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Grazing and Symmetry-Breaking Bifurcations Induced Oscillations in a Switched System Composed of Duffing and van der Pol Oscillators

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Abstract: By introducing a switching scheme related to the state and time, a typical switched model alternating between a Duffing oscillator and van der Pol oscillator is established to explore the typical dynamical behaviors as well as the mechanism of the switched system. Shooting methods to locate the limit cycle and specify bifurcation sets are described by defining an appropriate Poincaré map. Different types of multiple-Focus/Cycle and single-Focus/Cycle period oscillations in the system can be observed. Symmetry-breaking, period-doubling, and grazing bifurcation curves are obtained in the plane of bifurcation parameters, dividing the parameters plane into several regions corresponding to different kinds of oscillations. Meanwhile, based on the numerical simulation and bifurcation analysis, the mechanisms of several typical dynamical behaviors observed in different regions are presented.

Keywords: switched system; mixed-mode oscillation; nonsmooth evolution; bifurcation mechanism; transformed phase portrait

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1. Introduction

Switched dynamic systems, which are an important class of nonsmooth systems, are described by the interplay between two or more continuous- or discrete-time subsystems, with a switching rule that is defined according to the critical values of the state variable or depends on the fixed time [1,2]. The main motivation to study a switched system is based on the fact that many natural systems, such as biological systems [3], manipulator robots [4], power electronics [5], and automobiles and locomotives with different gears [6] use switches and can be described in terms of switched systems. Meanwhile, the behavior of a switched system not only involves each subsystem but also is related to the switching scheme. Due to the fact that a switching scheme is inherently nonlinear, the whole system may lead to complex dynamic behaviors, such as periodic switching oscillations or even chaotic oscillations, etc. [7–9]. Since the existence of switches is widespread, and many interesting dynamical phenomena are found in switching systems, several scholars have focused on the switched system, reporting many results in recent years [10–13].

At present, the research into switched systems mainly focuses on stability analysis and control analysis. There are two basic problems in the study of the stability of a switched system. One is to study the stability of the system under an arbitrary switched scheme, based on the common Lyapunov function method. For example, Zhang [14] presented a recursive searching method of the common Lyapunov function (CLF) for the robust stable matrix set, based on which a sliding mode controller was designed, so that the system state remained on the sliding mode surface from the initial time instant. Zhu et al. [15] analyzed the issues with the stability of linear time-invariant switched systems with stable/unstable multiple equilibrium points and studied the design of a global asymptotic region-stabilizing controller. Kernani et al. [16] studied pole assignment stabilization with memory state

feedback controllers for switched nonlinear time-varying delay systems based on novel common Lyapunov functions (CLFs), aggregation techniques, proprieties, and the Borne-Gentina criterion. The other is to research the stability of the system under a given switching scheme based on the multi-Lyapunov function method and average dwell time method. For example, Xie et al. [17] analyzed passivity and passive control for a cascade nonlinear system consisting of two switched systems and developed a asymptotic stability criterion for the cascade system without requiring the passivity of subsystem in the whole state interval. Liao et al. [18] investigated a class of nonlinear impulsive multidimensional switched systems and studied the passivity conditions of such systems by adopting the transition-dependent average dwell time and multiple Lyapunov functions. Kivilcim et al. [19] presented sufficient conditions for the near global stability of nonlinear switched systems consisting of both stable and unstable subsystems by the techniques from the stability analysis of the switched systems combined with the multiple Lyapunov density approach. In control analysis, how to control systems to achieve the stabilization control of the system or to find the optimal control of the appropriate switching control strategy is theoretically a very attractive task in switched systems. For example, from composite multiple Lyapunov-like functions and intermittently checking the direction of the worst state, Li et al. [20] designed a new intermittent compensation switching scheme to stabilize the switching nonlinear system. Bezborodov et al. [21] studied the issue of designing a switching scheme, which may ensure the stability of planar linear switched systems when the time delay affects the switching instants and proposed a switching table construction method based on minimum convergence speed.

