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Minimum Aberration Split-Plot Designs Focusing on the Whole Plot Factors

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Abstract: In some experiments, the levels of some factors are difficult to change; then fractional factorial split-plot (FFSP) designs are suitable for selection. In an FFSP design, the factors are divided into two classes—the whole plot (WP) and subplot (SP) factors. In some experiments, the selection of the levels of the WP factors can affect that of the SP factors, which attracts much attention to the WP factors. Paying more attention to the WP factors, a new optimality criterion for selecting such FFSP designs is proposed. The robustness of the proposed method is discussed. The construction method of the optimal designs under the new criterion is studied.

Keywords: fractional factorial design; split-plot design; minimum aberration

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1. Introduction

The study of optimal fractional factorial (FF) designs under the minimum aberration (MA) criterion has received significant attention over the last few decades, motivated by the wide applicability in industrial and agricultural experiments. When an FF experiment is performed, the order of the experimental runs is usually required to be completely randomized. However, it is sometimes difficult or expensive to change the levels of certain factors in an experiment. Fractional factorial split-plot (FFSP) designs which involve a two-phase randomization represent a practical design option in such situations. In general, the factors in an FFSP design are divided into two types. The factors with hard-to-change levels are called whole plot (WP) factors, and the rest are called subplot (SP) factors. If the WP and SP factors have different importance, their difference must be taken into account when choosing the optimal FFSP design.

To choose the optimal FF design, Box and Hunter [1] proposed the maximum resolution criterion, which indicates that a good design should have the maximum resolution. However, in most cases there are several designs with the same maximum resolution and the maximum resolution criterion cannot distinguish them. Hence, the MA criterion was proposed to choose optimal FF designs by Fries and Hunter [2]. They assumed that the experimenter believes initially that main effects are more important than two-factor interactions, that two-factor interactions are more important than three-factor interactions, and so forth. Wu and Hamada [3] summarized these as the effect hierarchy principle. Cheng et al. [4] explored the model robustness of minimum aberration designs. Chen and Wu [5] gave the construction method of MA 2^{n-k} designs with $k \leq 4$. Tang and Wu [6] developed the method of complementary designs to construct 2^{n-k}designs with large k. Huang et al. [7] extended the MA criterion to FFSP designs and discussed the method of constructing MA-FFSP designs. Bingham and Sitter [8] gave a construction method of the MA FFSP designs and tabulated a catalog of two-level MA FFSP designs with 8 and 16 runs. Bingham and Sitter [9] listed MA FFSP designs with up to 32 runs. Bingham and Sitter [10] discussed the impact of randomization restrictions on the choice of FFSP designs and developed theoretical results on MA FFSP designs. Yang et al. [11] extended the results of Bingham and Sitter [10] to multi-level designs. Bingham et al. [12] considered the construction of the optimal FFSP design with few WP factors. Yang and Lin [13] considered split-plot designs with factors of three, more than three, or mixed levels and tabulated a catalog of mixed-level MA FFSP designs with 18 and 36 runs. Since the WP and SP factors no longer have interchangeability, frequently, there exist several nonisomorphic MA FFSP designs. Cheng and Mukerjee [14] and Mukerjee and Fang [15] explored a criterion of minimum secondary aberration (MSA), called the MSA-FFSP criterion, which considered a secondary wordlength pattern and significantly narrowed the class of competing nonisomorphic MA FFSP designs. Ai and Zhang [16] constructed the MSA-FFSP designs in terms of consulting designs.

We should note that the underlying assumption of MA criterion is that all factors are of equal interest. It is not always the case. Box and Jones [17] and Bingham and Sitter [18] investigated the applicability of split-plot designs for robust product experimentation, focusing on the WP-by-SP interactions. The following example from Montgomery [19] shows two scenarios that WP factors are more important than SP factors. The experiments considered the effect of five factors on the uniformity in a single-wafer plasma etching process. The response variable of primary interest was the resulting uniformity data, and thus smaller is better. The experiments had three factors that were relatively difficult to change on the etching tool, each at two levels: A = electrode gap, B = gas flow, and C = pressure. It also had two factors that were easy to change from run to run, each at two levels: D = time and E = radio frequency power. The experimenters used a $2^{(3+2)-(0+1)}$ design with factors A, B, and C in the whole plots and factors D and E in the subplots. The design generator was E = ABCD. They assumed that all interactions beyond order two were negligible. A half-normal plot of the effects in Montgomery [19] identified five significant effects, depending on visual interpretation. The effects that they identified as active were A, B, E, AB and AE. The two-factor interaction graphs indicated that the treatment combination A high, B low, and E low or A low, B high, and E high would produce low levels of the uniformity response. On the one hand, the factors on the etching tool, namely WP factors, are difficult to change during etching, the experimenters may pay more attention to them. In fact, experimenters often pay more attention to the WP factors since the WP factors are usually difficult or expensive to change the levels. On the other hand, the experimenter usually randomly chooses one of the level combinations of the WP factors and then run all the SP factor settings while keeping the WP factors fixed in an FFSP experiment. Therefore, the experimenters focused on the WP factors in this experiment since the selection of the levels of A affected that of E. Tichon et al. [20] considered five design scenarios that may be important to practitioners, and the setting "WP factors are more important than SP factors" was one of the five scenarios they considered. Wang et al. [21] proposed the minimum aberration of type WP (WP-MA) criterion for the experiments with the WP factors being more important than the SP factors. Zhao and Zhao [22] constructed WP-MA FFSP designs via complementary sets.

