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A New Best Proximity Point Result with an Application to Nonlinear Fredholm Integral Equations

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Abstract: In the current paper, we first introduce a new class of contractions via a new notion called p-cyclic contraction mapping by combining the ideas of cyclic contraction mapping and p-contraction mapping. Then, we give a new definition of a cyclically 0-complete pair to weaken the completeness condition on the partial metric spaces. Following that, we prove some best proximity point results for p-cyclic contraction mappings on $D \cup E$ where (D, E) is a cyclically 0-complete pair in the setting of partial metric spaces. Hence, we generalize and unify famous and well-known results in the literature of metric fixed point theory. Additionally, we present some nontrivial examples to compare our results with earlier. Finally, we investigate the sufficient conditions for the existence of a solution to nonlinear Fredholm integral equations by the results in the paper.

Keywords: best proximity point; p-cyclic contractions; nonlinear Fredholm integral equations

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1. Introduction

Banach [1] proved a fixed point result, which is known as Banach contraction principle, in 1922. In this result, it has been shown that every self mapping $\Upsilon : \Lambda \to \Lambda$ on a complete metric space (Λ, ρ) such that there is $q \in [0, 1)$ such that

$$\rho(\Upsilon\varkappa, \Upsilon\zeta) \leq q\rho(\varkappa, \zeta)$$

for all $\varkappa, \zeta \in \Lambda$ has a unique fixed point in Λ.

Banach contraction principle has been considered the beginning of metric fixed point theory. Then, many authors have generalized and improved it in different ways [2–4]. In this sense, Popescu [5] introduced the concept of p-contraction mapping and obtained a fixed point result for these mappings. According to this result, every p-contraction mapping $\Upsilon : \Lambda \to \Lambda$ on a complete metric space (Λ, ρ), that is, there exists q in $[0, 1)$ such that

$$\rho(\Upsilon\varkappa, \Upsilon\zeta) \leq q[\rho(\varkappa, \zeta) + |\rho(\varkappa, \Upsilon\varkappa) - \rho(\zeta, \Upsilon\zeta)|]$$

for all $\varkappa, \zeta \in \Lambda$ has a unique fixed point.

Recently, Kirk et al. [6] has obtained a new generalization of Banach contraction principle via a new concept of cyclic mapping. In their result, the cyclic mapping may not be continuous, unlike the Banach's result. This is the important feature of their result. Then, many researchers have studied to obtain some fixed point results for cyclic mappings [7,8].

On the other hand, one of the interesting generalizations of Banach contraction principle has been obtained by taking into account nonself mappings. Let $\emptyset \neq D, E$ be subsets of a metric space (Λ, ρ) and $\Upsilon : D \to E$ be a nonself mapping. If $D \cap E = \emptyset$, Υ cannot have a fixed point. Then, since $\rho(\varkappa, \Upsilon\varkappa) \geq \rho(D, E)$ for all $\varkappa \in D$, it is reasonable to search a point $\varkappa$ satisfying $\rho(\varkappa, \Upsilon\varkappa) = \rho(D, E)$. This point is said to be a best proximity point of Υ. Note that a best proximity point of Υ is an optimal solution for the problem $\min_{x \in D} \rho(\varkappa, \Upsilon\varkappa)$. Additionally, a best proximity point turns into a fixed point in the case of $D = E = \Lambda$. Therefore, many authors have studied on this topic [9–15].

Taking into account the ideas of both best proximity point and cyclic mapping the famous concept of cyclic contraction mapping was introduced by Eldred and Veeremani [16]. In this way, these ideas were unified.

Until now, some properties such as bounded compactness have been used to guarantee the existence of best proximity points for cyclic contraction mappings. Recently, introducing a nice notion called cyclically completeness, Karpagam and Agrawal [17] show the existence of best proximity point of a cyclic contraction mapping without using the property of bounded compactness. Then, many authors have obtained some best proximity point results with the help of this concept [18,19].

In 1994, motivated by the experience of computer science, Matthews, Ref. [20], relaxed the condition $\varkappa = \zeta$ implies $\rho(\varkappa, \zeta) = 0$ in a metric space (Λ, ρ) by introducing the partial metric spaces. Following that, many authors obtained both various fixed point results and best proximity point results in the settings of partial metric spaces [21–23]. Very recently, Romaguera [24] introduced the concept of 0-complete partial metric space. Hence, a weaker form of completeness on partial metric spaces has been obtained.