Up to now, much of the previous literature has focused on the stability and controllability analysis or design of the switching conditions with different requirements. However, many problems such as dynamical behaviors, bifurcations associated with the switching conditions, and the mechanism of complexity with the variation in the parameters are seldom researched. Switched systems introduce many new characteristics, especially strong nonlinearity and singularity caused by the non-differentiability or discontinuity of vector fields [22,23]. Therefore, many dynamic characteristics of nonsmooth systems can not be treated by the ordinary smooth dynamic system theory, and special theories and methods need to be developed.

Duffing and van der Pol oscillators, which the simple nonlinear systems, may exhibit typical dynamical behaviors such as multiple states and self-excited oscillations. In order to study the complex dynamical characteristics as well as the evolution mechanism of the system with switching rules, a mathematical model alternating between Duffing and van der Pol oscillators with certain switch conditions related to the state variable and the fixed time is considered. Then, according to the property of the Poincaré map of the periodic orbit and the tangency of the trajectory of the vector field and the switching surface, critical boundaries related to different forms of bifurcations are obtained. Subsequently, based on numerical simulation and bifurcation analysis the two parameter bifurcation sets of the switched system and the mechanisms of several typical dynamical behaviors observed in different regions are presented.

2. Mathematical Model and Method

2.1. Hybrid System

The autonomous van der Pol system, called subsystem S_1 , can be described as

$$S_1 : \quad \ddot{x} - \beta(1 - x^2)\dot{x} + x = 0. \quad (1)$$

The stable limit cycle, LC_R , can be obtained, which indicates periodic vibration occurs in the vector field. Another system called the Duffing oscillator, expressed as S_2 , can be written in the form

$$S_2 : \quad \ddot{x} + \delta\dot{x} - \alpha x + x^3 = 0, \quad (2)$$

which may contain two stable focuses $E_{\pm} = (\pm\sqrt{\alpha}, 0)$ and a saddle $E_0 = (0, 0)$ as $\delta > 0, \alpha > 0$ on the condition $\sqrt{\delta^2 - 8\alpha} > 0$. When switches are introduced between the two subsystems, it may lead to movement alternating among the different types of behaviors. Here, the switching rules, which are defined according to the combination of the state variable and the fixed time are considered. Suppose that the switched system begins to be controlled by subsystem S_1 from an initial point X_0 , when the trajectory in continuous state space hits the following local surface

$$\Sigma_1 = \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid q(X) = x - x_{ref} = 0\}, \quad (3)$$

at $t = \tau_1$, the switched system will be redefined and controlled by subsystem S_2 for a fixed period of T , where x_{ref} is the fixed value related to the state. After a period of time $T + \tau_1$, the switched system may revert to subsystem S_1 until the state variable x hits the surface Σ_1 again, and the motion then continues as above, where $y = \dot{x}$. Since the whole system alternates between subsystem S_1 and S_2 , another local surface can be defined, expressed by

$$\Sigma_2 = \bigcup_{i=1}^{\infty} \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \mid t = iT + \sum_{j=0}^{i+1} \tau_j\}, \quad (4)$$

where $\tau_0 = 0$ and $(\tau_j : j = 1, 2, \dots)$ represent each time that the switch system runs in subsystem S_1 . Therefore, the system with the switching schemes described above can be written as follows

$$\begin{cases} \dot{x} = y \equiv f_1, \\ \dot{y} = -\mu x - \sigma(x^2 - 1)y \equiv g_1, \end{cases} \quad t \in \left[iT + \sum_{j=0}^i \tau_j, iT + \sum_{j=0}^{i+1} \tau_j \right],$$

$$\begin{cases} \dot{x} = y \equiv f_2, \\ \dot{y} = \alpha x - x^3 - \delta y \equiv g_2, \end{cases} \quad t \in \left[iT + \sum_{j=0}^{i+1} \tau_j, (i+1)T + \sum_{j=0}^{i+1} \tau_j \right],$$

while the trajectory of the switched system is split into two parts: one is governed by subsystem S_1 , denoted by

$$X(t) = \Psi(t, X_{2i}) = (\psi_1(t, (x_{2i}, y_{2i})), \psi_2(t, (x_{2i}, y_{2i}))); \quad (5)$$