Although the WP-MA criterion is a good choice for selecting optimal FFSP designs, it still has some shortcomings. Firstly, the WP-MA criterion considers that all WP factor effects are more possibly significant than SP factor effects, even higher order WP factor effects are more possibly significant than lower order SP factor effects. Obviously, this contradicts the effect hierarchy principle. Secondly, the robustness of the WP-MA design is not tested. In addition, Box and Hunter [1] think that a good design should have the maximum resolution. The WP-MA design does not meet this requirement. If the levels of SP factors are affected by the selection of the levels of WP factors in some experiments, experimenter needs to focus on the WP factors firstly. In this paper, a new criterion that focuses on the WP factors is proposed under the following effect hierarchy principle:

i Lower-order factorial effects are more important than higher-order ones;
ii The WP factorial effects are more important than the SP factorial effects of the same order;
iii The WP/SP factorial effects of the same order are equally likely to be important.

The rest of the paper is organized as follows. Section 2 proposes the new criterion and introduces some useful results. Section 3 tests the robustness of the WS-MA design. Section 4 compares the new criterion with the WP-MA criterion. Section 5 is devoted to constructing optimal FFSP designs under the new criterion. Section 6 gives a conclusion. All proofs are given in Appendix A.

2. Preliminaries

Consider the setup of a factorial experiment, which involves n two-level factors. The n factors are divided into two groups, $n_1 (1 \le n_1 < n)$ WP factors and $n_2 (= n - n_1)$ SP factors. A defining word is called WP type if it involves only WP factors and SP type if it involves at least one SP factor. A regular FFSP design d, denoted as $2^{(n_1+n_2)-(k_1+k_2)}$, involves n_1 WP factors and n_2 SP factors and is determined by k_1 independent WP type defining words and k_2 independent SP type defining words. The group generated by the $k_1 + k_2$ independent defining words is called the defining contrast subgroup of d. Many concepts of FFSP designs remain the same as FF designs, such as defining relation and resolution. The following definition of isomorphism is from Mukerjee and Wu [23].

Definition 1. *Two $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs are isomorphic if the defining contrast subgroup of one design can be obtained from that of the other by permuting the WP factor labels and/or the SP factor labels.*

For a regular $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design d, let $A_{i,0}(d)$ and $A_{i,1}(d)$ be the numbers of WP and SP type defining words with length i, respectively, and $A_i(d)$ be the number of defining words with length i. Obviously, $A_i(d) = A_{i,0}(d) + A_{i,1}(d)$. The wordlength pattern of a $2^{(n_1+n_2)-(k_1+k_2)}$ design d is defined as:

$$W(d) = (A_1(d), A_2(d), \dots, A_{n_1+n_2}(d)). \tag{1}$$

Let r be the smallest integer i such that $A_i(d) \neq 0$, which is called the resolution of design d. A $2^{(n_1+n_2)-(k_1+k_2)}$ design d which sequentially minimizes $W(d)$ is called an MA design. Wang et al. [21] defined the following two n-dimensional sequences

$$W_1(d) = (A_{1,0}(d), A_{2,0}(d), \dots, A_{n_1,0}(d), 0, \dots, 0), \tag{2}$$

$$W_2(d) = (A_{1,1}(d), A_{2,1}(d), \dots, A_{n_1+n_2,1}(d)), \tag{3}$$

and called them WP and SP wordlength patterns of design d, respectively.

Definition 2. *Let d_1 and d_2 be two $2^{(n_1+n_2)-(k_1+k_2)}$ designs. Under the condition that WP factors are more important than SP factors, d_1 is said to have less aberration of type WP than d_2 if either (i)$A_{r,0}(d_1) < A_{r,0}(d_2)$ for the smallest integer r such that $A_{r,0}(d_1) \neq A_{r,0}(d_2)$ or (ii)$A_{i,0}(d_1) = A_{i,0}(d_2)$ for any i but $A_{r,1}(d_1) < A_{r,1}(d_2)$ for the smallest integer r such that $A_{r,1}(d_1) \neq A_{r,1}(d_2)$. An FFSP design d is called a minimum aberration design of type WP (WP-MA) if no other design has less aberration of type WP than d.*

Define the $2n$-dimensional sequence:

$$W_0(d) = (A_{1,0}(d), A_{1,1}(d), \dots, A_{n_1+n_2,0}(d), A_{n_1+n_2,1}(d)), \tag{4}$$

and call it the whole-subplot (WS) wordlength pattern to discriminate it from the wordlength pattern (1) and the WP and SP wordlength patterns (2) and (3).

Definition 3. *With $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs d_1 and d_2, let r be the smallest integer i such that $(A_{i,0}(d_1), A_{i,1}(d_1)) \neq (A_{i,0}(d_2), A_{i,1}(d_2))$. If (i) $A_{r,0}(d_1) < A_{r,0}(d_2)$ or (ii) $A_{r,0}(d_1) = A_{r,0}(d_2)$ but $A_{r,1}(d_1) < A_{r,1}(d_2)$, then d_1 is said to have less aberration of type WS than d_2. An*

FFSP design d is called a minimum aberration design of type WS (WS-MA) if no other design has less aberration of type WS than d.

The WP-MA criterion based on Definition 2 requires sequential minimization of $W_1(d)$ and $W_2(d)$. This means that it sacrifices most of the accuracy of the estimation of the SP factor effects while focusing on WP factors. In contrast, the WS-MA criterion based on Definition 3 minimizes $W_0(d)$ sequentially. Thus, it guarantees that a lower order factorial effect is always more important than a higher order factorial effect, even when the WP factor is concerned. Therefore, compared with the WP-MA designs, the WS-MA designs always have better resolution and robustness.

The following Lemma 1 is the basis for constructing the WS-MA FFSP designs.

