mathematics

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Estimates of Mild Solutions of Navier–Stokes Equations in Weak Herz-Type Besov–Morrey Spaces

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Abstract: The main goal of this article is to provide estimates of mild solutions of Navier–Stokes equations with arbitrary external forces in $\mathbb{R}^n$ for $n \geq 2$ on proposed weak Herz-type Besov–Morrey spaces. These spaces are larger than known Besov–Morrey and Herz spaces considered in known works on Navier–Stokes equations. Morrey–Sobolev and Besov–Morrey spaces based on weak-Herz space denoted as $W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}$ and $W\dot{K}^{\alpha}_{p,q}\mathcal{N}^{s}_{\mu,r}$, respectively, represent new properties and interpolations. This class of spaces and its developed properties could also be employed to study elliptic, parabolic, and conservation-law type PDEs.

Keywords: system of PDEs; function spaces; mild solutions; real interpolation; heat semigroup operator

MSC: 35Q30

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Citation: Abdulkadirov, R.; Lyakhov, P. Estimates of Mild Solutions of Navier–Stokes Equations in Weak Herz-Type Besov–Morrey Spaces. *Mathematics* **2022**, *10*, 680. https://doi.org/10.3390/math10050680

Academic Editor:Emma Previato

Received: 10 January 2022
Accepted: 19 February 2022
Published: 22 February 2022

1. Introduction

Let us consider $\mathbb{R}^n$ with $n \geq 2$, and a fixed interval with $0 < T < \infty$. The incompressible Navier–Stokes equations system in $\mathbb{R}^n \times (0, T)$ is written in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where vector values u and f denote the velocity of the fluid and external forces acting on the fluid, respectively. The scalar value p represents the pressure.

The nonstationary Navier–Stokes equations are invariant under the following change of scaling:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t),\ \forall t > 0.$$

In Refs. [1,2], the authors considered the problem of applying hybrid spaces, such as Besov–Morrey or Triebel–Lizorkin–Morrey spaces, to nonlinear PDEs, for instance, nonlinear heat and Navier–Stokes equations. Properties of mild solutions of PDE in Lebesgue [3] and Sobolev [4] spaces were investigated. Additionally, they were observed in spaces, such as Hardy [5], Besov [6], Triebel–Lizorkin [7], Morrey [8], Herz [9], and other spaces.

There are several works [7,10,11], where properties of the Besov–Morrey space were provided, and they also included related $K_{\theta,r}$-method real interpolations. The properties of the Besov-weak Herz space $BW\dot{K}^{\alpha,s}_{p,q,r}$ were explored in [12]. Herz-type Besov $\dot{K}^{\alpha,p}_{q}B^{s}_{\beta}$ and $\dot{K}^{\alpha,p}_{q}F^{s}_{\beta}$ Triebel–Lizorkin spaces were considered in [6,7]. These spaces were introduced to explore global solutions of NSE in the case that $f = 0$ and to prove the Jawerth–Franke embeddings, respectively. The unique maximally strong solution for the Navier–Stokes equations with $f \neq 0$ on corresponding Triebel–Lizorkin–Lorentz spaces was constructed in [13]. The application of hybrid and global spaces to nonlinear heat and Navier–Stokes equations was observed in [1]. In addition, properties of local spaces and their applications to mild solutions of NSE with $f \neq 0$ were researched in [2].

The main idea of this article came from researching mild solutions of (1) on Besov–Morrey spaces, which were investigated in [10], and exploring the NSE with $f = 0$, realized on (weak) Herz spaces in [14]. According to the results of Besov–Morrey and (weak) Herz spaces from [10,14], we imply estimates of mild solutions of NSE with $f \neq 0$ on weak Herz-type Besov–Morrey spaces. In this article, we propose weak Herz-type Besov–Morrey space in Definition 5. Then, we prove interpolations of offered spaces, and three estimates containing the heat semigroup operator are proved in Lemma 1, engaging an estimate on weak Herz space in Corollary 1.

The proposed weak Herz-type Besov–Morrey spaces were not attended to in other works, so proper interpolations, wavelets, atomic decomposition, and embeddings are not provided. Theorem 3 and Lemma 1 can be used to find new interpolations and wavelet characterizations and further establish relations with other global and hybrid spaces.

Let us denote a projection onto the divergence-free vector fields, so-called Leray projection $\mathbb{P}$, on both sides of the first equation of (1). Then, we study the simpler equation, where $\mathbb{P}\nabla p = 0$. $\mathbb{P}$ can be represented as

$$\mathbb{P} = (\mathbb{P}_{ij})_{1\leq i,j\leq n},\ \mathbb{P}_{ij} = \delta_{ij} + R_i R_j,\ i,j = 1,...,n,$$

where $(\delta_{ij})_{1\leq i,j\leq n}$ is the Kronecker symbol and $R_j, j = 1,...,n$ are the Riesz transforms that can be represented by using Fourier transform:

$$R_j f = \frac{\partial_j}{\sqrt{-\Delta}} f = \mathcal{F}^{-1}\left(-\frac{i\xi_j}{|\xi|}\mathcal{F}f\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\epsilon\to 0}\int_{\|y\|\geq\epsilon} f(y)\frac{x_i - y_j}{\|x-y\|^{n+1}}dy,$$
$$\mathcal{F}(f(x)) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx,\ \mathcal{F}^{-1}(f(x)) = \mathcal{F}(f(-x)),$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

From the Calderón–Zygmund operator theory, for $1 < p < \infty$, $0 \leq \mu < n$, the boundedness of Riesz transform R_j on the Morrey space $\mathcal{M}_{p,\mu}$ implies that $\mathbb{P}$ is bounded on $\mathcal{M}_{p,\mu}$, as it was remarked in [10].

