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Some Remarks on Strong Fuzzy Metrics and Strong Fuzzy Approximating Metrics with Applications in Word Combinatorics

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Abstract: Noticing that ordinary metrics do not present an adequate tool for the study of analytic problems of word combinatorics, as well as in the research of some problems related to theoretical computer science, we propose to use fuzzy metrics in this type of problems. Specifically, the so-called strong fuzzy metric seems to be more appropriate here. In the first part of the paper, we study some special classes of strong fuzzy metrics, topological and lattice properties of certain families of strong fuzzy metrics, and, more generally, strong k-fuzzy metrics. Noticing that one of the standard axioms of a strong fuzzy metric can be easily violated when applied in real situations, in the second part of the paper we introduce more general, approximating fuzzy metrics and illustrate their applicability with some numerical examples.

Keywords: strong fuzzy metrics; standard fuzzy metric; word combinatorics; approximating fuzzy metric; k-fuzzy metric

MSC: 54A40; 68R15; 54E35

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1. Introduction

In 1951, K. Menger [1] introduced the notion of a statistical metric. This concept was thoroughly studied and renamed as a probabilistic metric in [2]. Later, based on the definition of a probabilistic metric, I. Kramosil and J. Michálek [3] introduced the notion of a fuzzy metric. This concept, after a certain editorial modification carried out by M. Grabiec [4], is usually called now fuzzy metric, in the sense of Kramosil and Michálek, or KM-fuzzy metric for short (Definition 2). On the basis of KM-fuzzy metric, George and Veeramani [5,6] introduced an alternative concept of a fuzzy metric, known now as a fuzzy metric in the sense of George and Veeramani, or GV-fuzzy metric for short (Definition 4). Among the advantages of George–Veeramani's definition of a fuzzy metric are its better topological properties as well as a large number of special possible realizations of such fuzzy metrics. On the other hand, some constructions which are feasible within KM-fuzzy metrics are not realizable in the framework of GV-fuzzy metrics, in particular some constructions presented in this work, see Section 6. Although there are also other, essentially different, approaches to the concept of a fuzzy metric (see e.g., [7–10], etc.), at present, most research work in the field of fuzzy metrics is conducted in the context of KM- and GV-fuzzy metrics. In addition, in this paper, we take KM-fuzzy metrics approach as a basis.

Recently, some researchers, in particular people working in the field of automatic sequences, in stringology, in word combinatorics, and other related areas of mathematics and theoretical computer science, started to use analytical methods in order to investigate the structure of the universe of infinite words and languages. To realize these methods, different metrics on the universe of infinite words were introduced, the topologies and the convergence structure induced by these metrics were studied, and limits of sequences of words were studied. However, as far as our experience shows, ordinary metrics cannot be an appropriate tool for the study of problems of combinatorics on words; see comments in Section 1.1. Fuzzy metric—either in MK- or GV-version—seems more appropriate for this merit since the parameter t allows to reflect information about the string (infinite word) at the moment t of observation, or, differently stated, up to the length t of this string. However, instead of general fuzzy metrics, we use their special kind, namely, strong fuzzy metric (Definitions 3 and 5), introduced by A. Sapena and S. Morillas [11] and later studied and used by different authors. The principal difference between general fuzzy metrics and strong fuzzy metrics is in axiom $(4KM)$ in Definition 2, replaced by axiom (4^sKM) in Definition 3. When studying strings (infinite words) and obtaining information at some level (or number of letters or length, or time) t, we do not see it reasonable (or even possible) to coordinate transition from a level t to another level s (it addition to obvious monotonicity) by a special formula, as it is requested by axiom $(4KM)$. Therefore, we think that it is sensible to coordinate information at the same level as it is asked by axiom (4^sKM) and stick here to the use of strong fuzzy metrics.

The first goal of this paper is to contribute to the study of strong fuzzy metric spaces. We realize this goal in Sections 2–5. In Section 2 (Preliminaries), we present general information about fuzzy metrics, in particular strong fuzzy metrics. In Section 3, some classes of strong fuzzy metrics are studied; these classes are constructed from ordinary metrics on the basis of some known families of t-norms. Section 4 is devoted to the study of global properties, namely, lattice structure and topological location, of certain families of strong fuzzy metrics. In Sections 5 and 6 in this series, we conduct some observation about strong fuzzy k-metrics—the strong version of the so called k-fuzzy metrics; see [12,13].

Our first attempts to adjust fuzzy metrics for the use in word combinatorics were undertaken in [14,15]. Already in these papers we realized that for a more adequate description of the distance between words, along with "classical" strong fuzzy metrics, it is reasonable to rely also on their modifications constructed from certain fragments of fuzzy metrics. The difference of such "fragmentary" fuzzy metrics from ordinary strong fuzzy metrics is that in "fragmentary" fuzzy metrics, we receive the complete information about the string only at the "infinity" level $t = \infty$ of the string, which is when the information about a string on all levels t is available. Developing this idea in the present paper, we introduce the concept of a strong approximating fuzzy metric. The study of strong fuzzy approximating metrics, illustrating them with examples and discussion of their appropriateness for the description of the structure of infinite words, is the second principal goal of this paper; it is realized in Sections 6 and 7. In the last section, Conclusion, we discuss some perspectives for continuation of this work—both from theoretical point and in view of possible applications.

1.1. Discrepancy of Ordinary Metrics for the Problems of Word Combinatorics

One can find several different metrics on the universe of infinite words. The first one, considered, e.g., in [16–18], can be defined as follows: Given $x = (x_0, x_1, x_2, \dots x_n, \dots)$ and $y = (y_0, y_1, y_2, \dots y_n, \dots)$, where $x_n, y_n \in \{0, 1\}$ for all $n \in \mathbb{N} \cup \{0\}$, let

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{where } n = \min\{i : x_i \neq y_i\} \end{cases}$$

In our opinion, this metric does not give any meaningful information about actual "distance" between the words. For example, let $x = (1, 1, 1, 1, 1, 1, 1, \dots)$, $y = (0, 1, 1, 1, 1, 1, 1, \dots)$, and $z = (0, 0, 0, 0, 0, 0, 0, \dots)$ be infinite words. Then, obviously, $d(x, y) = d(x, z) = 1$. However, this means that in both cases, the distance between these infinite words is 1, i.e., the greatest possible value that this metric can achieve. In actuality, this means everything concerning distance is dictated by the first digits of the strings. However, comparing these words informally, one may feel that x is closer to y than z.

Another metric on the universe of infinite words X can be found in, e.g., [19]. For every $i \in \mathbb{N} \cup \{0\}$, we define a function $\chi_i : X \times X \to \{0,1\}$ by setting for given $x = (x_0, x_1, x_2, \ldots x_n, \ldots)$ and $y = (y_0, y_1, y_2, \ldots y_n, \ldots)$

$$\chi_i(x,y) = \begin{cases} 0 & \text{if } x_i = y_i \text{ where } i \text{ is the } i\text{-th coordinate of the word} \\ 1 & \text{if } x_i \neq y_i \text{ where } i \text{ is the } i\text{-th coordinate of the word} \end{cases}$$

Now let

$$\sigma(x,y) = \sum_{i=0}^{\infty} \frac{1}{2^i} \chi_i(x,y).$$

One can easily see that the function $\sigma : X \times X \to [0,1]$ thus defined is a metric (actually an ultrametric) on the universe of all infinite words. As different from the metric d described in the previous paragraph, it takes into account information about a word on the whole, and not only information about their prefixes. However, this metric also gives only accumulated information about the distance on the universe X and neglects all specific details of this information. For example, let x, y and z be the same words as in the previous paragraph. Then, $\sigma(y,z) = 1$, and $\sigma(x,z) = 1$ and, hence, also σ neglects the essential local difference between these words but just accumulates all information in one number. A similar approach to ours can be seen in [20,21], where authors use a modification of Levenshtein distance called heuristic distance. In this case, the distance is expressed as a percentage and is equivalent to our output values from an interval $[0;1]$.

Summing up the conclusions drawn from the previous examples, we infer that ordinary metrics cannot serve as an appropriate analytic tool for determining nearness-type relations between infinite words. Therefore, instead of ordinary metrics, we suggest to use fuzzy (pseudo)metrics. In our opinion, fuzzy (pseudo)metrics are a subtler tool if compared with ordinary (pseudo)metrics and, if properly defined, will give a more precise information about the distance related properties in the universe of infinite words.

2. Preliminaries

2.1. t-Norms

The concept of a t-norm was introduced in [1] and further studied by many authors. The fundamental monograph by E.P. Klement, R. Mesiar, and E. Pap [22] serves as the standard reference concerning the theory of t-norms.

