Article

Approximation of GBS Type q -Jakimovski-Leviatan-Beta Integral Operators in Bögel Space

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Abstract: In the present article, we introduce the bivariate variant of Beta integral type operators based on Appell polynomials via q -calculus. We study the local and global type approximation properties for these new operators. Next, we introduce the GBS form for these new operators and then study the degree of approximation by means of modulus of smoothness, mixed modulus of smoothness and Lipschitz class of Bögel continuous functions.

Keywords: Szász operators; q -integers; Appell polynomial; Jakimovski–Leviatan operators; simultaneous approximation; mixed-modulus of continuity; Bögel functions

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1. Introduction and Preliminaries

In 1950, very famous mathematician, Szász, introduced the operators known as Szász positive linear operators [1]. Suppose $v \in [0, \infty)$ and the class of all continuous functions on $[0, \infty)$ is $C[0, \infty)$, then, for all $f \in C[0, \infty)$, the Szász operators are defined as:

$$S_r(f; v) = e^{-rv} \sum_{k=0}^{\infty} \frac{(rv)^k}{k!} f\left(\frac{k}{r}\right). \quad (1)$$

Later, the development of Szász operators introduced by the identity of Appell polynomials [2] and an advance technique of Appell polynomials were been addressed by Jakimovski and Leviatan in 1969 [3] by:

$$X(v)e^{vx} = \sum_{k=0}^{\infty} \beta_k(x)v^k, \quad (2)$$

where $\beta_k(x) = \sum_{i=0}^k \alpha_i \frac{x^{m-i}}{(m-i)!}$ for $m \in \mathbb{N}$ and $X(v) = \sum_{k=0}^{\infty} \alpha_k v^k$, $X(1) \neq 0$. Let $E[0, \infty)$ denote the set of functions defined by $[0, \infty)$ such that $|f(x)| \leq \kappa e^{\gamma x}$, where κ, γ are positive constants. For further development, we prefer to see the Jakimovski–Leviatan types and other related articles [4–10].

The family of q -Appell polynomials introduced by Al-Salam (see [11,12]) by using the generating functions $X_q(t) = \sum_{m=0}^{\infty} X_{m,q} \frac{t^m}{[m]_q!}$, $X_q(1) \neq 0$, such that:

$$X_{m,q}(x) = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q A_{m-k,q} x^k, \quad (m \in \mathbb{N}), \quad (3)$$

where the q -differential $D_{q,x}(X_{m,q}(x)) = [m]_q X_{m-1,q}(x)$, $m = 1, 2, \dots$ and notice that $D_{q,x}(X_{1,q}(x)) = [1]_q X_{0,q}(x) = X_{0,q}$, with $X_{0,q}$ is a positive constant. In addition, the

generating function satisfies the equality $X_q(t)e_q(tx) = \sum_{m=0}^{\infty} X_{m,q}(x) \frac{t^m}{[m]_q!}$, $0 < q < 1$. Here, we note some basic formulae of q -calculus, thus, the q -integer is given by:

$$[\eta]_q = \begin{cases} \frac{1-q^\eta}{1-q}, & q \neq 1 \\ \eta, & q = 1 \end{cases} \text{ for } \eta \in \mathbb{N} \text{ and } [0]_q = 0. \quad (4)$$

For $|q| < 1$, the q -factorial $[\eta]_q!$ is defined by:

$$[\eta]_q! = \begin{cases} 1, & (\eta = 0) \\ \prod_{k=1}^{\eta} [k]_q, & (\eta \in \mathbb{N}). \end{cases} \quad (5)$$

In the standard approach, the exponential functions for q -calculus:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad (6)$$

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{[k]_q!}, \quad (7)$$

while the q -Jackson improper integral of the function f is given as:

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{m \in \mathbb{N}} f\left(\frac{q^m}{A}\right) \frac{q^m}{A}, \quad A \in \mathbb{R} - \{0\}, \quad (8)$$

where $\mathbb{R}$ is the set of real numbers.

The q -beta function $B_q(l, m)$ is defined as follows:

$$B_q(l, m) \int_0^1 t^{l-1} (1-qt)_q^{m-1} d_q t \quad (l > 0; m > 0) \quad (9)$$

and for $l > 1; m > 0$:

$$B_q(l, m) = \mathcal{K}(A, l) \int_0^{\infty/A} \frac{t^{l-1}}{(1+t)_q^{l+m}} d_q t = \frac{[l-1]_q}{[m]_q} B_q(l-1, m+1) \quad (10)$$

with the explicit form of:

$$\mathcal{K}(A, l+1) = q^l \mathcal{K}(A, l).$$

$$\mathcal{K}(A, l) = \frac{1}{A+1} A^l \left(1 + \frac{1}{A}\right)_q^l (1+A)_q^{1-l}, \quad (11)$$

while the Gamma function in the q -form is defined as follows:

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1} E_q(-qt) d_q t \quad (x > 0) \quad (12)$$

satisfying $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$, $\Gamma_q(1) = 1$, and

$$\Gamma_q(x) = \int_0^{\infty/1-q} t^{x-1} E_q(-qt) d_q t \quad (x > 0) \quad (13)$$

and a relation of the Beta and Gamma functions given as:

$$B_q(l, m) = \frac{\Gamma_q(l)\Gamma_q(m)}{\Gamma_q(l+m)}. \quad (14)$$

In the present context, we consider the recent development of the published article [13] and then construct the operators in terms of two variables supposing $0 \leq v_1, v_2 < \infty$. Finally, we study the approximation results in terms bivariate functions by modulus and mixed-modulus of continuity and Lipschitz maximal functions. Next, we also construct the GBS-type polynomial functions and obtain the approximation in terms of Bögel continuous functions. For more related concepts on these classes of functions, we prefer to see the recent published article by Nasiruzzaman et al. [14–19]. In addition, there are various operators in several functional spaces given by the authors: Mohiuddine et al. [20–23], Mursaleen et al. [24,25], Acar et al. [26], Kajla et al. [27], Özger et al. [28], and Rao et al. [29,30].

2. Operators and Their Associated Moments

Take $\mathcal{M}^2 = \{(v_1, v_2) : 0 \leq v_1 < \infty, 0 \leq v_2 < \infty\}$, and $C(\mathcal{M}^2)$ is the class of all continuous functions on $\mathcal{M}^2$ and satisfies the norm by $\|g\|_{C(\mathcal{M}^2)} = \sup_{(v_1, v_2) \in \mathcal{M}^2} |f(v_1, v_2)|$. Suppose $f \in C(\mathcal{M}^2)$. Then, for all $i = 1, 2$ such that $m_i \in \mathbb{N}$, $X_{r_i, q_i}([m_i]_{q_i} v_i) \geq 0$ and $X_{q_i}(1) \neq 0$, we define:

$$\mathcal{J}_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) = \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1}(v_1) \mathcal{V}_{m_2, q_2}^{r_2}(v_2) \int_0^{\infty/A_1} \int_0^{\infty/A_2} \mathcal{T}_{m_1, m_2}^{q_1, q_2}(t, s) f(q_1^{r_1} t, q_2^{r_2} s) d_{q_1} t d_{q_2} s, \quad (15)$$

where

$$\mathcal{U}_{m_1, r_1}^{q_1}(v_1) = \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \frac{R_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} \frac{\mathcal{K}_1(A_1, r_1 + 1)}{B_{q_1}(r_1 + 1, m_1)}, \quad (16)$$

$$\mathcal{V}_{m_2, r_2}^{q_2}(v_2) = \frac{e_{q_2}(-[m_2]_{q_2} v_2)}{X_{q_2}(1)} \frac{R_{r_2, q_2}([m_2]_{q_2} v_2)}{[r_2]_{q_2}!} \frac{\mathcal{K}_2(A_2, r_2 + 1)}{B_{q_2}(r_2 + 1, m_2)} \quad (17)$$

and

$$\mathcal{T}_{m_1, m_2}^{q_1, q_2}(t, s) = \frac{t^{r_1}}{(1+t)_{q_1}^{r_1+m_1+1}} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}}. \quad (18)$$

It is noticed that for all $i = 1, 2$, and $r_i > 1$, $m_i > 0$ we have:

$$B_{q_i}(r_i, m_i) = \mathcal{K}_i(A_i, r_i) \int_0^{\infty/A_i} \frac{u^{r_i-1}}{(1+u_i)_{q_i}^{r_i+m_i}} d_{q_i} u_i = \frac{[r_i-1]_{q_i}}{[m_i]_{q_i}} B_{q_i}(r_i-1, m_i+1), \quad (19)$$

with

$$\mathcal{K}_i(A_i, r_i+1) = q_i^{r_i} \mathcal{K}_i(A_i, r_i),$$

$$\mathcal{K}_i(A_i, r_i) = q_i^{\frac{r_i(r_i-1)}{2}}, \quad \mathcal{K}_i(A_i, 0) = 1.$$

Lemma 1. Operators $J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2)$ defined by (15) have the equality:

$$J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) = \mathcal{C}_{m_1, r_1}^{q_1} \left(\mathcal{D}_{m_2, r_2}^{q_2}(f; v_1, v_2) \right) = \mathcal{D}_{m_2, r_2}^{q_2} \left(\mathcal{C}_{m_1, r_1}^{q_1}(f; v_1, v_2) \right) \quad (20)$$

where

$$\mathcal{C}_{m_1, r_1}^{q_1}(f; v_1, v_2) = \sum_{r_1=0}^{\infty} \mathcal{U}_{m_1, r_1}^{q_1}(v_1) \int_0^{\infty/A_1} \frac{t^{r_1}}{(1+t)_{q_1}^{r_1+m_1+1}} f(q_1^{r_1} t, q_2^{r_2} s) d_{q_1} t \quad (21)$$

$$\mathcal{D}_{m_2, r_2}^{q_2}(f; v_1, v_2) = \sum_{r_2=0}^{\infty} \mathcal{V}_{m_2, r_2}^{q_2}(v_2) \int_0^{\infty/A_2} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}} f(q_1^{r_1} t, q_2^{r_2} s) d_{q_2} s. \quad (22)$$

Proof. We easily see that:

$$\begin{aligned}
 \mathcal{C}_{m_1, r_1}^{q_1} \left(\mathcal{D}_{m_2, r_2}^{q_2} (f; v_1, v_2) \right) &= \mathcal{C}_{m_1, r_1}^{q_1} \left(\sum_{r_2=0}^{\infty} \mathcal{V}_{m_2, r_2}^{q_2} (v_2) \int_0^{\infty/A_2} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}} f(q_1^{r_1} t, q_2^{r_2} s) d_{q_2} s \right) \\
 &= \sum_{r_2=0}^{\infty} \mathcal{V}_{m_2, r_2}^{q_2} (v_2) \mathcal{C}_{m_1, r_1}^{q_1} \left(\int_0^{\infty/A_2} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}} f(q_1^{r_1} t, q_2^{r_2} s) d_{q_2} s \right) \\
 &= \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1} (v_1) \mathcal{V}_{m_2, q_2}^{r_2} (v_2) \int_0^{\infty/A_1} \int_0^{\infty/A_2} \mathcal{T}_{m_1, m_2}^{q_1, q_2} (t, s) f(q_1^{r_1} t, q_2^{r_2} s) d_{q_1} t d_{q_2} s \\
 &= J_{m_1, m_2}^{q_1, q_2} (f; v_1, v_2).
 \end{aligned}$$

