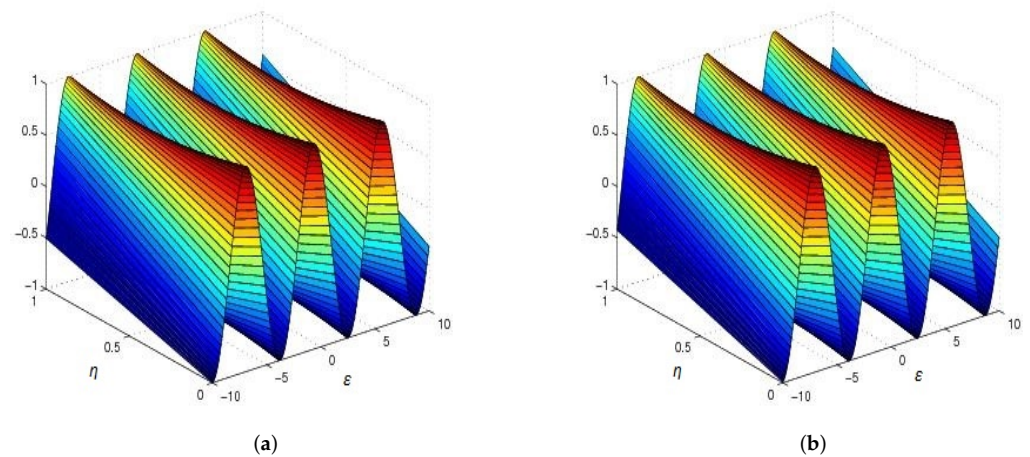


Figure 4. (a) NITM graph at fractional-order $\delta = 1.8$ of Example 2. (b) NITM plot at fractional-order $\delta = 1.6$ of Example 2.

Table 1. $\varphi(\varepsilon, \xi, \eta)$ Comparison of Exact solution, Our methods solution and Absolute Error (AE) of Example 4.

$\eta = 0.01$	Exact Solution	Our Methods Soltion	AE of Our Methods	AE of Our Methods	AE of Our Methods
(ε, ψ)	$\delta = 1$	$\delta = 1$	$\delta = 1$	$\delta = 0.9$	$\delta = 0.8$
0	0.000000000000	0.000000000000	0.000000000	0.000000000	0.000000000
0.1	0.03908475646	0.03908475647	$8.4147098480 \times 10^{-12}$	$5.8916348320 \times 10^{-6}$	$6.0179783350 \times 10^{-5}$
0.2	0.07777899096	0.07777899098	$1.6829419700 \times 10^{-11}$	$1.1724408770 \times 10^{-5}$	$1.1975827680 \times 10^{-4}$
0.3	0.11569608350	0.11569608350	0.00000000000	$1.7440075190 \times 10^{-5}$	$1.7814016480 \times 10^{-4}$
0.4	0.15245717900	0.15245717900	0.00000000000	$2.2981414070 \times 10^{-5}$	$2.3474221550 \times 10^{-4}$
0.5	0.18769497280	0.18769497280	$8.4147098480 \times 10^{-11}$	$2.8293115510 \times 10^{-5}$	$2.8899866580 \times 10^{-4}$
0.6	0.22105738040	0.22105738040	0.00000000000	$3.3322251000 \times 10^{-5}$	$3.4036760840 \times 10^{-4}$
0.7	0.25221105580	0.25221105590	$8.4147098480 \times 10^{-11}$	$3.8018332270 \times 10^{-5}$	$3.8833574600 \times 10^{-4}$
0.8	0.28084472170	0.28084472170	$8.4147098480 \times 10^{-11}$	$4.2334573540 \times 10^{-5}$	$4.3242369270 \times 10^{-4}$
0.9	0.30667227990	0.30667228000	$8.4147098480 \times 10^{-11}$	$4.6227807340 \times 10^{-5}$	$4.7219102240 \times 10^{-4}$
1.0	0.32943567010	0.32943567020	$8.4147098480 \times 10^{-11}$	$4.9659157730 \times 10^{-5}$	$5.0724039260 \times 10^{-4}$