The Navier–Stokes equations can be transformed into an integral formula

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)ds + B(u,u), \tag{2}$$

where

$$B(u,v) = -\frac{1}{2}\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v + v\otimes u)ds. \tag{3}$$

Functions that satisfy (2) are called mild solutions of the NSE.

Applying $\mathbb{P}$ to (1), we have

$$\begin{cases} \frac{\partial u}{\partial t} + Au + \mathbb{P}\nabla\cdot(u\otimes u) = \mathbb{P}f \ on \ (0,T), \\ u(0) = u_0, \end{cases} \tag{4}$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator.

In [15], mild solutions were constructed for $L^{\alpha,\infty}(0,\infty; L^{p,\infty}(\mathbb{R}^n))$, where $2/\alpha + n/p = 3$ and $\max\{1, n/3\} < p < \infty$. In [10], these properties were extended to homogeneous Besov–Morrey space $\dot{\mathcal{N}}^s_{q,\mu,r}(\mathbb{R}^n)$, and especially estimates of heat semigroup operator $e^{t\Delta}$. According to interpolations and Lemma 2.3 from [12] for Besov-weak Herz space, we prove the interpolation of the proposed weak Herz-type Besov–Morrey spaces. The motivation of our research is to propose new hybrid spaces (weak Herz-type Besov–Morrey spaces), which contain the properties of several global spaces (Herz, Besov–Morrey spaces), and explore mild solutions of the incompressible Navier–Stokes equations with $f \neq 0$. Mild solutions were researched on Besov–Morrey and Herz spaces with proper interpolations, embeddings, and estimates in [9,10], respectively. Herz-type Besov $\dot{K}_q^{\alpha,p}B_\beta^s$ and Triebel–Lizorkin spaces $\dot{K}_q^{\alpha,p}F_\beta^s$ were engaged in [6], and $BW\dot{K}_{p,q,r}^{\alpha,s}$ Besov-weak Herz spaces in [12]. In our manuscript, we explore weak Herz-type Besov–Morrey spaces $W\dot{K}_{p,q}^{\alpha}\dot{\mathcal{N}}_{\mu,r}^{s}(\mathbb{R}^n)$, which were not met in other publications. Therefore, it would be reasonable to provide their properties and study mild solutions of NSE on such spaces.

Our main results are

Theorem 1. *Let* $1 < q < \infty$, $0 \le \mu \le n$, $1 < \alpha < \alpha_0 < \infty$, $-\infty < s < s_0 < \infty$, *and* $-n/q < \gamma < \delta < n(1 - 1/q)$ *such that*

$$\frac{2}{\alpha_0} - s_0 = \frac{2}{\alpha} - s - 2. \tag{5}$$

Let $1 < p < \infty$ *satisfy*

$$\frac{n-\mu}{q} \le \frac{n-\mu}{p} < \frac{2}{\alpha_0} + \frac{n-\mu}{q}. \tag{6}$$

Suppose that a measurable function u *on* $\mathbb{R}^n \times (0,T)$ *is a mild solution of (4) and satisfies* $u_t, Au \in L^{\alpha,r}(0,T; W\dot{K}_{q,l}^{\gamma}\dot{\mathcal{N}}_{\mu,\infty}^{s})$ *with* $u(0) = u_0 \in W\dot{K}_{p,l}^{\delta}\dot{\mathcal{N}}_{\mu,r}^{k}$ *for* $k = 2 + (n-\mu)/p - (2/\alpha + (n-\mu)/q - s)$.

Then

$$u \in L^{\alpha_0,r}(0,T; W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,1}^{s_0}) \tag{7}$$

holds with estimate

$$\begin{aligned} \|u\|_{L^{\alpha_0,r}(0,T;W\dot{K}_{q,l}^{\gamma}\dot{\mathcal{N}}_{\mu,1}^{s_0})} &\le \|e^{t\Delta}\mathbb{P}u_0\|_{L^{\alpha_0,r}(0,T;W\dot{K}_{q,l}^{\gamma}\dot{\mathcal{N}}_{\mu,1}^{s_0})} \\ +C\Big(\|u_t\|_{L^{\alpha,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})} &+ \|Au\|_{L^{\alpha,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})}\Big), \\ \|e^{t\Delta}\mathbb{P}u_0\|_{L^{\alpha_0,r}(0,\infty;W\dot{K}_{q,l}^{\gamma}\dot{\mathcal{N}}_{\mu,1}^{s_0})} &\le C\|u_0\|_{W\dot{K}_{p,l}^{\delta}\dot{\mathcal{N}}_{\mu,r}^{k}}, \end{aligned} \tag{8}$$

where $C = C(n,\mu,q,\alpha,\gamma,\delta,s,p,r)$ *is independent of* u, u_0 *and* T.

