A Proof of a Conjecture on Bipartite Ramsey Numbers $B(2, 2, 3)$

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Abstract: The bipartite Ramsey number $B(n_1, n_2, \dots, n_t)$ is the least positive integer b , such that any coloring of the edges of $K_{b,b}$ with t colors will result in a monochromatic copy of K_{n_i, n_i} in the i -th color, for some i , $1 \leq i \leq t$. The values $B(2, 5) = 17$, $B(2, 2, 2, 2) = 19$ and $B(2, 2, 2) = 11$ have been computed in several previously published papers. In this paper, we obtain the exact values of the bipartite Ramsey number $B(2, 2, 3)$. In particular, we prove the conjecture on $B(2, 2, 3)$ which was proposed in 2015—in fact, we prove that $B(2, 2, 3) = 17$.

Keywords: Ramsey numbers; bipartite Ramsey numbers; Zarankiewicz number

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1. Introduction

The bipartite Ramsey number $B(n_1, n_2, \dots, n_t)$ is the least positive integer b , such that any coloring of the edges of $K_{b,b}$ with t colors will result in a monochromatic copy of K_{n_i, n_i} in the i -th color, for some i , $1 \leq i \leq t$. The existence of such a positive integer is guaranteed by a result of Erdős and Rado [1].

The Zarankiewicz number $z(K_{m,n}, t)$ is defined as the maximum number of edges in any subgraph G of the complete bipartite graph $K_{m,n}$, such that G does not contain $K_{t,t}$ as a subgraph. Zarankiewicz numbers and related extremal graphs have been studied by many authors, including Kóvari [2], Reiman [3], and Goddard, Henning, and Oellermann in [4].

The study of bipartite Ramsey numbers was initiated by Beineke and Schwenk in 1976 [5], and continued by others, in particular Exoo [6], Hattingh, and Henning [7]. The following exact values have been established: $B(2, 5) = 17$ [8], $B(2, 2, 2, 2) = 19$ [9], $B(2, 2, 2) = 11$ [6]. In the smallest open case for five colors, it is known that $26 \leq B(2, 2, 2, 2, 2) \leq 28$ [9]. One can refer to [2,9–14] and its references for further studies. Collins et al. in [8] showed that $17 \leq B(2, 2, 3) \leq 18$, and in the same source made the following conjecture:

Conjecture 1 ([8]). $B(2, 2, 3) = 17$.

We intend to get the exact value of the multicolor bipartite Ramsey numbers $B(2, 2, 3)$. We prove the following result:

Theorem 1. $B(2, 2, 3) = 17$.

In this paper, we are only concerned with undirected, simple, and finite graphs. We follow [15] for terminology and notations not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$, or simply by $\deg(v)$. The neighborhood $N_G(v)$ of a vertex v is the set of all vertices of G adjacent to v and satisfies $|N_G(v)| = \deg_G(v)$. The minimum and maximum degrees of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Additionally, the complete bipartite graph with bipartition (X, Y) , where $|X| = m$ and $|Y| = n$, is denoted by $K_{m,n}$. We use $[X, Y]$ to denote the set of edges between the bipartition (X, Y) of G . Let $G = (X, Y)$ be a bipartite graph

and $Z \subseteq X$ or $Z \subseteq Y$, the degree sequence of Z denoted by $D_G(Z) = (d_1, d_2, \dots, d_{|Z|})$, is the list of the degrees of all vertices of Z . The complement of a graph G , denoted by $\overline{G}$, is a graph with same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G . H is n -colorable to $(H_1, H_2, \dots, H_t)$ if there exists a t -coloring of the edges of H such that $H_i \not\subseteq H^i$ for each $1 \leq i \leq t$, where H^i is the spanning subgraph of H with edges of the i -th color.

2. Some Preliminary Results

To prove our main result—namely, Theorem 1—we need to establish some preliminary results. We begin with the following proposition:

Proposition 1 ([8,13]). *The following results about the Zarankiewicz number are true:*

- $z(K_{17,17}, 2) = 74$.
- $z(K_{16,17}, 2) \leq 71$.
- $z(K_{17,17}, 3) \leq 141$.
- $z(K_{16,17}, 3) \leq 133$.
- $z(K_{13,17}, 3) \leq 110$.
- $z(K_{12,17}, 3) \leq 103$.
- $z(K_{11,17}, 3) \leq 96$.

Proof of Proposition 1. By using the bounds in Table 3 and Table 4 of [8] and Table C.3 of [13], the proposition holds. $\square$

Theorem 2 ([8]). $17 \leq B(2, 2, 3) \leq 18$.

Proof of Theorem 2. The lower bound witness is found in Table 2 of [8]. The upper bound is implied by using the bounds in Table 3 and Table 4 of [8]. We know that $z(K_{18,18}, 2) = 81$, $z(K_{18,18}, 3) \leq 156$, and $2 \times 81 + 156 = 318 < 324 = |E(K_{18,18})|$. $\square$

Suppose that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$; in the following theorem, we specify some properties of the subgraph with color g . The properties are regarding $\Delta(G^g)$, $\delta(G^g)$, $E(G^g)$, and degree sequence of vertices X, Y in the induced graph with color g .

Theorem 3. *Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$, and $K_{3,3} \not\subseteq G^g$. So:*

- (a) $|E(G^g)| = 141$.
- (b) $\Delta(G^g) = 9$ and $\delta(G^g) = 8$.
- (c) $D_{G^g}(X) = D_{G^g}(Y) = (9, 9, 9, 9, 9, 8, 8, \dots, 8)$.

