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Stochastic Robustness of Delayed Discrete Noises for Delay Differential Equations

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Abstract: Stochastic robustness of discrete noises has already been proposed and studied in the previous work. Nevertheless, the significant phenomenon of delays is left in the basket both in the deterministic and the stochastic parts of the considered equation by the existing work. Stimulated by the above, this paper is devoted to studying the stochastic robustness issue of delayed discrete noises for delay differential equations, including the issues of robust stability and robust boundedness.

Keywords: robust boundedness; robust stability; delayed discrete noises; delay differential equations

MSC: 37H10; 93E15

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1. Introduction

Stochastic differential equations (SDEs) have been universally modeled to describe a great deal of real phenomena along with stochastic factor (e.g., [1–5]). Time delays are a significant and unavoidable factor for the reality. Thus, the stochastic delay differential equations (SDDEs) are well established with serial emphases on the asymptotic behavior; please refer to related Refs. [5–10] and the references therein. Generally, an SDDE

$$du(t) = f_0(u(t), u(t - \delta(t)), t)dt + g_0(u(t), u(t - \delta(t)), t)dB(t)$$

can be regarded as the stochastically perturbed system of deterministic delay differential equation (DDE) $du(t) = f_0(u(t), u(t - \delta(t)), t)dt$, where f_0 maps from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}^n$, g_0 maps from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}^{n \times m}$, $\delta(t) > 0$, $\check{\delta}(t) > 0$ are time-varying delays, $B(t)$ is a Brownian motion. Then, one must face the significant scheme of stochastic robustness: when this deterministic DDE is bounded/stable, how much stochastic perturbation $g_0(u(t), u(t - \check{\delta}(t)), t)\dot{B}(t)$ can be tolerated without the lost of boundedness/stability? One is referred to as stochastic robust boundedness, and the other as stochastic robust stability.

So far, there are serial assertions on asymptotic boundedness of SDDEs (e.g., [5,11–14]), but few assertions on stochastic robust boundedness are obtained. The pioneer work on stochastic robust boundedness was [15], which was further developed to functional case by [16]. Different from stochastic robust boundedness, many more assertions have been poured into the scheme of stochastic robust stability. On this topic, one can refer to Refs. [15,17–22] and the references therein.

It should be noted that, in almost all of the above assertions on stochastic robustness, stochastic perturbations were based on continuous states observations. Stimulated by Ref. [23], Feng and his coauthors [24] firstly proposed and studied the scheme of stochastic

robustness of discrete stochastic perturbations. Speaking in detail, for a bounded/stable deterministic differential equation

$$du(t) = f_1(u(t), t)dt,$$

discrete stochastic noise $g_1(u([t/h_0]h_0))\dot{B}(t)$ can be tolerated to keep the properties of boundedness/stability, where f_1 maps from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}^n$, g_1 maps from $\mathbb{R}^n$ to $\mathbb{R}^{n \times m}$, $h_0 > 0$ stands for the time interval of two consecutive observations, and $[a]$ stands for the biggest integer not greater than a . However, two aspects need to be further studied for Ref. [24], which are listed as follows:

(1) The underlying system considered by Ref. [24] is just the deterministic differential equations. In reality, the time delays phenomenon is widespread, which is always the source of instability and other dreadful behavior. Thus, it is necessary to consider deterministic differential equations with delays, particularly with time-varying delays [25]. This is the first goal of our paper.

(2) Stochastic perturbation in Ref. [24] was mainly based on discrete states observations at serial times $h_0, 2h_0, \dots$. Nevertheless, when observing the system states to make some estimations, time delays are unavoidable (e.g., data transmission). That is to say, making the observations not only needs to consider the discrete issue but also the delay issue. Hence, stochastic perturbations dependent on delayed discrete states observations (e.g., $u([t/h_0]h_0 - h_1)$) are more reasonable and practical than that in Ref. [24], where $h_1 > 0$ is a designed delay. Furthermore, studying the stochastic robustness issue of delayed discrete noises is necessary and significant, which is the second goal of our paper.

Motivated by the aforesaid aspects, this paper is devoted to studying the issue of stochastic robustness (including robust boundedness, robust H_∞ stability, robust asymptotic stability, and robust exponential stability) of delayed discrete noises for deterministic delay differential equations. So far, this paper is the first study on the stochastic robustness of delayed discrete noises, which makes up for the gap in this field.

Notations:

$h_0 > 0, h_1 > 0, h_2 > 0, h_3 > 0, \tau \in (0, 1)$: positive numbers.

$\mathbb{C}([-h_3, 0]; \mathbb{R}_+)$: the set of continuous functions $\xi : [-h_3, 0] \rightarrow \mathbb{R}^n$ with the norm $\|\xi\| = \sup_{-h_3 \leq s \leq 0} |\xi(s)|$.

$\delta_1(t) > 0$: positive time-varying delay with the boundedness h_2 and derivative $\frac{d(\delta_1(t))}{dt} \leq \tau$.

$|A| = \sqrt{\text{trace}(A^T A)}$: the trace norm of vector or matrix A .

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$: a complete probability space satisfying usual conditions.

$B(t) = (B_1(t), \dots, B_m(t))^T$: an m -dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

$\mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$: the set of functions $V(u, t) \geq 0$ with the continuity, twice differentiability about u , and once differentiability about t .

2. Problem Statement

An n -dimensional DDE

$$du(t) = f(u(t), u(t - \delta_1(t)), t)dt, \quad (1)$$

is concerned with initial value $\zeta = \{u(s) : -h_2 \leq s \leq 0\} \in \mathbb{C}([-h_2, 0]; \mathbb{R}_+)$, where f maps from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}^n$ with the local Lipschitz condition (LLC) and $f(0, 0, t) = 0$, $\forall t \in \mathbb{R}_+$. The following condition is assumed for f to ensure the boundedness and stability of DDE (1).