Lemma 1. *If a $2^{(n_1+n_2)-(k_1+k_2)}$ design d is a WS-MA design, then each of the $n_1 + n_2$ factors is involved in some defining word of d.*

The proof of Lemma 1 is similar to the proof of Theorem 4. We omit it to save space.

The following Theorem 1 proves that WS-MA designs have the maximum resolution. This is important to construct the WS-MA designs. The proof of Theorem 1 is given in Appendix A.

Theorem 1. *If a $2^{(n_1+n_2)-(k_1+k_2)}$ design d is a WS-MA design, then it has the same resolution as the MA $2^{(n_1+n_2)-(k_1+k_2)}$ design.*

3. Robustness of the WS-MA Designs

Cheng et al. [4] explored model robustness of MA designs with two different criteria: estimation capacity and the expected number of suspect two-factor interactions. Under the assumption that the number of active two-factor interactions is not too large, they showed that the MA two-level factorial designs are highly efficient with respect to two criteria for model robustness. We will test the robustness of the WS-MA design in a similar manner to Cheng et al. [4]. In this section, we only consider designs with resolution three or higher. All the interactions involving three or more factors are assumed to be negligible.

In a $2^{(n_1+n_2)-(k_1+k_2)}$ design d of resolution three or higher, $2^{n_1+n_2} - 2^{k_1+k_2}$ effects that appear outside the definition words are partitioned into $g = 2^{(n_1+n_2)-(k_1+k_2)} - 1$ alias sets. Clearly, $n_1 + n_2$ alias sets contain main effects. Let $f = g - (n_1 + n_2)$ and denote the alias sets not containing main effects by $M_1, \ldots, M_f$. Denote the rest by $M_{f+1}, \ldots, M_g$. For $1 \leq i \leq g$, let $m_i(d)$ be the number of two-factor interactions in M_i. Then we have the following equations from Cheng et al. [4]:

$$\sum_{i=1}^{f} m_i(d) = 3A_3(d) = 3(A_{3,0}(d) + A_{3,1}(d))$$

and

$$\sum_{i=1}^{g} m_i(d)^2 - \binom{n}{2} = 6A_4(d) = 6(A_{4,0}(d) + A_{4,1}(d)).$$

They proved that "a design has large estimation capacity if $\sum_{i=1}^{f} m_i(d)$ is as large as possible and the $m_i(d)s$ are as uniform as possible" and "if the number of active interactions is not large, a fractional factorial design will approximately minimize the expected number of suspect two-factor interactions if it minimizes $3A_3(d)$ and, subject to that condition, minimizes $\sum_{i=1}^{g} m_i(d)^2$".

Since the design d has resolution three or more, we have $A_1(d) = A_2(d) = 0$. By Definition 2, a WS-MA design sequentially minimizes $A_{3,0}(d)$, $A_{3,1}(d)$, $A_{4,0}(d)$ and $A_{4,1}(d)$. Then a WS-MA design of resolution three or higher approximately maximizes $\sum_{i=1}^{f} m_i(d)$, and among the designs approximately maximizing $\sum_{i=1}^{f} m_i(d)$ it approximately minimizes $\sum_{i=1}^{g} m_i(d)^2$. Thus, the WS-MA FFSP design is also robust.

4. Comparison with WP-MA Designs

Wang et al. (2019) proposed the WP-MA criterion for the experiments with the WP factors being more important than the SP factors. In this section, we will compare the WS-MA criterion with the WP-MA criterion. Some optimal designs under the WS-MA criterion are consistent with that under the WP-MA criterion. The results are summarized in Theorems 2 and 3.

Theorem 2. *For any WP-MA $2^{(n_1+n_2)-(k_1+k_2)}$ design d, let p be the largest integer i such that $A_{i,0}(d) \neq 0$. If $A_{j,1}(d) = 0, j = 1, \dots, p-2$, then d is also a WS-MA design.*

Theorem 3. *For any WP-MA $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design d, if it is an MA FF design, then it is also a WS-MA design.*

According to Theorems 2 and 3, we only need to construct WS-MA designs that may be different from WP-MA designs. This reduces the number of WS-MA designs that need to be constructed.

Theorem 4 shows that many WS-MA designs are different from the WP-MA designs.

Theorem 4. *For any WP-MA $2^{(n_1+n_2)-(k_1+1)}$ design d, let k be the smallest integer i such that $A_{i,0}(d) \neq 0$. If $A_{j,1}(d) = 0, j = k-1, \dots, n_1+n_2$, then there must exist a design which has less aberration of type WS than d.*

The following example shows the advantages of the WS-MA criterion over the WP-MA criterion on selecting FFSP designs when the WP factors are more important than the SP factors.

Example 1. *Consider $2^{(10+5)-(1+2)}$ designs under the three criteria above. Here $n_1 = 10$, $n_2 = 5, k_1 = 1, k_2 = 2$. Let $t_1, \dots, t_{10}$ denote the WP factors and $t_{11}, \dots, t_{15}$ denote the SP factors. The design d_{WS} with the defining relation*

$$
\begin{aligned}
I &= t_1t_2t_3t_4t_5t_6t_7t_8t_9 = t_1t_2t_3t_4t_5t_{12}t_{13}t_{14}t_{15} = t_1t_2t_3t_6t_7t_{10}t_{11}t_{14}t_{15} = t_1t_2t_3t_8t_9t_{10}t_{11}t_{12}t_{13} \\
&= t_4t_5t_6t_7t_{10}t_{11}t_{12}t_{13} = t_4t_5t_8t_9t_{10}t_{11}t_{14}t_{15} = t_6t_7t_8t_9t_{12}t_{13}t_{14}t_{15}
\end{aligned}
$$