Theorem 2. *Let* $1 < q < \infty, 1 < \alpha < \infty, 0 \le \mu < n$ *and* $-1 < s < \infty, 2/\alpha + (n-\mu)/q - s = 3$, $1 < p \le q$ *and* $1 \le r \le \infty$ *with* $\delta \in \mathbb{R}$, $1 < l \le \infty$ *and a measurable function* u *on* $\mathbb{R}^n \times (0,T)$ *is a mild solution of (4) with*

$$u_t, Au \in L^{\alpha,r}(0,T; W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s}),$$

$$u_0 \in W\dot{K}_{p,l}^{\delta}\dot{\mathcal{N}}_{\mu,r}^{-1+(n-\mu)/p}$$

it holds $\mathbb{P}(u \cdot \nabla u) \in L^{\alpha,r}(0,T; W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})$ *with the estimate*

$$\begin{aligned} \|\mathbb{P}(u\cdot\nabla u)\|_{L^{\alpha,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})} &\le C\Big(\|e^{t\Delta}\mathbb{P}u_0\|_{L^{\alpha_0,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,1}^{s_0})} \\ &+\|u_t\|_{L^{\alpha,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})} + \|Au\|_{L^{\alpha,r}(0,T;W\dot{K}_{q,l}^{\delta}\dot{\mathcal{N}}_{\mu,\infty}^{s})}\Big)^2, \end{aligned} \tag{9}$$

for some $\alpha < \alpha_0 < \infty$, $q < \infty$ *and* $s < s_0 < \infty$ *such that* $2/\alpha_0 + (n-\mu)/q - s_0 = 1$, *where* $C = C(n,\mu,\delta,q,\alpha,\alpha_0,p,r)$ *is a constant independent of* $0 < T \le \infty$.

Extension of (weak) Herz and Besov–Morrey spaces to (weak) Herz-type Besov–Morrey spaces allows enlarging their properties, especially embedding, interpolations and wavelet characterizations. Moreover, atomic partition and oscillations in [1,2] make it possible to receive useful estimates and properties of solutions of nonlinear PDEs and investigate the similar extension on Triebel–Lizorkin–Morrey spaces researching mild solutions of NSE with $f \neq 0$.

Our main contribution to the theory of Navier–Stokes equations is providing estimates in Theorems 1 and 2, which can state the maximal Lorentz regularity of a function u in $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}(\mathbb{R}^n)$. This allows us to approach establishing the unique existence of local strong or weak solutions to (1) for arbitrary large initial data u_0 and large external force f. The maximal Lorentz regularity is exploited for Besov–Morrey space in [10].

The current problems of nonlinear PDEs need new tools, such as embedding, wavelet characterization, real (K- and J-types), and complex interpolations. In our manuscript, we provide and prove K-real interpolations for Herz-type Besov–Morrey spaces, which allow us to imply useful estimates in Lemma 1 that engage not only the heat semigroup operator, but also the Leray projection. In [10] in Lemma 2.2 for Besov–Morrey spaces, Leray projection was not considered, while for weak Herz space, it was shown in [14] (Corollary 1). Combining Besov–Morrey and weak Herz spaces into Herz-type Besov–Morrey spaces allows us to imply new estimates in Lemma 1, by real interpolation of new proposed spaces.

The remaining of the paper is organized as follows. Section 2 is devoted to function spaces and some necessary statements from references. Section 3 defines weak Herz-type Besov–Morrey space and proves the interpolation of Theorem 3 and Lemma 1, providing essential properties and inequalities. Section 4 and 5 are devoted to proofs of Theorems 1 and 2, respectively.

2. Preliminaries

Let us define Herz spaces and weak Herz spaces from [12,14], respectively.

Definition 1. *Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. One defines the homogeneous Herz space $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ as*

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}); \|f\|_{\dot{K}^{\alpha}_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} := \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^p(A_k)}\right)^{1/q},$$

with the usual modification in the case $q = \infty$ and $A_k = \{x \in \mathbb{R}^n;\ 2^{k-1} \leq |x| < 2^k\}$.

Definition 2. *With the same conditions as in Definition 1, one defines the homogeneous weak Herz space $W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ as the space of measurable functions such that*

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^{p,\infty}(A_k)}\right)^{1/q} < \infty, & q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha}\|f\|_{L^{p,\infty}(A_k)}, & q = \infty. \end{cases}$$

The definition and the basic properties of Morrey and Besov-Morrey spaces were reviewed in [6,10,11].

From [13] we recall the definition of the Lorentz space that is applied in the proofs of the theorems.