Definition 1. *A t-norm is a binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *on the unit interval* $[0,1]$ *satisfying the following properties:*

(1tn) $*$ *is monotone:* $\alpha \leq \beta \Rightarrow \alpha * \gamma \leq \beta * \gamma$ *for all* $\alpha, \beta, \gamma \in [0,1]$*;*
(2tn) $*$ *is commutative:* $\alpha * \beta = \beta * \alpha$ *for all* $\alpha, \beta \in [0,1]$*;*
(3tn) $*$ *is associative:* $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$ *for all* $\alpha, \beta, \gamma \in [0,1]$*;*
(4tn) $\alpha * 1 = \alpha$ *for all* $\alpha \in [0,1]$.

In this paper, we consider a t-norm both as a binary operation $* : [0,1] \times [0,1] \to [0,1]$ defined above and as a two argument function $T : [0,1] \times [0,1] \to [0,1]$ related to $*$ by $T(\alpha, \beta) = \alpha * \beta$. We give preference to one of the entries depending on the situation and the context.

Example 1. *The following five important examples of t-norms are well known and can be found, e.g., in [2,22]):*

- *The minimum t-norm is defined by* $\alpha * \beta := \alpha \wedge \beta$ *where* $\wedge$ *denotes the operation of taking minimum in [0, 1].*
- *The product t-norm is defined by* $\alpha * \beta := \alpha \cdot \beta$ *where* $\cdot$ *is the product.*
- *The Łukasiewicz t-norm* $L = [0,1]$ *is defined by* $\alpha *_L \beta := \max(\alpha + \beta - 1, 0)$.
- *The Drastic t-norm is defined by*

$$T_D(\alpha,\beta) = \begin{cases} \alpha, & \text{if } \beta = 1, \\ \beta, & \text{if } \alpha = 1, \\ 0, & \text{otherwise} \end{cases}$$

- *The nilpotent minimum t-norm is defined by*

$$T_{nM}(\alpha,\beta) = \begin{cases} \min\{\alpha,\beta\}, & \text{if } \alpha+\beta > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1. *It is known and can be easily seen that $\alpha \wedge \beta \geq \alpha * \beta$ for every t-norm $*$. Hence $\wedge$ is the largest t-norm.*

2.2. Fuzzy (Pseudo)Metrics

The notion of a fuzzy metric was presented in 1975 by I. Kramosil and J. Michálek [3], taking as a base definition of a statistical metric in K. Menger's work [1]. Later, M. Grabiec [4] revised this definition in some sense "reducing its statistical origin". Only in this form are KM-fuzzy metrics now usually understood.

Definition 2 ([3,4]). *Let X be a set and $*$ a t-norm. A fuzzy KM-pseudometric on a set X on the base of t-norm $*$ is a mapping $M : X \times X \times \mathbb{R}^+ \to [0,1]$ where $\mathbb{R}^+ = [0,\infty)$, satisfying the following axioms:*

(0FKM) $M(x,y,0) = 0$ *for all* $x,y \in X$;
(1FKM) $M(x,y,t) = 1$ *for all* t *whenever* $x = y$;
(2FKM) $M(x,y,t) = M(y,x,t)$ *for all* $x,y \in X$, *for all* $t \in \mathbb{R}^+$;
(3FKM) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all* $x,y \in X$, *for all* $t \in \mathbb{R}^+$;
(4FKM) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is left continuous for all* $x,y \in X$.

A fuzzy pseudometric M is called a fuzzy metric if it satisfies a stronger form of axiom (1FKM):

(1′FKM) $M(x,y,t) = 1$ *for all* t *if and only if* $x = y$;

If needed to specify the role of the t-norm $*$ we view a fuzzy (pseudo)metric alternatively as the pair $(M,*)$.

The triple $(X,M,*)$ is called a fuzzy KM-(pseudo)metric space.

Definition 3 ([23]). *A fuzzy KM-(pseudo)metric M on a set X is called strong if, besides the axioms (0KM)– (2KM), the following stronger versions of axioms (3KM) and (4KM) are fulfilled*

(3^sFKM) $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$ *for all* $x,y,z \in X$ *and for all* $t \in \mathbb{R}^+$.
(4^sFKM) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is left continuous and increasing, (i.e.,* $t < s \Longrightarrow M(x,y,t) \leq M(x,y,s)$ *for all* $x,y \in X$.*).*

Remark 2. *Although the goal of introducing the strong version of a fuzzy metric in [11] was to replace axiom (3FKM) by axiom (3^sFKM), we decided to also modify axiom (4FKM), otherwise the rest of the axioms in Definition 2 will not guarantee that $M : X \times X \to \mathbb{R}^+$ is a fuzzy metric. The simplest example showing this fact was found by A. Sapenas and S. Morillas [11]. It is given by*

$$M(x,y,t) = \begin{cases} \frac{t^{-1}}{t^{-1}+|x-y|} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x,y \in \mathbb{R}$. As shown in [11], $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ satisfies axioms (0FKM), (1FKM), (2FKM), (3^sFKM) for the product t-norm, but fails to satisfy (3FKM). However, axioms (0FKM), (1FKM), (2FKM), (3^sFKM), and (4FKM) guarantee that M is a fuzzy metric, if additionally M is assumed to be increasing in the third argument and this is ensured by axiom (4^sFKM).

Although the main context of our work will be formed by strong KM-fuzzy (pseudo) metrics, some notes will be made also in concern of (strong) GV-fuzzy (pseudo)metrics. Therefore, we reproduce here the corresponding definitions.

Definition 4 ([5]). *Let X be a set and $*$ a continuous t-norm. A GV-fuzzy pseudometric on a set X is a mapping $M : X \times X \times \mathbb{R}_0^+ \to (0,1]$, where $\mathbb{R}_0^+ = (0,+\infty)$ (or a pair $(M,*)$ in order to specify the t-norm $*$ involved in its definition) such that :*

(0FGV) $M(x,y,t) > 0$ *for all $x,y \in X$ and all $s,t \in \mathbb{R}_0^+$;*
(1FGV) $M(x,y,t) = 1$ *whenever* $x = y$;
(2FGV) $M(x,y,t) = M(y,x,t)$ *for all $x,y \in X$ and all $s,t \in \mathbb{R}_0^+$;*
(3FGV) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all $x,y,z \in X$ and for all $s,t \in \mathbb{R}_0^+$;*
(4FGV) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is continuous for all $x,y \in X$.*

A GV-fuzzy pseudometric M is called a GV-fuzzy metric if it satisfies a stronger form of axiom (1FGV):

(1′FGV) $M(x,y,t) = 1$ *for all* t *if and only if* $x = y$;

The triple $(X,M,*)$ is called a GV-fuzzy metric space.

Definition 5 ([23]). *A GV-fuzzy (pseudo)metric m on X is called strong if it satisfies stronger versions of axiom (3FGV) and (4FGV):*

(3^sFGV) $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$ *for all $x,y,z \in X$, for all $t \in \mathbb{R}_0^+$.*
(4^sFGV) $M(x,y,-) : \mathbb{R}_0^+ \to [0,1]$ *is continuous and increasing (that is $t < s \Longrightarrow M(x,y,t) \leq M(x,y,s) \forall x,y \in X$.)*

Remark 3. *To compare KM and GV-fuzzy metrics axioms we restrict domain of a fuzzy metric to $\mathbb{R}_0^+$ as it is in case of GV-metrics. Then axioms (0FGV) and (2GV) coincide with axioms (0FKM) and (2FKM), respectively. However, axioms (4FGV) and (4FKM) and axioms (1FKM) and (1FGV) are different. In both cases, the George and Veeramani version is stronger than the Kramosil and Michálek one. The same remarks can be made for the strong versions of these definitions. Thus, if we restrict to the situation when $t > 0$, each GV-fuzzy metric is a KM-fuzzy metric, but not the converse. See also Remark 3.8 in [24] in this concern.*

In the sequel we use the term fuzzy pseudometric when it is not important which one of the versions, KM-fuzzy pseudometric or GV-fuzzy pseudometric, is under consideration.

3. Strongness of Standard Fuzzy Pseudometrics

In [5] the authors proposed a method allowing to construct from an arbitrary (pseudo) metric $d : X \times X \to \mathbb{R}^+$ a GV-fuzzy (pseudo)metric $M_d : X \times X \to (0,\infty)$ for the product t-norm. Later, this construction was developed for the case of an arbitrary continuous t-norm and the resulting fuzzy (pseudo)metric m_d called the standard (pseudo)metric induced by a metric d. Standard fuzzy (pseudo)metrics play an important role both as a broad source for constructing examples of fuzzy pseudometrics maintaining different prescribed properties and as an important link between the theories of metrics and fuzzy metrics.