On the other hand, we obtain $\mathcal{D}_{m_2, r_2}^{q_2} \left(\mathcal{C}_{m_1, r_1}^{q_1} (f; v_1, v_2) \right) = J_{m_1, m_2}^{q_1, q_2} (f; v_1, v_2)$. $\square$

Lemma 2. For all $i = 1, 2$, if we take $R_{r_i, q_i}([m_i]_{q_i} v_i) \geq 0$ and $X_{q_i}(1) \neq 0$, then for each $m_i \in \mathbb{N}$, we have:

$$\begin{aligned}
 \sum_{r_i=0}^{\infty} \frac{X_{r_i, q_i}([m_i]_{q_i} v_i)}{[r_i]_{q_i}!} &= X_{q_i}(1) e_{q_i}([m_i]_{q_i} v_i), \\
 \sum_{r_i=0}^{\infty} r_i \frac{X_{r_i, q_i}([m_i]_{q_i} v_i)}{[r_i]_{q_i}!} &= \left[[m_i]_{q_i} X_{q_i}(1) v_i + X'_{q_i}(1) \right] e_{q_i}([m_i]_{q_i} v_i), \\
 \sum_{r_i=0}^{\infty} r_i^2 \frac{X_{r_i, q_i}([m_i]_{q_i} v_i)}{[r_i]_{q_i}!} &= \left[[m_i]_{q_i}^2 X_{q_i}(1) v_i^2 + 2[m_i]_{q_i} X'_{q_i}(1) v_i + X''_{q_i}(1) \right] e_{q_i}([m_i]_{q_i} v_i), \\
 \sum_{r_i=0}^{\infty} r_i^3 \frac{X_{r_i, q_i}([m_i]_{q_i} v_i)}{[r_i]_{q_i}!} &= \left[[m_i]_{q_i}^3 X_{q_i}(1) v_i^3 + 3[m_i]_{q_i}^2 X'_{q_i}(1) v_i^2 + 3[m_i]_{q_i} X''_{q_i}(1) v_i + X'''_{q_i}(1) \right] e_{q_i}([m_i]_{q_i} v_i), \\
 \sum_{r_i=0}^{\infty} r_i^4 \frac{X_{r_i, q_i}([m_i]_{q_i} v_i)}{[r_i]_{q_i}!} &= \left[[m_i]_{q_i}^4 X_{q_i}(1) v_i^4 + 4[m_i]_{q_i}^3 X'_{q_i}(1) v_i^3 + 6[m_i]_{q_i}^2 X''_{q_i}(1) v_i^2 + 4[m_i]_{q_i} X'''_{q_i}(1) v_i + X^{(4)}_{q_i}(1) \right] e_{q_i}([m_i]_{q_i} v_i).
 \end{aligned}$$

Lemma 3. Let $i, j = \{0, 1, 2, 3, 4\}$ and $f(t, s) = \lambda_{i,j}$ be the test functions such that $\lambda_{i,j} = t^i s^j$. Then, for every $m_1, m_2 \in \mathbb{N} \setminus \{1, 2, 3, 4\}$, operators $J_{m_1, m_2}^{q_1, q_2}(\cdot; \cdot)$ have the following equalities:

$$\begin{aligned}
 (1) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2) &= \mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{0,0}; v_1, v_2) \\
 &= \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{0,0}; v_1, v_2) = 1; \\
 (2) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{1,0}; v_1, v_2) &= \mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) \\
 &= \frac{1}{q_1[m_1-1]_{q_1}} + \frac{1}{[m_1-1]_{q_1}} \left([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)} \right); \\
 (3) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,1}; v_1, v_2) &= \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2) \\
 &= \frac{1}{q_2[m_2-1]_{q_2}} + \frac{1}{[m_2-1]_{q_2}} \left([m_2]_{q_2} v_2 + \frac{X'_{q_2}(1)}{X_{q_2}(1)} \right); \\
 (4) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{2,0}; v_1, v_2) &= \mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{2,0}; v_1, v_2) \\
 &= \frac{(1+q_1)}{q_1^3[m_1-1]_{q_1}[m_1-2]_{q_1}} + \frac{(1+2q_1)}{q_1^2[m_1-1]_{q_1}[m_1-2]_{q_1}} \left([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{[m_1 - 1]_{q_1} [m_1 - 2]_{q_1}} \left([m_1]_{q_1}^2 v_1^2 + \frac{2[m_1]_{q_1} X'_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X''_{q_1}(1)}{X_{q_1}(1)} \right); \\
(5) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,2}; v_1, v_2) &= \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{0,2}; v_1, v_2) \\
&= \frac{(1 + q_2)}{q_2^3 [m_2 - 1]_{q_2} [m_2 - 2]_{q_2}} + \frac{(1 + 2q_2)}{q_2^2 [m_2 - 1]_{q_2} [m_2 - 2]_{q_2}} \left([m_2]_{q_2} v_2 + \frac{X'_{q_2}(1)}{X_{q_2}(1)} \right) \\
&+ \frac{1}{[m_2 - 1]_{q_2} [m_2 - 2]_{q_2}} \left([m_2]_{q_2}^2 v_2^2 + \frac{2[m_2]_{q_2} X'_{q_2}(1)}{X_{q_2}(1)} v_2 + \frac{X''_{q_2}(1)}{X_{q_2}(1)} \right); \\
(6) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{3,0}; v_1, v_2) &= \mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{3,0}; v_1, v_2) \\
&= \frac{1 + 2q_1 + 2q_1^2 + q_1^3}{q_1^4 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1}} \\
&+ \frac{(1 + 3q_1 + 4q_1^2 + 3q_1^3)}{q_1^3 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1}} \left(([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)}) \right) \\
&+ \frac{(1 + 2q_1 + 3q_1^2)}{q_1 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1}} \left([m_1]_{q_1}^2 v_1^2 + \frac{2[m_1]_{q_1} X'_{q_1}(1)}{X_{q_1}(1)} v_1 \right. \\
&\left. + \frac{X''_{q_1}(1)}{X_{q_1}(1)} \right) + \frac{q_1^2}{[m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1}} \left([m_1]_{q_1}^3 v_1^3 \right. \\
&\left. + 3[m_1]_{q_1}^2 \frac{X'_{q_1}(1)}{X_{q_1}(1)} v_1^2 + 3[m_1]_{q_1} \frac{X''_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X'''_{q_1}(1)}{X_{q_1}(1)} \right); \\
(7) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,3}; v_1, v_2) &= \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{0,3}; v_1, v_2) \\
&= \frac{1 + 2q_2 + 2q_2^2 + q_2^3}{q_2^4 [m_2 - 1]_{q_2} [m_2 - 2]_{q_2} [m_2 - 3]_{q_2}} \\
&+ \frac{(1 + 3q_2 + 4q_2^2 + 3q_2^3)}{q_2^3 [m_2 - 1]_{q_2} [m_2 - 2]_{q_2} [m_2 - 3]_{q_2}} \left(([m_2]_{q_2} v_2 + \frac{X'_{q_2}(1)}{X_{q_2}(1)}) \right) \\
&+ \frac{(1 + 2q_2 + 3q_2^2)}{q_2 [m_2 - 1]_{q_2} [m_2 - 2]_{q_2} [m_2 - 3]_{q_2}} \left([m_2]_{q_2}^2 v_2^2 + \frac{2[m_2]_{q_2} X'_{q_2}(1)}{X_{q_2}(1)} v_2 \right. \\
&\left. + \frac{X''_{q_2}(1)}{X_{q_2}(1)} \right) + \frac{q_2^2}{[m_2 - 1]_{q_2} [m_2 - 2]_{q_2} [m_2 - 3]_{q_2}} \left([m_2]_{q_2}^3 v_2^3 \right. \\
&\left. + 3[m_2]_{q_2}^2 \frac{X'_{q_2}(1)}{X_{q_2}(1)} v_2^2 + 3[m_2]_{q_2} \frac{X''_{q_2}(1)}{X_{q_2}(1)} v_2 + \frac{X'''_{q_2}(1)}{X_{q_2}(1)} \right); \\
(8) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{4,0}; v_1, v_2) &= \mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{4,0}; v_1, v_2) \\
&= \frac{1 + 3q_1 + 5q_1^2 + 6q_1^3 + 5q_1^4 + 3q_1^5 + q_1^6}{q_1^5 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1} [m_1 - 4]_{q_1}} \\
&+ \frac{(1 + 5q_1 + 10q_1^2 + 13q_1^3 + 12q_1^4 + 7q_1^5 + 2q_1^6)}{q_1^3 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1} [m_1 - 4]_{q_1}} \left([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)} \right) \\
&+ \frac{(1 + 3q_1 + 7q_1^2 + 9q_1^3 + 9q_1^4 + 6q_1^5)}{q_1 [m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1} [m_1 - 4]_{q_1}} \left([m_1]_{q_1}^2 v_1^2 \right. \\
&\left. + \frac{2[m_1]_{q_1} X'_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X''_{q_1}(1)}{X_{q_1}(1)} \right) \\
&+ \frac{(q_1^2 + 2q_1^3 + 2q_1^4 + 2q_1^5 + q_1^6 + 2q_1^7)}{[m_1 - 1]_{q_1} [m_1 - 2]_{q_1} [m_1 - 3]_{q_1} [m_1 - 4]_{q_1}} \left([m_1]_{q_1}^3 v_1^3 \right. \\
&\left. + 3[m_1]_{q_1}^2 \frac{X'_{q_1}(1)}{X_{q_1}(1)} v_1^2 + 3[m_1]_{q_1} \frac{X''_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X'''_{q_1}(1)}{X_{q_1}(1)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{q_1^6}{[m_1-1]_{q_1}[m_1-2]_{q_1}[m_1-3]_{q_1}[m_1-4]_{q_1}} \left([m_1]_{q_1}^4 v_1^4 + 4[m_1]_{q_1}^3 \frac{X'_{q_1}(1)}{X_{q_1}(1)} v_1^3 \right. \\
& \left. + 6[m_1]_{q_1}^2 \frac{X''_{q_1}(1)}{X_{q_1}(1)} v_1^2 + 4[m_1]_{q_1} \frac{X'''_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X^{(4)}_{q_1}(1)}{X_{q_1}(1)} \right); \\
(9) \quad J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,4}; v_1, v_2) &= \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{0,4}; v_1, v_2) \\
&= \frac{1 + 3q_2 + 5q_2^2 + 6q_2^3 + 5q_2^4 + 3q_2^5 + q_2^6}{q_2^5[m_2-1]_{q_2}[m_2-2]_{q_2}[m_2-3]_{q_2}[m_2-4]_{q_2}} \\
&+ \frac{(1 + 5q_2 + 10q_2^2 + 13q_2^3 + 12q_2^4 + 7q_2^5 + 2q_2^6)}{q_2^3[m_2-1]_{q_2}[m_2-2]_{q_2}[m_2-3]_{q_2}[m_2-4]_{q_2}} \left([m_2]_{q_2} v_2 + \frac{X'_{q_2}(1)}{X_{q_2}(1)} \right) \\
&+ \frac{(1 + 3q_2 + 7q_2^2 + 9q_2^3 + 9q_2^4 + 6q_2^5)}{q_2[m_2-1]_{q_2}[m_2-2]_{q_2}[m_2-3]_{q_2}[m_2-4]_{q_2}} \left([m_2]_{q_2}^2 v_2^2 \right. \\
&\left. + \frac{2[m_2]_{q_2} X'_{q_2}(1)}{X_{q_2}(1)} v_2 + \frac{X''_{q_2}(1)}{X_{q_2}(1)} \right) \\
&+ \frac{(q_2^2 + 2q_2^3 + 2q_2^4 + 2q_2^5 + q_2^6 + 2q_2^7)}{[m_2-1]_{q_2}[m_2-2]_{q_2}[m_2-3]_{q_2}[m_2-4]_{q_2}} \left([m_2]_{q_2}^3 v_2^3 \right. \\
&\left. + 3[m_2]_{q_2}^2 \frac{X'_{q_2}(1)}{X_{q_2}(1)} v_2^2 + 3[m_2]_{q_2} \frac{X''_{q_2}(1)}{X_{q_2}(1)} v_2 + \frac{X'''_{q_2}(1)}{X_{q_2}(1)} \right) \\
&+ \frac{q_2^6}{[m_2-1]_{q_2}[m_2-2]_{q_2}[m_2-3]_{q_2}[m_2-4]_{q_2}} \left([m_2]_{q_2}^4 v_2^4 + 4[m_2]_{q_2}^3 \frac{X'_{q_2}(1)}{X_{q_2}(1)} v_2^3 \right. \\
&\left. + 6[m_2]_{q_2}^2 \frac{X''_{q_2}(1)}{X_{q_2}(1)} v_2^2 + 4[m_2]_{q_2} \frac{X'''_{q_2}(1)}{X_{q_2}(1)} v_2 + \frac{X^{(4)}_{q_2}(1)}{X_{q_2}(1)} \right).
\end{aligned}$$