Definition 3. *Let (X, λ) be a measure space. Let f be a scalar-valued λ-measurable function and*

$$\lambda_{f(s)} = \lambda\{x : f(x) > s\}.$$

Then, the rearrangement function f^* of f is defined by: $f^*(t) = \inf\{\lambda_{f(s)} \leq t\}$. For any $1 < p < \infty$, the Lorentz spaces $L^{p,q}$ is defined by $L^{p,q}(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{C} : \|f\|_{L^{p,q}} < \infty\}$ where

$$\|f\|_{L^{p,q}} := \begin{cases} \frac{q}{p}\left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right) dt\right]^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0}\{t^{\frac{1}{p}} f^*(t)\}, & q = \infty. \end{cases}$$

In particular, $L^{p,\infty}$ agrees with the weak-L^p (Marcinkiewicz space) $L^{p*}(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{C} : \|f\|_{L^{p*}} < \infty\}$, equipped with the following quasi-norm $\|f\|_{L^{p*}} = \sup_{t>0} t\left[\lambda_f(t)\right]$.

Let us provide Proposition 2.2 and Corollary 2.1 in [14].

Proposition 1. *Let $1 < p \leq \infty$, $0 < q \leq \infty$ and $m \geq 0$. Suppose that $\phi \in L^{r,1} \cap L^\infty$, with $1 = 1/p + 1/r$, such that $|\phi(x)| \leq C_*|x|^{-m}$ for all $x \neq 0$. Then we have the following estimate:*

$$\|f * \phi\|_{W\dot{K}^{\beta}_{p,q}} \leq C\|f\|_{W\dot{K}^{\alpha}_{p,q}} \tag{10}$$

provided that one of the following cases holds:

(1) $0 < q \leq 1$, $-\frac{n}{p} < \beta \leq \alpha \leq n(1 - \frac{1}{p})$, $n - \alpha + \beta \leq m$ and $\beta + \frac{n}{p} < m$,
(2) $1 < q < \infty$, $-\frac{n}{p} < \beta \leq \alpha < n(1 - \frac{1}{p})$ and $n - \alpha + \beta \leq m$,
(3) $q = \infty$, $-\frac{n}{p} \leq \beta \leq \alpha < n(1 - \frac{1}{p})$, $n - \alpha + \beta \leq m$ and $n(1 - \frac{1}{p}) - \alpha < m$.

Some properties of the operators $e^{t\Delta}$ and $e^{t\Delta}\mathbb{P}$ are investigated and proved in [14], which gives us a necessary estimate.

Corollary 1. *Let $1 < p \leq \infty$, $0 < q \leq \infty$, $-n/p \leq \beta \leq \alpha \leq n(1 - 1/p)$. Then*

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}} \leq Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}}. \tag{11}$$

Let $B_r(x_0)$ be the open ball in $\mathbb{R}^n$ centered at x_0 and radius $r > 0$. The definition of the Morrey-type (weak) Herz space is provided in [14].

Definition 4. *Let $1 \leq p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $0 \leq \mu < n$, the Morrey-type (weak) Herz space $\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu(\mathbb{R}^n)$ ($W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu(\mathbb{R}^n)$) is defined to be the set of functions $f \in L^p(B_r(x_0))$ such that*

$$\|f\|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\mu/p}\|f\|_{\dot{K}^{\alpha}_{p,q}(B_r(x_0))} < \infty.$$

For weak Morrey-type Herz space, we substitute the norm of the $W\dot{K}^{\alpha}_{p,q}$ instead the norm of the $\dot{K}^{\alpha}_{p,q}$.

As in the [10,11] for $s \in \mathbb{R}$ and $1 \leq p < \infty$, the homogeneous weak Sobolev–Morrey-type Herz space $W\dot{K}^{\alpha}_{p,q}\mathcal{M}^s_\mu = (-\Delta)^{s/2} W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu$ is the Banach space with the norm

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^s_\mu} = \|(-\Delta)^{s/2} f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}.$$

Additionally, the Herz-type Sobolev space can be defined by means of the Riesz potential $I^s = (-\Delta)^{s/2}$, as in [16], defined as

$$I^s f(x) = 2^{-s}\pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^n} f(x-y)|y|^{-n+s} dy.$$

3. Weak Herz-Type Besov–Morrey Space and Its Properties

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwarz space and the tempered distributions space, respectively. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a non-negative radial function such that

$$supp(\phi) \subset \{\xi \in \mathbb{R}^n; 1/2 < |\xi| < 2\}$$

and

$$\sum_{k\in\mathbb{Z}} \phi_k(\xi) = 1,\ \forall \xi \neq 0,$$

where $\phi_k(\xi) = \phi(2^{-k}\xi)$.

Let us define the homogeneous weak Herz-type Besov–Morrey space.

Definition 5. *For $1 \le p < \infty$, $0 < q \le \infty$, $0 \le \mu < n$ and $s, \alpha \in \mathbb{R}$, the homogeneous weak Herz-type Besov–Morrey space $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ with $r \in [1,\infty]$ is the set of $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials, such that $\mathcal{F}^{-1}\phi_k * f \in W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu$ and*

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} = \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{ksr}\|\mathcal{F}^{-1}\phi_k * f\|^r_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{1/r} < \infty, & 1 \le r < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{F}^{-1}\phi_k * f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}, & r = \infty. \end{cases}$$

We denote the localization operators of the Littlewood–Paley decomposition as $\Delta_j f = \mathcal{F}^{-1}\phi_j * f$.