the other is determined by the behaviors of subsystem S_2 , i.e.,

$$X(t) = \Phi(t, X_{2i+1}) = (\phi_1(t, (x_{2i+1}, y_{2i+1})), \phi_2(t, (x_{2i+1}, y_{2i+1}))), \quad (6)$$

where $X_{2i}, X_{2i+1}, i = 0, 1, 2, \dots$ are the starting points of the two subsystems, respectively. Obviously, points $X_{2i}, i = 0, 1, 2, \dots$ are on the local surface Σ_1 , while points $X_{2i+1}, i = 0, 1, 2, \dots$ are on the local surface Σ_2 .

2.2. Limit Cycle and Poincaré Map

In order to locate the limit cycle and formulate the conditions for its bifurcation, we first define a Poincaré map. We consider the situation, where the trajectory of the whole system exhibits nearly periodic motion with repeating transitions between subsystems S_1 and S_2 alternately. Figure 1 is a schematic illustration that shows part of the trajectory crossing the two local surfaces Σ_1 and Σ_2 , alternately. Local maps from one section to the other one are defined as follows (see Figure 1)

$$\begin{aligned} P_1 : \Sigma_1 &\longrightarrow \Sigma_2 \\ X_0 &\longmapsto X_1 = \Phi(T, X_0) \\ P_2 : \Sigma_2 &\longrightarrow \Sigma_1 \\ X_1 &\longmapsto X_2 = \Psi(\tau_1, X_1) = \Psi(\tau_1, \Phi(T, X_0)). \end{aligned} \quad (7)$$

Figure 1. Schematic illustration for construction of the Poincaré map P from the local surface and local maps.

It should be noted here that the global coordinate $X_0 \in \Sigma_1 \subset \mathbb{R}^2$ is redundant to identify any point on the local surfaces. Thus, we define a local coordinate $\mathbf{u} \in \Sigma = \{(x, y) \mid x = x_{ref}, y = \mathbf{u}\}$ and choose the projection h and its inverse projection h^{-1} as follows

$$\begin{aligned} h : \Sigma_1 &\longrightarrow \Sigma, & (x_{ref}, y, t) &\longmapsto \mathbf{u}, \\ h^{-1} : \Sigma &\longrightarrow \Sigma_1, & \mathbf{u} &\longmapsto (x_{ref}, y, t). \end{aligned} \quad (8)$$

By treating Σ as a Poincaré section, the Poincaré map P can be defined as

$$\begin{aligned} P : \Sigma &\longrightarrow \Sigma \\ \mathbf{u} &\longmapsto h \circ \Psi \circ \Phi \circ h^{-1}(\mathbf{u}) = h\left(\Psi\left(\tau_1, \Phi(T, h^{-1}(\mathbf{u}))\right)\right). \end{aligned} \quad (9)$$

Similarly, a Poincaré map that indicates a limit cycle crossing the Poincaré map section m times is expressed by $P^m = h \circ (\Psi \circ \Phi)^m \circ h^{-1}$.

2.3. Stability and Local Bifurcation

A limit cycle of the whole system can be located by computing the related fixed point of the Poincaré map P , and the fixed point satisfies the following equation

$$F_{lc}(\mathbf{u}) \equiv P(\mathbf{u}) - \mathbf{u} = 0. \quad (10)$$

In general, the exact analytic expression of the Poincaré map P is unknown as is usual for a nonlinear system. The Jacobian matrix of DP required for numerical computation is given by

$$DP = \left(\frac{\partial \psi_2}{\partial x_{2i}} - \frac{g_1}{f_1} \frac{\partial \psi_1}{\partial x_{2i}} \right) \frac{\partial \phi_1}{\partial y_{2i-1}} + \left(\frac{\partial \psi_2}{\partial y_{2i}} - \frac{g_1}{f_1} \frac{\partial \psi_1}{\partial y_{2i}} \right) \frac{\partial \phi_2}{\partial y_{2i-1}}. \quad (11)$$

The fixed point in (10) can be numerically solved by using numerical methods, and its stability can be determined by the following characteristic equation

$$\chi(\mu) \equiv \det\left(\frac{\partial P}{\partial \mathbf{u}} - \mu I_{n-1}\right) = 0. \quad (12)$$

If all the exponents are located inside the unit circle on the complex plane, then the fixed point is stable. Furthermore, if one of the exponents crosses the unit circle outward from the inside by changing the parameters, the fixed point may become unstable corresponding to a codim-1 local bifurcation.