Proof. For $i, j = 0$, we take:

$$\begin{aligned}
J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2) &= \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1}(v_1) \mathcal{V}_{m_2, q_2}^{r_2}(v_2) \int_0^{\infty/A_1} \int_0^{\infty/A_2} \mathcal{T}_{m_1, m_2}^{q_1, q_2}(t, s) d_{q_1} t d_{q_2} s \\
&= \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1}(v_1) \mathcal{V}_{m_2, q_2}^{r_2}(v_2) \int_0^{\infty/A_1} \frac{t^{r_1}}{(1+t)_{q_1}^{r_1+m_1+1}} d_{q_1} t \int_0^{\infty/A_2} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}} d_{q_2} s \\
&= \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} \frac{B_{q_1}(r_1+1, m_1)}{B_{q_1}(r_1+1, m_1)} \\
&\times \frac{e_{q_2}(-[m_2]_{q_2} v_2)}{X_{q_2}(1)} \sum_{r_2=0}^{\infty} \frac{X_{r_2, q_2}([m_2]_{q_2} v_2)}{[r_2]_{q_2}!} \frac{B_{q_2}(r_2+1, m_2)}{B_{q_2}(r_2+1, m_2)} \\
&= 1.
\end{aligned}$$

For $i = 1, j = 0$, we obtain:

$$\begin{aligned}
J_{m_1, m_2}^{q_1, q_2}(\lambda_{1,0}; v_1, v_2) &= \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1}(v_1) \mathcal{V}_{m_2, q_2}^{r_2}(v_2) \int_0^{\infty/A_1} \frac{(q_1 t)^{r_1}}{(1+t)_{q_1}^{r_1+m_1+1}} d_{q_1} t \int_0^{\infty/A_2} \frac{s^{r_2}}{(1+s)_{q_2}^{r_2+m_2+1}} d_{q_2} s \\
&= \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} q_1^{r_1} \frac{\mathcal{K}_1(A_1, r_1+1)}{B_{q_1}(r_1+1, m_1)} \int_0^{\infty/A_1} \frac{t^{r_1+1}}{(1+t)_{q_1}^{r_1+m_1+1}} d_{q_1} t \\
&\times \frac{e_{q_2}(-[m_2]_{q_2} v_2)}{X_{q_2}(1)} \sum_{r_2=0}^{\infty} \frac{X_{r_2, q_2}([m_2]_{q_2} v_2)}{[r_2]_{q_2}!} \frac{B_{q_2}(r_2+1, m_2)}{B_{q_2}(r_2+1, m_2)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} q_1^{r_1} \frac{\mathcal{K}_1(A_1, r_1 + 1)}{B_{q_1}(r_1 + 1, m_1)} \frac{B_{q_1}(r_1 + 2, m_1 - 1)}{\mathcal{K}_1(A_1, r_1 + 2)} \\
&= \frac{1}{q_1[m_1 - 1]_{q_1}} \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} [r_1 + 1]_{q_1}.
\end{aligned}$$

By applying:

$$[r_1 + 1]_{q_1} = 1 + q_1[r_1]_{q_1}, \quad (23)$$

$$\begin{aligned}
J_{m_1, m_2}^{q_1, q_2}(\lambda_{1,0}; v_1, v_2) &= \frac{1}{q_1[m_1 - 1]_{q_1}} \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} \\
&+ \frac{1}{[m_1 - 1]_{q_1}} \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} [r_1]_{q_1} \\
&= \frac{1}{q_1[m_1 - 1]_{q_1}} + \frac{1}{[m_1 - 1]_{q_1}} \left([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)} \right)
\end{aligned}$$

For $i = 2, j = 0$, we have:

$$\begin{aligned}
J_{m_1, m_2}^{q_1, q_2}(\lambda_{2,0}; v_1, v_2) &= \frac{e_q(-[m]_q x)}{X_q(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m - 1]_{q_1} v_1)}{[r_1]_{q_1}!} q_1^{2r_1} \frac{\mathcal{K}_1(A_1, r_1 + 1)}{B_{q_1}(r_1 + 1, m_1)} \int_0^{\infty/A_1} \frac{t^{r_1+2}}{(1+t)_{q_1}^{r_1+m_1+1}} d_{q_1} t \\
&\times \frac{e_{q_2}(-[m_2]_{q_2} v_2)}{X_{q_2}(1)} \sum_{r_2=0}^{\infty} \frac{X_{r_2, q_2}([m_2]_{q_2} v_2)}{[r_2]_{q_2}!} \frac{B_{q_2}(r_2 + 1, m_2)}{B_{q_2}(r_2 + 1, m_2)} \\
&= \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} q_1^{2r_1} \frac{\mathcal{K}_1(A_1, r_1 + 1)}{\mathcal{K}_1(A_1, r_1 + 3)} \frac{B_{q_1}(r_1 + 3, m_1 - 2)}{B_{q_1}(r_1 + 1, m_1)} \\
&= \frac{1}{q_1^3[m_1 - 1]_{q_1}[m_1 - 2]_{q_1}} \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} [r_1 + 2]_{q_1} [r_1 + 1]_{q_1}.
\end{aligned}$$

We use the equality (23) and $[r_1 + 2]_{q_1} = 1 + q_1 + q_1^2[r_1]_{q_1}$:

$$\begin{aligned}
J_{m_1, m_2}^{q_1, q_2}(\lambda_{2,0}; v_1, v_2) &= \frac{1}{q_1^3[m_1 - 1]_{q_1}[m_1 - 2]_{q_1}} \frac{e_{q_1}(-[m_1]_{q_1} v_1)}{X_{q_1}(1)} \sum_{r_1=0}^{\infty} \frac{X_{r_1, q_1}([m_1]_{q_1} v_1)}{[r_1]_{q_1}!} \\
&\left((1 + q_1) + q_1(1 + 2q_1)[r_1]_{q_1} + q_1^3[r_1]_{q_1}^2 \right) \\
&= \frac{(1 + q_1)}{q_1^3[m_1 - 1]_{q_1}[m_1 - 2]_{q_1}} + \frac{(1 + 2q_1)}{q_1^2[m_1 - 1]_{q_1}[m_1 - 2]_{q_1}} \left([m_1]_{q_1} v_1 + \frac{X'_{q_1}(1)}{X_{q_1}(1)} \right) \\
&+ \frac{1}{[m_1 - 1]_{q_1}[m_1 - 2]_{q_1}} \left([m_1]_{q_1}^2 v_1^2 + \frac{2[m_1]_{q_1} X'_{q_1}(1)}{X_{q_1}(1)} v_1 + \frac{X''_{q_1}(1)}{X_{q_1}(1)} \right).
\end{aligned}$$

Similarly, we obtain the other identities. This completes the proof. $\square$

Lemma 4. Take $\Phi_{v_1, v_2}^{c, d}(t, s) = (t - v_1)^c (s - v_2)^d$ for all $c, d = \{0, 1, 2, 3, 4\}$. Then, for any $i = 1, 2$, $R_{r_i, q_i}([m_i]_{q_i} v_i) \geq 0$ and $X_{q_i}(1) \neq 0$, suppose $\left(\delta_{m_1, m_2}^{c, d} \right)^2 = J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{c, d}(t, s); v_1, v_2 \right)$.