The space $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ is Banach and in particular, $W\dot{K}^{0}_{p,p}\dot{\mathcal{N}}^{s}_{0,r}$ corresponds to the homogeneous Besov space with weak-Lebesgue space, which implies the $K_{\theta,r}$-method real-interpolation properties.

Theorem 3. *Let $\theta \in (0,\ 1)$, $s_0,\ s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $0 < p, q \neq \infty$ and $r, r_0, r_1 \in (0,\infty]$. Suppose $s = (1-\theta)s_0 + \theta s_1$, then*

$$W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r} = \left(W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_{\mu}, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_{\mu}\right)_{\theta,r} \tag{12}$$

and

$$W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r} = \left(W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,r_0}, W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s_1}_{\mu,r_1}\right)_{\theta,r}. \tag{13}$$

Proof. Let $f = f_0 + f_1$ with $f_i \in W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_i}_{\mu}$, $i = 0,1$. By using Lemma 2.3 from [12] for weak Herz-type Sobolev space, we note that $W\dot{K}^{\alpha,s}_{p,q}$ is an Herz-Sobolev space and it holds that

$$\begin{aligned} \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} &\le \|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}} + \|\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}} \\ &\le C'\left(\|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + \|\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right), \end{aligned}$$

where C' is a constant. Therefore,

$$\begin{aligned} \|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} &\le C\left(\|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + \|\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right) \\ &\le C\left(2^{-s_0 j}\|I^{s_0}\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + 2^{-s_1 j}\|I^{s_1}\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right) \\ &\le C\left(2^{-s_0 j}\|I^{s_0} f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + 2^{-s_1 j}\|I^{s_1} f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right) \\ &\le C2^{-s_0 j}\left(\|f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu} + 2^{(s_0-s_1)j}\|f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu}\right). \end{aligned}$$

It follows that

$$\|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \le C2^{-s_0 j}K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu).$$

Multiplying the previous inequality by 2^{js} and $s - s_0 = -\theta(s_0 - s_1)$, we obtain

$$2^{sj}\|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \le C(2^{(s_0-s_1)j})^{-\theta} K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu),$$

and then (see Lemma 3.1.3 from [17]) we can conclude that

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} \le C\|f\|_{(W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu)}. \tag{14}$$

To prove the reverse inequality of (14), note that by using Lemma 2.3 from [12], again we have

$$\begin{aligned}
2^{(s-s_0)j} J(2^{(s_0-s_1)j}, \Delta_j f, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu) = \\
2^{(s-s_0)j} \max\left(\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu}, 2^{(s_0-s_1)j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu}\right) \\
\le 2^{s_0 j} \max\left(2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}, 2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right) \\
\le 2^{sj} \max\left(\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}, \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right) \\
= 2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}.
\end{aligned}$$

Now the equivalence theorem (see Lemma 3.2.3 from [17]) leads us to

$$\|f\|_{(W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu)} \le C\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}.$$

In the remainder of the proof, we need to show that in fact $f \in W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ implies that $f \in W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu + W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu$. Suppose that $s_0 > s_1$ (without loss of generality). Using the decomposition $f = \sum_{j<0}\Delta_j f + \sum_{j\ge 0}\Delta_j f = f_0 + f_1$ and Lemma 2.3 from [12], we obtain

$$\begin{aligned}
\|f_0\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu} \le \sum_{j<0}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_\mu} \le \sum_{j<0} 2^{(s_0-s)j}2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \\
\le C\left(\sum_{j<0} 2^{(s_0-s)jr'}\right)^{1/r'}\left(\sum_{j<0} 2^{sjr}\|\Delta_j f\|^{r}_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{1/r} \le C\|f_0\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\|f_1\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu} \le \sum_{j\ge 0}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_\mu} \le \sum_{j\ge 0} 2^{(s_1-s)j}2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \\
\le C\left(\sum_{j\ge 0} 2^{(s_1-s)jr'}\right)^{1/r'}\left(\sum_{j\ge 0} 2^{sjr}\|\Delta_j f\|^{r}_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{1/r} \le C\|f_1\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}},
\end{aligned}$$

and then (12) is valid.

By using (12) and the reiteration theorem (see [17], Theorem 3.5.3 and its remark), we conclude that (13) is valid. □

Now we provide Lemma 1 for weak Herz-type Sobolev–Morrey and Besov–Morrey spaces.

Lemma 1. *Let $s, s_0 \in \mathbb{R}$, $1 < p < \infty$, $0 < q \le \infty$, $(s_0 - s) < 2$ with $s < s_0$, $0 \le \mu < n$, $-n/p < \beta < \alpha < n(1 - 1/p)$, then the following inequalities hold:*

$$(1)\ \|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s_0}_\mu} \le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_\mu} \tag{15}$$

for every $t > 0$ *and* $f \in W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}$.

$$(2)\ \|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,r}} \le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} \tag{16}$$

for every $t > 0$ *and* $f \in \mathcal{S}'/\mathcal{P}$.

$$(3)\ \|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,1}} \le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} \tag{17}$$

for every $t > 0$, $r \in [1,\infty]$ *and* $f \in \mathcal{S}'/\mathcal{P}$.