To specify a limit cycle, the position and its related local bifurcation can be obtained by the conditions as follows

$$\mathbf{F}_{lb}(\mathbf{u}, \alpha) = \begin{bmatrix} P(\mathbf{u}) - \mathbf{u} \\ \chi(\mu) \end{bmatrix} = \mathbf{0}, \quad (13)$$

where α is a bifurcation parameter. For the fixed periodic solution of the switched system, saddle-node or pitchfork bifurcation may occur when $\mu = 1$, period-doubling bifurcation may take place once $\mu = -1$, while Neimark-Sacker bifurcation may happen once $\mu = e^{i\theta}$.

These equations are simultaneously solved by Newton's method. The Jacobian matrix of F_{lb} can be written as follows

$$DF_{lb}(\mathbf{u}, \alpha) = \begin{bmatrix} \frac{\partial P}{\partial \mathbf{u}} - I_{n-1} & \frac{\partial P}{\partial \alpha} \\ \frac{\partial \chi(\mu)}{\partial \mathbf{u}} & \frac{\partial \chi(\mu)}{\partial \alpha} \end{bmatrix}. \quad (14)$$

2.4. Grazing Bifurcation

When a trajectory is tangent to the surface Σ_1 , as illustrated in Figure 2, a slight change in a parameter value may cause the trajectory to intersect with the local surface and induce a change in dynamics. This critical change is called grazing bifurcation. The kind of contact of the smooth vector fields $F_1 = (f_1, g_1)$ and $F_2 = (f_2, g_2)$ with Σ_1 are provided by the directional Lie derivatives

$$L_{F_i} q = \langle \nabla q, F_i \rangle = y \quad i = 1, 2, \quad (15)$$

where ∇q and $\langle \cdot, \cdot \rangle$ denote the gradient of the smooth function q and the canonical inner product, respectively. On Σ_1 we distinguish the following crossing regions, defined by

$$\begin{aligned} \Sigma_1^+ &= \{(x, y, t) \in \Sigma_1 \mid x = x_{ref}, y > 0\}, \\ \Sigma_1^- &= \{(x, y, t) \in \Sigma_1 \mid x = x_{ref}, y < 0\}, \end{aligned} \quad (16)$$

which implies that the grazing point on a trajectory is only possible on the boundary region

$$\Sigma_1^0 = \{(x, y, t) \in \Sigma_1 \mid x = x_{ref}, y = 0\}. \quad (17)$$

Figure 2. Schematic illustration for grazing trajectory tangent to section Σ_1 at X_0 . A trajectory crosses the section or does not touch the section Σ_1 when $\alpha_{grazing}$ is increased or decreased.

Next, we determine the conditions of the periodic grazing solution by considering the return time of a limit cycle as an unknown variable. Suppose that X_0 on Σ_1 is the initial point of a grazing trajectory, and the trajectory transversally intersects Σ_1 $(2m - 1)$ -times alternately; then, finally it becomes tangent to the local surface Σ_1 . For convenience, we define the following maps

$$\begin{aligned} \bar{P}_{2k-1} : \Sigma_1 &\longrightarrow \Sigma_1 \\ X_0 &\longmapsto X_{2k-1} = P_1 \circ (P_2 \circ P_1)^{k-1}(X_0) \\ \bar{P}_{2k} : \Sigma_1 &\longrightarrow \Sigma_1 \\ X_0 &\longmapsto X_{2k} = (P_2 \circ P_1)^k(X_0), \end{aligned} \quad (18)$$