Proof of Theorem 3. Assume that $X = \{x_1, x_2, \dots, x_{17}\}$, $Y = \{y_1, y_2, \dots, y_{17}\}$ is a partition set of $K = K_{17,17}$ and (G^r, G^b, G^g) is a 3-edge coloring of K , where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$, and $K_{3,3} \not\subseteq G^g$. Since $|E(K)| = 289$, if $|E(G^g)| \leq 140$ then $|E(\overline{G^g})| \geq 149$ —that is, either $|E(G^r)| \geq 75$ or $|E(G^b)| \geq 75$. In any case, by Proposition 1, either $K_{2,2} \subseteq G^r$ or $K_{2,2} \subseteq G^b$, a contradiction. Hence, assume that $|E(G^g)| \geq 141$. If $|E(G^g)| \geq 142$ then by Proposition 1, $K_{3,3} \subseteq G^g$, a contradiction again; that is, $|E(G^g)| = 141$ and part (a) is true.

To prove part (b), since $|E(G^g)| = 141$ by part (a), we can check that $\Delta(G^g) \geq 9$. Assume that there exists a vertex of $V(K)$ say x , such that $|N_{G^g}(x)| \geq 10$ —that is, $\Delta(G^g) \geq 10$. Consider x and set $G_1^g = G^g \setminus \{x\}$, hence by part (a), $|E(G_1^g)| \leq 141 - 10 = 131$. Therefore, since $|E(K_{16,17})| = 272$, so $|E(\overline{G_1^g})| \geq 141$ —that is, either $|E(G_1^r)| \geq 71$ or $|E(G_1^b)| \geq 71$. In any case, by Proposition 1 either $K_{2,2} \subseteq G_1^r \subseteq G^r$ or $K_{2,2} \subseteq G_1^b \subseteq G^b$, a contradiction. So, $\Delta(G^g) = 9$. To prove $\delta(G^g) = 8$, assume that $M = \{x \in X, |N_{G^g}(x)| = 9\}$ and $N = \{x \in X, |N_{G^g}(x)| = 8\}$; by part (a) one can say that $|M| \geq 5$, if $|M| = 6$, then $\delta(G^g) \leq 7$ —that is, there is a vertex of X (say x) such that $|N_{G^g}(x)| \leq 7$; therefore,

$|N| \leq 10$. If $|N| = 10$, then $|E(G^g[M \cup N, Y])| = 134$, so by Proposition 1, $K_{3,3} \subseteq G^g$, a contradiction. Now assume that $|N| \leq 9$, thus $|E(G^g)| \leq (6 \times 9) + (9 \times 8) + (2 \times 7) = 140$, a contradiction again. For $|M| = 7$ if $|N| \geq 6$, then $|E(G^g[M \cup N', Y])| = 111$, where $N' \subseteq N$ and $|N'| = 6$, so by Proposition 1, $K_{3,3} \subseteq G^g$, a contradiction. Hence assume that $|N| \leq 5$; therefore, $|E(G^g)| \leq (7 \times 9) + (5 \times 8) + (5 \times 7) = 138$, a contradiction again. For $|M| = 8$ if $|N| \geq 5$, then $|E(G^g[M \cup N', Y])| = 112$, where $N' \subseteq N$ and $|N'| = 5$; therefore, by Proposition 1, $K_{3,3} \subseteq G^g$, a contradiction, so assume that $|N| \leq 4$ —that is, $|E(G^g)| \leq (8 \times 9) + (4 \times 8) + (5 \times 7) = 139$, a contradiction again. For $|M| = 9$ if $|N| \geq 3$, then $|E(G^g[M \cup N', Y])| = 105$, where $N' \subseteq N$ and $|N'| = 3$, so by Proposition 1, $K_{3,3} \subseteq G^g$, a contradiction. Thus $|N| \leq 2$ —that is, $|E(G^g)| \leq (9 \times 9) + (2 \times 8) + (6 \times 7) = 139$, which is a contradiction again. For $|M| = 10$, if $|N| \geq 1$, then $|E(G^g[M \cup N', Y])| = 98$, where $N' \subseteq N$ and $|N'| = 1$; so, by Proposition 1 $K_{3,3} \subseteq G^g$, a contradiction. Thus, assume that $|N| = 0$, so $|E(G^g)| \leq (10 \times 9) + (7 \times 7) = 139$, a contradiction again. Therefore, $|M| = 5$ and $|N| = 12$ —that is, $\delta(G^g) = 8$, and part (b) is true.

Now, by parts (a) and (b) it is straightforward to say that $D_{G^g}(X) = D_{G^g}(Y) = (9, 9, 9, 9, 9, 8, 8, \dots, 8)$ —that is, part (c) is true, and this completes the proof. $\square$

3. Proof of the Main Theorem

In this section, by using the results of Section 2, we will prove the main theorem.

Suppose that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$. In the following theorem, we discuss the maximum number of common neighbors of $G^g(x)$ and $G^g(x')$ for $x, x' \in X$.

Theorem 4. Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$. Let $|N_{G^g}(x)| = 9$ and $N_{G^g}(x) = Y_1$; the following results are true:

- (a) For each $x \in X \setminus \{x_1\}$, we have $|N_{G^g}(x) \cap Y_1| \leq 5$.
- (b) Assume that $n = \sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_1|$, then $72 \leq n \leq 73$.

Proof of Theorem 4. Assume that $X = \{x_1, x_2, \dots, x_{17}\}$, $Y = \{y_1, y_2, \dots, y_{17}\}$ is a partition set of $K = K_{17,17}$, and (G^r, G^b, G^g) is a 3-edge coloring of K , where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$. Without loss of generality (W.l.g.) assume that $x = x_1$ and $Y_1 = \{y_1, \dots, y_9\}$. To prove part (a), by contrast assume that there exists a vertex of $X \setminus \{x_1\}$ (say x) such that $|N_{G^g}(x) \cap Y_1| \geq 6$. W.l.g., suppose that $x = x_2$ and $Y_2 = \{y_1, y_2, \dots, y_6\} \subseteq N_{G^g}(x_2)$. Since $K_{3,3} \not\subseteq G^g$, for each $x \in X \setminus \{x_1, x_2\}$, so $|N_{G^g}(x) \cap Y_2| \leq 2$ —that is, $\sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_2| \leq 6 + 6 + (15 \times 2) \leq 42$. Now, since $|E(G^g[X, Y_2])| \leq 42$, one can check that there exists at least one vertex of Y_2 (say y), such that $|N_{G^g}(y)| \leq 7$, a contradiction to part (c) of Theorem 3. Hence, $|N_{G^g}(x) \cap Y_1| \leq 5$ for each $x \in X \setminus \{x_1\}$ —that is, part (a) is true.