For all inequalities $\alpha - \beta \le s_0 - s$ *and* C *is a constant.*

Proof. (1) We use inequality

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s}_{\mu}} = \sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\mu/p}\|(-\Delta)^{s/2}e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}}$$

$$\le Ct^{-(\alpha-\beta)/2}\sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\mu/p}\|(-\Delta)^{s/2}f\|_{W\dot{K}^{\alpha}_{p,q}} = Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}}$$

from Corollary 2.1 (iv) in [14] for Herz-type Sobolev–Morrey spaces.

Now we use the Lemma 2.2 (i) from [10] and to get

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s_0}_{\mu}} \le C\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_0}_{\mu}} \le Ct^{-(s_0-s)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}}.$$

Finally, if $\alpha - \beta \le s_0 - s$, then we obtain

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s}_{\mu}} \le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}}.$$

(2) As in first part of this proof we can use Corollary 2.1 (iv) in [14] with respect to weak Herz space

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} = \left(\sum_{k\in\mathbb{Z}} 2^{ksr}\|\mathcal{F}^{-1}\phi_k * e^{t\Delta}\mathbb{P}f\|^{r}_{W\dot{K}^{\beta}_{p,q}\mathcal{M}_{\mu}}\right)^{1/r}.$$

Particularly we estimate the norm of the weak Herz-type Morrey space

$$\begin{aligned}\|\mathcal{F}^{-1}\phi_k * e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}_{\mu}} &\le C\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\mathcal{M}_{\mu}}\\ &\le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} &\le Ct^{-(\alpha-\beta)/2}\left(\sum_{k\in\mathbb{Z}} 2^{ksr}\|\mathcal{F}^{-1}(\phi_k * f)\|^{r}_{W\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}\right)^{1/r}\\ &\le Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}.\end{aligned} \tag{18}$$

Now by applying the Lemma 2.2 (ii) from [10] and properties of first part of this proof to (18), it implies that

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,r}} \le C\|e^{t\Delta}f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,r}} \le Ct^{-(s_0-s)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}.$$

Additionally, if $\alpha - \beta \le s_0 - s$, then we receive (16).

(3) Applying the K-method real interpolation (12):

$$W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r} = \left(W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_1}_{\mu}, W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s_2}_{\mu}\right)_{\theta,r},$$

$$W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r} = \left(W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s_1}_{\mu}, W\dot{K}^{\beta}_{p,q}\mathcal{M}^{s_2}_{\mu}\right)_{\theta,r}$$

for inequality (16) with $s = (1-\theta)s_1 + \theta s_2$, $s_1 \neq s_2$ we obtain

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{2s_0-s}_{\mu,\infty}} \leq Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty}}$$

and

$$\|e^{t\Delta}\mathbb{P}f\|_{W\dot{K}^{\beta}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty}} \leq Ct^{-(\alpha-\beta)/2}\|f\|_{W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty}}.$$

Using (13) for $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{2s_0-s}_{\mu,\infty}$ and $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty}$ with $(2s_0-s)(1-1/2)+s(1/2) = s_0$, it follows that $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s_0}_{\mu,1} = (W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{2s_0-s}_{\mu,\infty}, W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty})_{1/2,1}$ that yields (17). □

Example 1. *Let $\rho > 0$ be such that $B_\rho(0) \subset \mathbb{R}^n$ and set $f(x) = |x|^{-n}\chi_{B_\rho(0)}(x)$, for $n \geq 2$. This function satisfies the norm of weak Herz space and then Besov–Morrey spaces, which means that $f \in W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ for $\alpha > n - \frac{n}{p}$, $1 \leq p,q \leq \infty$, $0 \leq \mu < n$, $r \in [1,\infty]$ and $s \in \mathbb{R}$.*

Example 2. *In $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ we set $f(x) = |x|^{-n}(|\log|x||)^{-1}\chi_{B_{\frac{1}{2}}(0)}(x)$, where $f \in W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ for $\alpha = n - \frac{n}{p}$, $1 \leq p \leq \infty$, $1 < q \leq \infty$, $0 \leq \mu < n$, $r \in [1,\infty]$ and $s \in \mathbb{R}$.*

Examples 1 and 2 demonstrate functions belonging to weak Herz-type Besov–Morrey spaces that satisfy inequalities in Lemma 1.

In function space theory [18,19], it could be useful to provide a norm of $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$, defined by derivatives and differences, equivalent to the norm in Definition 5. In the case of Besov spaces, such an approach was used in [20,21], where the authors established the equivalence between the norms defined by Fourier analytic tools and by derivatives and differences, respectively.

Theorems 1 and 2 allow to provide the maximal Lorentz regularity theorem of Stokes and Navier–Stokes equations. They can help in establishing the unique existence of local strong solutions to Navier–Stokes equation on proposed weak Herz-type Besov–Morrey spaces, as it is made in [10] for homogeneous Besov–Morrey spaces and in [15] in Lorentz spaces.