Assumption 1. There are constants $a_i \geq 0, i = 1, 3, a_2 < 0, b_j \geq 0, j = 1, 2, 3$, satisfying $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$

$$\begin{aligned} \mathbf{u}^T \mathbf{f}(\mathbf{u}, \mathbf{v}, t) &\leq a_1 + a_2 |\mathbf{u}|^2 + a_3 |\mathbf{v}|^2, \\ |\mathbf{f}(\mathbf{u}, \mathbf{v}, t)| &\leq b_1 + b_2 |\mathbf{u}| + b_3 |\mathbf{v}|. \end{aligned} \quad (2)$$

Remark 1. Under Assumption 1 and some other conditions, DDE (1) is bounded and stable. For example, choosing Lyapunov function $|\mathbf{u}(t)|^q$, we can prove the H_∞ stability and asymptotic stability of DDE (1) similarly to the proof of Theorem 1 with $\mathbf{g} = 0$; Choosing Lyapunov function $e^{\gamma t} |\mathbf{u}(t)|^q$, we can prove the exponential stability of DDE (1) similarly to the proof of Theorem 2 with $\mathbf{g} = 0$ and the asymptotic boundedness similarly to the proof of Theorem 3 with $\mathbf{g} = 0$. The detailed proofs are left to readers.

As mentioned above, we will study the stochastic robustness of delayed discrete noise $\mathbf{g}(\mathbf{u}([t/h_0]h_0 - h_1))\dot{B}(t)$ satisfying the LLC and following Assumption 2 for DDE (1) in this section, where $h_0 > 0$ stands for the time interval of two consecutive observations, $h_1 > 0$ stands for the time interval between the system arrival instant and the observed instant.

Assumption 2. There are constants $c_k \geq 0, k = 1, 2$, satisfying, $\forall \mathbf{w} \in \mathbb{R}^n$

$$|\mathbf{g}(\mathbf{w})| \leq c_1 + c_2 |\mathbf{w}| \quad (3)$$

and $\mathbf{g}(0) = 0$.

In other words, the goal is to give the criteria on c_1, c_2, h_0 , and h_1 such that stochastically perturbed system

$$d\mathbf{u}(t) = \mathbf{f}(\mathbf{u}(t), \mathbf{u}(t - \delta_1(t)), t)dt + \mathbf{g}(\mathbf{u}([t/h_0]h_0 - h_1))dB(t), \quad (4)$$

retains the properties of boundedness or stability.

Remark 2. Under Assumption 2, $\mathbf{u}(t) = 0$ is the equilibrium solution of system (4). In addition, system (4) can be rewritten as

$$d\mathbf{u}(t) = \mathbf{f}(\mathbf{u}(t), \mathbf{u}(t - \delta_1(t)), t)dt + \mathbf{g}(\mathbf{u}(t - \delta_2(t) - h_1))dB(t), \quad (5)$$

where $\delta_2(t) = t - [t/h_0]h_0$. Hence, essentially speaking, three types of delays exist for system (4): time-varying delay $\delta_1(t)$, with the differentiability, is self-contained for (4); time-varying delay $\delta_2(t)$, without the differentiability, is natural interval between current instant t and observation instant $[t/h_0]h_0$; constant delay h_1 is intentionally designed. For the three types of delays, the property analyses are complex.

It is noted that $\delta_2(t) \leq h_0$ from Remark 2. Hence, system (4) is essentially an SDDE with bounded delays, and the global solution $\mathbf{u}(t)$ can be ensured for system (4) with $E|\mathbf{u}(t)|^2 < \infty (\forall t > 0)$ by Theorem 3.2 of Ref. [5], Assumptions 1 and 2.

For SDDE (4), the initial values $\{\mathbf{u}(s) : -[h_2 \vee (h_0 + h_1)] \leq s \leq 0\}$ are required. When $h_0 + h_1 \leq h_2$, it is okay. Otherwise, we need to give the values on $[-(h_0 + h_1), -h_2]$. Hence, it is assumed that $\mathbf{u}(t) = \mathbf{u}(-h_2), \forall t \in [-(h_0 + h_1), -h_2]$ and $\mathbf{f}(\mathbf{u}, \mathbf{v}, t) = \mathbf{f}(\mathbf{u}, \mathbf{v}, 0), \forall t \in [-[h_2 \vee (h_0 + h_1)], 0]$.

To further deduce the main assertions, for SDDE (4), the Itô's operator [5] of $V(\mathbf{u}, t) \in \mathbb{C}^{2,1}(R^n \times R_+; R_+)$ is cited as

$$\mathcal{L}V(\mathbf{u}, t) \triangleq V_t(\mathbf{u}, t) + V_u(\mathbf{u}, t)\mathbf{f}(\mathbf{u}, \mathbf{v}, t) + \frac{1}{2}\text{trace}[\mathbf{g}^T(\mathbf{w}, t)V_{uu}(\mathbf{u}, t)\mathbf{g}(\mathbf{w}, t)], \quad (6)$$

where $V_t(\mathbf{u}, t) = \frac{\partial V(\mathbf{u}, t)}{\partial t}$, $V_u(\mathbf{u}, t) = (\frac{\partial V(\mathbf{u}, t)}{\partial u_1}, \dots, \frac{\partial V(\mathbf{u}, t)}{\partial u_n})$, $V_{uu}(\mathbf{u}, t) = (\frac{\partial^2 V(\mathbf{u}, t)}{\partial u_i \partial u_j})_{n \times n}$.

Lemma 1 (Barbalat Lemma). Assume that $h(t)$ is uniformly continuous on $[0, \infty)$ with $\int_0^\infty h(t)dt < \infty$, then $\lim_{t \rightarrow \infty} h(t) = 0$.

3. Robust Stability

In this part, we will study the issue of stochastic robust stability of delayed discrete noises, which is one of the main goals.