Then, for all $i = 1, 2$, with $m_i \in \mathbb{N} \setminus \{1, 2\}$, we obtain:

$$\left(\delta_{m_1, m_2}^{c, d} \right)^2 = \begin{cases} \left(\frac{[m_i]_{q_i}^2}{[m_i - 1]_{q_i} [m_i - 2]_{q_i}} + 1 - \frac{2[m_i]_{q_i}}{[m_i - 1]_{q_i}} \right) v_i^2 \\ + \frac{1}{[m_i - 1]_{q_i}} \left(\frac{(1 + 2q_i)[m_i]_{q_i}}{q_i^2 [m_i - 2]_{q_i}} + \frac{2[m_i]_{q_i}}{[m_i - 2]_{q_i}} \frac{X'_{q_i}(1)}{X_{q_i}(1)} - \frac{2X'_{q_i}(1)}{X_{q_i}(1)} - \frac{2}{q_i} \right) v_i \\ + \frac{1}{q_i^2 [m_i - 1]_{q_i} [m_i - 2]_{q_i}} \left(\frac{(1 + q_i)}{q_i} + (1 + 2q_i) \frac{X'_{q_i}(1)}{X_{q_i}(1)} \right) \\ \text{for } i = 1, c = 2, d = 0 \text{ and } i = 2, d = 2, c = 0 \\ \left(\frac{[m_i]_{q_i}}{[m_i - 1]_{q_i}} - 1 \right) v_i + \frac{1}{[m_i - 1]_{q_i}} \left(\frac{1}{q_i} + \frac{X'_{q_i}(1)}{X_{q_i}(1)} \right) \\ \text{for } i = 1, c = 1, d = 0 \text{ and } i = 2, c = 0, d = 1. \end{cases} \quad (24)$$

Remark 1. Let $m_1, m_2 \in \mathbb{N}$, then, for any $\delta_{m_1}, \delta_{m_2} > 0$, we take:

$$\delta_{m_1, m_2}^{c, d} = \begin{cases} \sqrt{J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{c, d}(t, s); v_1, v_2 \right)} = \delta_{m_1} \text{ (suppose)} \\ \text{for } c = 2 \text{ and } d = 0. \\ \sqrt{J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{c, d}(t, s); v_1, v_2 \right)} = \delta_{m_2} \text{ (suppose)} \\ \text{for } d = 2 \text{ and } c = 0. \end{cases} \quad (25)$$

Lemma 5. For any $v_1, v_2 \in \mathcal{M}^2$ and sufficiently large $m_1, m_2 \in \mathbb{N} \setminus \{1, 2, 3, 4\}$, the operators $J_{m_1, m_2}^{q_1, q_2}(\cdot; \cdot)$ verify the following inequalities:

$$\begin{aligned} (1) \quad J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 0}(t, s); v_1, v_2 \right) &= O \left(\frac{1}{[m_1]_{q_1}^2} \right) (v_1 + 1)^2 \leq M_1 (v_1 + 1)^2 \text{ as } m_1, m_2 \rightarrow \infty; \\ (2) \quad J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 2}(t, s); v_1, v_2 \right) &= O \left(\frac{1}{[m_2]_{q_2}^2} \right) (v_2 + 1)^2 \leq M_2 (v_2 + 1)^2 \text{ as } m_1, m_2 \rightarrow \infty; \\ (3) \quad J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{4, 0}(t, s); v_1, v_2 \right) &= O \left(\frac{1}{[m_1]_{q_1}^4} \right) (v_1 + 1)^4 \leq M_3 (v_1 + 1)^4 \text{ as } m_1, m_2 \rightarrow \infty; \\ (4) \quad J_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 4}(t, s); v_1, v_2 \right) &= O \left(\frac{1}{[m_2]_{q_2}^4} \right) (v_2 + 1)^4 \leq M_4 (v_2 + 1)^4 \text{ as } m_1, m_2 \rightarrow \infty. \end{aligned}$$

3. Approximation in Weighted Space and Degree of Convergence

For any weight function ψ given, such that $\psi(v_1, v_2) = 1 + v_1^2 + v_2^2$, which satisfies the properties $B_\psi(\mathcal{M}^2) = \{f : |f(v_1, v_2)| \leq M_f \psi(v_1, v_2), \quad M_f > 0\}$, where $B_\psi(\mathcal{M}^2)$ is defined for the class of all bounded functions on $\mathcal{M}^2 = [0, \infty) \times [0, \infty)$. In addition, take $C^{(m)}(\mathcal{M}^2)$ as the m -times continuously differentiable functions on $\mathcal{M}^2 = \{(v_1, v_2) \in \mathcal{M}^2 : v_1, v_2 \in [0, \infty)\}$. The norm on B_ψ is equipped by $\|f\|_\psi = \sup_{v_1, v_2 \in \mathcal{M}^2} \frac{|f(v_1, v_2)|}{\psi(v_1, v_2)}$. Moreover, some classified functions are defined as follows:

$$C_\psi^m(\mathcal{M}^2) = \{f : f \in C_\psi(\mathcal{M}^2); \text{ such that } \lim_{(v_1, v_2) \rightarrow \infty} \frac{f(v_1, v_2)}{\psi(v_1, v_2)} = k_f < \infty\}, \quad (26)$$

$$C_\psi^0(\mathcal{M}^2) = \{f : f \in C_\psi^m(\mathcal{M}^2); \text{ such that } \lim_{(v_1, v_2) \rightarrow \infty} \frac{f(v_1, v_2)}{\psi(v_1, v_2)} = 0\}, \quad (27)$$

$$C_\psi(\mathcal{M}^2) = \{f : f \in B_\psi \cap C_\psi(\mathcal{M}^2)\}. \quad (28)$$

For all $f \in C_\psi^0(\mathcal{M}^2)$ and $\delta_{m_1}, \delta_{m_2} > 0$, the weighted modulus of continuity of function f , suppose $\omega_\psi(f; \delta_{m_1}, \delta_{m_2})$ be defined by:

$$\omega_\psi(f; \delta_{m_1}, \delta_{m_2}) = \sup_{(v_1, v_2) \in [0, \infty)} \sup_{0 \leq |\varphi_1| \leq \delta_{m_1}, 0 \leq |\varphi_2| \leq \delta_{m_2}} \frac{|f(v_1 + \varphi_1, v_2 + \varphi_2) - f(v_1, v_2)|}{\psi(v_1, v_2)\psi(\varphi_1, \varphi_2)}, \quad (29)$$

for any $\mu_1, \mu_2 > 0$, one has:

$$\omega_\psi(f; \mu_1 \delta_{m_1}, \mu_2 \delta_{m_2}) \leq 4(1 + \mu_1)(1 + \mu_2)(1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2)\omega_\psi(f; \delta_{m_1}, \delta_{m_2}), \quad (30)$$

$$\begin{aligned} |f(t, s) - f(v_1, v_2)| &\leq \psi(v_1, v_2)\psi(|t - v_1|, |s - v_2|)\omega_\psi(f; |t - v_1|, |s - v_2|) \\ &\leq (1 + v_1^2 + v_2^2)(1 + (t - v_1)^2)(1 + (s - v_2)^2)\omega_\psi(f; |t - v_1|, |s - v_2|). \end{aligned}$$

Theorem 1. Let $f \in C_\psi^0(\mathcal{M}^2)$. Then, for sufficiently large $m_1, m_2 \in \mathbb{N}$, the operators $J_{m_1, m_2}^{q_1, q_2}$ satisfy the inequality:

$$\begin{aligned} \frac{|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)|}{4(1 + v_1^2 + v_2^2)} &\leq \Psi_{v_1, v_2} \left(1 + O([m_1]_{q_1}^{-2})\right) \left(1 + O([m_2]_{q_2}^{-2})\right) \\ &\times \omega_\psi \left(f; O([m_1]_{q_1}^{-1}), O([m_2]_{q_2}^{-1})\right), \end{aligned}$$

where $\Psi_{v_1, v_2} = \left(1 + (v_1 + 1) + M_1(v_1 + 1)^2 + M_1(v_1 + 1)^3\right) \left(1 + (v_2 + 1) + M_2(v_2 + 1)^2 + M_2(v_2 + 1)^3\right)$, $M_1, M_2 > 0$ and $\delta_{m_i} = O\left(\frac{1}{[m_i]_{q_i}}\right)$, for $i = 1, 2$.

Proof. For all $\delta_{m_1}, \delta_{m_2} > 0$, we have:

$$\begin{aligned} |f(t, s) - f(v_1, v_2)| &\leq 4(1 + v_1^2 + v_2^2) \left(1 + (t - v_1)^2\right) \left(1 + (s - v_2)^2\right) \\ &\times \left(1 + \frac{|t - v_1|}{\delta_{m_1}}\right) \left(1 + \frac{|s - v_2|}{\delta_{m_2}}\right) (1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2)\omega_\psi(f; \delta_{m_1}, \delta_{m_2}) \\ &= 4(1 + v_1^2 + v_2^2)(1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2) \\ &\times \left(1 + \frac{|t - v_1|}{\delta_{m_1}} + (t - v_1)^2 + \frac{1}{\delta_{m_1}} |t - v_1| (t - v_1)^2\right) \\ &\times \left(1 + \frac{|s - v_2|}{\delta_{m_2}} + (s - v_2)^2 + \frac{|s - v_2|}{\delta_{m_2}} (s - v_2)^2\right) \omega_\psi(f; \delta_{m_1}, \delta_{m_2}). \end{aligned}$$

On applying the positive linear operator $J_{m_1, m_2}^{q_1, q_2}$ and taking into account the well-known Cauchy–Schwarz inequality:

$$\begin{aligned} \frac{|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)|}{4(1 + v_1^2 + v_2^2)} &\leq J_{m_1, m_2}^{q_1, q_2} \left(\left(1 + \frac{|t - v_1|}{\delta_{m_1}} + (t - v_1)^2 + \frac{1}{\delta_{m_1}} |t - v_1| (t - v_1)^2; v_1, v_2\right) \right. \\ &\times J_{m_1, m_2}^{q_1, q_2} \left(\left(1 + \frac{|s - v_2|}{\delta_{m_2}} + (s - v_2)^2 + \frac{|s - v_2|}{\delta_{m_2}} (s - v_2)^2; v_1, v_2\right) \right) \\ &\times (1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2)\omega_\psi(f; \delta_{m_1}, \delta_{m_2}) \\ &= (1 + v_1^2 + v_2^2)(1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2)\omega_\psi(f; \delta_{m_1}, \delta_{m_2}) \end{aligned}$$

$$\begin{aligned}
& \times \left(1 + \frac{1}{\delta_{m_1}} J_{m_1, m_2}^{q_1, q_2}(|t - v_1|; v_1, v_2) + J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2) \right. \\
& + \left. \frac{1}{\delta_{m_1}} J_{m_1, m_2}^{q_1, q_2}(|t - v_1| (t - v_1)^2; v_1, v_2) \right) \\
& \times \left(1 + \frac{1}{\delta_{m_2}} J_{m_1, m_2}^{q_1, q_2}(|s - v_2|; v_1, v_2) + J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2) \right. \\
& + \left. \frac{1}{\delta_{m_2}} J_{m_1, m_2}^{q_1, q_2}(|s - v_2| (s - v_2)^2; v_1, v_2) \right); \\
\frac{|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)|}{4(1 + v_1^2 + v_2^2)} & \leq (1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2) \omega_\psi(f; \delta_{m_1}, \delta_{m_2}) \\
& \times \left[1 + \frac{1}{\delta_{m_1}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)} + J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2) \right. \\
& + \left. \frac{1}{\delta_{m_1}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)} \sqrt{J_{m_1, m_2}^{q_1, q_2}((t - v_1)^4; v_1, v_2)} \right] \\
& \times \left[1 + \frac{1}{\delta_{m_2}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)} + J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2) \right. \\
& + \left. \frac{1}{\delta_{m_2}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)} \sqrt{J_{m_1, m_2}^{q_1, q_2}((s - v_2)^4; v_1, v_2)} \right].
\end{aligned}$$