To prove part (b), if $n \leq 71$, then by part (c) of Theorem 3, it can be checked that there exists at least one vertex of Y_1 (say y), such that $|N_{G^g}(y)| \leq 7$, a contradiction. Therefore, $n \geq 72$. Assume that $n \geq 74$ and let $D_{G^g}(Y_1) = (d_1, d_2, \dots, d_9)$. Since $\sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_1| \geq 74$, there exist at least two vertices of Y_1 (say y', y''), such that $|N_{G^g}(y')| = |N_{G^g}(y'')| = 9$. Since $n \geq 74$ and $|X \setminus \{x_1\}| = 16$, there exists at least one vertex of $X \setminus \{x_1\}$ (say x'), such that $|N_{G^g}(x') \cap Y_1| = 5$. W.l.g., suppose that $x' = x_2$ and $N_{G^g}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. Now we have the following claims:

Claim 1. For each $x \in X \setminus \{x_1, x_2\}$, we have $|N_{G^g}(x) \cap Y_2| = 2$ and $D_{G^g}(Y_2) = (8, 8, 8, 8, 8)$.

Proof of Claim 1. Since $K_{3,3} \not\subseteq G^g$ for each $x \in X \setminus \{x_1, x_2\}$, thus $|N_{G^g}(x) \cap Y_2| \leq 2$ —that is, $\sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_2| \leq 5 + 5 + (15 \times 2) \leq 40$. Now, since $|E(G^g[X, Y_2])| \leq 40$ and $|Y_2| = 5$, if there exists a vertex of X_1 (say x'), such that $|N_{G^g}(x) \cap Y_2| \leq 1$, then $|E(G^g[X, Y_2])| \leq$

39; therefore, there exists at least one vertex of Y_2 (say y), such that $|N_{G^s}(y)| \leq 7$, a contradiction to part (c) of Theorem 3. So, $|N_{G^s}(x) \cap Y_2| = 2$ and $\sum_{y \in Y_2} |N_{G^s}(y)| = 40$, therefore by part (c) of Theorem 3 $D_{G^s}(Y_2) = (8, 8, 8, 8, 8)$, and the proof of the claim is complete. $\square$

Claim 2. $D_{G^s}(X_1) = (5, 4, 4, \dots, 4)$ where $X_1 = X \setminus \{x_1\}$, in other word $|N_{G^s}(x_i) \cap Y_1| = 4$ for each $i \in \{3, 4, \dots, 17\}$.

Proof of Claim 2. By contradiction, assume that there exists a vertex of $X \setminus \{x_1, x_2\}$ (say x), such that $|N_{G^s}(x) \cap Y_1| = 5$. W.l.g. suppose that $x = x_3$ and $N_{G^s}(x_3) \cap Y_1 = Y_3$, now by Claim 1, $|N_{G^s}(x_3) \cap Y_2| = 2$. W.l.g., assume that $Y_3 = \{y_1, y_2, y_6, y_7, y_8\}$, thus by Claim 1, $D_{G^s}(Y_3) = (8, 8, 8, 8, 8)$ —that is, $|N_{G^s}(y)| = 8$ for each $y \in Y_1 \setminus \{y_9\}$. Since $\Delta(G^s) = 9$, we can check that $n = \sum_{i=1}^{i=17} |N_{G^s}(x_i) \cap Y_1| = \sum_{i=1}^{i=9} |N_{G^s}(y_i)| \leq (8 \times 8) + 9 = 73$, a contradiction. So, $D_{G^s}(X_1) = (5, 4, 4, \dots, 4)$, and the proof of the claim is complete. $\square$

Assume that $N_{G^s}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$, by Claim 1 $D_{G^s}(Y_2) = (8, 8, 8, 8, 8)$. Since there exist at least two vertices of Y_1 (say y', y''), such that $|N_{G^s}(y')| = |N_{G^s}(y'')| = 9$, thus $y', y'' \in \{y_6, y_7, y_8, y_9\}$. W.l.g., we can suppose that $y' = y_6$ and $N_{G^s}(y_6) = X_2 = \{x_1, x_3, \dots, x_{10}\}$. By Claim 2, $|N_{G^s}(x) \cap Y_1| = 4$ and $|N_{G^s}(x) \cap Y_2| = 2$ for each $x \in X_2 \setminus \{x_1\}$ —that is, $|N_{G^s}(x) \cap \{y_7, y_8, y_9\}| = 1$ for each $x \in X_2 \setminus \{x_1\}$. Since $|X_2 \setminus \{x_1\}| = 8$ and $|N_{G^s}(x) \cap \{y_7, y_8, y_9\}| = 1$, by the pigeon-hole principle, there exists a vertex of $\{y_7, y_8, y_9\}$ (say y), such that $|N_{G^s}(y) \cap X_2 \setminus \{x_1\}| \geq 3$. W.l.g., we can suppose that $y = y_7$ and $\{x_3, x_4, x_5\} \subseteq N_{G^s}(y_7) \cap X_2 \setminus \{x_1\}$. As $|Y_2| = 5$ and $|N_{G^s}(x_i) \cap Y_2| = 2$ for $i = 3, 4, 5$, there exist $i, i' \in \{3, 4, 5\}$, such that $|N_{G^s}(x_i) \cap N_{G^s}(x_{i'}) \cap Y_2| \neq 0$. W.l.g., suppose that $i = 3, i' = 4$ and $y_1 \in N_{G^s}(x_3) \cap N_{G^s}(x_4) \cap Y_2$. Therefore, $K_{3,3} \subseteq G^s[\{x_1, x_3, x_4\}, \{y_1, y_6, y_7\}]$, a contradiction. So, $n \leq 73$ and the proof of the theorem is complete. $\square$

In part (b) of Theorem 4, we showed that $72 \leq n = \sum_{i=1}^{i=17} |N_{G^s}(x_i) \cap Y_1| \leq 73$. Now we consider these two cases independently.