Theorem 1. Under Assumptions 1 and 2 with $a_1 = b_1 = c_1 = 0$, if there exist constants $q \geq 2$ and $\Delta \in (0, 1)$ satisfying

$$0 \leq c_2 < [(q-1)(\frac{q}{2} - 1 + 2^{q-1})]^{-\frac{1}{2}} [-qa_2 - (q-2)a_3 - 2a_3 \frac{1}{1-\tau}]^{\frac{1}{2}}, \quad (7)$$

$$0 < h_0 + h_1 \leq \left((\frac{q(q-1)}{2})^{-\frac{q}{2}} c_2^{-q} 4^{1-q} (1-\Delta) \right)^{\frac{2}{q}} \quad (8)$$

and

$$\theta_1 < (h_0 + h_1)^{\frac{q}{2}} < \theta_2, \quad (9)$$

then, solution $\mathbf{u}(t)$ of SDDE (4) is moment H_∞ stable and moment asymptotic stable, i.e.,

$$\int_0^\infty E|\mathbf{u}(t)|^q dt < \infty, \quad (10)$$

$$\limsup_{t \rightarrow \infty} E|\mathbf{u}(t)|^q = 0, \quad (11)$$

where θ_1, θ_2 with $\theta_1 < \theta_2$ represent the two solutions of

$$\begin{aligned} c_4(\theta) \triangleq & \frac{1}{\Delta} 2^{3q-3} (q-1) c_2^2 (b_2^q + b_3^q \frac{1}{1-\tau}) \theta^2 + \frac{1}{\Delta} 2^{3p-3} (q-1) (\frac{q(q-1)}{2})^{\frac{q}{2}} c_2^{q+2} \theta \\ & + [qa_2 + (q-2)a_3 + (q-1)c_2^2 (\frac{q}{2} - 1 + 2^{q-1}) + 2a_3 \frac{1}{1-\tau}] = 0. \end{aligned} \quad (12)$$

Please see the detailed proof of Theorem 1 in Appendix A.

Remark 3. Essentially, condition (7) is implied by condition (9). Condition (7) here is especially given as a simplified bound for parameter c_2 .

Theorem 1 has studied the H_∞ stability and asymptotic stability of SDDE (4), but it does not consider the convergent rate. In the following, it will be studied.

Theorem 2. Under the conditions of Theorem 1, the solution $\mathbf{u}(t)$ of SDDE (4) is moment exponentially stable and almost surely exponentially stable, i.e.,

$$\limsup_{t \rightarrow \infty} \frac{\log(E|\mathbf{u}(t)|^q)}{t} \leq -\gamma_0, \quad (13)$$

$$\limsup_{t \rightarrow \infty} \frac{\log |\mathbf{u}(t)|}{t} \leq -\frac{1}{q} \gamma_0, \quad a.s. \quad (14)$$

where $\gamma_0 = \gamma_1 \wedge \gamma_2$, $\gamma_1 = \frac{1}{h_0+h_1} \log [(\frac{q(q-1)}{2})^{-\frac{q}{2}} (h_0 + h_1)^{-\frac{q}{2}} c_2^{-q} 4^{1-q} (1-\Delta)]$ and γ_2 represents the positive solution of

$$\begin{aligned}
c_{10}(\gamma) \triangleq & \gamma + qa_2 + (q-2)a_3 + \frac{1}{\Delta}(h_0 + h_1)^q 2^{3q-3}(q-1)c_2^2 b_2^q e^{\gamma(h_0+h_1)} + (q-1)c_2^2 \left[\frac{q}{2} \right. \\
& -1 + 2^{q-1} + \frac{1}{\Delta} \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q 2^{3p-3} e^{\gamma(h_0+h_1)} \left. \right] + \left(2a_3 + \frac{1}{\Delta} 2^{3q-3}(q-1)c_2^2 (h_0 \right. \\
& \left. + h_1)^q b_3^q e^{\gamma(h_0+h_1)} \right) \frac{e^{\gamma h_2}}{1-\tau} = 0.
\end{aligned} \tag{15}$$

Please see the detailed proof of Theorem 2 in Appendix B.

Remark 4. It is noted that $c_{10}(\gamma)$ is monotonically increasing. Hence, from the definition of γ_2 and $c_{10}(0) = c_4((h_0 + h_1)^{\frac{q}{2}}) < 0$, we have $c_{10}(\gamma) < 0$ for $\gamma \in (0, \gamma_2)$. Furthermore, from the definition of γ_1 , we have $c_{10}(\gamma) < 0$ for $\gamma \in (0, \gamma_0)$.

Furthermore, the following assertion, as the applications of obtained assertions (i.e., Theorems 1 and 2), can be proposed.

Corollary 1. Under conditions (7)–(9), H_∞ stable/asymptotically stable/exponentially stable DDE (1) with Assumption 1 and $a_1 = b_1 = 0$ can tolerate the delayed discrete noise $\mathbf{g}(\mathbf{u}([t/h_0]h_0 - h_1))\dot{\mathbf{B}}(t)$ with Assumption 2 and $c_1 = 0$.

4. Robust Boundedness

In the above section, under Assumptions 1 and 2 with $a_1 = b_1 = c_1 = 0$, the scheme of robust stability of delayed discrete noises, in means of H_∞ stability, asymptotic stability and exponential stability, has been studied. Then, without the constraint of $a_1 = b_1 = c_1 = 0$, what assertions can be obtained? That is the goal of this part, i.e., the scheme of robust boundedness of delayed discrete noises.

Theorem 3. Under Assumptions 1 and 2, if there exist constants $q \geq 2$, $\Delta \in (0, 1)$ and $\beta > 0$ satisfying

$$0 \leq c_2 < [(q-1)(q-2+2^q)]^{-\frac{1}{2}} [-qa_2 - 2(q-2)\beta - (q-2)a_3 - 2a_3 \frac{1}{1-\tau}]^{\frac{1}{2}}, \tag{16}$$

$$0 < h_0 + h_1 \leq \left(\left(\frac{q(q-1)}{2} \right)^{-\frac{q}{2}} c_2^{-q} 8^{1-q} (1-\Delta) \right)^{\frac{2}{q}} \tag{17}$$

and

$$\theta_3 < (h_0 + h_1)^{\frac{q}{2}} < \theta_4, \tag{18}$$

then, solution $\mathbf{u}(t)$ of SDDE (4) is asymptotically bounded, i.e.,

$$\limsup_{t \rightarrow \infty} E|\mathbf{u}(t)|^q = \frac{c_{12}(\gamma_3)}{\gamma_3}, \tag{19}$$