is the WS-MA design. The design d_{WP} with the defining relation

$$
\begin{aligned}
I &= t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10} = t_1t_2t_3t_4t_9t_{11}t_{12}t_{14} = t_1t_2t_5t_6t_9t_{12}t_{13}t_{15} = t_3t_4t_5t_6t_{11}t_{13}t_{14}t_{15} \\
&= t_5t_6t_7t_8t_{10}t_{11}t_{12}t_{14} = t_3t_4t_7t_8t_{10}t_{12}t_{13}t_{15} = t_1t_2t_7t_8t_9t_{10}t_{11}t_{13}t_{14}t_{15}
\end{aligned}
$$

is the WP-MA design. The design d_{MA} with the defining relation

$$
\begin{aligned}
I &= t_1t_2t_3t_4t_7t_8t_9t_{10} = t_1t_2t_3t_4t_5t_6t_{13}t_{14}t_{15} = t_1t_2t_5t_6t_7t_8t_{11}t_{12} = t_1t_2t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15} \\
&= t_3t_4t_7t_8t_{11}t_{12}t_{13}t_{14}t_{15} = t_3t_4t_5t_6t_9t_{10}t_{11}t_{12} = t_5t_6t_7t_8t_9t_{10}t_{13}t_{14}t_{15}
\end{aligned}
$$

is an MA design.

Table 1 lists the wordlength patterns of the three designs above. Clearly, d_{WS} and d_{MA} have the same wordlength pattern, hence d_{WS} is also an MA design.

In an FFSP design, SP effects in the SP alias sets are tested against the SP level error while other effects are tested against the WP level error. An SP alias set is one which contains no WP effects. Since the WP level error is typically larger than the SP level error, a good FFSP design should avoid confounding of the most important effects of the SP factors with any WP type effect. One such measure is given by the secondary wordlength pattern of an FFSP design d which is denoted as $W^*(d) = (B_1(d), B_2(d), \dots, B_{n_1+n_2}(d))$, where $B_i(d)$ is the number of the ith-order SP type *effects aliased with WP type effects. See Mukerjee and Fang [15] for more details. Table 2 provides the secondary wordlength patterns of the three designs. Clearly, d_{WS} and d_{WP} have the approximate secondary wordlength patterns which are significantly better than that of d_{MA}. So, in $2^{(10+5)-(1+2)}$ designs, the WS-MA design is also the approximate optimal design under MSA criterion.*

Table 1. $W(d)$ of d_{WS}, d_{WP} and d_{MA}.

d	$W(d)$
d_{WS}	$(0,0,0,0,0,0,0,3,4,0,0,0,0,0,0)$
d_{WP}	$(0,0,0,0,0,0,0,5,0,2,0,0,0,0,0)$
d_{MA}	$(0,0,0,0,0,0,0,3,4,0,0,0,0,0,0)$

Table 2. $W^*(d)$ of d_{WS}, d_{WP} and d_{MA}.

d	$W^*(d)$
d_{WS}	$(0,0,4,42,200,570,1080,1425,1341,900,420,130,24,2,0)$
d_{WP}	$(0,0,4,42,200,570,1080,1423,1344,899,420,130,24,2,0)$
d_{MA}	$(0,2,22,110,332,680,1014,1162,1076,834,530,262,92,20,2)$

In view of the different application situations of WS-MA criterion and WP-MA criterion, we propose which criteria should be used in the following two scenarios:

1. When the experimenters have no prior information about the importance of the factors, or they have some prior information about the significance of the WP factors and do not want to sacrifice too much of the accuracy of the estimation of the SP factor effects, the WS-MA criterion or MA criterion should be used.
2. When the experimenters have some prior information about the significance of the WP factors, and they do not care too much about the effects of SP factors, the WP-MA criterion should be used.

5. Construction of WS-MA FFSP Designs

When k_1 and k_2 are small, the construction of WS-MA designs is relatively easy since the defining contrast subgroup of a $2^{(n_1+n_2)-(k_1+k_2)}$ design has few elements. Construction of WS-MA $2^{(n_1+n_2)-(k_1+k_2)}$ designs with small k_1 and k_2 is discussed in this section. We use $1,\ldots,n_1+n_2$ to denote the factors in the following.

5.1. WS-MA $2^{(n_1+n_2)-(k_1+k_2)}$ Designs with $k_1=0$ or $k_2=0$

Theorem 5. *Regarding a $2^{(n_1+n_2)-(0+k_2)}$ design d_0 as an FF 2^{n-k_2} design d, where $n=n_1+n_2$, if the 2^{n-k_2} design d is an MA design, then d_0 is a WS-MA design.*

Theorem 6. *An FFSP $2^{(n_1+n_2)-(k_1+0)}$ design is a WS-MA design if and only if its WP factors constitute an MA $2^{n_1-k_1}$ design.*

Theorems 5 and 6 can be easily derived from (4) and Definition 3. Note that $W_1(d)$ ($W_2(d)$) is a sequence of zeros when k_1 (k_2) equals 0. Therefore, by Theorem 2, the WS-MA design is the same as the WP-MA design when $k_1=0$ or $k_2=0$.

5.2. WS-MA $2^{(n_1+n_2)-(k_1+k_2)}$ Designs with $k_1=1,2,3$ and $k_2=1$

This section gives the construction of $2^{(n_1+n_2)-(k_1+1)}$ designs with $k_1=1,2,3$. From Section 4, we need only to consider the construction of WS-MA designs which do not satisfy the conditions of Theorems 2 and 3.