3.1. The Case That $n = 73$

In the following theorem, we prove that in any 3-edge coloring of $K_{17,17}$ (say (G^r, G^b, G^s)), where $K_{2,2} \not\subseteq G^r, K_{2,2} \not\subseteq G^b$, if there exists a vertex of $V(K)$ (say x), such that $|N_{G^s}(x)| = 9$ and $\sum_{x_i \in X \setminus \{x\}} |N_{G^s}(x_i) \cap N_{G^s}(x)| = 64$, then $K_{3,3} \subseteq G^s$.

Theorem 5. Assume that (G^r, G^b, G^s) is a 3-edge coloring of $K = K_{17,17}$, such that $K_{2,2} \not\subseteq G^r, K_{2,2} \not\subseteq G^b$. Assume that there exists a vertex of $V(K)$ (say x), such that $|N_{G^s}(x)| = 9$. If $\sum_{i=1}^{i=17} |N_{G^s}(x_i) \cap Y_1| = 73$ where $Y_1 = N_{G^s}(x)$, then $K_{3,3} \subseteq G^s$.

Proof of Theorem 5. By contradiction, assume that $K_{3,3} \not\subseteq G^s$. Therefore, by Theorem 3 and Theorem 4, we have the following results:

- $|E(G^s)| = 141$.
- $\Delta(G^s) = 9$ and $\delta(G^s) = 8$.
- $D_{G^s}(X) = D_{G^s}(Y) = (9, 9, 9, 9, 9, 8, 8, \dots, 8)$.
- For each $x' \in X \setminus \{x\}$ we have $|N_{G^s}(x) \cap N_{G^s}(x')| \leq 5$.
- If $A = \{x \in X, |N_{G^s}(x)| = 9\}$, then $|A| = 5$ and $72 \leq \sum_{y \in N_{G^s}(x)} |N_{G^s}(y)| \leq 73$, for each $x \in A$.

Assume that $X = \{x_1, x_2, \dots, x_{17}\}$, $Y = \{y_1, y_2, \dots, y_{17}\}$ is the partition set of $K = K_{17,17}$, and (G^r, G^b, G^s) is a 3-edge coloring of K , where $K_{2,2} \not\subseteq G^r, K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^s$.

W.l.g., assume that $x = x_1$, $Y_1 = \{y_1, y_2, \dots, y_9\}$, and $n = \sum_{i=1}^{17} |N_{G^s}(x_i) \cap Y_1| = 73$. Since $n = 73$, by (c) we can say that $D_{G^s}(Y_1) = (d_1, d_2, \dots, d_9) = (9, 8, 8, \dots, 8)$ —that is, there exists a vertex of Y_1 (say y), such that $|N_{G^s}(y)| = 9$. By (d), $|N_{G^s}(x_1) \cap N_{G^s}(x)| \leq 5$ for each $x \in X \setminus \{x_1\}$. Set $C = \{x \in X, |N_{G^s}(x) \cap N_{G^s}(x_1)| = 5\}$. Now by argument similar to the proof of Claim 1, we have the following claim:

Claim 3. Assume that $x \in C$ and $N_{G^s}(x) \cap Y_1 = Y'$, then for each $x' \in X \setminus \{x_1, x\}$, we have $|N_{G^s}(x') \cap Y'| = 2$ and $D_{G^s}(Y') = (8, 8, 8, 8, 8)$.

Here there exists a claim about $|C|$ as follows:

Claim 4. $|C| \leq 2$.

Proof of Claim 4. By contradiction, assume that $|C| \geq 3$. W.l.g., suppose that $\{x_2, x_3, x_4\} \subseteq C$ and $N_{G^s}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. By Claim 3, $|N_{G^s}(x) \cap Y_2| = 2$ for each $x \in X \setminus \{x_1, x_2\}$. W.l.g., suppose that $N_{G^s}(x_3) \cap Y_1 = Y_3 = \{y_1, y_2, y_6, y_7, y_8\}$. Since $x_4 \in C$ and $|N_{G^s}(x_4) \cap Y_i| = 2$ for $i = 2, 3$, $y_9 \in N_{G^s}(x_4) \cap Y_1$. Hence, for each $y \in Y_1$, there is at least one $i \in \{2, 3, 4\}$ such that $y \in N_{G^s}(x_i)$; therefore, by Claim 3, $D_{G^s}(Y_1) = (8, 8, 8, 8, 8, 8, 8, 8, 8)$, which is in contrast to $\sum_{i=1}^{17} |N_{G^s}(x_i) \cap Y_1| = \sum_{i=1}^9 |N_{G^s}(y_i)| = 73$, so $|C| \leq 2$. $\square$

Now by considering $|C|$ there are three cases as follows:

Case 1: $|C| = 0$. Since $n = 73$, $|Y_1| = 9$ and $|C| = 0$, $D_{G^s}(X \setminus \{x_1\}) = (4, 4, \dots, 4, 4)$, $D_{G^s}(Y_1) = (9, 8, 8, 8, 8, 8, 8, 8, 8)$, $\sum_{i=1}^{17} |N_{G^s}(x_i) \cap Y'| = 68$, and $D_{G^s}(Y') = (9, 9, 9, 9, 8, 8, 8, 8, 8)$, where $Y' = Y \setminus Y_1$. Set $B = \{y \in Y', |N_{G^s}(y)| = 9\}$, so $|B| = 4$. Now we are ready to prove the following claim:

Claim 5. There exists a vertex of $A \setminus \{x_1\}$ (say x), such that:

$$\sum_{y \in N_{G^s}(x)} |N_{G^s}(y)| \geq 74,$$

in which $A = \{x \in X, |N_{G^s}(x)| = 9\}$.