In this section, we first are interested whether the standard fuzzy (pseudo)metric is strong depending on the t-norm used in its definition. First, we recall the definition of the standard fuzzy metric (slightly modified in order to be appropriate also for KM-version of fuzzy (pseudo)metrics).

Definition 6. *Given an (ordinary) pseudometric $d : X \times X \to [0,+\infty)$ and a t-norm $* : [0,1] \times [0,1] \to [0,1]$, the standard fuzzy pseudometric $M_d : X \times X \times \mathbb{R}^+ \to (0,1]$ induced by d is defined by*

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

It is known and easy to see that m_d is indeed a fuzzy pseudometric for the minimum t-norm, and hence (by Remark 1) also for every t-norm.

Since the standard fuzzy pseudometric is obviously increasing and continuous on the parameter t, the only problem we have to consider is whether the axiom (3^sKM) is satisfied for m_d. In order to follow a certain consistency here we start with considering some known families of t-norms.

Recall that the family of Hamacher t-norms is defined by

$$T_H^\lambda(x,y) = \begin{cases} 0, & \text{if } \lambda = x = y = 0, \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)}, & \text{otherwise} \end{cases}$$

where $\lambda \in [0,+\infty)$ is a parameter.

Theorem 1. *For every pseudometric $d : X \times X \to [0,+\infty)$, standard fuzzy pseudometric is strong for every Hamacher t-norm T_H^λ.*

Proof. We have to prove that

$$m_d(x,z,t) \geq T_H^\lambda(m_d(x,y,t), m_d(y,z,t))$$

for any $x,y,z \in X$ and $t \in [0,+\infty)$.

In order to simplify the entry in the proof, here and in the sequel we denote $d(x,y) = a$, $d(y,z) = b$, $d(x,z) = c$ and rely on the inequality $c \leq a+b$ justified by the triangle axiom of the pseudometric d. Thus, we have to prove

$$\frac{t}{t+c} \geq \frac{\frac{t}{t+a} \cdot \frac{t}{t+b}}{\lambda+(1-\lambda)\left(\frac{t}{t+a}+\frac{t}{t+b}-\frac{t^2}{(t+a)(t+b)}\right)}$$

By a simplification, this inequality is equivalent to the following one

$$\frac{t}{t+c} \geq \frac{t^2}{\lambda ta + tb + ta + t^2} = \frac{t}{\lambda a + a + b + t} \iff$$
$$\iff \lambda bt + ta + tb + t^2 \geq t^2 + ct \iff \lambda b + b + a \geq c$$

The last one is obvious by the properties of the metric d. □

Since the product t-norm is a specific case of the Hamacher t-norm in case the parameter $\lambda = 1$, from this theorem we obtain the following (actually well-known, see, e.g., [11,25]) corollary.

Corollary 1. *The standard pseudometric for a product t-norm is strong.*

Another important family of t-norms are Weber t-norms defined for a parameter $\lambda \in (-1;+\infty)$ by

$$T_W^\lambda(x,y) = max\{\tfrac{x+y-1+\lambda xy}{1+\lambda}, 0\}, \text{if } \lambda \in (-1;+\infty).$$

Theorem 2. *For every metric $d : X \times X \to [0,+\infty)$ the standard fuzzy pseudometric is strong for every Weber t-norm T_W^λ.*

Proof. We have to prove that

$$M_d(x,z,t) \geq T_W(M_d(x,y,t), M_d(y,z,t)),$$

that is the inequality

$$\frac{t}{c+t} \geq \frac{\frac{t}{a+t} + \frac{t}{b+t} - 1 + \frac{\lambda t^2}{(a+t)(b+t)}}{1+\lambda}.$$

By obvious simplifications we reduce it

$$\frac{t}{c+t} \geq \frac{ta + t^2 + bt + t^2 - ab - at - bt - t^2 + \lambda t^2}{(a+t)(b+t)(1+\lambda)} \iff$$

$$\iff t^2(a+b-c) + \lambda t^2(a+b-c) + abt + \lambda abt + abc + abt \geq 0$$

The last inequality is obvious since $\lambda > -1$. $\square$

In case $\lambda = 0$ we have Łukasiewicz t-norm $T_{Luk}(x,y) = max\{x+y-1,0\}$, and from Theorem 2 we obtain the following known, see, e.g., [25] fact:

Corollary 2. *Standard fuzzy metric $M_d(x,y,t)$ is strong in case of the Łukasiewicz t-norm $T_{Luk}(x,y) = max\{x+y-1,0\}$.*

Theorem 3. *Standard fuzzy pseudometric for the drastic t-norm T_D is strong.*

Proof. To prove that

$$m_d(x,z,t) \geq T_D(m_d(x,y,t), m_d(y,z,t)) \text{ that is } \tfrac{t}{t+c} \geq T_D\left(\tfrac{t}{t+a}, \tfrac{t}{t+b}\right)$$

we consider several cases:

- If $a \neq 0$ and $b \neq 0$, then left side of inequality is equal to 0
- If $a = 0$ and $b \neq 0$, then $c \leq b$ and $\frac{t}{t+c} \geq \frac{t}{t+b}$. We similarly reason if $b = 0$ and $a \neq 0$
- If $a = b = 0$, then $c = 0$ and $1 \geq 1$.

$\square$

Notice that standard fuzzy pseudometrics in case of some important t-norms are not strong.

Example 2. *Standard fuzzy pseudometric for the minimum t-norm generally is not strong. Indeed, if $d(x,z) > max(d(x,y), d(y,z))$ then $M_d(x,z,t) < M_d(x,y,t) \wedge M_d(y,z,t)$ for every $t \in \mathbb{R}^+$.*

Example 3. *Standard fuzzy pseudometric for the nilpotent minimum t-norm generally is not strong.*

Proof. Assume the opposite

$$\frac{t}{t+d(x,z)} \geq T_{nM}\left(\frac{t}{t+d(x,y)}, \frac{t}{t+d(y,z)}\right)$$

and suppose $d(x,z) = \frac{1}{2}, d(x,y) = \frac{1}{3}, d(y,z) = \frac{1}{3}$ and choose $t = 2$. Then

$$\tfrac{t}{t+b} + \tfrac{t}{t+c} = \tfrac{12}{7} > 1 \text{ and } \min\left\{\tfrac{t}{t+b}, \tfrac{t}{t+c}\right\} = \tfrac{t}{t+b} = \tfrac{6}{7}.$$

Thus, in this case we obtain

$$\tfrac{t}{t+a} \geq T_{nM}\left(\tfrac{t}{t+b}, \tfrac{t}{t+c}\right) = \min\left\{\tfrac{t}{t+b}, \tfrac{t}{t+c}\right\}, \text{ i.e., } \tfrac{4}{5} \geq \tfrac{6}{7}.$$

The obtained contradiction completes the proof. $\square$

Strongness of Standard Fuzzy k-Pseudometrics

Let $k \geq 1$ be a constant and X be a set. Generalizing the concept of a (pseudo)metric, Bakhtin and Czervik (independently) introduced the notion which is now known by metric-type structure, a b-(pseudo)metric or a k-(pseudo)metric. We stick here to the last term:

Definition 7 ([26–28])**.** *Let $k \in [1,\infty)$. A k-(pseudo)metric on a set X is a mapping $d : X \times X \to \mathbb{R}^+$ such that*

(1Mk) $d(x,y) = 0 \iff x = y$;
(2Mk) $d(x,y) = d(y,x)\ \forall x,y \in X$;
(3Mk) $d(x,z) \leq k \cdot (d(x,y) + d(y,z))\ \forall x,y,z \in X$.

Obviously, we return to the definition of a metric if $k = 1$, while in case $k < 1$, the definition makes no sense.

Example 4. *In the paper [13], the following scheme for constructing k-pseudometrics for a given $k \geq 1$ from ordinary pseudometrics was suggested. Let $\varphi_k : \mathbb{R}^+ \to \mathbb{R}^+$ be a strongly increasing continuous function such that $\varphi_k(0) = 0$ and $\varphi_k(a+b) \leq k \cdot \varphi_k(a) + k \cdot \varphi_k(b)$ for all $a, b \in \mathbb{R}^+$. A series of k-(pseudo)metrics can be obtained from an ordinary (pseudo)metric by the following construction; see, e.g., [13]. Let $k \geq 1$ be a fixed constant and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing mapping such that $\varphi(0) = 0$ and $\varphi_k(a+b) \leq k \cdot \varphi_k(a) + k \cdot \varphi_k(b)$ for all $a, b \in \mathbb{R}^+$. Now, for an arbitrary (pseudo)metric $d : X \times X \to \mathbb{R}^+$ on a set X, by setting*

$$d_\varphi(x,y) = (\varphi \circ d)(x,y) \quad x, y \in X$$

we obtain a k-(pseudo)metric d_φ on this set.