where $k = 1, 2, \dots, m$. Obviously, based on the switching scheme, the traveling time from X_{2k} to X_{2k+1} is a fixed time T , while the traveling time from X_{2k+1} to X_{2k+2} is denoted by $\tau_{k+1}(X_{2k+1})$ for $k = 0, 1, 2, \dots, m - 1$. Then, the cumulative traveling time from X_0 to X_{2m} is denoted by

$$\bar{\tau}_{2k}(X_0) = \bar{\tau}_{2k-1}(X_0) + \tau_k(X_{2k-1}), \quad (19)$$

$$\bar{\tau}_{2k-1}(X_0) = \bar{\tau}_{2k-2}(X_0) + T, \quad (20)$$

where $k = 1, 2, \dots, m$, $\bar{\tau}_0(X_0) \equiv 0$. The above setting provides a map dependent on the total time τ and the initial condition X_0 as follows

$$\bar{P}(\tau, X_0) = \Psi\left(\tau - \bar{\tau}_{2m-1}(X_0), \Phi(T, X_{2m-2})\right), \quad (21)$$

where $\tau > \bar{\tau}_{2m-1}(X_0)$. Upon the map $\bar{P}$, the condition of a grazing bifurcation of an m -fold periodic oscillation can be formulated as follows

$$\mathbf{F}_{gb}(X_0, \tau, \alpha) = \begin{bmatrix} \bar{P}(\tau, X_0) - X_0 \\ q(\bar{P}(\tau, X_0)) \\ \frac{\partial q}{\partial X} \cdot F_2(\bar{P}(\tau, X_0)) \end{bmatrix} = \mathbf{0}. \quad (22)$$

The first equation indicates that a trajectory starting from X_0 on Σ_1 may return back to X_0 leading to an m -folded limit cycle with period τ . The second equation represents that the trajectory is located on Σ_1 at time τ . The third equation means that the trajectory is tangent to the surface Σ_1 at time τ , where α is a bifurcation parameter.

3. Results

In this section, based on the numerical method introduced in the previous section, we fix some of the parameters at $\beta = 1.0$, $\delta = 1.0$, and $T = 4.5$, and take parameters λ and x_{ref} as bifurcation to investigate the dynamical evolution of the oscillator. Since different initial conditions may influence the structure of the attractors of the switched system, all the tests presented in this paper have been performed for the initial condition $(x_0, y_0) = (0.5, 0.1)$.

Figure 3 shows a bifurcation diagram of the limit cycles in the (x_{ref}, λ) plane, indicating the parameter region of no switching together with the bifurcation curves of limit cycles. Both side regions correspond to no switching, where the whole system is only governed by subsystem S_1 .

Figure 3. Bifurcation diagram of the limit cycles in the (x_{ref}, λ) plane. Period-doubling, symmetry-breaking, and grazing bifurcations of the limit cycle are denoted by PD , SB , and GB , respectively.

Let us examine in detail the oscillation and bifurcation mechanisms of the limit cycle. The period 2 oscillation exists in the region I . For example, when $x_{ref} = 0.53$, $\lambda = 1.2$, the fixed points and the exponent can be computed at $\mathbf{u}_1 = (0.53, -1.007) \in \Sigma_1^-$, $\mathbf{u}_2 = (0.53, 2.003) \in \Sigma_1^+$ and $\mu = 0.673$, respectively, indicating that a stable period 2 trajectory crossing the surface Σ_1 twice can be observed (see in Figure 4a the light gray trajectory). When $x_{ref} = -0.53$, $\lambda = 1.2$, another stable period 2 trajectory (see in Figure 4a the blue trajectory) can be obtained, the fixed points of which are $\mathbf{u}_1 = (-0.53, 1.007) \in \Sigma_1^-$ and

$u_2 = (-0.53, -2.003) \in \Sigma_1^+$. It can be seen that the two stable period 2 trajectories are center symmetric around the origin of the coordinates, which implies that when the parameters are taken as axially symmetric around the line $x_{ref} = 0.0$, the periodic oscillations may center symmetrically around the origin of the coordinates. Meanwhile, based on the dynamics of the subsystems, one can see that the vector fields of the whole system may alternate between the transient process of the stable focuses $E_{\pm}$ and the stable limit cycle LC_R , forming the multiple-Focus/Cycle period 2 oscillation in region I.