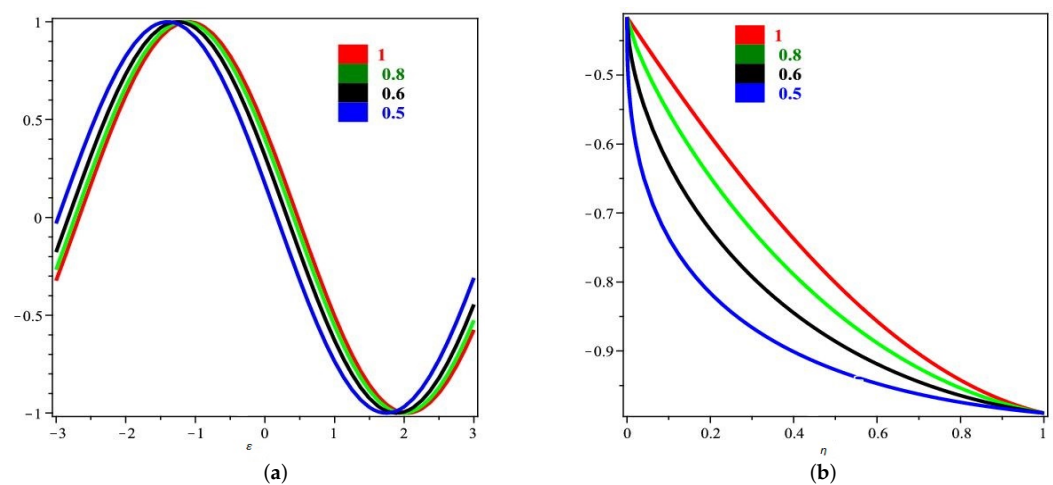


Figure 5. (a) The analytical solution graph of different fractional-order δ of Example 2. (b) The analytical solution graph of different fractional-order δ of Example 2.

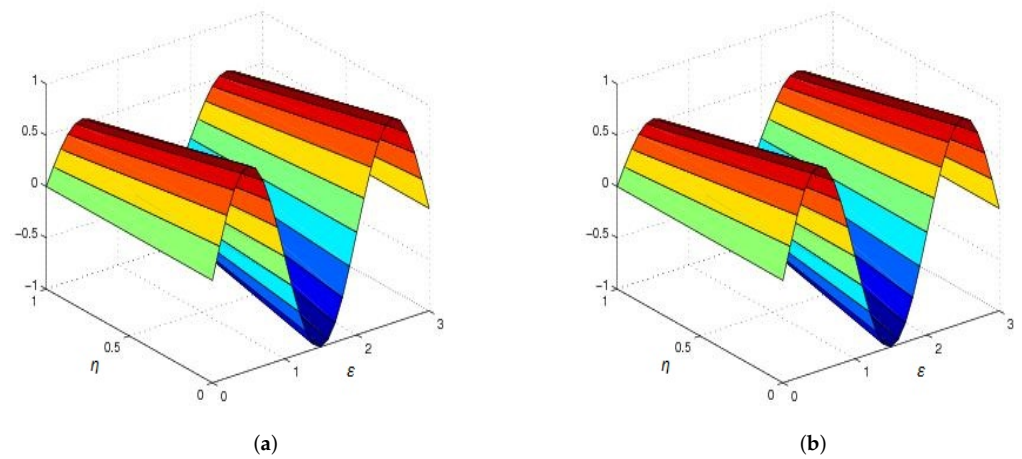


Figure 6. (a) The actual solution graph of Example 3. (b) The analytical result of figure of Example 3.

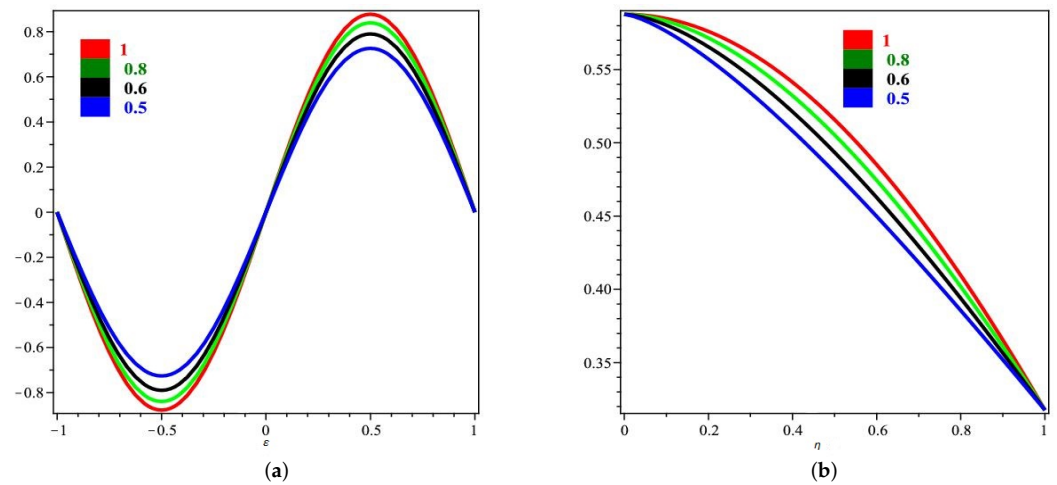


Figure 7. (a) The analytical solution graph of different fractional-order δ of Example 3. (b) The analytical solution graph of different fractional-order δ of Example 3.