Consider the construction of the WS-MA $2^{(n_1+n_2)-(1+1)}$ designs first. Let $n_1+n_2=3m+r$ for $0\le r<3$. By Theorem 2, we just consider the case of $n_2\le n_1/2$. For $i=1,2,3$, define

$$B_i = \begin{cases} (im-m+1)(im-m+2)\cdots(im)(3m+i), & i \le r, \\ (im-m+1)(im-m+2)\cdots(im), & r < i \le 3. \end{cases} \tag{5}$$

By $n_1 = 3m + r - n_2$ and $n_2 \le n_1/2$, it's easy to get $n_2 \le m$. Thus, the n_2 SP factors can be labeled as $2m+1, \ldots, 2m+n_2$ and arranged in B_3.

Theorem 7. *The $2^{(n_1+n_2)-(1+1)}$ design d_0 with $n_2 \le n_1/2$ and the defining relation $I = B_1B_2 = B_1B_3 = B_2B_3$, where B_1, B_2, B_3 are given in (5), is a WS-MA design.*

Next consider the construction of the WS-MA $2^{(n_1+n_2)-(2+1)}$ designs. Let $n_1 + n_2 = 7m + r$ for $0 \le r < 7$. Similarly, we just consider the case of $n_2 \le n_1/6$. For $i = 1, \ldots, 7$, define

$$B_i = \begin{cases} (im-m+1)(im-m+2)\cdots(im)(6m+i), & i \le r, \\ (im-m+1)(im-m+2)\cdots(im), & r < i \le 7. \end{cases}$$

Similar to (5), we can get $n_2 \le m$. Thus, the n_2 SP factors can be labeled as $6m+1, \ldots, 6m+n_2$ and arranged in B_7.

When $r \ne 3$, let

$$B = \{B_1B_2B_6B_7, B_1B_4B_5B_6, B_3B_4B_6B_7\}. \tag{6}$$

When $r = 3$, switch B_4 and B_7 in (6). Then we have the following theorem.

Theorem 8. *The $2^{(n_1+n_2)-(2+1)}$ design d_0 with $n_2 \le n_1/6$, whose defining contrast subgroup is generated by the three words in (6), is a WS-MA design.*

Next, consider the construction of the WS-MA $2^{(n_1+n_2)-(3+1)}$ designs. Let $n_1 + n_2 = 15m + r$ for $0 \le r < 15$. By Theorems 2 and 3, we just consider the case of $n_2 \le n_1/14$. When $r = 5$ and $n_2 \le m-1$, $i = 1, \ldots, 15$, define

$$B_i = \begin{cases} (im-m+1)(im-m+2)\cdots(im)(15m+i-1), & i = 1,2,3, \\ (im-m+1)(im-m+2)\cdots(im)(15m+i-2), & i = 5,6,7, \\ (im-m+1)(im-m+2)\cdots(im), & i = 4,8,\ldots,14, \\ (im-m+1)(im-m+2)\cdots(im-1), & i = 15. \end{cases}$$

In other cases, define

$$B_i = \begin{cases} (im-m+1)(im-m+2)\cdots(im)(15m+i), & i \le r, \\ (im-m+1)(im-m+2)\cdots(im), & r < i \le 15. \end{cases}$$

Similar to (5), the n_2 SP factors can be labeled as $14m+1, \ldots, 14m+n_2$ and arranged in B_{15}.

When $r \ne 5,6,7,10,11$ and when $r = 5$ and $n_2 \le m-1$, let

$$\begin{aligned} B = \{&B_{15}B_{14}B_{12}B_9B_8B_7B_6B_1, B_{15}B_{13}B_{11}B_9B_8B_7B_5B_2, \\ &B_{15}B_{14}B_{11}B_{10}B_8B_6B_5B_3, B_{15}B_{13}B_{12}B_{10}B_7B_6B_5B_4\}. \end{aligned} \tag{7}$$

When $r = 5$ and $n_2 = m$, switch B_5 and B_{15} in (7). When $r = 6,7$, switch B_8 and B_{15} in (7). When $r = 10$, switch B_{14} and B_{15} in (7). When $r = 11$, switch B_{12} and B_{15} in (7).

Theorem 9. *The $2^{(n_1+n_2)-(3+1)}$ design d_0 with $n_2 \le n_1/14$, whose defining contrast subgroup is generated by the four words in (7), is a WS-MA design.*

By Theorems 2 and 3, Theorems 7–9 construct all WS-MA designs that may be different from WP-MA designs.

5.3. WS-MA $2^{(n_1+n_2)-(1+2)}$ Designs

Let $n_1 + n_2 = 7m + r$, $0 \le r < 7$. We consider the case of $n_2 \le 3[n_1/4]$, where $[n_1/4]$ is the integer part of $n_1/4$. For $i = 1, \ldots, 7$, define:

$$B_i = \begin{cases} (im-m+1)(im-m+2)\cdots(im)(7m+i), & i \le r, \\ (im-m+1)(im-m+2)\cdots(im), & i > r. \end{cases} \quad (8)$$

By $n_1 = 7m + r - n_2$ and $n_2 \le 3[n_1/4]$, we can get $n_2 \le 3m + 2$. When $n_2 \le 3m$, the n_2 SP factors can be labeled as $4m+1, \ldots, 4m+[(n_2+2)/3]$, $5m+1, \ldots, 5m+[(n_2+1)/3]$, $6m+1, \ldots, 6m+[n_2/3]$. When $n_2 = 3m+1$, the n_2 SP factors can be labeled as $4m+1, \ldots, 7m, 7m+5$. When $n_2 = 3m+2$, the n_2 SP factors can be labeled as $4m+1, \ldots, 7m, 7m+5, 7m+6$. Then, all SP factors are arranged in B_5, B_6 and B_7.