In this paper, we aim to extend and unify some famous results in the literature of metric fixed point theory, such as the main results of Eldred-Veeremani [16] and Popescu [5]. Hence, we first introduce a new class of contractions via a new notion called p-cyclic contraction mapping by combining the ideas of cyclic contraction mapping and p-contraction mapping. Then, we give a new definition of a cyclically 0-complete pair to weaken the completeness condition on the partial metric spaces. Following, we prove some best proximity point results for p-cyclic contraction mappings on $D \cup E$ where (D, E) is a cyclically 0-complete pair in a partial metric space. Additionally, we present some nontrivial examples to show the effectiveness of our work. Finally, we investigate the sufficient conditions for the existence of a solution to nonlinear Fredholm integral equations by the results in the paper.

2. Preliminaries

In this section, we give some definitons, lemmas and theorems which are important in our main result. We begin this section with the following result for cyclic mappings, which was obtained by Kirk et al. [6].

Theorem 1 ([6]). *Let $\emptyset \neq D, E$ be closed subsets of a complete metric space (Λ, ρ) and $\Upsilon : D \cup E \to D \cup E$ be a cyclic mapping, that is, $\Upsilon(D) \subseteq E$ and $\Upsilon(E) \subseteq D$. Then, Υ has a fixed point in $D \cap E$ if there is $q \in [0, 1)$ such that*

$$\rho(\Upsilon\varkappa, \Upsilon\zeta) \leq q\rho(\varkappa, \zeta)$$

for all $\varkappa \in D$ and $\zeta \in E$.

Taking into account $D \cap E = \emptyset$ in Theorem 1 the famous concept of cyclic contraction mapping was introduced by Eldred and Veeremani [16]. Then, they obtain a best proximity point result as follows:

Definition 1 ([16]). *Let $\emptyset \neq D, E$ be subsets of a metric space (Λ, ρ) and $\Upsilon : D \cup E \to D \cup E$ be a cyclic mapping. Then, $\Upsilon : D \cup E \to D \cup E$ is called cyclic contraction mapping if there exists q in $[0, 1)$ such that*

$$\rho(\Upsilon\varkappa, \Upsilon\zeta) \leq q\rho(\varkappa, \zeta) + (1 - q)\rho(D, E)$$

for all $\varkappa \in D$ and $\zeta \in E$.

Theorem 2 ([16]). *Let (Λ, ρ) be a metric space, $\emptyset \neq D, E \subseteq \Lambda$ where D, E are closed and $\Upsilon : D \cup E \to D \cup E$ be a cyclic contraction mapping. If either D or E is boundedly compact, then Υ has a best proximity point in $D \cup E$.*

Next, we recall the concept of cyclically completeness.

Definition 2 ([17]). *Let (Λ, ρ) be a metric space and $\emptyset \neq D, E \subseteq \Lambda$. A sequence $\{\varkappa_n\}$ in $D \cup E$ with $\{\varkappa_{2n}\} \subset D$ and $\{\varkappa_{2n+1}\} \subset E$ is called a cyclically Cauchy sequence if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ satisfying*

$$\rho(\varkappa_n, \varkappa_m) < \rho(D, E) + \varepsilon,$$

for all $n, m \geq n_0$ with m is odd, n is even.

Definition 3 ([17]). *A pair (D, E) of subsets of a metric space is said to be cyclically complete if for every cyclically Cauchy sequence $\{\varkappa_n\}$ in $D \cup E$, either $\{\varkappa_{2n}\}$ or $\{\varkappa_{2n+1}\}$ are convergent.*

Now, we give the definition of the partial metric space and its topological properties.

Definition 4 ([20]). *Let $\Lambda \neq \emptyset$ and $\theta : \Lambda \times \Lambda \to [0, \infty)$ be a mapping satisfying the following conditions:*

(θ_1) $\theta(\varkappa, \varkappa) = \theta(\varkappa, \zeta) = \theta(\zeta, \zeta)$ *if and only if* $\varkappa = \zeta$,
(θ_2) $\theta(\varkappa, \varkappa) \leq \theta(\varkappa, \zeta)$,
(θ_3) $\theta(\varkappa, \zeta) = \theta(\zeta, \varkappa)$,
(θ_4) $\theta(\varkappa, \eta) \leq \theta(\varkappa, \zeta) + \theta(\zeta, \eta) - \theta(\zeta, \zeta)$.

for all $\varkappa, \zeta, \eta \in \Lambda$. Then, θ is called a partial metric. Additionally, (Λ, θ) is called a partial metric space.