In [12,13], the GV-fuzzy version of a k-(pseudo)metric was introduced. Below, we present this definition in the format of KM-fuzzy (pseudo)metrics.

Definition 8 ([12,13])**.** *A fuzzy k-pseudometric on a set X is a pair $(M, *)$ where $*$ is a continuous t-norm and $M : X \times X \times \mathbb{R}^+ \to [0,1]$ is a mapping satisfying the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:*

(0FKMk) $M(x,y,0) = 0$ *for all* $x, y \in X$;
(1FKMk) $M(x,y,t) = 1$ *for all* t *whenever* $x = y$;
(2FKMk) $M(x,y,t) = M(y,x,t)$ *for all* $x, y \in X$, *for all* $t \in \mathbb{R}^+$;
(3FKMk) $M(x,z,k(t+s)) \geq M(x,y,t) * M(y,z,s)$ *for all* $x, y \in X$, *for all* $t \in \mathbb{R}^+$;
(4FKMk) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is left continuous for all* $x, y \in X$.

The triple $(X, m, *)$ is a called a fuzzy k-pseudometric space.

If the axioms (3FKMk) and (4FKMk) are replaced, respectively, by axioms (3^sFKMk) and (4^sFKMk),

(3^sFKMk) $M(x,z,kt) \geq M(x,y,t) * M(y,z,t)$;
(4^sFKMk) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is left continuous and increasing for all* $x, y \in X$.

Then M is called a strong fuzzy k-pseudometric.

Patterned after the construction of the standard fuzzy pseudometric induced by a metric set (see Definition 6), we present here the construction of a fuzzy k-pseudometric from a k-pseudometric.

Theorem 4. *Let $d : X \times X \to \mathbb{R}_0^+$ be a k-pseudometric. Then the mapping $M_d : X \times X \times \mathbb{R}^+ \to (0,1]$ defined by*

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

is a fuzzy k-pseudometric for the minimum t-norm and hence (by Remark 1) for any continuous t-norm.

Proof. The validity of axioms (0FKMk), (1FKMk), (2FKMk) and (4FKMk) for $M_d(x,y,t)$ is obvious. Hence, to prove this statement, we have to verify axiom (3FKMk), that is to show that

$$\frac{t}{t+d(x,y)} \wedge \frac{s}{s+d(y,z)} \leq \frac{k(t+s)}{k(t+s)+d(x,z)} \quad \forall x, y, z \in X \text{ and } \forall s, t > 0.$$

Since d is a k-pseudometric and hence $d(x,z) \leq k(d(x,y) + d(y,z))$, we replace the inequality to be proved by a stronger inequality

$$\frac{t}{t+d(x,y)} \wedge \frac{s}{s+d(x,y)} \leq \frac{k(t+s)}{k(t+s)+k(d(x,y)+d(y,z))} = \frac{t+s}{(t+s)+(d(x,y)+d(y,z))}.$$

Without loss of generality we assume that $\frac{t}{t+d(x,y)} \leq \frac{s}{s+d(y,z)}$, and therefore we have to show that

$$\frac{t}{t+d(x,y)} \leq \frac{t+s}{t+s+(d(x,y)+d(y,z))}$$

We prove this inequality straightforwardly just by noticing that, as it follows from the assumption $\frac{t}{t+d(x,y)} \leq \frac{s}{s+d(y,z)}$, we have $t \cdot (y,z) \leq s \cdot d(x,y)$. $\square$

The question whether the standard k-fuzzy metric M_d induced by a k-metric is a strong one is subtler. However, for a certain kind of k-metrics, we have the following general result.

Theorem 5. *Let $* : [0,1] \times [0,1]$ be a continuous t-norm and $d : X \times X \to \mathbb{R}^+$ be a pseudometric. If the standard fuzzy pseudometric $(M_d, *)$ is strong for the pseudometric $d : X \times X \to \mathbb{R}^+$ and $d_\varphi : X \times X \to \mathbb{R}^+$ is defined as in Example 4, then the standard fuzzy k-pseudometric $(M_{d_\varphi}, *)$ is strong.*

Proof. The validity of axioms (0FKMk), (1FKMk), (2FKMk) and (4FKMk) for M_{d_φ} is obvious. Referring to construction given in Theorem 4 we have to prove only the validity of (3^sFKMk), that is to show that

$$M_{d_\varphi}(x,z,k \cdot t) = \frac{k \cdot t}{k \cdot t + d_\varphi(x,z)} \geq \frac{t}{t+d_\varphi(x,z)} * \frac{t}{t+d_\varphi(x,z)} = M_{d_\varphi}(x,y,t) * M_{d_\varphi}(y,z,t)$$

for any $x,y,z \in X$ and any $t,s \in \mathbb{R}^+$. Now, applying the inequality

$$d_\varphi(x,z) \leq k \cdot (d_\varphi(x,y) + d_\varphi(y,z)) \quad \forall x,y,z \in X$$

provided by the properties required for the mapping φ, we replace the provable inequality by a stronger one:

$$\frac{t}{t + \frac{1}{k} \cdot d_\varphi(x,z)} \geq \frac{t}{t+d_\varphi(x,y)} * \frac{t}{t+d_\varphi(y,z)}.$$

However, this inequality can be proved verbatim repeating the reasoning which was used when proving axiom (3^sFKM) in the definition of the standard fuzzy pseudometric M_d. Recall that we have assumed that for the metric d the corresponding standard fuzzy metric M_d is strong. $\square$

Now we present a construction allowing to obtain a new strong fuzzy k-pseudometric from a given one on the basis of the product t-norm (cf similar construction in case of strong fuzzy pseudometrics, [11]).

Let $M : X \times X \times \mathbb{R}^+ \to [0,1]$ be a strong fuzzy k-pseudometric for the product t-norm. Then the mapping $N : X \times X \times \mathbb{R}^+ \to [0,1]$ defined by

$$N(x,y,t) = \frac{t + M(x,y,t)}{t+1} \; \forall x,y \in X, \; \forall t > 0$$

is also a strong fuzzy k-pseudometric. Since the validity of axioms (0FKMk), (1FKMk), (2FKMk), and (4^sFKMk), for $N : X \times X \to [0,1]$ are ensured by the corresponding axioms for $M : X \times X \times \mathbb{R}^+ \to [0,1]$, we have to establish only axiom (3^sFKMk), that is the inequality

$$\frac{t + M(x,y,t)}{t+1} \cdot \frac{t + M(y,z,t)}{t+1} \leq \frac{kt + M(x,z,kt)}{kt+1}.$$

It will follow from the stronger inequality

$$\frac{t + M(x,y,t)}{t+1} \cdot \frac{t + M(y,z,t)}{t+1} \leq \frac{t + M(x,y,kt)}{t+1},$$

which, in its turn, can be reduced to the inequality

$$t \cdot M(x,y,t) + t \cdot M(y,z,t) + M(x,y,t) \cdot M(y,z,t) \leq t + t \cdot M(x,z,kt) + M(x,z,kt).$$

The last inequality can be easily established recalling that $M(x,y,t) \cdot M(y,z,t) \leq M(x,z,kt)$ by axiom (3^sFKMk) and noticing that $M(x,y,t) + M(y,z,t) \leq 1 + M(x,y,t) \cdot M(y,z,t)$.

4. Topological and Lattice Structure of Some Families of Strong Fuzzy Metric Spaces

Different from the objectives of research in the previous section, here we focus on the study of global properties of some families of t-norms and fuzzy metrics in relation with the strongness property. Specifically, in the first subsection, we describe the location properties of the family $\mathfrak{S}$ of all t-norms which ensure the strongness of the standard fuzzy metric in the family $\mathfrak{T}$ of all continuous t-norms. In turn, in the second subsection we study the global properties of the family $\mathfrak{M}_T$ of all fuzzy metrics which are strong for a given t-norm T.

4.1. Some Remarks on t-Norms That Ensure Strongness of Standard Fuzzy Metrics

Let $\mathfrak{T} = \{T : \{T : [0,1] \times [0,1] \to [0,1]\}$ be the set of all t-norms $T : [0,1] \times [0,1] \to [0,1]$. We define a partial order on the set $\mathfrak{T}$ by setting $T_1 \leq T_2$ for $T_1, T_2 \in \mathfrak{T}$ if and only if $T_1(\alpha,\beta) \leq T_2(\alpha,\beta)$ for any $\alpha,\beta \in [0,1]$. It is easy to see that $(\mathfrak{T},\leq)$ is a partially ordered set with the minimum t-norm $T(\alpha,\beta) = \alpha \wedge \beta$ as the top element of $(\mathfrak{T},\leq)$ and the drastic t-norm as its bottom element. Further, let $\mathfrak{C} \subseteq \mathfrak{T}$ be the set of all continuous t-norms $T : [0,1] \times [0,1] \to [0,1]$ and let $\mathfrak{S}$ denote the family of all continuous t-norms which ensures that for each metric $d : X \times X \to \mathbb{R}^+$ the induced standard fuzzy metric $M(x,y,t) = \frac{t}{t+d(x,y)}$ is strong. We consider $(\mathfrak{C},\leq)$ and $(\mathfrak{S},\leq)$ with the order induced from $(\mathfrak{T},\leq)$.