where the expression of $c_{12}(\gamma)$ is given in the proof, $\gamma_3 = \gamma_4 \wedge \gamma_5$, $\gamma_4 = \frac{1}{h_0+h_1} \log \left[\left(\frac{q(q-1)}{2} \right)^{-\frac{q}{2}} (h_0 + h_1)^{-\frac{q}{2}} c_2^{-q} 8^{1-q} (1-\Delta) \right]$, γ_5 represents the positive solution of

$$\begin{aligned}
c_{13}(\gamma) \triangleq & \gamma + qa_2 + 2(q-2)\beta + (q-2)a_3 + \frac{1}{\Delta}(h_0 + h_1)^q 2^{2q-1} 3^{q-1} (q-1)c_2^2 b_2^q e^{\gamma(h_0+h_1)} \\
& + (q-1)c_2^2 \left[q-2+2^q + \frac{1}{\Delta} \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q 2^{3p-2} e^{\gamma(h_0+h_1)} \right] + \left(2a_3 \right. \\
& \left. + \frac{1}{\Delta} 2^{2q-1} 3^{q-1} (q-1)c_2^2 (h_0 + h_1)^q b_3^q e^{\gamma(h_0+h_1)} \right) \frac{e^{\gamma h_2}}{1-\tau} = 0,
\end{aligned} \tag{20}$$

and θ_3, θ_4 with $\theta_3 < \theta_4$ represent the two solutions of

$$c_{16}(\theta) \triangleq \frac{1}{\Delta} 2^{2q-1} 3^{q-1} (q-1) c_2^2 (b_2^q + b_3^q \frac{1}{1-\tau}) \theta^2 + \frac{1}{\Delta} 2^{3p-2} (q-1) (\frac{q(q-1)}{2})^{\frac{q}{2}} c_2^{q+2} \theta + [qa_2 + 2(q-2)\beta + (q-2)a_3 + (q-1)c_2^2(q-2+2^q) + 2a_3 \frac{1}{1-\tau}] = 0. \quad (21)$$

Please see the detailed proof of Theorem 3 in Appendix C.

Remark 5. Similarly to Remark 3, condition (16) is implied by condition (18) and given as a simplified bound for parameter c_2 .

Remark 6. Similarly to Remark 4, from the monotonically increasing property of $c_{13}(\gamma)$, the definitions of γ_4, γ_5 and $c_{13}(0) = c_{16}((h_0 + h_1)^{\frac{q}{2}}) < 0$, we have $c_{13}(\gamma) < 0$ for $\gamma \in (0, \gamma_5)$ and even for $\gamma \in (0, \gamma_3)$.

Similarly to Corollary 1, the assertion on the robust stability can be yielded.

Corollary 2. Under conditions (16)–(18), asymptotically bounded DDE (1) with Assumption 1 can tolerate the delayed discrete noise $\mathbf{g}(\mathbf{u}([t/h_0]h_0 - h_1))\dot{B}(t)$ with Assumption 2.

Remark 7. Compared with the obtained assertions on stochastic robustness in Refs. [15–22], the novelty and significance of our theory are as follows.

(1) Stochastic perturbations here are based on delayed discrete states observations rather than continuous states observations in Refs. [15–22];

(2) It is noted that the methods in Refs. [15–22] can not be applied to deal with discrete term $\mathbf{u}([t/h_0]h_0 - h_1)$. Hence, the Lyapunov functionals $W_1(\mathbf{u}(t), t)$ of (A2), $W_2(\mathbf{u}(t), t)$ of (A9) and $W_3(\mathbf{u}(t), t)$ of (A13) are especially designed to deal with the discrete term.

5. An Example

Example

See the 1-dimensional DDE

$$du(t) = [c - 5u(t) + u(t - \delta_1(t))]dt, \quad (22)$$

with initial value ζ , where $\delta_1(t) = \frac{1}{10}(4 + \sin t)$, c is a constant.

Here, we mainly study the stochastic robustness of delayed discrete noise $c_{15}\mathbf{u}([t/h_0]h_0 - h_1)\dot{B}(t)$ for DDE (22), where $c_{15} > 0, h_0 > 0, h_1 > 0$. Calculations yield that $\delta_1(t) \leq h_2 = 0.5$ and $\frac{d(\delta_1(t))}{dt} = 0.1 \cos t \leq \tau = 0.1$.

When $c = 0$, it is easily shown that DDE (22) is asymptotically stable (see Figure 1). For $q = 2.1$ and $\Delta = 0.75$, we can take parameters h_0, h_1 with $h_0 + h_1 = 0.05, c_{15} = 1$ such that conditions (7)–(9) hold. Calculations yield that $\gamma_1 = 1.659904, \gamma_2 = 0.2656423, \gamma_0 = 0.2656423$. Hence, by [5], Theorem 1, and Theorem 2, the stochastically perturbed system

$$du(t) = [c - 5u(t) + u(t - \delta_1(t))]dt + c_{15}\mathbf{u}([t/h_0]h_0 - h_1)dB(t) \quad (23)$$

has a global solution $\mathbf{u}(t)$ and

$$\limsup_{t \rightarrow \infty} \frac{\log |\mathbf{u}(t)|}{t} \leq -0.1264963, \quad a.s. \quad (24)$$

which can be verified by Figure 2.

Figure 1. Trajectory of $u(t)$ for DDE (22) with $c = 0$.

Figure 2. Stochastic trajectories of $u(t)$, $\log(|u(t)|)/t$ for system (23) with $c = 0$.

$$\int_0^\infty E|u(t)|^{2.1} dt < \infty, \quad (25)$$

$$\limsup_{t \rightarrow \infty} E|u(t)|^{2.1} = 0, \quad (26)$$

$$\limsup_{t \rightarrow \infty} \frac{\log(E|u(t)|^{2.1})}{t} \leq -0.2656423, \quad (27)$$

which can be verified by Figure 3.

Figure 3. $\int_0^t E|u(t)|^{2.1} dt$, $E|u(t)|^{2.1}$, $\log(E|u(t)|^{2.1})/t$ for system (23) with $c = 0$ and sample size 200.