It can be easily seen that every metric space is a partial metric space, but the converse may not be true (see for more details [25–28]). Now, assume that θ is a partial metric on Λ. Then, there is an T_0 topology τ_θ on Λ. Additionally, the family open θ-balls

$$\{B_\theta(\varkappa, \varepsilon) : \varkappa \in \Lambda, \varepsilon > 0\}$$

is a base for the topology τ_θ where

$$B_\theta(\varkappa, \varepsilon) = \{\zeta \in \Lambda : \theta(\varkappa, \zeta) < \theta(\varkappa, \varkappa) + \varepsilon\}$$

for all $\varkappa \in \Lambda$ and $\varepsilon > 0$. If we take a sequence $\{\varkappa_n\} \subseteq \Lambda$ and $\varkappa \in \Lambda$, then it is clear that $\{\varkappa_n\}$ converges to $\varkappa$ w.r.t. τ_θ if and only if

$$\lim_{n\to\infty} \theta(\varkappa_n, \varkappa) = \theta(\varkappa, \varkappa).$$

The sequence $\{\varkappa_n\}$ is said to be Cauchy sequence if $\lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m)$ exists and is finite. Additionally, (Λ, θ) is said to be a complete partial metric space if every Cauchy sequence $\{\varkappa_n\}$ converges to a point $\varkappa$ in Λ w.r.t. τ_θ such that

$$\lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m) = \theta(\varkappa, \varkappa).$$

Definition 5 ([24]). *Let (Λ, θ) be a partial metric space and $\{\varkappa_n\}$ be a sequence in Λ.*

(i) *$\{\varkappa_n\}$ is called 0-Cauchy sequence if*

$$\lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m) = 0.$$

(ii) (Λ, θ) *is called* 0*-complete partial metric space if every* 0*-Cauchy sequence converges to a point* $\varkappa$ *in* Λ *w.r.t.* τ_θ *such that*

$$\lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m) = \theta(\varkappa, \varkappa).$$

If θ is a partial metric on Λ, then the mapping $\rho_\theta : \Lambda \times \Lambda \to [0, \infty)$ defined by

$$\rho_\theta(\varkappa, \zeta) = 2\theta(\varkappa, \zeta) - \theta(\varkappa, \varkappa) - \theta(\zeta, \zeta)$$

for all $\varkappa, \zeta \in \Lambda$ is an ordinary metric on Λ.

The following lemma shows the relation between a partial metric θ and ordinary metric ρ_θ.

Lemma 1. *Let* (Λ, θ) *be a partial metric space.*

(i) $\{\varkappa_n\}$ *is a Cauchy sequence in* (Λ, θ) *if and only if* $\{\varkappa_n\}$ *is a Cauchy sequence in* (Λ, ρ_θ)*,*
(ii) (Λ, θ) *is a complete partial metric space if and only if* (Λ, ρ_θ) *is a complete metric space,*
(iii) *Given a sequence* $\{\varkappa_n\}$ *in* Λ *and* $\varkappa \in \Lambda$*. Then, we have*

$$\lim_{n\to\infty} \rho_\theta(\varkappa_n, \varkappa) = 0 \Longleftrightarrow \theta(\varkappa, \varkappa) = \lim_{n\to\infty} \theta(\varkappa_n, \varkappa) = \lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m).$$

3. Main Results

We first give the definition of p-cyclic contraction mapping on partial metric spaces.

Definition 6. *Let* $\emptyset \neq D, E$ *be subsets of a partial metric space* (Λ, θ) *and* $\Upsilon : D \cup E \to D \cup E$ *be a cyclic mapping. Then,* Υ *is said to be p-cyclic contraction mapping if there is q in* $[0, 1)$ *such that*

$$\theta(\Upsilon\varkappa, \Upsilon\zeta) \leq q\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} + (1 - q)\theta(D, E) \tag{1}$$

for all $\varkappa \in D$ *and* $\zeta \in E$ *where* $\theta(D, E) = \inf\{\theta(\varkappa, \zeta) : \varkappa \in D \text{ and } \zeta \in E\}$*.*

The following example shows that the classes of cyclic contractions and p-contractions are proper subsets of the class of p-cyclic contractions.