Proof of Claim 5. $D_{G^s}(X_1) = (4, 4, \dots, 4, 4)$ and $D_{G^s}(Y_1) = (9, 8, 8, 8, 8, 8, 8, 8, 8)$ for each $x \in A \setminus \{x_1\}$; thus:

$$\sum_{y \in N_{G^s}(x) \cap Y_1} |N_{G^s}(y)| \geq 32$$

As $|Y'| = 8$, $|B| = 4$, and $|N_{G^s}(x_i) \cap Y'| = 5$ for each $x \in A \setminus \{x_1\}$, there exists at least one vertex of $A \setminus \{x_1\}$ (say x), such that $|N_{G^s}(x) \cap B| \geq 2$, otherwise $K_{3,3} \subseteq G^s[A, Y' \setminus B]$, a contradiction. Hence, w.l.g., suppose that $x_2 \in A$, where $|N_{G^s}(x_2) \cap B| \geq 2$. So:

$$\sum_{y \in N_{G^s}(x_2) \cap Y'} |N_{G^s}(y)| \geq 42.$$

That is,

$$\sum_{y \in N_{G^s}(x_2)} |N_{G^s}(y)| = \sum_{y \in N_{G^s}(x_2) \cap Y'} |N_{G^s}(y)| + \sum_{y \in N_{G^s}(x_2) \cap Y_1} |N_{G^s}(y)| \geq 42 + 32 = 74.$$

Now by considering x_2 and $N_{G^s}(x_2)$ and by (e) (or part (b) of Theorem 4) $K_{3,3} \subseteq G^s$, a contradiction again. $\square$

Case 2: $|C| = 1$. W.l.g., suppose that $C = \{x_2\}$, $N_{G^s}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. By Claim 3, $|N_{G^s}(x_2) \cap N_{G^s}(x) \cap Y_1| = 2$ for each $x \in X \setminus \{x_1, x_2\}$ and $|N_{G^s}(y_i)| = 8$ for each $i \in \{1, 2, \dots, 5\}$. Since there exists a vertex of Y_1 named y , such that $|N_{G^s}(y)| = 9$, w.l.g. we can suppose that $y = y_6$ and $N_{G^s}(y_6) = \{x_1, x_3, x_4, \dots, x_{10}\}$. Since $n = 73$ and $|C| = 1$, $D_{G^s}(X_1) = (5, 4, 4, \dots, 4, 3)$ —that is, there exist at least seven vertices of $N_{G^s}(y_6) \setminus \{x_1\}$ (say $X_3 = \{x_3, x_4, \dots, x_9\}$), such that $|N_{G^s}(x) \cap Y_1| = 4$ for each $x \in X_3$. Since $|X_3| = 7$, $|Y_2| = 5$, $|N_{G^s}(x) \cap Y_1| = 4$ and $|N_{G^s}(x_i) \cap Y_2| = 2$ for each $x \in X_3$, $|N_{G^s}(x) \cap \{y_7, y_8, y_9\}| = 1$ for each $x \in X_3$. Therefore, by the pigeon-hole principle there exists a vertex of $\{y_7, y_8, y_9\}$ (say y'), such that $|N_{G^s}(y') \cap X_3| \geq 3$. W.l.g., suppose that $y' = y_7$ and $\{x_3, x_4, x_5\} \subseteq N_{G^s}(y_7)$. Therefore, since $|Y_2| = 5$, there exists $i, i' \in \{3, 4, 5\}$ such that $|N_{G^s}(x_i) \cap N_{G^s}(x_{i'}) \cap Y_2| \neq 0$. W.l.g., suppose that $i = 3, i' = 4$ and $y_1 \in N_{G^s}(x_3) \cap N_{G^s}(x_4) \cap Y_2$. Therefore, $K_{3,3} \subseteq G^s[\{x_1, x_3, x_4\}, \{y_1, y_6, y_7\}]$, which is a contradiction.

Case 3: $|C| = 2$. W.l.g., suppose that $C = \{x_2, x_3\}$, $N_{G^s}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. By Claim 3, $|N_{G^s}(x_2) \cap N_{G^s}(x_3) \cap Y_1| = 2$. So, w.l.g. we can suppose that $N_{G^s}(x_3) \cap Y_1 = Y_3 = \{y_1, y_2, y_6, y_7, y_8\}$. Now, by Claim 3, $|N_{G^s}(y_i)| = 8$ for each $i \in \{1, 2, \dots, 8\}$. Since there is a vertex of Y_1 named y , such that $|N_{G^s}(y)| = 9$, $y = y_9$. W.l.g., we can assume that $N_{G^s}(y_9) = X_2 = \{x_1, x_4, x_5, \dots, x_{11}\}$. Since $n = 73$ and $|C| = 2$, $D_{G^s}(X_1) = (5, 5, 4, 4, \dots, 4, 3, 3)$ —that is, there exist two vertices of X (say x, x'), such that $|N_{G^s}(x) \cap Y_1| = 3$. If $|N_{G^s}(y_9) \cap \{x, x'\}| \leq 1$, then there exist at least seven vertices of $N_{G^s}(y_9) \setminus \{x_1\}$, such that $|N_{G^s}(x) \cap Y_1| = 4$; in this case, the proof is the same as Case 1. Hence, assume that $x, x' \in N_{G^s}(y_9)$. Since $|N_{G^s}(x) \cap Y_2| = |N_{G^s}(x') \cap Y_2| = 2$, one can check that $|N_{G^s}(x) \cap \{y_6, y_7, y_8\}| = |N_{G^s}(x') \cap \{y_6, y_7, y_8\}| = 0$. Assume that $X_i = N_{G^s}(y_i)$ for $i = 6, 7, 8$. Since $|X_i| = 8$ and $x, x' \notin X_i$, then for each $x \in X_i \setminus \{x_1\}$ we have $|N_{G^s}(x) \cap Y_1| = 4$. Therefore, by considering $X_i \setminus \{x_1\}$ and y_i for each $i \in \{6, 7, 8\}$, the proof is the same as Case 1 and $K_{3,3} \subseteq G^s$, a contradiction again.