When $c = 2$, it is also shown that DDE (22) is asymptotically bounded (see Figure 4). For $q = 2.1$, $\Delta = 0.65$, and $\beta = 0.0001$, we can take parameters h_0, h_1 with $h_0 + h_1 = 0.035$, $c_{15} = 1$, such that conditions (16)–(18) hold. Calculations yield that $\gamma_4 = 0.9003872$, $\gamma_5 =$

0.471875, $\gamma_3 = 0.471875$. Hence, by [5] and Theorem 3, the solution $u(t)$ of stochastically perturbed system (23) has

$$\limsup_{t \rightarrow \infty} E|u(t)|^{2.1} \leq 5.958494, \quad (28)$$

which can be verified by Figure 5.

Figure 4. Trajectory of $u(t)$ for DDE (22) with $c = 2$.

Figure 5. Trajectories of $u(t)$ and $E|u(t)|^{2.1}$ for system (23) with $c = 2$ and sample size 200.

6. Conclusions

This paper mainly studies the stochastic robustness, including the issues of robust stability and robust boundedness, of delayed discrete noises for deterministic DDEs with time-varying delays, which makes up for the gap of stochastic robustness. In our theory, it is necessary to design parameters h_0 and h_1 together. However, designing these two parameters independently is more reasonable, which will be our future work.

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Appendix A. Proof of Theorem 1

Proof. The global solution $\mathbf{u}(t)$ has been ensured. For integer $i > i_0$, set $\alpha_i = \inf\{t \geq 0 : |\mathbf{u}(t)| \geq i\}$, where sufficiently large positive constant $i_0 > \sup_{-[\delta_1(0) \vee (h_0+h_1)] \leq s \leq 0} |\mathbf{u}(s)|$. Obviously, α_i is increasing and $\lim_{i \rightarrow \infty} \alpha_i = \infty$. Let $t_i = t \wedge \alpha_i$.

Inequality (8) ensures $(\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q}{2}}c_2^q4^{q-1} \leq 1 - \Delta$ and $2^{2q-2}(q-1)c_2^2 + \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2(\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q}{2}}c_2^q4^{q-1} \leq \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2$. and inequality (9) ensures

$$c_4((h_0+h_1)^{\frac{q}{2}}) = qa_2 + (q-2)a_3 + \frac{1}{\Delta}(h_0+h_1)^q2^{3q-3}(q-1)c_2^2b_2^q + (q-1)c_2^2[\frac{q}{2} - 1 + 2^{q-1} + \frac{1}{\Delta}(\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q}{2}}c_2^q2^{3p-3}] + (2a_3 + \frac{1}{\Delta}2^{3q-3}(q-1)c_2^2(h_0+h_1)^qb_3^q)\frac{1}{1-\tau} < 0. \quad (A1)$$

Write $\mathbf{f}(\mathbf{u}(t), \mathbf{u}(t - \delta_1(t)), t)$, $\mathbf{g}(\mathbf{u}([t/h_0]h_0 - h_1))$ as $\tilde{\mathbf{f}}(t)$, $\tilde{\mathbf{g}}(t)$ for short, respectively. Let

$$W_1(\mathbf{u}(t), t) = |\mathbf{u}(t)|^q + \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2V_1(\mathbf{u}(t), t). \quad (A2)$$

where $V_1(\mathbf{u}(t), t) = \int_{t-(h_0+h_1)}^t \int_s^t ((h_0+h_1)^{q-1}|\tilde{\mathbf{f}}(r)|^q + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q-2}{2}}|\tilde{\mathbf{g}}(r)|^q)drds$.

Calculations show that

$$\begin{aligned} \frac{d}{dt}V_1(\mathbf{u}(t), t) &= (h_0+h_1)^q|\tilde{\mathbf{f}}(t)|^q + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q}{2}}|\tilde{\mathbf{g}}(t)|^q - L_1(t) \\ &\leq (h_0+h_1)^q2^{q-1}(b_2^q|\mathbf{u}(t)|^q + b_3^q|\mathbf{u}(t - \delta_1(t))|^q) \\ &\quad + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q}{2}}c_2^q|\mathbf{u}(t - \delta_2(t) - h_1)|^q - L_1(t), \end{aligned} \quad (A3)$$

where $L_1(t) = \int_{t-(h_0+h_1)}^t ((h_0+h_1)^{q-1}|\tilde{\mathbf{f}}(r)|^q + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0+h_1)^{\frac{q-2}{2}}|\tilde{\mathbf{g}}(r)|^q)dr$.

Compute the Itô's operator to $|\mathbf{u}(t)|^q$ and obtain that

$$\begin{aligned} &\mathcal{L}|\mathbf{u}(t)|^q \\ &= q|\mathbf{u}(t)|^{q-2}\mathbf{u}^T(t)\mathbf{f}(\mathbf{u}(t), \mathbf{u}(t - \delta_1(t)), t) + \frac{1}{2}tr\{\mathbf{g}^T(\mathbf{u}(t - \delta_2(t) - h_1))[q(q-2)|\mathbf{u}(t)|^{q-4}\mathbf{u}(t)\mathbf{u}^T(t) + q|\mathbf{u}(t)|^{q-2}]\mathbf{g}(\mathbf{u}(t - \delta_2(t) - h_1))\} \\ &\leq q|\mathbf{u}(t)|^{q-2}[\mathbf{u}^T(t)\mathbf{f}(\mathbf{u}(t), \mathbf{u}(t - \delta_1(t)), t) + \frac{q-1}{2}|\mathbf{g}(\mathbf{u}(t - \delta_2(t) - h_1))|^2] \\ &\leq qa_2|\mathbf{u}(t)|^q + qa_3|\mathbf{u}(t)|^{q-2}|\mathbf{u}(t - \delta_1(t))|^2 + \frac{1}{2}q(q-1)c_2^2|\mathbf{u}(t)|^{q-2}|\mathbf{u}(t - \delta_2(t) - h_1)|^2. \end{aligned} \quad (A4)$$