Take into account Lemma 5 and for all $i = 1, 2$, if we choose $\delta_{m_i} = O\left(\frac{1}{[m_i]_{q_i}}\right)$, then:

$$\begin{aligned}
\frac{|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)|}{4(1 + v_1^2 + v_2^2)} & \leq (1 + \delta_{m_1}^2)(1 + \delta_{m_2}^2) \omega_\psi(f; \delta_{m_1}, \delta_{m_2}) \\
& \times \left[1 + \frac{1}{\delta_{m_1}} O\left(\frac{1}{[m_1]_{q_1}}\right) (v_1 + 1) + O\left(\frac{1}{[m_1]_{q_1}^2}\right) (v_1 + 1)^2 \right. \\
& + \left. \frac{1}{\delta_{m_1}} O\left(\frac{1}{[m_1]_{q_1}}\right) (v_1 + 1) O\left(\frac{1}{[m_1]_{q_1}^2}\right) (v_1 + 1)^2 \right] \\
& \times \left[1 + \frac{1}{\delta_{m_2}} O\left(\frac{1}{[m_2]_{q_2}}\right) (v_2 + 1) + O\left(\frac{1}{[m_2]_{q_2}^2}\right) (v_2 + 1)^2 \right. \\
& + \left. \frac{1}{\delta_{m_2}} O\left(\frac{1}{[m_2]_{q_2}}\right) (v_2 + 1) O\left(\frac{1}{[m_2]_{q_2}^2}\right) (v_2 + 1)^2 \right] \\
& \leq 4 \left(1 + O\left(\frac{1}{[m_1]_{q_1}^2}\right) \right) \left(1 + O\left(\frac{1}{[m_2]_{q_2}^2}\right) \right) \\
& \times \omega_\psi\left(f; O\left(\frac{1}{[m_1]_{q_1}}\right), O\left(\frac{1}{[m_2]_{q_2}}\right)\right) \\
& \times \left[1 + (v_1 + 1) + M_1(v_1 + 1)^2 + M_1(v_1 + 1)^3 \right] \\
& \times \left[1 + (v_2 + 1) + M_2(v_2 + 1)^2 + M_2(v_2 + 1)^3 \right].
\end{aligned}$$

This completes the proof. $\square$

Lemma 6 ([31,32]). For the positive sequence of operators $\{L_{m_1, m_2}\}_{m_1, m_2 \geq 1}$, which act as $C_\psi \rightarrow B_\psi$, then there exists some positive K such that:

$$\|X_{m_1, m_2}(\psi; v_1, v_2)\|_\psi \leq K.$$

Theorem 2 ([31,32]). For the positive sequence of operators $\{X_{m_1, m_2}\}_{m_1, m_2 \geq 1}$ acting as $C_\psi \rightarrow B_\psi$ defined earlier, the following conditions are satisfied:

$$\begin{aligned} (1) \quad & \lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}(1; v_1, v_2) - 1\|_\psi = 0; \\ (2) \quad & \lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}(t; v_1, v_2) - v_1\|_\psi = 0; \\ (3) \quad & \lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}(s; v_1, v_2) - v_2\|_\psi = 0; \\ (4) \quad & \lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}((t^2 + s^2); v_1, v_2) - (v_1^2 + v_2^2)\|_\psi = 0. \end{aligned}$$

Then, for all $f \in C_\psi^0$:

$$\lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}f - f\|_\psi = 0$$

and there exists another function $f \in C_\psi \setminus C_\psi^0$, such that:

$$\lim_{m_1, m_2 \rightarrow \infty} \|X_{m_1, m_2}f - f\|_\psi \geq 1.$$

Theorem 3. If $f \in C_\psi^0(\mathcal{M}^2)$, then we have:

$$\lim_{m_1, m_2 \rightarrow \infty} \|J_{m_1, m_2}^{q_1, q_2}(f) - f\|_\psi = 0.$$

Proof.

$$\begin{aligned} \|J_{m_1, m_2}^{q_1, q_2}(\psi; v_1, v_2)\|_\psi &= \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{|J_{m_1, m_2}^{q_1, q_2}(1 + v_1^2 + v_2^2; v_1, v_2)|}{1 + v_1^2 + v_2^2} \\ &= 1 + \sup_{(v_1, v_2) \in \mathcal{M}^2} \left[\frac{1}{1 + v_1^2 + v_2^2} \left| \left(J_{m_1, m_2}^{q_1, q_2}(v_1^2; v_1, v_2) + J_{m_1, m_2}^{q_1, q_2}(v_2^2; v_1, v_2) \right) \right| \right] \\ &= 1 + \left| \frac{[m_1]_{q_1}^2}{[m_1 - 1]_{q_1}[m_2 - 1]_{q_1}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{v_1^2}{1 + v_1^2 + v_2^2} \\ &\quad + \left| \frac{(1 + 2q_1) + 2q_1^2 \frac{X'_{q_1}(1)}{X_{q_1}(1)}}{q_1^2 [m_1 - 1]_{q_1} [m_2 - 1]_{q_1}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{v_1}{1 + v_1^2 + v_2^2} \\ &\quad + \left| \frac{(1 + 2q_1) + q_1(1 + 2q_1) \frac{X'_{q_1}(1)}{X_{q_1}(1)} + q_1^3 \frac{X''_{q_1}(1)}{X_{q_1}(1)}}{q_1^3 [m_1 - 1]_{q_1} [m_2 - 1]_{q_1}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{1}{1 + v_1^2 + v_2^2} \\ &\quad + \left| \frac{[m_2]_{q_2}^2}{[m_1 - 1]_{q_2} [m_2 - 1]_{q_2}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{v_2^2}{1 + v_1^2 + v_2^2} \\ &\quad + \left| \frac{(1 + 2q_2) + 2q_2^2 \frac{X'_{q_2}(1)}{X_{q_2}(1)}}{q_2^2 [m_1 - 1]_{q_2} [m_2 - 1]_{q_2}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{v_2}{1 + v_1^2 + v_2^2} \\ &\quad + \left| \frac{(1 + 2q_2) + q_2(1 + 2q_2) \frac{X'_{q_2}(1)}{X_{q_2}(1)} + q_2^3 \frac{X''_{q_2}(1)}{X_{q_2}(1)}}{q_2^3 [m_1 - 1]_{q_2} [m_2 - 1]_{q_2}} \right| \sup_{(v_1, v_2) \in \mathcal{M}^2} \frac{1}{1 + v_1^2 + v_2^2}. \end{aligned}$$

Now, for all $m_1, m_2 \in \mathbb{N} \setminus \{1, 2\}$, there exists a positive constant K such that:

$$\|J_{m_1, m_2}^{q_1, q_2}(\psi; v_1, v_2)\|_\psi \leq K.$$

Therefore, in order to prove Theorem 3, it is sufficient to take from the Lemma 3 and Theorem 2. Thus, we aimed to prove of Theorem 3. $\square$

Here, we obtain the approximations of our new operators by the use of the second-order modulus of continuity. For this purpose, if any $f \in C(\mathcal{M}^2)$ and $\delta > 0$, then the second-order modulus of continuity in terms of partial continuity is given by:

$$\omega(f; \delta_{m_1}, \delta_{m_2}) = \sup\{|f(t, s) - f(v_1, v_2)| : (t, s), (v_1, v_2) \in \mathcal{M}^2\} \quad (31)$$

with $|t - v_1| \leq \delta_{m_1}$, $|s - v_2| \leq \delta_{m_2}$ with the partial modulus of continuity defined as:

$$\omega_1(f; \delta_{m_1}) = \sup_{0 \leq v_1, v_2 \leq 1} \sup_{|x_1 - x_2| \leq \delta_{m_1}} \{|f(x_1, v_2) - f(x_2, v_2)|\}, \quad (32)$$

$$\omega_2(f; \delta_{m_2}) = \sup_{0 \leq v_1, v_2 \leq 1} \sup_{|y_1 - y_2| \leq \delta_{m_2}} \{|f(v_1, y_1) - f(v_1, y_2)|\}. \quad (33)$$

While for any function φ and $\delta^* > 0$, one has taken the modulus of continuity of order one $\omega^*(f; \delta^*)$, and it is defined such that $\lim_{\delta^* \rightarrow 0+} \omega^*(\varphi; \delta^*) = 0$ and:

$$\omega^*(\varphi; \delta^*) = \sup_{|v_1 - v_2| \leq \delta^*} |\varphi(v_1) - \varphi(v_2)|; \quad v_1, v_2 \in [0, \infty), \quad (34)$$

$$|\varphi(v_1) - \varphi(v_2)| \leq \left(1 + \frac{|v_1 - v_2|}{\delta^*}\right) \omega^*(\varphi; \delta^*). \quad (35)$$

Theorem 4. If $f \in C(\mathcal{M}^2)$, then we obtain the inequality for the operators $J_{m_1, m_2}^{q_1, q_2}$:

$$|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| \leq 2 \left(\omega_1(f; \delta_{m_1}^{2,0}) + \omega_2(f; \delta_{m_1, m_2}^{0,2}) \right).$$

Proof. For proof of Theorem 4, we use the Cauchy–Schwarz inequality, therefore, we obtain here:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq J_{m_1, m_2}^{q_1, q_2}(|f(t, s) - f(v_1, v_2)|; v_1, v_2) \\ &\leq J_{m_1, m_2}^{q_1, q_2}(|f(t, s) - f(v_1, s)|; v_1, v_2) \\ &\quad + J_{m_1, m_2}^{q_1, q_2}(|f(v_1, s) - f(v_1, v_2)|; v_1, v_2) \\ &\leq J_{m_1, m_2}^{q_1, q_2}(\omega_1(f; |t - v_1|); v_1, v_2) + J_{m_1, m_2}^{q_1, q_2}(\omega_2(f; |s - v_2|); v_1, v_2) \\ &\leq \omega_1(f; \delta_{m_1}) \left(1 + \delta_{m_1}^{-1} J_{m_1, m_2}^{q_1, q_2}(|t - v_1|; v_1, v_2)\right) \\ &\quad + \omega_2(f; \delta_{m_2}) \left(1 + \delta_{m_2}^{-1} J_{m_1, m_2}^{q_1, q_2}(|s - v_2|; v_1, v_2)\right) \\ &\leq \omega_1(f; \delta_{m_1}) \left(1 + \frac{1}{\delta_{m_1}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)}\right) \\ &\quad + \omega_2(f; \delta_{m_2}) \left(1 + \frac{1}{\delta_{m_2}} \sqrt{J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)}\right), \end{aligned}$$

where $\delta_{m_1} = \delta_{m_1, m_2}^{2,0} = \sqrt{J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)}$ and $\delta_{m_2} = \delta_{m_1, m_2}^{0,2} = \sqrt{J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)}$, which then led to results. $\square$