The properties of Herz-type Besov–Morrey spaces, such as the interpolations in Theorem 3 and the inequalities in Lemma 1, can be also used to study other nonlinear PDEs. For example, a mathematical model of waves on shallow water surfaces described by Korteweg-de Vries equation [22]; the Keller–Segel system [23] presents a cellular chemotaxis model; and Fokker–Planck equations [24] demonstrate models of anomalous diffusion processes. Developing atomic decomposition, oscillations, real and complex interpolations can advance the study of the $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ spaces, especially observing them not only with the Fourier approach ([25]), but by the finite difference approach, in the same fashion of Besov spaces in [26,27].

4. Proofs of the Theorem 1

Let u be a mild solution of the NSE (1). Then, by using representation (4), we obtain

$$f(t) = u_t + Au, \tag{19}$$

as in [10]. Then it holds that $u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau)d\tau$. By (19), we have $f \in L^{\alpha,r}(0,T; W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})$ with the estimate

$$\|f\|_{L^{\alpha,r}(0,T;W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})} \le \|u_t\|_{L^{\alpha,r}(0,T;W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})} + \|Au\|_{L^{\alpha,r}(0,T;W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})}.$$

As $s < s_0$ by Lemma 1 (ii), we have that

$$\begin{aligned}\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}f(\tau)d\tau\right\|_{W\dot{K}^{\gamma}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1}} &\le \int_0^t \left\|e^{(t-\tau)\Delta}\mathbb{P}f(\tau)\right\|_{W\dot{K}^{\gamma}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1}} d\tau \\ &\le C\int_0^t (t-\tau)^{-(s_0-s)/2}\|f(\tau)\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}} d\tau \\ &\le C\int_0^t (t-\tau)^{-(\delta-\gamma)/2}\|f(\tau)\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}} d\tau \\ &\le C\int_0^t (t-\tau)^{\sigma-1}\|f(\tau)\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}} d\tau, \end{aligned} \tag{20}$$

with $\sigma - 1 = -\frac{1}{2}(s_0 - s)$. Since $2/\alpha_0 - s_0 = 2/\alpha - s - 2$ and $1 < \alpha < \alpha_0$, we obtain

$$\sigma = 1 - \frac{1}{2}(s_0 - s) = \frac{1}{\alpha} - \frac{1}{\alpha_0} < \frac{1}{\alpha},$$

which yields $0 < \sigma < 1$ and $1 < \alpha < 1/\sigma$. Being $1/\alpha_0 = 1/\alpha - \sigma$ and applying Proposition 3.1 from [10] we obtain the following inequality

$$\begin{aligned}\left\|\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}f(\tau)d\tau\right\|_{W\dot{K}^{\gamma}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1}}\right\|_{L^{\alpha_0,r}(0,T)} &\le C\left\|\|f\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}}\right\|_{L^{\alpha,r}(0,T)} \\ &\le C\left\|\|f\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}}\right\|_{L^{\alpha,r}(0,T)}. \end{aligned} \tag{21}$$

Then we obtain

$$\begin{aligned}\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}f(\tau)d\tau\right\|_{L^{\alpha_0,r}(0,T;W\dot{K}^{\gamma}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1})} &\le C\Big(\|u_t\|_{L^{\alpha,r}(0,T;W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})} \\ &\quad + \|Au\|_{L^{\alpha,r}(0,T;W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})}\Big),\end{aligned}$$

where $C = C(n, q, s, \alpha, s_0, \alpha_0, r, \delta, \gamma)$ is independent of T. Now we need to show that $e^{t\Delta}\mathbb{P}u_0 \in L^{\alpha_0,r}(W\dot{K}^{\gamma}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1})$ for $u_0 \in W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k}_{\mu,r}$. We assume that

$$p \le q,\ k = (n-\mu)/p - (2/\alpha + (n-\mu)/q - s - 2) < s_0,$$

and hence there exist k_1, k_2 and $0 < \theta < 1$ with $k_1 < k < k_2 < s_0$ such that $k = (1-\theta)k_1 + \theta k_2$. Lemma 1 (iii) implies

$$\|e^{t\Delta}\mathbb{P}u_0\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1}} \le Ct^{-\frac{1}{2}\left(\frac{n-\mu}{p} - \frac{n-\mu}{q}\right) - \frac{1}{2}(s_0 - k_i)}\|u_0\|_{W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k_i}_{\mu,\infty}},$$

where $i = 1,2$ for any $t > 0$. The real interpolations $\left(W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k_1}_{\mu,\infty}, W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k_2}_{\mu,\infty}\right)_{\theta,r} = W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k}_{\mu,r}$ and $(L^{\alpha_1,\infty}(0,\infty), L^{\alpha_2,\infty}(0,\infty))_{\theta,r} = L^{\alpha_0,r}(0,\infty)$ implied by $(1-\theta)\alpha_1 + \theta\alpha_2 = \frac{1}{2}((n-\mu)/p - (n-\mu)/q) + \frac{1}{2}(s_0 - k) = \frac{1}{\alpha_0}$ lead to the following estimate:

$$\left\| \left\| e^{t\Delta}\mathbb{P}u_0 \right\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1}} \right\|_{L^{\alpha_0,r}(0,\infty)} \leq C\|u_0\|_{W\dot{K}^{\delta}_{p,l}\dot{\mathcal{N}}^{k}_{\mu,r}},$$

where $C = C(n,q,\alpha,s,\alpha_0,s_0,r,\gamma,\delta)$. This proves Theorem 1.