Then, we have that

$$\begin{aligned}
& \mathcal{L}W_1(\mathbf{u}(t), t) \\
= & \mathcal{L}|\mathbf{u}(t)|^q + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \frac{d}{dt} V_1(\mathbf{u}(t), t) \\
\leq & qa_2|\mathbf{u}(t)|^q + a_3(q-2)|\mathbf{u}(t)|^q + 2a_3|\mathbf{u}(t - \delta_1(t))|^q + \frac{1}{2}(q-1)(q-2)c_2^2|\mathbf{u}(t)|^q \\
& + (q-1)c_2^2|\mathbf{u}(t - \delta_2(t) - h_1)|^q + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left[(h_0 + h_1)^q 2^{q-1} (b_2^q |\mathbf{u}(t)|^q \right. \\
& \left. + b_3^q |\mathbf{u}(t - \delta_1(t))|^q) + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q |\mathbf{u}(t - \delta_2(t) - h_1)|^q - L_1(t) \right] \\
\leq & \left[qa_2 + a_3(q-2) + \frac{1}{2}(q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 (h_0 + h_1)^q 2^{q-1} b_2^q + 2^{q-1} \left((q-1)c_2^2 \right. \right. \\
& \left. \left. + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q \right) \right] |\mathbf{u}(t)|^q + \left[2a_3 + \frac{1}{\Delta} 2^{2q-2}(q-1) \right. \\
& \cdot c_2^2 (h_0 + h_1)^q 2^{q-1} b_3^q \left. \right] |\mathbf{u}(t - \delta_1(t))|^q + 2^{q-1} \left[(q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} \right. \\
& \left. \cdot (h_0 + h_1)^{\frac{q}{2}} c_2^q \right] |\mathbf{u}(t - \delta_2(t) - h_1)|^q - \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 L_1(t). \tag{A5}
\end{aligned}$$

It is noted that, from Burkholder–Davis–Gundy inequality, $\forall s \geq 0$,

$$\begin{aligned}
& E|\mathbf{u}(s) - \mathbf{u}(s - \delta_2(s) - h_1)|^p \\
\leq & 2^{q-1} E \left| \int_{s-\delta_2(s)-h_1}^s \tilde{f}(r) dr \right|^q + 2^{q-1} E \left| \int_{s-\delta_2(s)-h_1}^s \tilde{g}(r) dB(r) \right|^q \\
\leq & 2^{q-1} E \int_{s-\delta_2(s)-h_1}^s \left((h_0 + h_1)^{q-1} |\tilde{f}(r)|^q + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q-2}{2}} |\tilde{g}(r)|^q \right) dr, \tag{A6}
\end{aligned}$$

and that, $\forall t \geq 0$,

$$\int_0^t |\mathbf{u}(s - \delta_1(s))|^q ds \leq \frac{1}{1-\tau} \int_{-h_2}^t |\mathbf{u}(r)|^q dr. \tag{A7}$$

From Itô's formula [5], inequalities (A5)–(A7), we yield, for $t \geq 0$,

$$\begin{aligned}
& EW_1(\mathbf{u}(t_i), t_i) \\
= & EW_1(\mathbf{u}(0), 0) + E \int_0^{t_i} \mathcal{L}W_1(\mathbf{u}(s), s) ds \\
\leq & EW_1(\mathbf{u}(0), 0) + \left[qa_2 + a_3(q-2) + \frac{1}{2}(q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 (h_0 + h_1)^q 2^{q-1} b_2^q \right. \\
& \left. + 2^{q-1} \left((q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q \right) \right] E \int_0^{t_i} |\mathbf{u}(s)|^q ds \\
& + \left[2a_3 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 (h_0 + h_1)^q 2^{q-1} b_3^q \right] \frac{1}{1-\tau} E \int_{-h_2}^{t_i} |\mathbf{u}(s)|^q ds + 2^{q-2} \left[(q-1)c_2^2 \right. \\
& \left. + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q \right] E \int_0^{t_i} \int_{s-\delta_2(s)-h_1}^s \left((h_0 + h_1)^{q-1} |\tilde{f}(r)|^q \right. \\
& \left. + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q-2}{2}} |\tilde{g}(r)|^q \right) dr ds - \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 E \int_0^{t_i} L_1(s) ds \\
\leq & c_3 + c_4 ((h_0 + h_1)^{\frac{q}{2}}) E \int_0^{t_i} |\mathbf{u}(s)|^q ds, \tag{A8}
\end{aligned}$$

where $c_3 = EW_1(\mathbf{u}(0), 0) + \left[2a_3 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 (h_0 + h_1)^q 2^{q-1} b_3^q \right] \frac{1}{1-\tau} E \int_{-h_2}^0 |\mathbf{u}(s)|^q ds$.

Then, inequality (A8) implies that, as $i \rightarrow \infty$,

$$E|u(t)|^q \leq c_3, \quad \forall t \geq 0$$

and the assertion (10) as $t \rightarrow \infty$ and $i \rightarrow \infty$.

Furthermore, from Itô's formula [5] and (A4), one has that, $\forall 0 \leq s_1 < s_2$,

$$\begin{aligned} & E|u(s_2)|^q - E|u(s_1)|^q \\ &= E \int_{s_1}^{s_2} \mathcal{L}|u(s)|^q ds \\ &\leq \int_{s_1}^{s_2} [c_5 + c_6 E|u(s)|^q + c_7 E|u(s - \delta_1(s))|^q + c_8 E|u(s - \delta_2(s) - h_1)|^q] ds \\ &\leq c_9(s_2 - s_1), \end{aligned}$$

which implies the uniform continuity for $E|u(t)|^q$, where $c_i, i = 5, \dots, 9$ are some positive constants. Hence, the assertion (11) can be obtained by Barbalat Lemma (i.e., Lemma 1). $\square$

Appendix B. Proof of Theorem 2

Proof. Notations in the proof of Theorem 1 are still employed. The definition of γ_0 and inequality (8) can ensure that there is $\gamma \in (0, \gamma_0)$ such that $(\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}}c_2^q 4^{q-1}e^{\gamma(h_0+h_1)} \leq 1 - \Delta$, $2^{2q-2}(q-1)c_2^2 + \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2(\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}}c_2^q 4^{q-1}e^{\gamma(h_0+h_1)} \leq \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2$, and $c_{10}(\gamma) < 0$. Let

$$W_2(u(t), t) = e^{\gamma t}|u(t)|^q + \frac{1}{\Delta}2^{2q-2}(q-1)c_2^2 V_2(u(t), t), \quad (\text{A9})$$

where $V_2(u(t), t) = \int_{t-(h_0+h_1)}^t e^{\gamma(s+h_0+h_1)} \int_s^t \left((h_0 + h_1)^{q-1} |\tilde{f}(r)|^q + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0 + h_1)^{\frac{q-2}{2}} |\tilde{g}(r)|^q \right) dr ds$.