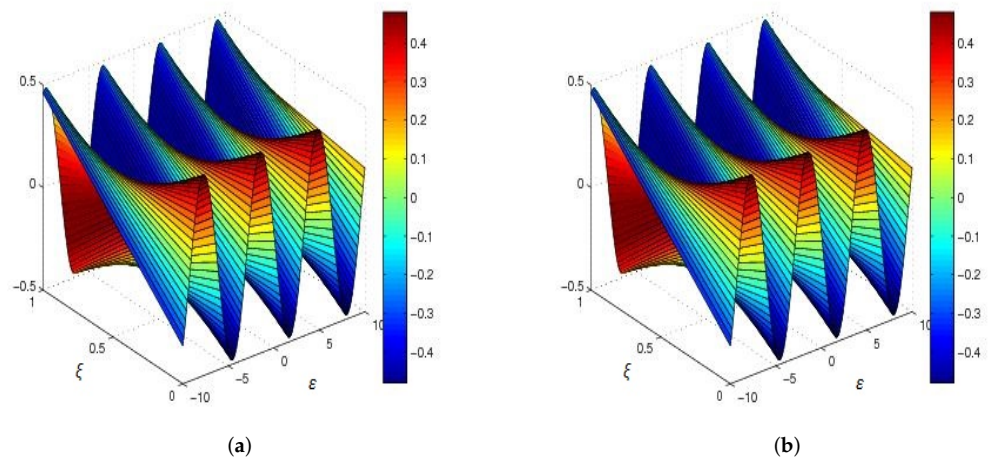


Figure 8. (a) The actual result graph Example 4. (b) The analytical result of plot of Example 4.

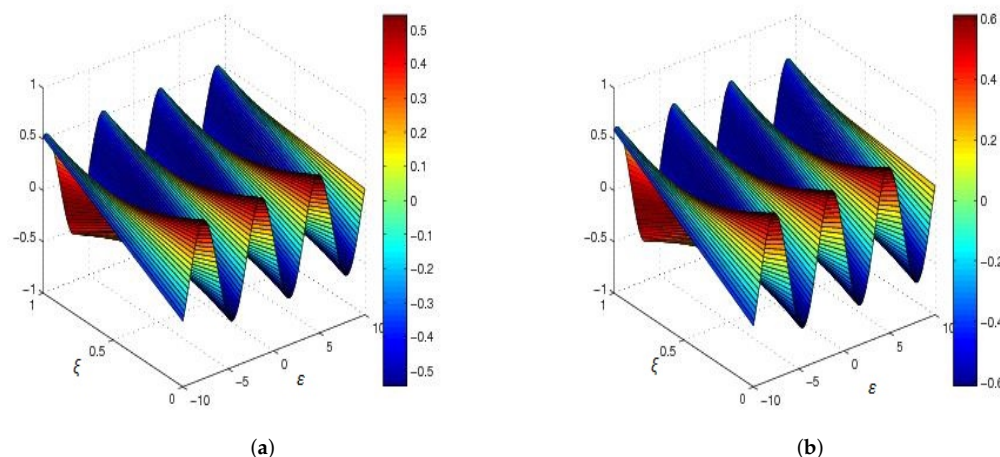


Figure 9. (a) The analytical solution of fractional-order graph $\delta = 1.8$ of Example 4. (b) The analytical solution of fractional-order graph $\delta = 1.6$ of Example 4.

7. Conclusions

The iterative transform approach is used to solve fractional-order third-order dispersive partial differential equations in this paper. The resulting findings' graphical representations have been completed. The improved accuracy of the recommended procedure is clearly demonstrated by this depiction of the acquired findings. The solutions obtained for fractional systems are closely akin to their exact result. It has been demonstrated that fractional answers may be converted to integer-order solutions. The proposed method's key themes include fewer computations and improved precision. It was later developed to solve various fractional-order linear and nonlinear partial differential equations by the researchers.

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