Therefore, by Cases 1, 2, and 3 the assumption does not hold—that is, $K_{3,3} \subseteq G^s$ and this completes the proof of the theorem. $\square$

3.2. The Case That $n = 72$

In the following theorem, we prove that in any 3-edge coloring of $K_{17,17}$ (say (G^r, G^b, G^s)), where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$, if there exists a vertex of $V(K)$ (say x), such that $|N_{G^s}(x)| = 9$ and $\sum_{x_i \in X \setminus \{x\}} |N_{G^s}(x_i) \cap N_{G^s}(x)| = 63$, then $K_{3,3} \subseteq G^s$.

Theorem 6. Assume that (G^r, G^b, G^s) is a 3-edge coloring of $K = K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$. Suppose that there exists a vertex of $V(K)$ (say x), such that $|N_{G^s}(x)| = 9$. If $\sum_{i=1}^{17} |N_{G^s}(x_i) \cap Y_1| = 72$, where $Y_1 = N_{G^s}(x)$, then $K_{3,3} \subseteq G^s$.

Proof of Theorem 6. By contradiction, assume that $K_{3,3} \not\subseteq G^s$. Therefore, by Theorems 3 and 4, we have the following results:

- $|E(G^s)| = 141$.
- $\Delta(G^s) = 9$ and $\delta(G^s) = 8$.
- $D_{G^s}(X) = D_{G^s}(Y) = (9, 9, 9, 9, 9, 8, 8, \dots, 8)$.
- For each $x \in X \setminus \{x_1\}$, we have $|N_{G^s}(x) \cap Y_1| \leq 5$.
- If $A = \{x \in X, |N_{G^s}(x)| = 9\}$, then $|A| = 5$ and $72 \leq \sum_{y \in N_{G^s}(x)} |N_{G^s}(y)| \leq 73$, for each $x \in A$.

Assume that $X = \{x_1, x_2, \dots, x_{17}\}$, $Y = \{y_1, y_2, \dots, y_{17}\}$ is a partition set of $K = K_{17,17}$, and (G^r, G^b, G^s) is a 3-edge coloring of K , where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^s$. W.l.g., assume that $x = x_1$, $Y_1 = \{y_1, y_2, \dots, y_9\}$, and $n = \sum_{i=1}^{17} |N_{G^s}(x_i) \cap Y_1| = 72$. Since

$n = 73$, by (c) we can say that $D_{G^s}(Y_1) = (d_1, d_2, \dots, d_9) = (8, 8, 8, \dots, 8)$. Set $C = \{x \in X, |N_{G^s}(x) \cap N_{G^s}(x_1)| = 5\}$. Define D and E as follows:

$$D = \{x \in X \setminus \{x_1\}, \text{ such that } |N_{G^s}(x) \cap Y_1| = 5\}$$

$$E = \{x \in X \setminus \{x_1\}, \text{ such that } |N_{G^s}(x) \cap Y_1| = 3\}.$$

Here we have a claim about $|D|$ and $|E|$ as follows:

Claim 6. $|D| \leq 3$ and $|E| \leq 4$.

Proof of Claim 6. By contradiction, suppose that $|D| \geq 4$. W.l.g., assume that $\{x_2, x_3, x_4, x_5\} \subseteq D$, $N_{G^s}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. Now, by Claim 3, $|N_{G^s}(x) \cap Y_2| = 2$ for each $x \in X \setminus \{x_1, x_2\}$. W.l.g., we can suppose that $N_{G^s}(x_3) \cap Y_1 = Y_3 = \{y_1, y_2, y_6, y_7, y_8\}$. Consider $N_{G^s}(x_i) \cap Y_1$ ($i = 4, 5$). Since $|N_{G^s}(x_i) \cap Y_j| = 2$ ($i = 4, 5, j = 2, 3$) and $x_i \in A$, $|N_{G^s}(x_i) \cap \{y_3, y_4, y_5\}| = 2$, $|N_{G^s}(x_i) \cap \{y_6, y_7, y_8\}| = 2$, and $y_9 \in N_{G^s}(x_i)$ for $i = 4, 5$; otherwise, if there exists a vertex of $\{x_4, x_5\}$ (say x), such that $|N_{G^s}(x_i) \cap \{y_1, y_2\}| \neq 2$, then $K_{3,3} \subseteq G^s[\{x_1, x_i, x\}, Y_1]$ for some $i \in \{1, 2\}$, a contradiction. Therefore, since $|\{y_3, y_4, y_5\}| = |\{y_6, y_7, y_8\}| = 3$ and $x_4, x_5 \in A$, by the pigeon-hole principle $|N_{G^s}(x_4) \cap N_{G^s}(x_5) \cap \{y_3, y_4, y_5\}| \geq 1$ and $|N_{G^s}(x_4) \cap N_{G^s}(x_5) \cap \{y_6, y_7, y_8\}| \geq 1$. W.l.g., we can suppose that $y_3, y_6 \in N_{G^s}(x_4) \cap N_{G^s}(x_5)$, since $y_9 \in N_{G^s}(x_4) \cap N_{G^s}(x_5)$, so $K_{3,3} \subseteq G^s[\{x_1, x_4, x_5\}, \{y_3, y_6, y_9\}]$, a contradiction. Therefore, $|D| \leq 3$. Now, as $\sum_{i=2}^{i=17} |N_{G^s}(x_i) \cap Y_1| = 63$ and $|D| \leq 3$, we can say that $|E| \leq 4$ and the proof of the claim is complete. $\square$