Example 1. *Let* $\Lambda = \left\{(\varkappa_1, \varkappa_2, \cdots, \varkappa_n, \cdots) \subseteq \mathbb{R} : \varkappa_1 \geq 0 \text{ and } \sup_{i\in\mathbb{N}} |\varkappa_i| < \infty\right\}$ *and* $\theta : \Lambda \times \Lambda \to \mathbb{R}$ *be a function defined by*

$$\theta(\varkappa, \zeta) = max\{\varkappa_1, \zeta_1\} + \sup_{i\geq 2}|\varkappa_i - \zeta_i|$$

for $\varkappa = (\varkappa_1, \varkappa_2, \cdots, \varkappa_n, \cdots)$, $\zeta = (\zeta_1, \zeta_2 \cdots, \zeta_n, \cdots) \in \Lambda$*. It is clear that* (Λ, θ) *is a partial metric space. Consider the following subsets*

$$D = \{(t, 1, \cdots, 1, \cdots) : t \in [0, 1]\} \cup \{(1, -1, \cdots, -1, \cdots)\},$$

and

$$E = \{(t, 0, \cdots, 0, \cdots) : t \in [0, 1]\} \cup \{(0, 2, \cdots, 2, \cdots)\}.$$

Then, we have $\theta(D, E) = 1$*. Let define a mapping* $\Upsilon : D \cup E \to D \cup E$ *by*

$$\Upsilon\varkappa = \begin{cases} (\frac{t}{2}, 0, \cdots, 0, \cdots) & , \quad \varkappa = (t, 1, \cdots, 1, \cdots) \in D \\ (0, 0, \cdots, 0, \cdots) & , \quad \varkappa = (1, -1, \cdots, -1, \cdots) \\ (1, -1, \cdots, -1, \cdots) & , \quad \varkappa = (0, 2, \cdots, 2, \cdots) \\ (\frac{t}{2}, 1, \cdots, 1, \cdots) & , \quad \varkappa = (t, 0, \cdots, 0, \cdots) \in E \end{cases}.$$

Now, we shall show that Υ *is a p-cyclic contraction mapping. Then, we have the following four conditions:*

Case 1: Let $\varkappa = (t, 1, \cdots, 1, \cdots) \in D$ *and* $\zeta = (s, 0, \cdots, 0, \cdots) \in E$. *In this case, we have*

$$\begin{aligned}\theta(\Upsilon\varkappa, \Upsilon\zeta) &= max\left\{\frac{t}{2}, \frac{s}{2}\right\} + 1 \\ &= \frac{1}{2}(max\{t, s\} + 1) + \frac{1}{2} \\ &\leq \frac{1}{2}\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} + \left(1 - \frac{1}{2}\right)\theta(D, E).\end{aligned}$$

Case 2: Let $\varkappa = (t, 1, \cdots, 1, \cdots) \in D$ *and* $\zeta = (0, 2, \cdots, 2, \cdots) \in E$. *In this case, we have*

$$\begin{aligned}\theta(\Upsilon\varkappa, \Upsilon\zeta) &= 2 \\ &\leq \frac{1}{2}(1 + t + 3 - t) + \frac{1}{2} \\ &= \frac{1}{2}\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} + \left(1 - \frac{1}{2}\right)\theta(D, E).\end{aligned}$$

Case 3: Let $\varkappa = (1, -1, \cdots, -1, \cdots) \in D$ *and* $\zeta = (t, 0, \cdots, 0, \cdots) \in E$. *In this case, we have*

$$\begin{aligned}\theta(\Upsilon\varkappa, \Upsilon\zeta) &= \frac{t}{2} + 1 \\ &\leq \frac{1}{2}.2 + \frac{1}{2} \\ &\leq \frac{1}{2}\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} + \left(1 - \frac{1}{2}\right)\theta(D, E).\end{aligned}$$

Case 4: Let $\varkappa = (1, -1, \cdots, -1, \cdots) \in D$ *and* $\zeta = (0, 2, \cdots, 2, \cdots) \in E$. *In this case, we have*

$$\begin{aligned}\theta(\Upsilon\varkappa, \Upsilon\zeta) &= 2 \\ &\leq \frac{1}{2}.4 + \frac{1}{2} \\ &\leq \frac{1}{2}\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} + \left(1 - \frac{1}{2}\right)\theta(D, E).\end{aligned}$$