5. Proofs of the Theorem 2

Let $\alpha < \alpha_0 < \infty$ and $s_0 \in \mathbb{R}$ so that $\alpha_0 = 2\alpha$, $max\{(n-\mu)/p - 1/\alpha, s+2-1/\alpha\} < (n-\mu)/q$, $max\{s+1, (n-\mu)/p - 1\} < s_0$ and

$$2/\alpha_0 + (n-\mu)/q - s_0 = 1.$$

Since $0 < s+1 < s_0$ by hypothesis, then there exists $\sigma \in \mathbb{R}$ such that

$$0 < \sigma < s_0 - (s+1).$$

As in [10], we obtain

$$\|\mathbb{P}\nabla\cdot(u\otimes u)\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty}} = \|\mathbb{P}(u\otimes u)\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s+1}_{\mu,\infty}}$$

$$\leq C\|u\otimes u\|_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s+1}_{\mu,\infty}} \leq C\|u\|^2_{W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,\infty}},$$

namely

$$\|\mathbb{P}\nabla\cdot(u\otimes u)\|_{L^{\alpha,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})} \leq C\|u\|^2_{L^{\alpha_0,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,\infty})},$$

where $C = C(n,\mu,q,\alpha,\alpha_0,r,\delta)$ is a constant independent of $0 < T \leq \infty$. Since $p \leq q$, $(n-\mu)/p - 1 < s_0$, and $2/\alpha_0 + (n-\mu)/q - s_0 = 1$, we have

$$(n-\mu)/q \leq (n-\mu)/p < s_0 + 1 = 2/\alpha_0 + (n-\mu)/q,$$

$$k = 2 + (n-\mu)/p - (2/\alpha + (n-\mu)/q - s) = -1 + (n-\mu)/p.$$

Hence from Theorem 1,

$$\|u\|_{L^{\alpha,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,\infty})} \leq C\Big(\|e^{t\Delta}\mathbb{P}u_0\|_{L^{\alpha_0,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s_0}_{\mu,1})}$$

$$+\|u_t\|_{L^{\alpha,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})} + \|Au\|_{L^{\alpha,r}(W\dot{K}^{\delta}_{q,l}\dot{\mathcal{N}}^{s}_{\mu,\infty})}\Big),$$

where $C = C(n,\mu,q,\alpha,\alpha_0,r,\delta)$ independent of $0 < T \leq \infty$. This proves Theorem 2.

6. Conclusions

This article focused on mild solutions of the incompressible Navier–Stokes equations with external forces on $\mathbb{R}^n$ for $n \geq 2$ on Herz-type Besov–Morrey spaces. We introduced real interpolations on $W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}$ and $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ and discussed some useful properties, which were proved in Theorem 3. The inequalities in Lemma 1 were extended from $W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$, $\mathcal{M}^{s}_{\mu}(\mathbb{R}^n)$, and $\dot{\mathcal{N}}^{s}_{p,q,r}(\mathbb{R}^n)$ into $W\dot{K}^{\alpha}_{p,q}\mathcal{M}^{s}_{\mu}(\mathbb{R}^n)$ and $W\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}(\mathbb{R}^n)$. Applying such properties, we achieved some estimates for mild solutions of Navier–Stokes equations, described in Theorems 1 and 2.

The function spaces theory propagates not only for nonlinear PDEs and abstract harmonic analysis, but for global and geometric analysis. For example, Besov and Triebel–Lizorkin spaces are defined on the Riemannian manifold, Lie groups, and fractals. Weak

Herz-type Besov–Morrey spaces can be applied, for instance, in Riemannian geometry, global and geometric analysis, pseudo-differential operator theory, and approximation theory.

The provided estimates can be helpful to explore mild solutions of Navier–Stokes equations and imply the existence and uniqueness of weak and strong solutions. Theorems 1.2–1.4 from [10] show the uniqueness of strong solutions for Navier–Stokes equations, from properties of mild solutions on Besov–Morrey spaces. Future works could focus on obtaining some features of weak Herz-type Besov–Morrey spaces, such as their interpolations, atomic decompositions, and representation via finite differences. Combining (weak) Herz and Triebel–Lizorkin–Morrey spaces may be useful for further studying nonlinear PDEs.

Author Contributions: Conceptualization, R.A.; Formal analysis, R.A.; Funding acquisition, P.L.; Investigation, R.A.; Methodology, P.L.; Project administration, P.L.; Resources, R.A.; Supervision, P.L. Writing—original draft, R.A.; Writing—review & editing, P.L. All authors have read and agreed to the published version of the manuscript.

Funding: The work of Ruslan Abdulkadirov is supported by the North-Caucasus Center for Mathematical Research under agreement №075-02-2022-892 with the Ministry of Science and Higher Education of the Russian Federation. The work of Pavel Lyakhov is supported by the Ministry of Science and Higher Education of the Russian Federation 'Goszadanie' №075-01024-21-02 from 29 September 2021 (project FSEE-2021-0015).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the anonymous reviewers for their valuable comments, which significantly improved the quality of the article.

Conflicts of Interest: The authors declare no conflict of interest.

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