Theorem 6. *The set $\mathfrak{S}$ is a lowerset in $(\mathfrak{C},\leq)$: that is, if a t-norm $T_1 \in \mathfrak{S}$, a t-norm $T_2 \in \mathfrak{C}$ and $T_2 \leq T_1$ then $T_2 \in \mathfrak{S}$.*

Proof. The construction $M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t \neq 0, \\ 0, & \text{otherwise} \end{cases}$ defines a fuzzy pseudometric $(M_d,*)$ for every pseudometric d and for any continuous t-norm $*$ (see Definition 6). Therefore, we have to verify only the strongness property in respect of the t-norm T_2. We do this as follows. Since $T_1 \in \mathfrak{S}$, for each pseudometric $d : X \times X \to \mathbb{R}^+$, all $x,y,z \in X$ and $t \in [0,+\infty)$ we have

$$\frac{t}{t+d(x,y)} \geq T_1\left(\frac{t}{t+d(x,z)}, \frac{t}{t+d(z,y)}\right) \geq T_2\left(\frac{t}{t+d(x,z)}, \frac{t}{t+d(z,y)}\right) \square$$

The opposite statement is obviously false: the standard fuzzy pseudometric induced by the minimum t-norm is not strong.

□

Concerning the topological issues, we consider $\mathfrak{T}$ and its subset $\mathfrak{C}$ endowed with topology of uniform convergence $\mathcal{T}_u$.

Theorem 7. *The set $\mathfrak{S}$ is a closed in the space $\mathfrak{C}$ induced by the topology of uniform convergence $\mathcal{T}_u$.*

Proof. Recall first that the topology of uniform convergence in this situation is the topology defined by the family of all open balls:

$$B(T_0,\varepsilon) = \{T \in \mathfrak{C} \mid |(T(a,b) - T_0(a,b)| \; \forall a,b \in [0,1] \text{ for all } T_0 \in \mathfrak{C}, \text{ and all } \varepsilon > 0\}$$

and a set is closed in $(\mathfrak{C},\mathcal{T}_u)$ if and only if it contains the limits of all its convergent sequences which lay in $\mathfrak{C}$.

Therefore, let $\{T_i \mid i \in \mathbb{N}\} \subseteq \mathfrak{S}$ and $\lim_{i\to\infty} T_i = T$ in $(\mathfrak{C},\mathcal{T}_u)$. We have to show that $T \in \mathfrak{S}$. Since the axioms (0FKM) - (2FKM) obviously hold, we have to prove the last two axioms.

Let $x, y, z \in X$ be fixed, $t \in \mathbb{R}^+$ and $d(x,y) = a, d(y,z) = b$ and $d(x,z) = c$. We have that

$$\tfrac{t}{t+c} \geq T_i(\tfrac{t}{t+a}, \tfrac{t}{t+b}) \text{ for every } i \in \mathbb{N}.$$

Since $\lim_{i\to\infty} T_i = T$, we obtain the requested inequality

$$\tfrac{t}{t+c} \geq T(\tfrac{t}{t+a}, \tfrac{t}{t+b})$$

by the definition of the pointwise convergence topology, and hence, moreover, in the topology of uniform convergence; thus, (3^sFKM) is justified.

To justify axiom (4^sFKM) for T notice that its continuity follows from the continuity of all T_i and uniform convergence of the sequence $\{T_i \mid i \in \mathbb{N}\}$ while monotonicity of T obviously follows from monotonicity of all T_i. □

Since $\mathfrak{C}$ is closed in $\mathfrak{T}$ in the topology of uniform convergence, we obtain the following corollary from the previous theorem:

Theorem 8. *The set $\mathfrak{S}$ is a closed in the space $\mathfrak{T}$ endowed with the topology of uniform convergence $\mathcal{T}_u$.*

4.2. Some Remarks about the Set of Strong Fuzzy Metrics for a Fixed t-Norm

Let $T : [0,1] \times [0,1] \to [0,1]$ be a fixed continuous t-norm and $\mathfrak{M}_T$ be the set of all fuzzy pseudometrics $M : X \times X \to \mathbb{R}^+$, which are strong with respect of this t-norm. We are interested in topological and lattice properties of the set $\mathfrak{M}_T$ in the set $\mathfrak{F}$ of all function $f : X \times X \times \mathbb{R}^+ \to [0,1]$ endowed with order relation $\leq$ defined by

$$F_1 \leq F_2 \iff \forall x, y \in X, \forall t \in \mathbb{R}^+ F_1(x,y,t) \leq F_2(x,y,t) \ \forall F_1, F_2 \in \mathfrak{M}_T$$

and the topology of uniform convergence. Recall that the base for the topology of uniform convergence in this situation is defined by the family of open balls

$$B(F_0, \varepsilon) = \{F \in \mathfrak{F} :\mid F(x,y,t) - F_0(x,y,t) \mid < \varepsilon \ \forall x, y \in X, \ \forall t \in \mathbb{R}^+\}$$

for all $F_0 \in \mathfrak{F}$ and $\varepsilon \in (0,1)$.

Recall also that the set $S \subseteq \mathfrak{F}$ is closed in this topology if, and only if, for every sequence $\{F_n(x,y,t)\}_{n\in\mathbb{N}} \subset S$:

$$\lim_{n\to\infty} F_n(x,y,t) = F_0(x,y,t) \implies F_0(x,y,t) \in S.$$

Theorem 9. *The set $\mathfrak{M}_T$ is closed in $\mathfrak{F}$; that is, if for every $x, y \in X, t \in \mathbb{R}^+$, the sequence $M_n(x,y,t) \subset \mathfrak{M}_T$ converges to $F(x,y,t)$ uniformly with respect to t, then $F(x,y,t) \in \mathfrak{M}_T$.*

Proof. Let $\lim_{n\to\infty} M_n = F_0$.
Since $M_n(x,y,t) \in [0,1]$ for all $x, y \in X, t \in \mathbb{R}^+$, (by 0FKM) then $F_0(x,y,t) \in [0,1]$ for all $x, y \in X, t \in \mathbb{R}^+$ and hence (0FMK) holds for F_0.
Since $M_n(x,x,t) = 1$ for every $x \in X, t \in \mathbb{R}^+$ by (1FKM), then $F_0(x,x,t)$ for every $x \in X, t \in \mathbb{R}^+$ and hence (1FMK) holds for F_0.
Since $M_n(x,y,t) = M_n(y,x,t)$ for every $x \in X, t \in \mathbb{R}^+$ by (2FKM), then $F_0(x,y,t) = F_0(y,x,t)$ for every $x, y \in X, t \in \mathbb{R}^+$ and hence (2FMK) holds for F_0.

To obtain (3^sFMK) for the mapping F_0, that is to prove that

$$F_0(x,z,t) \geq T\big(F_0(x,y,t), F_0(y,z,t)\big),$$

we refer to the continuity of the t-norm and the inequality

$$M_n(x,y,t) \geq T\big(M_n(x,y,t), M_n(y,z,t)\big) \text{ for all } n \in \mathbb{N}$$

and the reasoning is as follows:

$$F_0(x,z,t) = \lim_{n\to\infty} M_n(x,z,t) \geq \lim_{n\to\infty} \big(T(m_n(x,y,t), M_n(y,z,t))\big) =$$

$$T\big(\lim_{n\to\infty} M_n(x,y,t), \lim_{n\to\infty} M_n(y,z,t)\big) = T\big(F_0(x,y,t), F_0(y,z,t)\big).$$

To show (4^sFKM) for the mapping F_0, notice that left semicontinuity of $F_0(x,y,t)$ in respect of t for every $x, y \in X$ is ensured by the uniform convergence of the sequence $M_n(x,y,t)$. Besides $F_0(x,y,t)$ is increasing by the third argument since every $M_n(x,y,t)$ is increasing by the third argument. □

Remark 4. *Topology of uniform continuity on $\mathfrak{F}$ is needed only to ensure the left semicontinuity of the limit function F_0. For all other properties of the limit function F_0, it is sufficient to consider $\mathfrak{F}$ in the topology of pointwise convergence.*

Theorem 10. *If $M_1, M_2 \in \mathfrak{M}_T$, then $M = M_1 \wedge M_2 \in \mathfrak{M}_T$.*