Let

$$B = \{B_1B_2B_6B_7, B_1B_4B_5B_6, B_3B_4B_6B_7\}. \quad (9)$$

Theorem 10. *The $2^{(n_1+n_2)-(1+2)}$ design d_0 with $n_2 \le 3[n_1/4]$, whose defining contrast subgroup is generated by the three words in (9), is a WS-MA design.*

Theorem 10 constructs some WS-MA designs. By Theorem 2, only one case needs to be considered, that is $n_1 = 4m+3$ and $n_2 = 3m+1$.

Now consider the case of $n_1 = 4m+3$ and $n_2 = 3m+1$. In this case, the n_2 SP factors are $2m+1, \ldots, 5m, 7m+3$. Clearly, all SP factors are arranged in B_3, B_4 and B_5 by (8).

Theorem 11. *The $2^{(n_1+n_2)-(1+2)}$ design d_0 with $n_1 = 4m+3$ and $n_2 = 3m+1$, whose defining contrast subgroup is generated by the three words in (9), is a WS-MA design.*

By Theorems 2, Theorems 10 and 11 construct all WS-MA designs that may be different from WP-MA designs.

6. Conclusions

The FFSP designs are widely used when the levels of some factors are very difficult or expensive to change or control. Sometimes the selection of the levels of the WP factors affects that of the SP factors. This requires the experimenters to pay more attention to WP factors. However, the experimenters perhaps do not want to sacrifice too much of the accuracy of the estimation of SP factor effects. In this paper, we first propose a criterion for selecting $2^{(n_1+n_2)-(k_1+k_2)}$ designs, that is, the minimum aberration of type WS (WS-MA). Then we test the robustness of the WS-MA design and compare the WS-MA criterion with the other criteria. Finally, construction methods of WS-MA FFSP designs with small k_1 and k_2 are studied.

In some situations, there are many factors to be tested. Then designs with larger k_1 and k_2 are needed. Note that the WS-MA designs are not unique sometimes. The new criterion is needed to further discriminate them. We look forward to exploring the problems in future research.

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Abbreviations

The following abbreviations are used in this manuscript:

FF	Fractional factorial
MA	Minimum aberration
FFSP	Fractional factorial split-plot
WP	Whole plot
SP	Sub plot
MSA	Minimum secondary aberration
WP-MA	Minimum aberration of type WP
WS-MA	Minimum aberration of type WS

Appendix A

Lemmas A1 from Wang et al. [21] will be helpful. The proof of Lemma A2 is similar to that of Lemma 3.2.1 in Mukerjee and Wu [23]. We omitted it to save space here.

Lemma A1. *For any* $2^{(n_1+n_2)-(k_1+k_2)}$ *design,*
(a) $\sum_{i=1}^{n} A_{i,0} + \sum_{i=1}^{n} A_{i,1} = 2^k - 1$,
(b) $\sum_{i=1}^{n} iA_{i,0} + \sum_{i=1}^{n} iA_{i,1} = n2^{k-1}$,
(c) *either all the defining words have even lengths or* 2^{k-1} *of them have odd lengths, where* $n = n_1 + n_2, k = k_1 + k_2$.

Lemma A2. *For any* $2^{(n_1+n_2)-(k_1+k_2)}$ *design,*
(a) $\sum_{i=1}^{n_1} A_{i,0} = 2^{k_1} - 1$,
(b) $\sum_{i=1}^{n_1} iA_{i,0} = n_1' 2^{k_1-1}$,
(c) *either all the WP defining words have even lengths or* 2^{k_1-1} *of them have odd lengths, where* n_1' *is a positive integer with* $n_1' \le n_1$.

Proof of Theorem 1. Suppose that d_1 is a WS-MA FFSP design with resolution r_1, d_2 is an MA FFSP design with resolution r_2, and $r_2 > r_1$. This means that $(A_{i,0}(d_1), A_{i,1}(d_1)) = (A_{i,0}(d_2), A_{i,1}(d_2)) = (0,0)$ for $0 \le i < r_1$ and $(A_{r_1,0}(d_1), A_{r_1,1}(d_1)) \ne (A_{r_1,0}(d_2), A_{r_1,1}(d_2)) = (0,0)$. By Definition 2, it is not difficult to conclude that d_2 has less aberration of type WS than d_1. This contradiction completes the proof. □

Proof of Theorem 2. Let d_1 be a WS-MA design with a different WS wordlength pattern from the WP-MA design d_2. Suppose the WS wordlength pattern of d_1 is

$$W_0(d_1) = (A_{1,0}(d_1), A_{1,1}(d_1), \ldots, A_{n_1+n_2,0}(d_1), A_{n_1+n_2,1}(d_1)).$$

Let r be the smallest integer i such that $(A_{i,0}(d_1), A_{i,1}(d_1)) \ne (A_{i,0}(d_2), A_{i,1}(d_2))$. By Definition 3, we have (i) $A_{r,0}(d_1) < A_{r,0}(d_2)$ or (ii) $A_{r,0}(d_1) = A_{r,0}(d_2), A_{r,1}(d_1) < A_{r,1}(d_2)$.

For (i). By the definition of resolution, we have $A_{j,0}(d_1) = A_{j,0}(d_2), j = 1, \ldots, r-1$. Since $A_{r,0}(d_1) < A_{r,0}(d_2)$, the design d_1 has less aberration of type WP than d_2. This contradicts the assumption that d_2 is a WP-MA design.