When the parameters change from region I to the line $x_{ref} = 0.0$, the period 2 oscillations on both sides of the line $x_{ref} = 0.0$ may collide with each other and suddenly change to an unstable period oscillation, which is also a multiple-Focus/Cycle oscillation, but it is center symmetric around the origin of coordinates with itself, via symmetry breaking bifurcation. The typical unstable symmetric multiple-Focus/cycle period 2 trajectory is presented in Figure 4b.

Figure 4. (a,b) Typical multiple-Focus/Cycle period 2 oscillations and (c,d) typical single-Focus/Cycle period 1 oscillations for $\lambda = 1.2$. (a) $x_{ref} = \pm 0.53$; (b) $x_{ref} = 0.0$; (c) $x_{ref} = \pm 0.7$; (d) $x_{ref} = \pm 1.67$.

Crossing the bifurcation curve of PD from the region I to II, the multiple-Focus/Cycle period 2 oscillation may become unstable at PD, leading to a stable period 1 oscillation via reversed period-doubling bifurcation. The typical phase portraits in region II are presented in Figure 4c. As shown in Figure 4c, it can be seen that the trajectory of blue period 1 oscillation may alternate between the transient process of the stable focus E_- and the stable limit cycle LC_D , while the gray period 1 oscillation may alternate between the transient process of another stable focus E_+ and the stable limit cycle LC_D . These type of oscillations are called single-Focus/Cycle period 1 oscillations.

In region III, another single-Focus/Cycle period 1 oscillation is obtained, as shown in Figure 4d, from which we find that the structure of the periodic switching may differ from the period 1 oscillations in region II, although the switching attractors still alternate between stable focus E_- and stable limit cycle LC_R or stable focus E_+ and stable limit cycle LC_R . The phenomenon can be understood by analyzing the change in the location of the fixed point between the crossing regions Σ_1^- and Σ_1^+ . For $\lambda = 1.2$ and x_{ref} decreasing from -1.09 to -1.0954 , the switched system behaves as a single-Focus/Cycle period 1 oscillation

with the fixed point on Σ_1^+ (typical phase diagram see Figure 4c, the blue trajectory). However, when x_{ref} decreases through $-\sqrt{\lambda} = -1.0954$, the fixed point suddenly jumps to the crossing region Σ_1^- (see Table 1), resulting another type of stable single-Focus/Cycle period 1 oscillation (see Figure 4d, the blue trajectory). When x_{ref} increases through $\sqrt{\lambda} = 1.0954$, the fixed point suddenly jumps to the crossing region Σ_1^+ from Σ_1^- (see Table 1), the period 1 oscillation (see Figure 4c, light gray trajectory) may change to another period 1 oscillation (see Figure 5d, light gray trajectory). Based on the discussion, it can be seen that these two types of single-Focus/Cycle period 1 oscillations may be divided by the curve $\lambda = x_{ref}^2$ (see in Figure 3, the parameter in which one of the stable focuses $E_{\pm}$ happens to be on the surface Σ_1).

Table 1. The situation of the fixed point on the crossing regions for $\lambda = 1.2$ and different x_{ref} .

x_{ref}	Fixed Point	Crossing Region	x_{ref}	Fixed Point	Crossing Region
-1.09	(-1.09, 0.10821)	Σ_1^+	1.09	(1.09, -0.10821)	Σ_1^-
-1.0948	(-1.0948, 0.0436)	Σ_1^+	1.0948	(1.0948, -0.0436)	Σ_1^-
-1.0952	(-1.0952, 0.0219)	Σ_1^+	1.0952	(1.0952, -0.0219)	Σ_1^-
-1.0954	(-1.0954, 0.0109)	Σ_1^+	1.0954	(1.0954, -0.0109)	Σ_1^-
-1.0955	(-1.0955, -2.5560)	Σ_1^-	1.0955	(1.0955, 2.5560)	Σ_1^+
-1.1	(-1.1, -2.5384)	Σ_1^-	1.1	(1.1, 2.5384)	Σ_1^+