Similarly to (A3), it shows that

$$\begin{aligned} \frac{d}{dt} V_2(u(t), t) &= e^{\gamma t} \left[\left((h_0 + h_1)^q |\tilde{f}(t)|^q + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}} |\tilde{g}(t)|^q \right) e^{\gamma(h_0+h_1)} - L_1(t) \right] \\ &\leq e^{\gamma t} \left[\left((h_0 + h_1)^q 2^{q-1} (b_2^q |u(t)|^q + b_3^q |u(t - \delta_1(t))|^q) \right. \right. \\ &\quad \left. \left. + (\frac{q(q-1)}{2})^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}} c_2^q |u(t - \delta_2(t) - h_1)|^q \right) e^{\gamma(h_0+h_1)} - L_1(t) \right]. \end{aligned} \quad (\text{A10})$$

Please see the meaning of $L_1(t)$ as above.

Compute the Itô's operator to $W_2(u(t), t)$, then

$$\begin{aligned} & \mathcal{L}W_2(u(t), t) \\ &= e^{\gamma t} \left[\mathcal{L}|u(t)|^q + \gamma |u(t)|^q + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \frac{d}{dt} V_2(u(t), t) \right] \\ &\leq e^{\gamma t} \left\{ \left[\gamma + qa_2 + a_3(q-2) + \frac{1}{2}(q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2(h_0 + h_1)^q 2^{q-1} b_2^q e^{\gamma(h_0+h_1)} \right. \right. \\ &\quad \left. \left. + 2^{q-1} \left((q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right) \right] |u(t)|^q + \left[2a_3 \right. \right. \\ &\quad \left. \left. + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2(h_0 + h_1)^q 2^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] |u(t - \delta_1(t))|^q + 2^{q-1} \left[(q-1)c_2^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}}(h_0 + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right] |u(t) - u(t - \delta_2(t) - h_1)|^q \right. \\ &\quad \left. - \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 L_1(t) \right\}. \end{aligned} \quad (\text{A11})$$

Noting, $\forall t \geq 0$,

$$\int_0^t e^{\gamma s} |\mathbf{u}(s - \delta_1(s))|^q ds \leq \frac{e^{\gamma h_2}}{1 - \tau} \int_{-h_2}^t |\mathbf{u}(r)|^q dr,$$

then, similarly to (A8), it follows, by (A6),

$$\begin{aligned} & EW_2(\mathbf{u}(t_i), t_i) \\ &= EW_2(\mathbf{u}(0), 0) + E \int_0^{t_i} \mathcal{L}W_2(\mathbf{u}(s), s) ds \\ &\leq EW_2(\mathbf{u}(0), 0) + \left[\gamma + qa_2 + a_3(q-2) + \frac{1}{2}(q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2(h_0 \right. \\ &\quad \left. + h_1)^{q-1} b_2^q e^{\gamma(h_0+h_1)} + 2^{q-1} \left((q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} \right. \right. \\ &\quad \left. \left. \cdot c_2^q e^{\gamma(h_0+h_1)} \right) \right] E \int_0^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds + \left[2a_3 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2(h_0 + h_1)^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] \\ &\quad \cdot \frac{e^{\gamma h_2}}{1 - \tau} E \int_{-h_2}^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds + 2^{2q-2} \left[(q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 \right. \\ &\quad \left. + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right] E \int_0^{t_i} e^{\gamma s} \int_{s-\delta_2(s)-h_1}^s \left((h_0 + h_1)^{q-1} |\tilde{\mathbf{f}}(r)|^q \right. \\ &\quad \left. + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q-2}{2}} |\tilde{\mathbf{g}}(r)|^q \right) dr ds - \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2 E \int_0^{t_i} e^{\gamma s} L_1(s) ds \\ &\leq c_{11}(\gamma) + c_{10}(\gamma) E \int_0^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds, \end{aligned} \quad (\text{A12})$$

where $c_{11}(\gamma) = EW_2(\mathbf{u}(0), 0) + \left[2a_3 + \frac{1}{\Delta} 2^{2q-2}(q-1)c_2^2(h_0 + h_1)^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] \frac{e^{\gamma h_2}}{1 - \tau} E \int_{-h_2}^0 e^{\gamma s} |\mathbf{u}(s)|^q ds$.

Hence, the assertion (13) is obtained, as $i \rightarrow \infty$ and $\gamma \rightarrow \gamma_0$.

Furthermore, for the linear growth conditions of SDDE (4), assertion (13) implies assertion (14) by Theorem 6.2 of Ref. [5]. $\square$

Appendix C. Proof of Theorem 3

Proof. Similarly, there is $\gamma \in (0, \gamma_3)$ such that $\left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q 8^{q-1} e^{\gamma(h_0+h_1)} \leq 1 - \Delta$, $2^{2q-1}(q-1)c_2^2 + \frac{1}{\Delta} 2^{2q-1}(q-1)c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q 8^{q-1} e^{\gamma(h_0+h_1)} \leq \frac{1}{\Delta} 2^{2q-1}(q-1)c_2^2$, $c_{16}((h_0 + h_1)^{\frac{q}{2}}) < 0$ and $c_{13}(\gamma) < 0$. Let

$$W_3(\mathbf{u}(t), t) = e^{\gamma t} |\mathbf{u}(t)|^q + \frac{1}{\Delta} 2^{2q-1}(q-1)c_2^2 V_2(\mathbf{u}(t), t), \quad (\text{A13})$$

Please see the meaning of $V_2(\mathbf{u}(t), t)$ as above.