For (ii). (a) When $r < p-1$, we have $A_{r,1}(d_2) = 0$ and then $A_{r,1}(d_1) \ge A_{r,1}(d_2)$ which contradicts $A_{r,1}(d_1) < A_{r,1}(d_2)$.

(b) When $r = p-1$, if $A_{r+1,0}(d_1) < A_{r+1,0}(d_2)$, then d_1 has less aberration of type WP than d_2 due to $A_{i,0}(d_1) = A_{i,0}(d_2), i = 1, \ldots, r$. This contradicts with the assumption that d_2 is a WP-MA design. If $A_{r+1,0}(d_1) = A_{r+1,0}(d_2)$, since $A_{i,0}(d_2) = 0, i = r+2, \ldots, n_1+n_2$, we have $\sum_{i=1}^{r+1} A_{i,0}(d_1) = \sum_{i=1}^{r+1} A_{i,0}(d_2) = 2^{k_1} - 1$. So $A_{i,0}(d_1) = 0, i = r+2, \ldots, n_1+n_2$ and $W_1(d_1) = W_1(d_2)$. Since $A_{r,1}(d_1) < A_{r,1}(d_2)$, the design d_1 has less aberration of type

WP than d_2 which also contradicts with the assumption that d_2 is a WP-MA design. If $A_{r+1,0}(d_1) > A_{r+1,0}(d_2)$, then $\sum_{i=1}^{n} A_{i,0}(d_1) \geq \sum_{i=1}^{r+1} A_{i,0}(d_1) > \sum_{i=1}^{r+1} A_{i,0}(d_2) = 2^{k_1} - 1$, which contradicts Lemma A2(a).

(c) When $r > p - 1$, similar to the case of $A_{r+1,0}(d_1) = A_{r+1,0}(d_2)$ in (b), we have $W_1(d_1) = W_1(d_2)$. Since $A_{r,1}(d_1) < A_{r,1}(d_2)$, d_1 has less aberration of type WP than d_2 which again contradicts with the assumption that d_2 is a WP-MA design. □

Proof of Theorem 3. Suppose d_1 is a WS-MA design and d_2 is a WP-MA design. Let r be the smallest integer i such that $(A_{i,0}(d_1), A_{i,1}(d_1)) \neq (A_{i,0}(d_2), A_{i,1}(d_2))$. From Definition 3, we have (i) $A_{r,0}(d_1) < A_{r,0}(d_2)$ or (ii) $A_{r,0}(d_1) = A_{r,0}(d_2)$, $A_{r,1}(d_1) < A_{r,1}(d_2)$.

If (i) holds, then $A_{j,0}(d_1) = A_{j,0}(d_2), j = 1, \ldots, r-1$. Thus d_1 has less aberration of type WP than d_2 since $A_{r,0}(d_1) < A_{r,0}(d_2)$. This contradicts with the assumption that d_2 is a WP-MA design.

If (ii) holds, then d_1 has less aberration than d_2. This contradicts with the condition that d_2 is an MA FF design in Theorem 3. □

Proof of Theorem 4. Suppose d_1 is a WP-MA $2^{(n_1+n_2)-(k_1+1)}$ design and w_1 is a WP factor of d_1. Suppose the independent defining words of d_1 are $W_1, \ldots, W_{k_1}, S$, where W_i $(i = 1, \ldots, k_1)$ and S denote the WP and SP type defining words, respectively. By Definition 2, the WP part of d_1 is an MA $2^{n_1-k_1}$ design and hence every WP factor is involved in some WP type defining word by Lemma 2.5.1 in Mukerjee and Wu [23]. It is obvious that S contains at least one WP factor. Otherwise, we can construct a design which has less aberration of type WP than d_1 by adding a WP factor to S. Without loss of generality, suppose w_1 is involved in S. Consider d_2 determined by deleting the letter w_1 from the WP type defining words of d_1. Then we can check that d_2 has less aberration of type WS than d_1. □

Theorems 5 and 6 can be easily derived from (4) and Definition 3.

The proofs of the Theorems 7–11 are similar. We only give the proof of $r = 4$ and 5 in Theorem 9 to save space.

Proof of Theorem 9. For $r = 4$, d_0 has the WS wordlength pattern

$$W_0(d_0) = (0, \ldots, 0, 4, 6, 0, 0, 4, 1, 0, \ldots, 0),$$

where $A_{8m+1,1}(d_0) = 4, A_{8m+2,0}(d_0) = 6, A_{8m+3,1}(d_0) = 4, A_{8m+4,0}(d_0) = 1$. We can directly check that d_0 has the same resolution as the MA 2^{n-4} design, where $n = n_1 + n_2$. To prove that d_0 is an WS-MA design, we only need to consider the $2^{(n_1+n_2)-(3+1)}$ designs d that might have less aberration of type WS than d_0.

According to Lemmas A1 and A2, design d needs to satisfy the following conditions:

- $\sum_{i=1}^{n} A_{i,0}(d) + \sum_{i=1}^{n} A_{i,1}(d) = 15$,
- $\sum_{i=1}^{n} iA_{i,0}(d) + \sum_{i=1}^{n} iA_{i,1}(d) = 8n$,
- either all the defining words have even lengths or eight of them have odd lengths,
- $\sum_{i=1}^{n_1} A_{i,0}(d) = 7$,
- $\sum_{i=1}^{n_1} iA_{i,0}(d)$ is divisible by four, and
- either all the WP defining words have even lengths or four of them have odd lengths.