Now, by considering $|D|$, there are three cases as follows:

Case 1: $|D| = 0$. Since $n = 72$ and $|D| = 0$, $D_{G^s}(X \setminus \{x_1\}) = (4, 4, \dots, 4, 3)$, $D_{G^s}(Y_1) = (8, 8, 8, 8, 8, 8, 8, 8, 8)$, $\sum_{i=1}^{i=17} |N_{G^s}(x_i) \cap Y'| = 69$ and $D_{G^s}(Y') = (9, 9, 9, 9, 9, 8, 8, 8)$, where $Y' = Y \setminus Y_1$. Set $B = \{y \in Y', |N_{G^s}(y)| = 9\}$, hence $|B| = 5$. Now, we have the following claim:

Claim 7. *There exists a vertex of $A \setminus \{x_1\}$ (say x), such that:*

$$\sum_{y \in N_{G^s}(x)} |N_{G^s}(y)| \geq 75,$$

in which $A = \{x \in X, |N_{G^s}(x)| = 9\}$.

Proof of Claim 7. Since $D_{G^s}(X_1) = (4, 4, \dots, 4, 3)$ and $D_{G^s}(Y_1) = (8, 8, 8, 8, 8, 8, 8, 8, 8)$, so for at least three vertices of $A \setminus \{x_1\}$,

$$\sum_{y \in N_{G^s}(x) \cap Y_1} |N_{G^s}(y)| \geq 32.$$

Therefore, since $|N_{G^s}(x_i) \cap Y'| = 5$ for each $x \in A \setminus \{x_1\}$ and $D_{G^s}(Y') = (9, 9, 9, 9, 9, 8, 8, 8)$, there exists at least one vertex of $A \setminus \{x_1\}$ (say x), such that $|N_{G^s}(x) \cap B| \geq 3$; otherwise, $K_{3,3} \subseteq G^s[A, Y' \setminus B]$, a contradiction. Hence, w.l.g., suppose that $x_2 \in A$ and $|N_{G^s}(x_2) \cap B| \geq 3$; therefore:

$$\sum_{y \in N_{G^s}(x) \cap Y'} |N_{G^s}(y)| \geq 3 \times 9 + 2 \times 8 = 43.$$

That is, we have:

$$\sum_{y \in N_{G^s}(x_2)} |N_{G^s}(y)| = \sum_{y \in N_{G^s}(x_2) \cap Y'} |N_{G^s}(y)| + \sum_{y \in N_{G^s}(x) \cap Y_1} |N_{G^s}(y)| \geq 43 + 32 = 75.$$

Now, by considering x_2 and $N_{G^8}(x_2)$ and by (e) (or by part (b) of Theorem 4), $K_{3,3} \subseteq G^8$, a contradiction again. $\square$

Case 2: $|D| = 1$ (for the case that $|D| = 2$, the proof is same). W.l.g., assume that $D = \{x_2\}$, $N_{G^8}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. Since $n = 72$, $|D| = 1$ and $|N_{G^8}(x) \cap Y_1| \leq 5$, $|E| = 2$. As $|N_{G^8}(x) \cap Y_2| = 2$ for each $x \in X \setminus \{x_1, x_2\}$ and $|E| = 2$, there exists a vertex of $\{y_6, y_7, y_8, y_9\}$ (say y), such that for each vertex of $N_{G^8}(y) \cap X \setminus \{x_1\}$ (say x), $|N_{G^8}(x) \cap Y_1| = 4$. W.l.g., we can suppose that $y = y_6$, $N_{G^8}(y_6) \cap X \setminus \{x_1\} = \{x_3, x_4, \dots, x_9\}$. Since $|N_{G^8}(y_6) \cap X \setminus \{x_1\}| = 7$ and $|N_{G^8}(x) \cap Y_2| = 2$ for each $x \in N_{G^8}(y_6) \cap X \setminus \{x_1\}$, $|N_{G^8}(x) \cap \{y_7, y_8, y_9\}| = 1$. Therefore, by the pigeon-hole principle there exists a vertex of $\{y_7, y_8, y_9\}$ (say y'), such that $|N_{G^8}(y_6) \cap N_{G^8}(y') \cap X \setminus \{x_1\}| \geq 3$. W.l.g., suppose that $y' = y_7$ and $\{x_3, x_4, x_5\} \subseteq N_{G^8}(y_6) \cap N_{G^8}(y_7) \cap X \setminus \{x_1\}$. Therefore, since $|Y_2| = 5$ and $|N_{G^8}(x) \cap Y_2| = 2$, there exist at least two vertices of $\{x_3, x_4, x_5\}$ (say x', x''), such that $|N_{G^8}(x') \cap N_{G^8}(x'') \cap Y_2| \neq 0$. W.l.g., suppose that $x' = x_3, x'' = x_4$ and $y_1 \in N_{G^8}(x_3) \cap N_{G^8}(x_4)$. Therefore, $K_{3,3} \subseteq G^8[\{x_1, x_3, x_4\}, \{y_1, y_6, y_7\}]$, a contradiction.