Next, we obtain the convergence by use of class of the Lipschitz maximal in bivariate form for any function $f \in C(\mathcal{M}^2)$. For any positive constant K and $0 \leq \rho_1, \rho_2 \leq 1$ the Lipschitz maximal function on any set $\mathcal{I} \times \mathcal{I} \subset \mathcal{M}^2$ is given by:

$$\begin{aligned} \mathcal{L}_{\rho_1, \rho_2}(\mathcal{I} \times \mathcal{I}) &= \left\{ f : \sup (1+t)^{\rho_1} (1+s)^{\rho_2} (f_{\rho_1, \rho_2}(t, s) - f_{\rho_1, \rho_2}(v_1, v_2)) \right. \\ &\quad \left. \leq K \frac{1}{(1+v_1)^{\rho_1}} \frac{1}{(1+v_2)^{\rho_2}} \right\}. \end{aligned}$$

$$f_{\rho_1, \rho_2}(t, s) - f_{\rho_1, \rho_2}(v_1, v_2) = \frac{|f(t, s) - f(v_1, v_2)|}{|t - v_1|^{\rho_1} |s - v_2|^{\rho_2}}; \quad (t, s), (v_1, v_2) \in \mathcal{M}^2. \quad (36)$$

Theorem 5. For all $f \in \mathcal{L}_{\rho_1, \rho_2}(\mathcal{I} \times \mathcal{I})$, if $0 \leq \rho_1, \rho_2 \leq 1$, then we obtain:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq K \left\{ \left((d(v_1, \mathcal{I}))^{\rho_1} + (\delta_{m_1, m_2}^{2,0})^{\frac{\rho_1}{2}} \right) \left((d(v_2, \mathcal{I}))^{\rho_2} + (\delta_{m_1, m_2}^{0,2})^{\frac{\rho_2}{2}} \right) \right. \\ &\quad \left. + (d(v_1, \mathcal{I}))^{\rho_1} (d(v_2, \mathcal{I}))^{\rho_2} \right\}, \end{aligned}$$

where $\delta_{m_1, m_2}^{2,0}$ and $\delta_{m_1, m_2}^{0,2}$ are defined by Theorem 4 and $K > 0$.

Proof. Take into account $|v_1 - x_0| = d(v_1, \mathcal{I})$ and $|v_2 - y_0| = d(v_2, \mathcal{E})$. For any $(v_1, v_2) \in \mathcal{M}^2$ and $(x_0, y_0) \in \mathcal{I} \times \mathcal{I}$, we let $d(v_1, \mathcal{I}) = \inf\{|v_1 - v_2| : v_2 \in \mathcal{I}\}$. Thus, we can write here:

$$|f(t, s) - f(v_1, v_2)| \leq K |f(t, s) - f(x_0, y_0)| + |f(x_0, y_0) - f(v_1, v_2)|. \quad (37)$$

Applying $J_{m_1, m_2}^{q_1, q_2}$, we obtain:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq J_{m_1, m_2}^{q_1, q_2}(|f(v_1, v_2) - f(x_0, y_0)| + |f(x_0, y_0) - f(v_1, v_2)|) \\ &\leq K J_{m_1, m_2}^{q_1, q_2}(|t - x_0|^{\rho_1} |s - y_0|^{\rho_2}; v_1, v_2) \\ &\quad + K |v_1 - x_0|^{\rho_1} |v_2 - y_0|^{\rho_2}. \end{aligned}$$

For all $U, V \geq 0$ and $\rho \in [0, 1]$, we know the inequality $(U + V)^\rho \leq U^\rho + V^\rho$. Then, it is easy to obtain:

$$\begin{aligned} |t - x_0|^{\rho_1} &\leq |t - v_1|^{\rho_1} + |v_1 - x_0|^{\rho_1}, \\ |s - y_0|^{\rho_2} &\leq |s - v_2|^{\rho_2} + |v_2 - y_0|^{\rho_2}. \end{aligned}$$

Therefore:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq K J_{m_1, m_2}^{q_1, q_2}(|t - v_1|^{\rho_1} |s - v_2|^{\rho_2}; v_1, v_2) \\ &\quad + K |v_1 - x_0|^{\rho_1} J_{m_1, m_2}^{q_1, q_2}(|s - v_2|^{\rho_2}; v_1, v_2) \\ &\quad + K |v_2 - y_0|^{\rho_2} J_{m_1, m_2}^{q_1, q_2}(|t - v_1|^{\rho_1}; v_1, v_2) \\ &\quad + 2K |v_1 - x_0|^{\rho_1} |v_2 - y_0|^{\rho_2} J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2), \end{aligned}$$

by the use of the Hölder inequality, we obtain the equality:

$$\begin{aligned} J_{m_1, m_2}^{q_1, q_2}(|t - v_1|^{\rho_1} |s - v_2|^{\rho_2}; v_1, v_2) &= C_{m_1, r_1}^{q_1}(|t - v_1|^{\rho_1}; v_1, v_2) \mathcal{D}_{m_2, r_2}^{q_2}(|s - v_2|^{\rho_2}; v_1, v_2) \\ &\leq \left(J_{m_1, m_2}^{q_1, q_2}(|t - v_1|^2; v_1, v_2) \right)^{\frac{\rho_1}{2}} \left(J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2) \right)^{\frac{2-\rho_1}{2}} \\ &\quad \times \left(J_{m_1, m_2}^{q_1, q_2}(|s - v_2|^2; v_1, v_2) \right)^{\frac{\rho_2}{2}} \left(J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2) \right)^{\frac{2-\rho_2}{2}}. \end{aligned}$$

Thus, it is easy to obtain:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq K \left(\delta_{m_1, m_2}^{2,0} \right)^{\frac{\rho_1}{2}} \left(\delta_{m_1, m_2}^{0,2} \right)^{\frac{\rho_2}{2}} + 2K (d(v_1, \mathcal{I}))^{\rho_1} (d(v_2, \mathcal{I}))^{\rho_2} \\ &\quad + K (d(v_1, \mathcal{I}))^{\rho_1} \left(\delta_{m_1, m_2}^{2,0} \right)^{\frac{\rho_2}{2}} + K (d(v_2, \mathcal{I}))^{\rho_2} \left(\delta_{m_1, m_2}^{0,2} \right)^{\frac{\rho_1}{2}}. \end{aligned}$$

This completes the proof. $\square$

Theorem 6. Suppose $f \in C'(\mathcal{M}^2)$. Then, for all $(v_1, v_2) \in \mathcal{M}^2$, the operators $J_{m_1, m_2}^{q_1, q_2}$ have the equality:

$$|J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| \leq \|f_{v_1}\|_{C(\mathcal{M}^2)} \left(\delta_{m_1, m_2}^{2,0}\right)^{\frac{1}{2}} + \|f_{v_2}\|_{C(\mathcal{M}^2)} \left(\delta_{m_1, m_2}^{0,2}\right)^{\frac{1}{2}},$$

where $\delta_{m_1, m_2}^{2,0}$ and $\delta_{m_1, m_2}^{0,2}$ are given by Theorem 4.

Proof. For any $f \in C'(\mathcal{M}^2)$ and fixed $(v_1, v_2) \in \mathcal{M}^2$, we take:

$$f(t, s) - f(v_1, v_2) = \int_{v_1}^t f_q(q, s) dq + \int_{v_2}^s f_\mu(v_1, \mu) d\mu. \quad (38)$$

Apply operators $J_{m_1, m_2}^{q_1, q_2}$ on both sides, then:

$$J_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) - f(v_1, v_2) = J_{m_1, m_2}^{q_1, q_2}\left(\int_{v_1}^t f_q(q, s) dq; v_1, v_2\right) + J_{m_1, m_2}^{q_1, q_2}\left(\int_{v_2}^s f_\mu(v_1, \mu) d\mu; v_1, v_2\right). \quad (39)$$

The sup-norm on $\mathcal{M}^2$ gives us:

$$\left|\int_{v_1}^t f_q(q, s) dq\right| \leq \int_{v_1}^t |f_q(q, s)| dq \leq \|f_{v_1}\|_{C(\mathcal{M}^2)} |t - v_1| \quad (40)$$

$$\left|\int_{v_2}^s f_\mu(v_1, \mu) d\mu\right| \leq \int_{v_2}^s |f_\mu(v_1, \mu)| d\mu \leq \|f_{v_2}\|_{C(\mathcal{M}^2)} |s - v_2|. \quad (41)$$

Taking into account (39)–(41), it is easy to obtain:

$$\begin{aligned} |J_{m_1, m_2}^{q_1, q_2}(f(v_1, v_2); v_1, v_2) - f(v_1, v_2)| &\leq J_{m_1, m_2}^{q_1, q_2}\left(\left|\int_{v_1}^t f_q(q, s) dq\right|; v_1, v_2\right) \\ &+ J_{m_1, m_2}^{q_1, q_2}\left(\left|\int_{v_2}^s f_\mu(v_1, \mu) d\mu\right|; v_1, v_2\right) \\ &\leq \|f_{v_1}\|_{C(\mathcal{M}^2)} J_{m_1, m_2}^{q_1, q_2}(|t - v_1|; v_1, v_2) \\ &+ \|f_{v_2}\|_{C(\mathcal{M}^2)} J_{m_1, m_2}^{q_1, q_2}(|s - v_2|; v_1, v_2) \\ &\leq \|f_{v_1}\|_{C(\mathcal{M}^2)} \left(J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2) J_{m_1, m_2}^{q_1, q_2}(1; v_1, v_2)\right)^{\frac{1}{2}} \\ &+ \|f_{v_2}\|_{C(\mathcal{M}^2)} \left(J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2) J_{m_1, m_2}^{q_1, q_2}(1; v_1, v_2)\right)^{\frac{1}{2}} \\ &= \|f_{v_1}\|_{C(\mathcal{M}^2)} \left(\delta_{m_1, m_2}^{2,0}\right)^{\frac{1}{2}} + \|f_{v_2}\|_{C(\mathcal{M}^2)} \left(\delta_{m_1, m_2}^{0,2}\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 7. For an arbitrary function $f \in C(\mathcal{M}^2)$, we define the auxiliary operators as:

$$S_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) = J_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) + f(v_1, v_2) - f\left(C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2), \mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{1,0}; v_1, v_2)\right).$$

Then, for all $f \in C'(\mathcal{M}^2)$, the operators $S_{m_1, m_2}^{q_1, q_2}$ verify the inequality:

$$\begin{aligned} S_{m_1, m_2}^{q_1, q_2}(f; t, s) - f(v_1, v_2) &\leq \left\{ \delta_{m_1, m_2}^{2,0} + \delta_{m_1, m_2}^{0,2} + \left(C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) - v_1\right)^2 \right. \\ &\quad \left. + \left(\mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{1,0}; v_1, v_2) - v_2\right)^2 \right\} \|f\|_{C^2(\mathcal{M}^2)}. \end{aligned}$$

where $C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2)$ and $D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2)$ are defined by Lemma 3.