Proof. Properties (0FMK)–(2FMK) for the fuzzy pseudometric M are ensured by the corresponding properties for fuzzy pseudometrics M_1 and M_2. To prove (3^sFMK) for M let $x, y, z \in X$ and $t \in \mathbb{R}^+$. Then

$$\begin{gathered}M(x,z,t) = M_1(x,z,t) \wedge M_2(x,z,t) \geq \\ \big(M_1(x,y,t) * M_1(y,z,t)\big) \wedge \big(M_2(x,y,t) * M_2(y,z,t)\big) \geq \\ \big(M_1(x,y,t) \wedge M_2(x,y,t)\big) * \big(M_1(y,z,t) \wedge M_2(y,z,t)\big) = M(x,y,t) * M(y,z,t).\end{gathered}$$

The first one of the above inequalities is ensured by axiom (3^sFMK), which holds for M_1 and M_2. The second one is ensured by the inequality

$$(a * b) \wedge (c * d) \geq (a \wedge c) * (b \wedge d), a, b, c, d \in [0,1]$$

that is true for every t-norm $*$. It can be proved as follows:

$$a \geq a \wedge c, b \geq b \wedge d \Longrightarrow a * b \geq (a \wedge c) * (b \wedge d);$$

$$c \geq a \wedge c, d \geq b \wedge b \Longrightarrow c * d \geq (a \wedge c) * (b \wedge d);$$

From the above two inequalities we obtain the required

$$(a * b) \wedge (c * d) \geq (a \wedge c) * (b \wedge d), a, b, c, d \in [0,1]$$

from the last two inequalities.

Lastly, the left semicontinuity for M and its increase, that is property (4^sFMK) is ensured by the corresponding property satisfied by M_1 and M_2. □

Theorem 11. *$\mathfrak{M}_T$ is a complete meet semilattice. Its bottom element is $M_\perp$, defined by*

$$M_\perp(x,y,t) = M_\perp(x,y) = \begin{cases} 1, & \textit{if } x = y \textit{ and } t > 0, \\ 0, & \textit{otherwise} \end{cases}$$

Proof. From Theorems 9 and 10 it follows that the meet $\bigwedge \mathcal{M}$ of every subset of $\mathcal{M} \subseteq \mathfrak{M}_T$ belongs to $\mathfrak{M}_T$. Therefore, to complete the proof we have to show that $M_\perp$ is a strong fuzzy pseudometric and that it is the bottom element in $\mathfrak{M}_T$. That is, $M_\perp$ is less than or equal to any other strong fuzzy pseudometric.

Since the properties (0FKM), (1FKM), (2FKM), and (4^sFKM) for $M_\perp$ are obvious, we have to prove that (3^sFKM) holds, that is, to show that $M(x,z) \geq T(M(x,y), M(y,z))$ for all $x, y \in X$. We obtain it as follows:

- If $x = z$, then $M(x,x) = 1 \geq T(M(x,y), M(y,z))$

- If $x \neq z$, then either $x \neq y$ or $y \neq z$. Therefore either $M(x,y) = 0$ or $M(y,z) = 0$. So $M(x,z) = 0 \geq T(M(x,y), M(y,z)) = 0$.

Lastly, we have to prove that if $M^*(x,y,t)$ is a fuzzy pseudometric (specifically, strong), then $M(x,y,t) \leq M^*(x,y,t)$ for all $x, y \in X,\ t \in \mathbb{R}^+$. However, from the definition of a fuzzy metric we obtain that $M^*(x,x,t) = 1$, so the inequality holds if $x = y$. Otherwise, if $x \neq y$, we obtain $M(x,y,t) = 0 \leq M^*(x,y,t)$. □

Remark 5. *Different to meet, the join of even two strong fuzzy metrics needs not be strong fuzzy metric. We show this for the case of the product t-norm. Similar counterexamples can be constructed for drastic t-norm, minimum t-norm, T_{H_0} t-norm, and many other cases.*

Proof. Let M_1 and M_2 be strong pseudometrics with respect to the product t-norm and let $x, y, z \in X$ and $t \in \mathbb{R}^+$ Further, let $M(x,y,t) = M_1(x,y,t) \vee M_2(x,y,t)$. We denote

$$M_1(x,y,t) = \alpha_1,\ M_1(x,z,t) = \alpha_2\ M_1(z,y,t) = \alpha_3$$
$$M_2(x,y,t) = \beta_1,\ M_2(x,z,t) = \beta_2\ M_2(z,y,t) = \beta_3.$$

M_1 and M_2 are strong fuzzy metrics in case of the product t-norm, and hence

$$M_1(x,y,t) = \alpha_1 \geq M_1(x,z,t) \cdot M_1(z,y,t) = \alpha_2 \cdot \alpha_3; \tag{1}$$

$$M_2(x,y,t) = \beta_1 \geq M_2(x,z,t) \cdot M_2(z,y,t) = \beta_2 \cdot \beta_3. \tag{2}$$

If M would be a strong fuzzy metric, then

$$M(x,y,t) \geq M(x,z,t) \cdot M(z,y,t),$$

i.e.,

$$\alpha_1 \vee \beta_1 \geq (\alpha_2 \vee \beta_2) \cdot (\alpha_3 \vee \beta_3).$$

From (1) and (2) we obtain

$$\alpha_1 \vee \beta_1 \geq (\alpha_2 \cdot \alpha_3) \vee (\beta_2 \cdot \beta_3).$$

Suppose that this inequality stands and let

$$\alpha_2 = \tfrac{1}{2}, \alpha_3 = \tfrac{1}{6}, \beta_2 = \tfrac{1}{4} \text{ and } \beta_3 = \tfrac{1}{3}.$$

Then, we have that $\alpha_1 \geq \frac{1}{12}$ and $\beta_1 \geq \frac{1}{12}$, and suppose that $\alpha_1 = \beta_1 = \frac{1}{12}$. We obtain that

$$\alpha_1 \vee \beta_1 = \tfrac{1}{12} \geq \left(\tfrac{1}{2} \vee \tfrac{1}{4}\right) \cdot \left(\tfrac{1}{6} \vee \tfrac{1}{3}\right) = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6}.$$

The obtained contradiction completes the proof. □

5. Fuzzy Approximating Metrics and Strong Fuzzy Approximating Metrics

Although strong fuzzy metrics fit well when studying global problems of words combinatorics, for example, considering such questions as topological and lattice-type properties of arrays of words, they are not always satisfactory in applications for problems that involve computation of actual distance between two infinite words. The problem is that in practice of computation, words usually are not available as given at present but appear in the process of computation. We interpret this computation as the procedure along parameter $t \in \mathbb{R}^+$, that is, along the third argument in the definition of a strong fuzzy pseudometric. Under this interpretation axiom (FKM1) is too strong: given a string $x = (x_0, x_1, \ldots, x_n, \ldots)$ at the stage $t \in \mathbb{R}^+$, we have compared this string only until the $[t]^{th}$ coordinate and we cannot confirm yet that $M(x,x,t) = 1$. On the other hand, "at the infinity", we have information about all elements of the string and therefore it is natural to request that $\lim_{t\to\infty} M(x,x,t) = 1$ for every $x \in X$. Besides, when comparing x and y at every step t, thus having information up to t on both strings and not knowing yet whether $x = y$, we obviously have only relation $M(x,x,t) \geq M(x,y,t)$. Note also that we cannot be sure that the equality $M(x,y,t) = 1$ for every $t < \infty$ means that $x = y$, since the whole information is obtained only at the "∞". We view these observations as justification for the following definitions.

Definition 9. *A (KM-)fuzzy approximating pseudometric on a set X is a mapping $M : X \times X \times \mathbb{R}^+ \to [0,1]$ satisfying the following axioms*

(0FAKM) $M(x,y,0) = 0 \; \forall x,y \in X$;
(1FAKM) $M(x,x,t) \geq M(x,y,t) \; \forall x,y \in X$;
(2FAKM) *If* $x,y \in X$ *then* $\lim_{t\to\infty} M(x,y,t) = 1$ *whenever* $x = y$;
(3FAKM) $M(x,y,t) = M(y,x,t) \; \forall x,y \in X, \; \forall t \in \mathbb{R}^+$
(4FAKM) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s) \forall x,y,z \in X, \; \forall t,s \in \mathbb{R}^+$
(5FAKM) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is lower semicontinuous for all* $x,y \in X$.

Definition 10. *A strong (KM-)fuzzy approximating pseudometric on a set X is a mapping $M : X \times X \times \mathbb{R}^+ \to [0,1]$ satisfying axioms (0FAKM)–(3FAKM) and the following modified versions of axioms (4FAM) and (5FAM)*

(4^sFAKM) $M(x,z,t) \geq M(x,y,t) * M(y,z,t) \forall x,y,z \in X, \; \forall t \in \mathbb{R}^+$
(5^sFAKM) $M(x,y,-) : \mathbb{R}^+ \to [0,1]$ *is lower semicontinuous and increasing for all* $x,y \in X$.