Case 3: $|D| = 3$. W.l.g., suppose that $D = \{x_2, x_3, x_4\}$, $N_{G^8}(x_2) \cap Y_1 = Y_2 = \{y_1, \dots, y_5\}$. By Claim 3, $|N_{G^8}(x_2) \cap N_{G^8}(x_3) \cap Y_1| = 2$. W.l.g., we can assume that $N_{G^8}(x_3) \cap Y_1 = Y_3 = \{y_1, y_2, y_6, y_7, y_8\}$. Since $x_4 \in D$ and $|N_{G^8}(x_4) \cap Y_i| = 2$ for $i = 2, 3$, $y_9 \in N_{G^8}(x_4)$. If $|N_{G^8}(x_4) \cap \{y_1, y_2\}| \neq 0$, as $|N_{G^8}(x_2) \cap N_{G^8}(x_4) \cap Y_1| = 2$ and $x_4 \in D$, one can check that $|N_{G^8}(x_4) \cap \{y_6, y_7, y_8\}| = 2$ —that is, $K_{3,3} \subseteq G^8[\{x_1, x_3, x_4\}, Y_1]$, a contradiction. Hence, $|N_{G^8}(x_4) \cap \{y_1, y_2\}| = 0$. Therefore, $|N_{G^8}(x_4) \cap \{y_3, y_4, y_5\}| = 2$ and $|N_{G^8}(x_4) \cap \{y_6, y_7, y_8\}| = 2$. W.l.g., we can suppose that $N_{G^8}(x_4) \cap Y_1 = Y_4 = \{y_3, y_4, y_6, y_7, y_9\}$. Since $|D| = 3$, so $|E| = 4$. W.l.g., suppose that $E = \{x_5, x_6, x_7, x_8\}$. Here, we have a claim as follows:

Claim 8. $|N_{G^8}(y_9) \cap E| = 0$.

Proof of Claim 8. By contradiction, suppose that $|N_{G^8}(y_9) \cap E| \neq 0$. Assume that $x_5 \in N_{G^8}(y_9) \cap E$ —that is, $x_5 y_9 \in E(G^8)$. Since $x_5 \in E$ and $\{x_2, x_3, x_4\} = D$, by Claim 3, $|N_{G^8}(x_5) \cap N_{G^8}(x_i)| = |N_{G^8}(x_5) \cap Y_i| = 2$ for $i = 2, 3, 4$. Consider $N_{G^8}(x_5) \cap Y_2$, assume that $N_{G^8}(x_5) \cap Y_2 = \{y', y''\}$, if $\{y', y''\} = \{y_1, y_2\}$, then $|N_{G^8}(x_5) \cap Y_4| = 1$, a contradiction. Therefore, we can assume that $|\{y', y''\} \cap \{y_1, y_2\}| \leq 1$. If $|\{y', y''\} \cap \{y_1, y_2\}| = 0$, then $|N_{G^8}(x_5) \cap Y_3| = 0$, and if $|\{y', y''\} \cap \{y_1, y_2\}| = 1$, then $|N_{G^8}(x_5) \cap Y_3| \leq 1$. In any case there exists a vertex of D (say x'), such that $|N_{G^8}(x_5) \cap N_{G^8}(x')| = 1$, a contradiction. So, the assumption does not hold and the claim is true. $\square$

Therefore, by Claim 8, since $|N_{G^8}(y_9) \cap D| = 0$, we can say that for any vertex of $N_{G^8}(y_9) \cap X \setminus \{x_1\}$ (say x), $|N_{G^8}(x) \cap Y_1| \geq 4$; therefore, by considering Y_2 and y_9 , as $|N_{G^8}(y_9) \cap X \setminus \{x_1\}| = 7$ and $|N_{G^8}(x) \cap Y_1| \geq 4$ for each $x \in N_{G^8}(y_9) \cap X \setminus \{x_1\}$, the proof is similar to Case 1, a contradiction.

Therefore, by Cases 1, 2, and 3 the assumption does not hold—that is, $K_{3,3} \subseteq G^8$ and the proof of the theorem is complete. $\square$

Now, combining Theorems 3–6 yields the proof of Theorem 1.

4. Discussion

There are several papers in which the bipartite Ramsey numbers have been studied. In this paper, we proved the conjecture on $B(2, 2, 3)$, which was proposed in 2015 and states that $B(2, 2, 3) = 17$. We proved this conjecture by a combinatorial argument with no computer calculations. This is significant because computing the exact value of Ramsey numbers is a challenge. To approach the proof of this conjecture, we proved four theorems as follows:

1. Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$. Hence, we have:

(a) $|E(G^g)| = 141$.

- (b) $\Delta(G^g) = 9$ and $\delta(G^g) = 8$.
- (c) $D_{G^g}(X) = D_{G^g}(Y) = (9, 9, 9, 9, 9, 8, 8, \dots, 8)$.
2. Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$ and $K_{3,3} \not\subseteq G^g$. Let $|N_{G^g}(x)| = 9$ and $N_{G^g}(x) = Y_1$, the following results are true:
- (a) For each $x \in X \setminus \{x_1\}$, we have $|N_{G^g}(x) \cap Y_1| \leq 5$.
- (b) Assume that $n = \sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_1|$, then $72 \leq n \leq 73$.
3. Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K = K_{17,17}$, such that $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$. Assume that there exists a vertex of $V(K)$ (say x), such that $|N_{G^g}(x)| = 9$. If $\sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_1| = 73$, where $Y_1 = N_{G^g}(x)$, then $K_{3,3} \subseteq G^g$.
4. Assume that (G^r, G^b, G^g) is a 3-edge coloring of $K = K_{17,17}$, where $K_{2,2} \not\subseteq G^r$, $K_{2,2} \not\subseteq G^b$. Assume that there exists a vertex of $V(K)$ (say x), such that $|N_{G^g}(x)| = 9$. If $\sum_{i=1}^{i=17} |N_{G^g}(x_i) \cap Y_1| = 72$, where $Y_1 = N_{G^g}(x)$, then $K_{3,3} \subseteq G^g$.
- One might also be able to compute $B(n_1, \dots, n_m)$ for small i, n_i like $B(2, 3, 3, 3)$ or $B(3, 3, 3, 3)$ in the future, using the idea of proofs laid out in this paper.

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