Proof. By the use of Lemma 3, it is easy to obtain $S_{m_1, m_2}^{q_1, q_2}(1; v_1, v_2) = 1$, $S_{m_1, m_2}^{q_1, q_2}(t - v_1; v_1, v_2) = 0$ and $S_{m_1, m_2}^{q_1, q_2}(s - v_2; v_1, v_2) = 0$. In addition, for any $f \in C'(\mathcal{M}^2)$ from the Taylor series expansions, we obtain:

$$\begin{aligned} f(t, s) - f(v_1, v_2) &= \frac{\partial f(v_1, v_2)}{\partial v_1}(t - v_1) + \int_{v_1}^t (t - \chi) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi \\ &+ \frac{\partial f(v_1, v_2)}{\partial v_2}(s - v_2) + \int_{v_2}^s (s - \sigma) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma. \end{aligned}$$

On applying $S_{m_1, m_2}^{q_1, q_2}$, we see that:

$$\begin{aligned} &S_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) - S_{m_1, m_2}^{q_1, q_2}(f(v_1, v_2)) \\ &= S_{m_1, m_2}^{q_1, q_2}\left(\int_{v_1}^t (t - \chi) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi; v_1, v_2\right) + S_{m_1, m_2}^{q_1, q_2}\left(\int_{v_2}^s (s - \sigma) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma; v_1, v_2\right) \\ &= J_{m_1, m_2}^{q_1, q_2}\left(\int_{v_1}^t (t - \chi) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi; v_1, v_2\right) + J_{m_1, m_2}^{q_1, q_2}\left(\int_{v_2}^s (s - \sigma) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma; v_1, v_2\right) \\ &- \int_{v_1}^{C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2)} \left(C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) - \chi\right) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi \\ &- \int_{v_2}^{D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2)} \left(D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2) - \sigma\right) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma. \end{aligned}$$

From this hypothesis, we easily obtain:

$$\left| \int_{v_1}^t (t - \chi) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi \right| \leq \int_{v_1}^t \left| (t - \chi) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi \right| \leq \|f\|_{C^2(\mathcal{M}^2)} (t - v_1)^2,$$

$$\left| \int_{v_2}^s (s - \sigma) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma \right| \leq \int_{v_2}^s \left| (s - \sigma) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma \right| \leq \|f\|_{C^2(\mathcal{M}^2)} (s - v_2)^2,$$

$$\begin{aligned} &\left| \int_{v_1}^{C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2)} \left(C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) - \chi\right) \frac{\partial^2 f(\chi, v_2)}{\partial \chi^2} d\chi \right| \\ &\leq \|f\|_{C^2(\mathcal{M}^2)} \left(C_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) - v_1\right)^2 \end{aligned}$$

$$\begin{aligned} &\left| \int_{v_2}^{D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2)} \left(D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2) - \sigma\right) \frac{\partial^2 f(v_1, \sigma)}{\partial \sigma^2} d\sigma \right| \\ &\leq \|f\|_{C^2(\mathcal{M}^2)} \left(D_{m_2, r_2}^{q_2}(\lambda_{0,1}; v_1, v_2) - v_2\right)^2. \end{aligned}$$

Thus:

$$|S_{m_1, m_2}^{q_1, q_2}(f; t, s) - f(v_1, v_2)| \leq \left\{ J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2) + J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2) \right\}$$

$$+ \left(\mathcal{C}_{m_1, r_1}^{q_1}(\lambda_{1,0}; v_1, v_2) - v_1 \right)^2 + \left(\mathcal{D}_{m_2, r_2}^{q_2}(\lambda_{1,0}; v_1, v_2) - v_2 \right)^2 \Big\} \|f\|_{C^2(\mathcal{M}_2)},$$

which completes the desired proof of Theorem 7. $\square$

4. GBS-Type Approximation in Bögel Space

Suppose $\mathcal{M}_1 \times \mathcal{M}_2 = \{(v_1, v_2) : 0 \leq v_1, v_2 < \infty\}$. Take any function $f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$ for real compact intervals of $\mathcal{M}_1 \times \mathcal{M}_2$. For all $(t, s), (v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ suppose $\Delta_{(t,s)}^* f(v_1, v_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta_{(t,s)}^* f(v_1, v_2) = f(t, s) - f(t, v_2) - f(v_1, s) + f(v_1, v_2). \quad (42)$$

If at any point $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, the function $f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$ defined on $\mathcal{M}_1 \times \mathcal{M}_2$, then $\lim_{(t,s) \rightarrow (v_1, v_2)} \Delta_{(t,s)}^* f(v_1, v_2) = 0$.

If set of all the space of all Bögel-continuous (B -continuous) denoted by $C_B(\mathcal{M}_1 \times \mathcal{M}_2)$ on $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ and be defined such that $C_B(\mathcal{M}_1 \times \mathcal{M}_2) = \{f, \text{ such that } f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R} \text{ is } f, B\text{-bounded on } \mathcal{M}_1 \times \mathcal{M}_2\}$. Next, the Bögel-differentiable function on $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ be $f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$ and limit exists finite defined by:

$$\lim_{(t,s) \rightarrow (v_1, v_2), t \neq v_1, s \neq v_2} \frac{1}{(t - v_1)(s - v_2)} \left(\Delta_{(t,s)}^* \right) = D_B f(v_1, v_2) < \infty. \quad (43)$$

Let the classes of all Bögel-differentiable function denoted by $D_\psi f(v_1, v_2)$ and be $D_\psi(\mathcal{M}_1 \times \mathcal{M}_2) = \{f, \text{ such that } f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R} \text{ is } f, B\text{-differentiable on } \mathcal{M}_1 \times \mathcal{M}_2\}$. Suppose the function f is B -bounded on D and be $f : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$, then for all $(t, s), (v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ there exists positive constant M such that $|\Delta_{(t,s)}^* f(v_1, v_2)| \leq M$. The classes of all B -continuous function is called a B -bounded on $\mathcal{M}_1 \times \mathcal{M}_2$, when $\mathcal{M}_1 \times \mathcal{M}_2$ is compact subset. Let $B_\psi(\mathcal{M}_1 \times \mathcal{M}_2)$ denote the classes of all B -bounded function defined on $\mathcal{M}_1 \times \mathcal{M}_2$, which equipped the norm on B as $\|f\|_B = \sup_{(t,s), (v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2} |\Delta_{(t,s)}^* f(v_1, v_2)|$. As we know to approximate the degree for a set of all B -continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So, we let for all (t, s) and $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ and $f \in B_\psi(\mathcal{M}_1 \times \mathcal{M}_2)$, the mixed-modulus of continuity of function f such that $\omega_B : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$ and be defined such as:

$$\omega_B(f; \delta_1, \delta_2) = \sup \{ \Delta_{(t,s)}^* f(v_1, v_2) : |t - v_1| \leq \delta_1, |s - v_2| \leq \delta_2 \}. \quad (44)$$

For any $\mathcal{M}_1 \times \mathcal{M}_2 = [0, \infty) \times [0, \infty)$, we suppose the classes of all B -continuous function defined on $\mathcal{M}_1 \times \mathcal{M}_2$ denoted by $C_\psi(\mathcal{M}_1 \times \mathcal{M}_2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{M}_1 \times \mathcal{M}_2$ be $C(\mathcal{M}_1 \times \mathcal{M}_2)$. For further details on space of Bögel functions and GBS-type operators related to this paper, we propose the article [33–38].

Let $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ and $m_1, m_2 \in \mathbb{N}$. Then, for all $f \in C(\mathcal{M}_1 \times \mathcal{M}_2)$, we define the GBS-type new positive linear operators $K_{m_1, m_2}^{q_1, q_2}$ for the operators $J_{m_1, m_2}^{q_1, q_2}$. Thus, we suppose:

$$K_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) = J_{m_1, m_2}^{q_1, q_2} \left(f(t, v_2) + f(v_1, s) - f(t, s); v_1, v_2 \right). \quad (45)$$

More precisely, the generalized GBS operator for the bivariate function is defined as follows:

$$K_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) = \sum_{r_1, r_2=0}^{\infty} \mathcal{U}_{m_1, q_1}^{r_1}(v_1) \mathcal{V}_{m_2, q_2}^{r_2}(v_2) \int_0^{\infty/A_1} \int_0^{\infty/A_2} Q_{v_1, v_2}(t, s) f(q_1^{r_1} t, q_2^{r_2} s) d_{q_1} t d_{q_2} s, \quad (46)$$

where $Q_{v_1, v_2}(t, s) = \mathcal{T}_{m_1, m_2}^{q_1, q_2}(t, s) P_{v_1, v_2}(t, s)$ and $P_{v_1, v_2}(t, s) = (f(t, v_2) + f(v_1, s) - f(t, s))$.

Theorem 8. For all $f \in C_\psi(\mathcal{M}_1 \times \mathcal{M}_2)$, the operators $K_{m_1, m_2}^{q_1, q_2}$ verify that:

$$|K_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) - f(v_1, v_2)| \leq 4\omega_B\left(f; \delta_{m_1, m_2}^{2,0}, \delta_{m_1, m_2}^{0,2}\right),$$

where $\delta_{m_1, m_2}^{2,0}$ and $\delta_{m_1, m_2}^{0,2}$ are defined by Theorem 4.

Proof. Let (t, s) and $(v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$. For all $m_1, m_2 \in \mathbb{N}$ and $\delta_{m_1}, \delta_{m_2} > 0$, it follows that:

$$\begin{aligned} |\Delta_{(v_1, v_2)}^* f(t, s)| &\leq \omega_B(f; |t - v_1|, |s - v_2|) \\ &\leq \left(1 + \frac{t - v_1}{\delta_{m_1}}\right) \left(1 + \frac{s - v_2}{\delta_{m_2}}\right) \omega_B(f; \delta_{m_1}, \delta_{m_2}). \end{aligned}$$

From (45) and the well-known Cauchy–Schwarz inequality, we easily conclude that:

$$\begin{aligned} |K_{m_1, m_2}^{q_1, q_2}(f(t, s); v_1, v_2) - f(v_1, v_2)| &\leq J_{m_1, m_2}^{q_1, q_2}(|\Delta_{(v_1, v_2)}^* f(t, s)|; v_1, v_2) \\ &\leq \left(J_{m_1, m_2}^{q_1, q_2}(\lambda_{0,0}; v_1, v_2) + \frac{1}{\delta_{m_1}} \left(J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)\right)^{\frac{1}{2}}\right. \\ &\quad + \frac{1}{\delta_{m_2}} \left(J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)\right)^{\frac{1}{2}} \\ &\quad + \left.\frac{1}{\delta_{m_1}} \left(J_{m_1, m_2}^{q_1, q_2}((t - v_1)^2; v_1, v_2)\right)^{\frac{1}{2}}\right) \\ &\quad \times \frac{1}{\delta_{m_2}} \left(J_{m_1, m_2}^{q_1, q_2}((s - v_2)^2; v_1, v_2)\right)^{\frac{1}{2}} \omega_B(f; \delta_{m_1}, \delta_{m_2}). \end{aligned}$$

In the view of Theorem 4, we easily obtain our results. $\square$

If we let $x = (t, s)$, $y = (v_1, v_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, then the Lipschitz function in terms of B -continuous functions is defined by:

$$Lip_M^\xi = \left\{f \in C(\mathcal{M}_1 \times \mathcal{M}_2) : |\Delta_{(v_1, v_2)}^* f(x, y)| \leq M \|x - y\|^\xi\right\}, \quad (47)$$

where M is a positive constant, $0 < \xi \leq 1$, and the Euclidean norm $\|x - y\| = \sqrt{(t - v_1)^2 + (s - v_2)^2}$.