A reader can easily reformulate GV-versions of these definitions.

Remark 6. *Comparing Definitions 9 and 10 with definitions of a KM-fuzzy pseudometric and strong KM-fuzzy pseudometric, respectively, notice first that the principal revision of the definition of a KM-fuzzy pseudometric is that we generalized axiom (1FKM) by splitting it into two axioms (1FAKM) and (2FAKM); the intuitive meaning of this splitting is explained above. We do not have to revise axioms (2FKM) and (3FKM) that appear as axioms (3FAKM) and (4FAKM) in the Definitions 9 and 10 since they reflect information at finite steps $[t]^{th}$ and hence are operating with the information already received at this step. We do not have to also revise axioms (4^sFKM) and (4FKM) that appear now as axioms (5FAKM) and (5^sFAKM) respectively since they are given already in the global way, that is, for each specific $t \in [0,\infty)$.*

Remark 7. *In [14], where our first attempt to apply fuzzy metrics for description of distance between infinite words was undertaken, we introduced the notion of a fragmentary fuzzy (pseudo)metric, and the name "fragmentary" was justified by their construction from fragments of (pseudo)metrics on the set of infinite words. Later, in [15], we defined φ-fuzzy (pseudo)metrics, generalizing fragmentary fuzzy (pseudo)metrics. One can easily show that fragmentary and φ-fuzzy pseudometrics can be obtained as special kind of GV-fuzzy approximating metrics.*

6. Some Examples of Application of Strong Fuzzy Approximating Metrics in Words Combinatorics

Theorem 12. *Let (X,d) be an pseudometric space and define a mapping $m : X \times X \times \mathbb{R}^+ \to [0;1]$ by*

$$m(x,y,t) = \tfrac{t-d(x,y)}{t+100} \vee 0.$$

Then $m(x,y,t)$ is a strong (KM-)fuzzy approximating pseudometric in case of the Łukasiewicz t-norm T_{Luk}.

Proof. We have to prove that

$$m(x,z,t) \geq T_{Luk}(m(x,y,t), m(y,z,t)),$$

i.e.,

$$\frac{t-c}{t+100} \vee 0 \geq max\Big\{\Big(\frac{t-a}{t+100} \vee 0\Big) + \Big(\frac{t-b}{t+100} \vee 0\Big) - 1; 0\Big\}.$$

If $t \leq a$ (similarly, if $t \leq b$), then we have

$$\frac{t-c}{t+100} \vee 0 \geq max\left\{\frac{t-b}{t+100} - 1; 0\right\} = 0$$

If $t > a$ and $t > b$ then two options need to be examined:

1. If $c \geq t$ then we have

$$0 \geq \tfrac{t-a-b-100}{t+100} \iff t \leq a+b+100,$$

   which stands as $a+b \geq c \geq t$.
2. If $c < t$ then we have

$$\tfrac{t-c}{t+100} \geq \tfrac{t-a-b-100}{t+100} \iff c \leq a+b+100,$$

   which stands as $a+b \geq c$.

□

Corollary 3. *Let (X, d) be an pseudometric space and define a mapping $m : X \times X \times \mathbb{R}^+ \to [0; 1]$ by*

$$m(x, y, t) = \tfrac{t-d(x,y)}{t+100} \vee 0.$$

Then $m(x, y, t)$ is a strong (KM-)fuzzy approximating pseudometric in case of the drastic t-norm T_D.

Notice that some important t-norms generally do not give a strong (KM-)fuzzy approximating pseudometric, which is defined by the mapping $m(x, y, t) = \frac{t-d(x,y)}{t+100} \vee 0$.

Example 5. *Let (X, d) be an pseudometric space and define a mapping $m : X \times X \times \mathbb{R}^+ \to [0; 1]$ by*

$$m(x, y, t) = \tfrac{t-d(x,y)}{t+100} \vee 0.$$

Then $m(x, y, t)$ generally is not a strong (KM-)fuzzy approximating pseudometric in case of the product t-norm T_{prod}.

Proof. Let us assume the opposite, i.e., $m(x, y, t)$ is a strong (KM-)fuzzy approximating pseudometric in case of the product t-norm T_{prod}. We have to prove that

$$m(x, y, t) \geq T_{prod}(m(x, z, t), m(z, y, t)),$$

i.e.,

$$\frac{t-c}{1+t} \vee 0 \geq \left(\frac{t-a}{1+t} \vee 0\right) \cdot \left(\frac{t-b}{1+t} \vee 0\right)$$

which is not true, if $c = \frac{1}{5}$, $t = \frac{1}{5}$, $a = \frac{1}{10}$ and $b = \frac{1}{10}$: contradiction. □

Let X be the set of infinite words. We define a sequence

$$\{d_n \mid n \in \mathbb{N} \cup \{0\}\}$$

of pseudometrics on X as follows. Let $x = (x_0, x_1, x_2, \ldots), y = (y_0, y_1, y_2, \ldots) \in X$ and let $\chi_i(x, y) = 0$ if $x_i = y_i$ and $\chi_i(x, y) = 1$ if $x_i \neq y_i$. We define:
$d_0(x, y) = \left(\frac{5}{6} + \frac{2}{3}\right)\chi_0(x, y)$;
$d_1(x, y) = \left(\frac{5}{6} + \frac{2}{3}\right)\chi_0(x, y) + \left(\frac{5}{6+1} + \frac{2}{3}\right)\chi_1(x, y)$;
$d_2(x, y) = \left(\frac{5}{6} + \frac{2}{3}\right)\chi_0(x, y) + \left(\frac{5}{6+1} + \frac{2}{3}\right)\chi_1(x, y) + \left(\frac{5}{6+2} + \frac{2}{3}\right)\chi_2(x, y)$;
...
$d_n(x, y) = \sum_{i=0}^{n}\left(\frac{5}{6+i} + \frac{2}{3}\right)\chi_i(x, y)$;
...

Proposition 1. *Every d_n is a pseudometric.*

Basing on this sequence of pseudometrics and referring to Theorem 12 we construct the sequence of strong (KM-)fuzzy approximating pseudometrics in case of the Łukasiewicz t-norm T_{Luk} on the set X of all right-infinite words:
$\mu_0(x,y,t) = \frac{t-d_0(x,y)}{t+100} \vee 0;$
$\mu_1(x,y,t) = \frac{t-d_1(x,y)}{t+100} \vee 0;$
$\mu_2(x,y,t) = \frac{t-d_2(x,y)}{t+100} \vee 0;$
$\ldots;$
$\mu_n(x,y,t) = \frac{t-d_n(x,y)}{t+100} \vee 0;$
$\ldots$

Further, we define the following family of mappings:
$m_0(x,y,t) = \mu_0(x,y,t);$
$m_1(x,y,t) = m_0(x,y,1) \vee \mu_1(x,y,t);$
$m_2(x,y,t) = m_1(x,y,2) \vee \mu_2(x,y,t);$
$\ldots;$
$m_n(x,y,t) = m_{n-1}(x,y,n) \vee \mu_n(x,y,t);$
$\ldots$

Unfortunately, we are not able to prove or disclaim that these mappings are strong (KM-)fuzzy approximating pseudometrics in case of the Łukasiewicz t-norm T_{Luk} on the set X of infinite words. Nevertheless, we can state the following obvious statement.

Proposition 2. *Mappings $m_n : X \times X \times \mathbb{R}^+ \to [0,1]$ are strong (KM-)fuzzy approximating pseudometrics in case of the drastic t-norm T_D on the set X of infinite words.*

Finally, we construct a mapping $m : X \times X \times \mathbb{R}^+ \to (0,1]$ as follows:

$$m(x,y,t) = \begin{cases} m_0(x,y,t) & \text{if } 0 < t \leq 1 \\ m_1(x,y,t) & \text{if } 1 < t \leq 2 \\ m_2(x,y,t) & \text{if } 2 < t \leq 3 \\ \ldots & \\ m_n(x,y,t) & \text{if } n < t \leq n+1 \\ \ldots & \end{cases}$$

Theorem 13. *The mapping $m : \times X \times \mathbb{R}^+ \to [0,1]$ is a strong (KM-)fuzzy approximating pseudometric in case of the drastic t-norm T_D.*

The proof is straightforward from Proposition 2.