Under Assumptions 1 and 2, for $\beta > 0$,

$$\begin{aligned} & \mathcal{L}|\mathbf{u}(t)|^q \\ &\leq qa_1 |\mathbf{u}(t)|^{q-2} + qa_2 |\mathbf{u}(t)|^q + qa_3 |\mathbf{u}(t)|^{q-2} |\mathbf{u}(t - \delta_1(t))|^2 + q(q-1)c_1^2 |\mathbf{u}(t)|^{q-2} \\ &\quad + q(q-1)c_2^2 |\mathbf{u}(t)|^{q-2} |\mathbf{u}(t - \delta_2(t) - h_1)|^2 \\ &\leq 2a_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + (q-2)\beta |\mathbf{u}(t)|^q + qa_2 |\mathbf{u}(t)|^q + (q-2)a_3 |\mathbf{u}(t)|^q + 2a_3 |\mathbf{u}(t - \delta_1(t))|^q + 2(q \\ &\quad - 1)^{\frac{q}{2}} c_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + (q-2)\beta |\mathbf{u}(t)|^q + (q-2)(q-1)c_2^2 |\mathbf{u}(t)|^q + 2(q-1)c_2^2 |\mathbf{u}(t - \delta_2(t) - h_1)|^q. \end{aligned} \quad (\text{A14})$$

Similarly to (A10), it shows that

$$\begin{aligned} \frac{d}{dt} V_2(\mathbf{u}(t), t) &\leq e^{\gamma t} \left[\left((h_0 + h_1)^{q-1} (b_1^q + b_2^q |\mathbf{u}(t)|^q + b_3^q |\mathbf{u}(t - \delta_1(t))|^q) + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 \right. \right. \\ &\quad \left. \left. + h_1)^{\frac{q}{2}} 2^{q-1} (c_1^q + c_2^q |\mathbf{u}(t - \delta_2(t) - h_1)|^q) \right) e^{\gamma(h_0+h_1)} - L_1(t) \right]. \end{aligned} \quad (\text{A15})$$

Hence, we yield that

$$\begin{aligned} &\mathcal{L}W_3(\mathbf{u}(t), t) \\ &\leq e^{\gamma t} \left\{ 2a_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + 2(q-1)^{\frac{q}{2}} c_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_1^q e^{\gamma(h_0+h_1)} \right. \\ &\quad + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} 2^{q-1} c_1^q e^{\gamma(h_0+h_1)} + \left[\gamma + qa_2 + 2(q-2)\beta + a_3(q \right. \\ &\quad \left. - 2) + (q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_2^q e^{\gamma(h_0+h_1)} + 2^{q-1} \left(2(q-1) c_2^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{\Delta} 2^{3q-2} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right) \right] |\mathbf{u}(t)|^q + \left[2a_3 + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 \right. \\ &\quad \left. + h_1)^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] |\mathbf{u}(t - \delta_1(t))|^q + 2^{q-1} \left[2(q-1) c_2^2 + \frac{1}{\Delta} 2^{3q-2} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 \right. \\ &\quad \left. + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right] |\mathbf{u}(t) - \mathbf{u}(t - \delta_2(t) - h_1)|^q - \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 L_1(t) \left. \right\}. \end{aligned} \quad (\text{A16})$$

Furthermore, it follows that,

$$\begin{aligned} &EW_3(\mathbf{u}(t_i), t_i) \\ &\leq EW_3(\mathbf{u}(0), 0) + \left[2a_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + 2(q-1)^{\frac{q}{2}} c_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_1^q \right. \\ &\quad \cdot e^{\gamma(h_0+h_1)} + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} 2^{q-1} c_1^q e^{\gamma(h_0+h_1)} \left. \right] E \int_0^{t_i} e^{\gamma s} ds + \left[\gamma \right. \\ &\quad \left. + qa_2 + 2(q-2)\beta + a_3(q-2) + (q-1)(q-2)c_2^2 + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_2^q \right. \\ &\quad \left. \cdot e^{\gamma(h_0+h_1)} + 2^{q-1} \left(2(q-1) c_2^2 + \frac{1}{\Delta} 2^{3q-2} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q e^{\gamma(h_0+h_1)} \right) \right] \\ &\quad \cdot E \int_0^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds + \left[2a_3 + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] \frac{e^{\gamma h_2}}{1-\tau} \\ &\quad \cdot E \int_{-h_2}^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds + 2^{2q-2} \left[2(q-1) c_2^2 + \frac{1}{\Delta} 2^{3q-2} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} c_2^q \right. \\ &\quad \left. \cdot e^{\gamma(h_0+h_1)} \right] E \int_0^{t_i} e^{\gamma s} \int_{s-\delta_2(s)-h_1}^s \left((h_0 + h_1)^{q-1} |\tilde{\mathbf{f}}(r)|^q \right. \\ &\quad \left. + \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q-2}{2}} |\tilde{\mathbf{g}}(r)|^q \right) dr ds - \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 E \int_0^{t_i} e^{\gamma s} L_1(s) ds \\ &\leq c_{14}(\gamma) + \frac{c_{12}(\gamma)}{\gamma} (e^{\gamma t_i} - 1) + c_{13}(\gamma) E \int_0^{t_i} e^{\gamma s} |\mathbf{u}(s)|^q ds, \end{aligned} \quad (\text{A17})$$

where $c_{14}(\gamma) = EW_3(\mathbf{u}(0), 0) + \left[2a_3 + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_3^q e^{\gamma(h_0+h_1)} \right] \frac{e^{\gamma h_2}}{1-\tau} E \int_{-h_2}^0 e^{\gamma s} \cdot |\mathbf{u}(s)|^q ds$ and $c_{12}(\gamma) = 2a_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + 2(q-1)^{\frac{q}{2}} c_1^{\frac{q}{2}} \beta^{-\frac{q-2}{2}} + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 (h_0 + h_1)^{q-1} b_1^q e^{\gamma(h_0+h_1)} + \frac{1}{\Delta} 2^{2q-1} (q-1) c_2^2 \left(\frac{q(q-1)}{2} \right)^{\frac{q}{2}} (h_0 + h_1)^{\frac{q}{2}} 2^{q-1} c_1^q e^{\gamma(h_0+h_1)}$.

This implies the assertion (19), as $i \rightarrow \infty$ and $\gamma \rightarrow \gamma_3$. $\square$

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