Theorem 9. For all $f \in Lip_M^\xi$, the operators $K_{m_1, m_2}^{q_1, q_2}$ satisfy:

$$|K_{m_1, m_2}^{q_1, q_2}(f(x, y); v_1, v_2) - f(v_1, v_2)| \leq M \left(\delta_{m_1, m_2}^{2,0} + \delta_{m_1, m_2}^{0,2}\right)^{\frac{\xi}{2}},$$

where $\delta_{m_1, m_2}^{2,0}$ and $\delta_{m_1, m_2}^{0,2}$ are defined by Theorem 4.

Proof. We easily see that:

$$\begin{aligned} K_{m_1, m_2}^{q_1, q_2}(f(x, y); v_1, v_2) &= J_{m_1, m_2}^{q_1, q_2}(f(v_1, y) + f(x, v_2) - f(x, s); v_1, v_2) \\ &= J_{m_1, m_2}^{q_1, q_2}\left(f(v_1, v_2) - \Delta_{(v_1, v_2)}^* f(x, s); v_1, v_2\right) \end{aligned}$$

$$= f(v_1, v_2) - J_{m_1, m_2}^{q_1, q_2} \left(\Delta_{(v_1, v_2)}^* f(x, s); v_1, v_2 \right).$$

Therefore:

$$\begin{aligned} |K_{m_1, m_2}^{q_1, q_2}(f(x, y); v_1, v_2) - f(v_1, v_2)| &\leq J_{m_1, m_2}^{q_1, q_2} \left(|\Delta_{(v_1, v_2)}^* f(x, y)|; v_1, v_2 \right) \\ &\leq M J_{m_1, m_2}^{q_1, q_2} \left(\|x - y\|^\xi; v_1, v_2 \right) \\ &\leq M \left(J_{m_1, m_2}^{q_1, q_2} \left(\|x - y\|^2; v_1, v_2 \right) \right)^{\frac{\xi}{2}} \\ &\leq M \left(J_{m_1, m_2}^{q_1, q_2} \left((t - v_1)^2; v_1, v_2 \right) + J_{m_1, m_2}^{q_1, q_2} \left((s - v_2)^2; v_1, v_2 \right) \right)^{\frac{\xi}{2}}. \end{aligned}$$

□

Theorem 10. For all $f \in D_\psi(\mathcal{M}_1 \times \mathcal{M}_2)$ and $D_B f \in B(\mathcal{M}_1 \times \mathcal{M}_2)$, we obtain:

$$\begin{aligned} |K_{m_1, m_2}^{q_1, q_2}(f; v_1, v_2) - f(v_1, v_2)| &\leq C \left\{ 1 + 3 \|D_B f\|_\infty \right\} (v_1 + 1)(v_2 + 1) \\ &\quad + \left\{ 1 + \sqrt{M_2}(v_1 + 1) + \sqrt{M_1}(v_2 + 1) \right\} (v_1 + 1)(v_2 + 1) \\ &\quad \times \omega_{mixed} \left(D_B f; O\left(\frac{1}{[m_1]_{q_1}}\right), O\left(\frac{1}{[m_2]_{q_2}}\right) \right), \end{aligned}$$

where C is any positive constant.

Proof. Suppose $\rho \in (v_1, t)$, $\xi \in (v_2, s)$ and

$$\Delta_{(v_1, v_2)}^* f(t, s) = (t - v_1)(s - v_2) D_B f(\rho, \xi),$$

$$D_B f(\rho, \xi) = \Delta_{(v_1, v_2)}^* D_B f(\rho, \xi) + D_B f(\rho, y) + D_B f(x, \xi) - D_B f(v_1, v_2).$$

For all $D_B f \in B(\mathcal{M}_1 \times \mathcal{M}_2)$, it follows that:

$$\begin{aligned} |K_{m_1, m_2}^{q_1, q_2}(\Delta_{(v_1, v_2)}^* f(t, s); v_1, v_2)| &= |K_{m_1, m_2}^{q_1, q_2}((t - v_1)(s - v_2) D_B f(\rho, \xi); v_1, v_2)| \\ &\leq K_{m_1, m_2}^{q_1, q_2} \left(|t - v_1| |s - v_2| |\Delta_{(v_1, v_2)}^* D_B f(\rho, \xi)|; v_1, v_2 \right) \\ &\quad + K_{m_1, m_2}^{q_1, q_2} \left(|t - v_1| |s - v_2| (|D_B f(\rho, v_2)| \right. \\ &\quad \left. + |D_B f(v_1, \xi)| + |D_B f(v_1, v_2)|); v_1, v_2 \right) \\ &\leq K_{m_1, m_2}^{q_1, q_2} \left(|t - v_1| |s - v_2| \right. \\ &\quad \left. \times \omega_{mixed}(D_B f; |\rho - v_1|, |\xi - v_2|); v_1, v_2 \right) \\ &\quad + 3 \|D_B f\|_\infty K_{m_1, m_2}^{q_1, q_2}(|t - v_1| |s - v_2|; v_1, v_2). \end{aligned}$$

Here, ω_{mixed} is the mixed-modulus of continuity, and it follows that:

$$\begin{aligned} \omega_{mixed}(D_B f; |\rho - v_1|, |\xi - v_2|) &\leq \omega_{mixed}(D_B f; |t - v_1|, |s - v_2|) \\ &\leq \left(1 + \delta_{m_1}^{-1} |t - v_1|\right) \left(1 + \delta_{m_2}^{-1} |s - v_2|\right) \omega_{mixed}(D_B f; \delta_{m_1}, \delta_{m_2}). \end{aligned}$$

Therefore, it is obvious that:

$$\begin{aligned}
 |K_{m_1, m_2}^*(f; v_1, v_2) - f(v_1, v_2)| &= |\Delta_{(v_1, v_2)}^* f(t, s); v_1, v_2| \\
 &\leq 3 \|D_B f\|_\infty \left(K_{m_1, m_2}^{q_1, q_2} \left((t - v_1)^2 (s - v_2)^2; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &+ \left(K_{m_1, m_2}^{q_1, q_2} (|t - v_1| |s - v_2|; v_1, v_2) \right. \\
 &+ \delta_{m_1}^{-1} K_{m_1, m_2}^{q_1, q_2} \left((t - v_1)^2 |s - v_2|; v_1, v_2 \right) \\
 &+ \delta_{m_2}^{-1} K_{m_1, m_2}^{q_1, q_2} (|t - v_1| (s - v_2)^2; v_1, v_2) \\
 &+ \delta_{m_1}^{-1} \delta_{m_2}^{-1} K_{m_1, m_2}^{q_1, q_2} \left((t - v_1)^2 (s - v_2)^2; v_1, v_2 \right) \omega_{mixed}(D_B f; \delta_{m_1}, \delta_{m_2}); \\
 |K_{m_1, m_2}^*(f; v_1, v_2) - f(v_1, v_2)| &= |\Delta_{(v_1, v_2)}^* f(t, s); v_1, v_2| \\
 &\leq 3 \|D_B f\|_\infty \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &+ \left\{ \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} \right. \\
 &+ \delta_{m_1}^{-1} \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{4, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} + \delta_{m_2}^{-1} \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 4}; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &\left. + \delta_{m_1}^{-1} \delta_{m_2}^{-1} K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 2}; v_1, v_2 \right) \right\} \omega_{mixed}(D_B f; \delta_{m_1}, \delta_{m_2})
 \end{aligned}$$

which follows that:

$$\begin{aligned}
 |K_{m_1, m_2}^*(f; v_1, v_2) - f(v_1, v_2)| &= 3 \|D_B f\|_\infty \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 0}; v_1, v_2 \right) K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &+ \left\{ \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 0}; v_1, v_2 \right) K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} \right. \\
 &+ \delta_{m_1}^{-1} \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{4, 0}; v_1, v_2 \right) K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 2}; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &+ \delta_{m_2}^{-1} \left(K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 0}; v_1, v_2 \right) K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 4}; v_1, v_2 \right) \right)^{\frac{1}{2}} \\
 &\left. + \delta_{m_1}^{-1} \delta_{m_2}^{-1} K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{2, 0}; v_1, v_2 \right) K_{m_1, m_2}^{q_1, q_2} \left(\Phi_{v_1, v_2}^{0, 2}; v_1, v_2 \right) \right\} \\
 &\times \omega_{mixed}(D_B f; \delta_{m_1}, \delta_{m_2}).
 \end{aligned}$$

From Lemma 5, we demonstrate:

$$\begin{aligned}
 |K_{m_1, m_2}^*(f; v_1, v_2) - f(v_1, v_2)| &\leq 3 \|D_B f\|_\infty \left(\sqrt{M_1 M_2} (v_1 + 1)(v_2 + 1) \right) \\
 &+ \left\{ \left(\sqrt{M_1 M_2} (v_1 + 1)(v_2 + 1) \right) \right. \\
 &+ \delta_{m_1}^{-1} \left(\sqrt{M_2} O \left(\frac{1}{[m_1]_{q_1}} \right) (v_1 + 1)^2 (v_2 + 1) \right) \\
 &+ \delta_{m_2}^{-1} \left(\sqrt{M_1} O \left(\frac{1}{[m_2]_{q_2}} \right) (v_2 + 1)^2 (v_1 + 1) \right) \\
 &\left. + \delta_{m_1}^{-1} \delta_{m_2}^{-1} \left(O \left(\frac{1}{[m_1]_{q_1}} \right) O \left(\frac{1}{[m_2]_{q_2}} \right) (v_1 + 1)(v_2 + 1) \right) \right\}
 \end{aligned}$$

$$\times \omega_{mixed}(D_B f; \delta_{m_1}, \delta_{m_2}),$$

where, in the view of Theorem 1, we have $\delta_{m_i} = O\left(\frac{1}{[m_i]_{q_i}}\right)$ for $i = 1, 2$, which completes the proof of Theorem 10. $\square$

5. Conclusions

In our investigation, we introduced the bivariate form of the previous operators [13] and then obtained the GBS-type approximation in the Bögel continuous function. These types of constructions of operators give an extended form of Jakimovski–Leviatan–Beta type q -integral operators given by [13]. We studied the bivariate properties of our new operators by the use of the modulus of continuity and the mixed-modulus of continuity and then obtained the degree of approximations for Lipschitz-class maximal functions. Moreover, we are mentioning here that the convergence at a certain point by GBS-type operators defined by (46) is more generalized than the bivariate operator (15) and the operator defined by [13]. Finally, in our future work, we will study the statistical approximation of these operators, as well as the approximation properties, by using power series methods, including some shape-preserving properties of these types operators.

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