Example 6. *Let us go back to that counterexample and let*

$$x = (1,0,0,0,\ldots.), y = (0,1,1,1,..), z = (0,0,0,0,\ldots).$$

Previously, we obtained $\sigma(x,z) = \sigma(y,z)$. We start with a strong (KM-)fuzzy approximating pseudometric $m(x,z,t)$. In this case,

$$\chi_0(x,z) = 1 \text{ and } \chi_i(x,z) = 0 \; \forall i = 1,2,\ldots$$

Let us remind that

$$d_n(x,z) = \textstyle\sum_{i=0}^{n} \chi_i(x,z) \cdot \left(\frac{5}{i+6} + \frac{2}{3}\right).$$

Therefore

$$d_i(x,z) = \tfrac{3}{2} \; \forall i = 0,1,2,\ldots$$

Finally, from a strong (KM-)fuzzy approximating pseudometric $m(x,y,t)$, we obtain that

$$m(x,z,t) = \frac{t-\frac{3}{2}}{t+100} \vee 0, \forall \text{ t} \in \mathbb{R}^+$$

and

$$\lim_{t\to\infty} m(x,z,t) = \lim_{t\to\infty}\left(\frac{t-\frac{3}{2}}{t+100} \vee 0\right) = 1.$$

Now consider $y = (0,1,1,1,..)$, $z = (0,0,0,0,\dots)$. In this case,

$$\chi_0(y,z) = 0 \text{ and } \chi_i(y,z) = 1\ \forall i = 1,2,\dots.$$

Let us remind that

$$d_n(y,z) = \sum_{i=0}^{n} \chi_i(y,z) \cdot \left(\frac{5}{i+6} + \frac{2}{3}\right);$$

Therefore,
$d_0(y,z) = 0$
$d_1(y,z) = \left(\frac{5}{7} + \frac{2}{3}\right)$
$d_2(y,z) = \left(\frac{5}{7} + \frac{2}{3}\right) + \left(\frac{5}{8} + \frac{2}{3}\right)$
$d_3(y,z) = \left(\frac{5}{7} + \frac{2}{3}\right) + \left(\frac{5}{8} + \frac{2}{3}\right) + \left(\frac{5}{9} + \frac{2}{3}\right)$
...
$d_n(y,z) = \sum_{i=1}^{n}\left(\frac{5}{6+i} + \frac{2}{3}\right)$
...

Now, we calculate

$$\lim_{t\to\infty} m(y,z,t) = \lim_{t\to\infty}\left(\frac{t-\sum_{i=1}^{\lfloor t\rfloor}\left(\frac{5}{6+i}+\frac{2}{3}\right)}{t+100} \vee 0\right)$$
$$= \lim_{t\to\infty}\left(\frac{t-\frac{2}{3}\lfloor t\rfloor-\sum_{i=1}^{\lfloor t\rfloor}\frac{5}{6+i}}{t+100} \vee 0\right) = \lim_{t\to\infty}\left(\frac{\frac{1}{3}\lfloor t\rfloor+\{t\}-\sum_{i=1}^{\lfloor t\rfloor}\frac{5}{6+i}}{1+t} \vee 0\right),$$

where $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function. Before we go further, we will refer to one result. For the indication of this result we are grateful to E. M. Miķelsons.

Theorem 14. *If $x \geq 1$ we have*

$$\sum_{n\leq x} \frac{1}{n} = log x + C + o\left(\frac{1}{n}\right)$$

where C is Euler's constant.

From Theorem 14 we have

$$\sum_{i=1}^{\lfloor t\rfloor} \frac{5}{6+i} = 5\left(\sum_{i=1}^{\lfloor t\rfloor} \frac{1}{i} - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)\right)$$
$$= 5\sum_{i=1}^{\lfloor t\rfloor} \frac{1}{i} - \frac{49}{4} = 5log\lfloor t\rfloor + 5C - \frac{49}{4} + o\left(\frac{1}{\lfloor t\rfloor}\right)$$

If we place this expression back into limit we obtain

$$\lim_{t\to\infty} \frac{\frac{1}{3}\lfloor t\rfloor+\{t\}-5log\lfloor t\rfloor-5C+\frac{49}{4}-o\left(\frac{1}{\lfloor t\rfloor}\right)}{t+100} \vee 0 = \frac{1}{3}.$$

Corollary 4. *If we have*

$$x = (1,0,0,0,\dots.), y = (0,1,1,1,..), z = (0,0,0,0,\dots),$$

then

$$\lim_{t\to\infty} m(x,z,t) = 1 > \frac{1}{3} = \lim_{t\to\infty} m(y,z,t),$$

which shows that infinite word z is estimated "closer" to x than to y. It is natural as words y and z coincide only in the first position, but words x and z do not coincide only in the first position.

Remark 8. *The defined strong (KM-)fuzzy approximating pseudometric in Theorem 12 can be generalized as* $m(x,y,t) = \frac{t-d(x,y)}{t+c} \vee 0$*, where* $c \in \mathbb{R}^+$*. The choice of this constant c depends on the context of specific applications. If we want to find a real "nearness-type" relation between two infinite words, the choice of the constant c depends on an importance of the prefix of the word. For example, if we take* $c = 1$*, then the outcome of this metric for two words with the same letters in the first position will be at least one half. If we take* $c = 100$ *(as in our case), then the outcome will just be at least* $\frac{1}{101}$*. Therefore, the greater the constant c is, the lower the meaning of the prefix and vice versa.*

Remark 9. *The defined pseudometrics* $d_n(x,y)$ *in the construction can be generalized as* $d_n(x,y) = \sum_{i=0}^{n}\left(\frac{n^2-m^2}{mn+i} + \frac{m}{n}\right)\chi_i(x,y)$*, where* $m,n \in \mathbb{R}^+$ *and* $m < n$*. If we consider two pairs* (m_1,n_1) *and* (m_2,n_2) *with* $\frac{m_1}{n_1} > \frac{m_2}{n_2}$*, then in the case of a pair* (m_2,n_2) *we attach more importance for prefixes, but in the case of a pair* (m_1,n_1)*, we attach less importance for prefixes.*

7. Conclusions

As it was stated in the introduction, the main goal of our study in this paper, as well as in two previous works [14,15], is the use of fuzzy metrics for description of the structure of the family of infinite words. In order to realize this goal, in the first part of the paper (Sections 3 and 4), we study families of strong fuzzy pseudometrics, assuming *strong* fuzzy pseudometrics (and their modifications) are better suited to describing structure of word families than general ones. Noticing that the axiom $M(x,x,t) = 1$ for every $x \in X$ and every $t > 0$ assumed in the definition of a (strong) fuzzy (pseudo)metric can be violated for some (justified by specific examples) choices of this "pseudometric", we introduce a more flexible notion of a fuzzy strong approximating metric in Section 5. Examples of the use of fuzzy strong approximating metrics for the description of the distance between infinite words are presented in Section 6.

We foresee several directions, both theoretical and practical ones, in which the research started in this work can be continued. The following are the directions where we are planning to continue this work.

- To consider strong fuzzy pseudometric spaces and strong fuzzy approximating metrics as categories, In particular, investigate products, coproducts, and other operations in these categories. To study interrelations between these categories.
- To study the (fuzzy) topology, induced by (strong) fuzzy approximating metrics. Specifically, extend (as far as possible) the results about (fuzzy) topology induced by fuzzy metrics obtained in [5,6,29] et al. for the case of (strong) fuzzy approximating metrics.
- As an important problem to be investigated in our future work, we consider the study of interrelations between our fuzzy approximating (in particular strong) metrics with partial and especially fuzzy partial metrics. An attentive reader probably will notice some similarity between our approximating metrics on one side and partial and especially fuzzy partial metric on the other. Partial metrics were introduced in 1994 by Matthews [30] and now are the focus of interest for some mathematicians and theoretical computer scientists (see, e.g., the survey [31]). Based on the concept of a partial metric, V. Gregori, J-J. Minana, and D. Miravet [32] introduced the concept of a fuzzy partial metric. Many researchers working in theoretical computer science showed serious interest in partial metrics, and recently also in fuzzy partial metrics in view of their perspectives of the use in domain theory and some other areas of theoretical computer science. An attentive reader of our paper will probably notice its certain common features with partial and fuzzy partial metrics, and this is not a surprise, since the idea of both approaches when applied to evaluation of two infinite strings is that the result will not be achieved immediately or at some step, but in the process of comparing these strings. On the other hand, we apply essentially different

approaches to realize this evaluation. It is one of our principal goals for future work to investigate the relations, in particular, on the categorical level, between these theories.

- We illustrated the opportunities provided by strong fuzzy approximating metrics by some examples and comments in Section 6. We view this material only as the first step in the developing methods for the study of the problems of words combinatorics. This work will be continued in particular in the next work (in preparation) where fuzzy approximating metrics based on different t-norms and parameters will be used and the obtained results will be analyzed for a series of numerical examples.

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