When x_{ref} decreases along the horizontal arrowed line $\alpha = 1.2$, the fixed point on the crossing region Σ_1^- (see Figure 5a fixed point $\mathbf{u}^*$) may gradually reach the boundary region Σ_1^0 at $x_{ref} = -1.98797$ (see Table 2), which implies that the single-Focus/Cycle period 1 oscillation may become unstable at $x_{ref} = -1.98797$, resulting a unstable period 1 oscillation via grazing bifurcation GB^1 (see Figure 3). A typical unstable single-Focus/Cycle period 1 grazing oscillation is shown in Figure 5b (the blue trajectory). By further increasing x_{ref} , the fixed point on Σ_1^0 will return back to the crossing region Σ_1^- (see Figure 5c, fixed point $\mathbf{u}^*$) and then reach Σ_1^0 again at $x_{ref} = -1.9958$ (see Table 2), indicating grazing bifurcation GB^2 (see Figure 3) takes place at $x_{ref} = -1.9958$. Another typical unstable single-Focus/Cycle period 1 grazing oscillation is obtained in Figure 5d. When x_{ref} increases along the horizontal arrowed line $\alpha = 1.2$, the unstable single-Focus/Cycle period 1 grazing oscillations (the light grey trajectories in Figure 5b,d), which are center symmetric around the origin of the coordinates with the blue grazing trajectories in Figure 5b,d, respectively, are obtained. The mechanism of bifurcation can also be explored through the analysis of the change in the location of the fixed point between the different regions, seen in Table 2.

Table 2. The situation of the fixed point on the crossing regions for $\lambda = 1.2$ and different x_{ref} .

x_{ref}	Fixed Point	Crossing Region	x_{ref}	Fixed Point	Crossing Region
-1.9871	(-1.9871, -0.0598)	Σ_1^-	1.9871	(1.9871, 0.0598)	Σ_1^+
-1.9877	(-1.9877, -0.0388)	Σ_1^-	1.9877	(1.9877, 0.0388)	Σ_1^+
-1.9879	(-1.9879, -0.00182)	Σ_1^-	1.9879	(1.9879, 0.00182)	Σ_1^+
-1.98797	(-1.98797, 0.0)	Σ_1^0	1.98797	(1.98797, 0.0)	Σ_1^0
-1.988	(-1.988, -0.1770)	Σ_1^-	1.988	(1.988, 0.1770)	Σ_1^+
-1.995	(-1.995, -0.061)	Σ_1^-	1.995	(1.995, 0.061)	Σ_1^+
-1.9957	(-1.9957, -0.0191)	Σ_1^-	1.9957	(1.9957, 0.0191)	Σ_1^+
-1.9958	(-1.9958, 0.0)	Σ_1^0	1.9958	(1.9958, 0.0)	Σ_1^0

Figure 5. (a,c) two types of stable single-Focus/Cycle period 1 oscillations and (b,d) two types of unstable single-Focus/Cycle period 1 grazing oscillations for $\lambda = 1.2$, where the subfigures give a clear understanding of the rectangular region. (a) $x_{ref} = \pm 1.80$; (b) $x_{ref} = \pm 1.9879$; (c) $x_{ref} = \pm 1.99$; (d) $x_{ref} = \pm 1.9958$.

4. Conclusions

Based on a switching scheme related to the state and time, a typical switched model alternating between a Duffing oscillator and a van der Pol oscillator was established, which exhibited complex combined movements, such as multiple-Focus/Cycle and single-Focus/Cycle periodic oscillations. The existence of these periodic oscillations were located by solving the fixed point of the Poincaré map. Conducting bifurcation analysis on the Poincaré map and considering the tangency of the trajectory of the vector field and the switching surface, two-parameter bifurcation diagrams were obtained. This study shows that symmetry-breaking bifurcation, period-doubling bifurcation, and grazing bifurcation play an important role in the transition from one type of regular oscillation to another oscillation. The results we obtained in this paper may help to understand different oscillation behaviors and the associated mechanisms of the switched system.

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