Under the above conditions, we obtain the following two WS wordlength patterns of possible designs with less aberration of type WS than d_0:

$$W_0(d) = (0, \ldots, 0, 4, 2, 4, 4, 0, 1, 0, \ldots, 0), \tag{A1}$$

where $A_{8m+1,1}(d) = 4, A_{8m+2,0}(d) = 2, A_{8m+2,1}(d) = 4, A_{8m+3,0}(d) = 4, A_{8m+4,0}(d) = 1$,

$$W_0(d) = (0, \ldots, 0, 4, 3, 4, 3, 0, 0, 0, 1, 0, \ldots, 0), \tag{A2}$$

where $A_{8m+1,1}(d) = 4, A_{8m+2,0}(d) = 3, A_{8m+2,1}(d) = 4, A_{8m+3,0}(d) = 3, A_{8m+5,0}(d) = 1$.

We first consider the pattern (A1). Since the sum of the lengthes of the shortest SP type defining word and the longest WP type defining word is $16m+5>n$, there is at least a WP factor shared by one of the shortest SP type defining words and the longest WP type defining word. Let l denote such a WP factor. By deleting all the words containing l, the remaining words define a $2^{(n_1'+n_2')-(2+1)}$ design d_1 with $n_1' \leq n_1$ and $n_2' \leq n_2$. From Lemma A1(c), there are 3 defining words of length $8m+2$. By Lemma A1(a) and (b) and Lemma A2(a), we have

$$\sum_{i=1}^{n_1'+n_2'} A_{i,0}(d_1) + \sum_{i=1}^{n_1'+n_2'} A_{i,1}(d_1) = 7,$$

$$\sum_{i=1}^{n_1'+n_2'} iA_{i,0}(d_1) + \sum_{i=1}^{n_1'+n_2'} iA_{i,1}(d_1) = 4(n_1'+n_2'),$$

$$\sum_{i=1}^{n_1'} A_{i,0}(d_1) = 3.$$

Only two wordlength patterns of type WS of design d_1 satisfy the above three equations:

$$W_0(d_1) = (0,\ldots,0,3,2,1,1,0,\ldots,0),$$

where $A_{8m+1,1}(d_1)=3$, $A_{8m+2,0}(d_1)=2$, $A_{8m+2,1}(d_1)=1$, $A_{8m+3,0}(d_1)=1$ and

$$W_0(d_1) = (0,\ldots,0,1,0,3,3,0,\ldots,0),$$

where $A_{8m+1,1}(d_1)=1$, $A_{8m+2,1}(d_1)=3$, $A_{8m+3,0}(d_1)=3$. They all violate Lemma A2(c). Hence, there is no design having the WS wordlength pattern (A1). Similarly, we can prove that there is no design having the WS wordlength pattern (A2).

For $r=5$ and $n_2 \leq m-1$, d_0 has the WS wordlength pattern

$$W_0(d_0) = (0,\ldots,0,3,8,0,0,3,0,0,0,1,0\ldots,0),$$

where $A_{8m+2,0}(d)=3$, $A_{8m+2,1}(d_0)=8$, $A_{8m+4,0}(d_0)=3$, $A_{8m+6,0}(d_0)=1$. The conditions are similar to those in $r=4$ and we obtain the following WS wordlength pattern of possible designs with less aberration of type WS than d_0

$$W_0(d) = (0,\ldots,0,2,8,0,0,5,0,\ldots,0), \tag{A3}$$

where $A_{8m+2,0}(d)=2$, $A_{8m+2,1}(d)=8$, $A_{8m+4,0}(d)=5$.

Now we prove that there is no design having WS wordlength pattern (A3). Let l be a WP factor that appears in one of the two shortest WP defining words but not in the other. By deleting all the defining words containing l, the remaining defining words define a $2^{(n_1'+n_2')-(2+1)}$ design d_1 with $n_1' \leq n_1$ and $n_2' \leq n_2$. By Lemma A1(a) and Lemma A2(a), only one wordlength pattern of type WS of possible design d_1 meets the conditions:

$$W_0(d_1) = (0,\ldots,0,1,4,0,0,2,0,\ldots,0),$$

where $A_{8m+2,0}(d_1)=1$, $A_{8m+2,1}(d_1)=4$, $A_{8m+4,0}(d_1)=2$. This violates Lemma A1(b).

For $r=5$ and $n_2=m$, d_0 has the WS wordlength pattern

$$W_0(d_0) = (0,\ldots,0,4,6,0,0,3,2,0,\ldots,0),$$

where $A_{8m+2,0}(d_0) = 4, A_{8m+2,1}(d_0) = 6, A_{8m+4,0}(d_0) = 3, A_{8m+4,1}(d_0) = 2$. The conditions are also similar to those in $r = 4$ and we obtain the following two WS wordlength patterns of possible designs with less aberration of type WS than d_0:

$$W_0(d) = (0, \ldots, 0, 2, 8, 0, 0, 5, 0, \ldots, 0), \tag{A4}$$

where $A_{8m+2,0}(d) = 2, A_{8m+2,1}(d) = 8, A_{8m+4,0}(d) = 5$,

$$W_0(d) = (0, \ldots, 0, 3, 8, 0, 0, 3, 0, 0, 0, 1, 0 \ldots, 0), \tag{A5}$$

where $A_{8m+2,0}(d) = 3, A_{8m+2,1}(d) = 8, A_{8m+4,0}(d) = 3, A_{8m+6,0}(d) = 1$.

Similar to (A3), we can prove that there is no design having the WS wordlength pattern (A4). For (A5), we have

$$\sum_{i=1}^{n_1} iA_{i,0}(d) = 56m + 24 = 4 \times (14m + 6) > 4 \times (14m + 5),$$

which violates Lemma A2(b). So, there is no design having the WS wordlength pattern (A5). □

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