Hence, Υ *is a p-cyclic contraction mapping for* $q = \frac{1}{2}$. *However,* Υ *is not neither a cyclic contraction mapping nor a p-contraction mapping. If we take* $\varkappa = (0, 1, \cdots, 1, \cdots) \in D$ *and* $\zeta = (0, 2, \cdots, 2, \cdots) \in E$, *then we have*

$$\theta(\Upsilon\varkappa, \Upsilon\zeta) = 2 > 1 = q\theta(\varkappa, \zeta) + (1 - q)\theta(D, E)$$

for all $q \in [0, 1)$ *which implies that* Υ *is not a cyclic contraction mapping. Additionally, if we take* $\varkappa = (0, 1, \cdots, 1, \cdots) \in D$ *and* $\zeta = (0, 0, \cdots, 0, \cdots) \in E$, *then we have*

$$\theta(\Upsilon\varkappa, \Upsilon\zeta) = 1 > q = q\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\}$$

for all $q \in [0, 1)$, *which implies that* Υ *is not a p-contraction mapping.*

Then, we restate the definition of cyclically Cauchy sequence in the settings of partial metric spaces.

Definition 7. *Let* $\emptyset \neq D, E$ *be subsets of a partial metric space* (Λ, θ). *A sequence* $\{\varkappa_n\}$ *in* $D \cup E$ *with* $\{\varkappa_{2n}\} \subseteq D$ *and* $\{\varkappa_{2n+1}\} \subseteq E$ *is called a cyclically Cauchy sequence if for each* $\varepsilon > 0$ *there is* $n_0 \in \mathbb{N}$ *such that*

$$\theta(\varkappa_n, \varkappa_m) < \theta(D, E) + \varepsilon$$

for all $n, m \geq n_0$ with m is odd, n is even.

Note that, if $\theta(D, E) = 0$ in Definition 7, then the definition of cyclically Cauchy sequence turns into the definition of 0-Cauchy sequence.

Now, we introduce the definition of cyclically 0-complete pair in a partial metric spaces.

Definition 8. *Let $\emptyset \neq D, E$ be subsets of a partial metric space (Λ, θ). A pair (D, E) is said to be cyclically 0-complete pair if for every cyclically Cauchy sequence $\{\varkappa_n\}$ in $D \cup E$, either the sequence $\{\varkappa_{2n}\}$ has a convergent subsequence $\{\varkappa_{2n_k}\}$ to a point $\varkappa^* \in D$ w.r.t. τ_θ such that*

$$\lim_{k,l \to \infty} \theta(\varkappa_{2n_k}, \varkappa_{2n_l}) = \theta(\varkappa^*, \varkappa^*) = 0 \tag{2}$$

or $\{\varkappa_{2n+1}\}$ has a convergent subsequence $\{\varkappa_{2n_k+1}\}$ to a point $\zeta^* \in E$ w.r.t. τ_θ such that

$$\lim_{k,l \to \infty} \theta(\varkappa_{2n_k+1}, \varkappa_{2n_l+1}) = \theta(\zeta^*, \zeta^*) = 0. \tag{3}$$

Remark 1. *If D or E is a closed subset of 0-complete partial metric space (Λ, θ) and $\theta(D, E) = 0$, then (D, E) is a cyclically 0-complete pair. However, if (D, E) is a cyclically 0-complete pair, then D and E are not necessarily 0-complete. The following example shows this fact.*

Example 2. *Let $\Lambda = [0, \infty) \times \mathbb{R}$ and $\theta : \Lambda \times \Lambda \to [0, \infty)$ be a function defined as*

$$\theta(\varkappa, \zeta) = \varkappa_1 + \zeta_1 + |\varkappa_2 - \zeta_2|$$

for all $\varkappa = (\varkappa_1, \varkappa_2), \zeta = (\zeta_1, \zeta_2) \in \Lambda$. Then, (Λ, θ) is a partial metric space. Let's take the subsets $D = [0, 1) \times (-1, 0]$ and $E = [0, 1) \times [1, 2)$ of Λ. In this case, we have $\theta(D, E) = 1$. Now, we claim that (D, E) is a cyclically 0-complete pair. For this, let's take a cyclically Cauchy sequence $\{\varkappa_n\}$ in $D \cup E$ with $\{\varkappa_{2n}\} \subseteq D$ and $\{\varkappa_{2n+1}\} \subseteq E$. Then, for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\theta(\varkappa_n, \varkappa_m) < 1 + \varepsilon$$

for all $n, m \geq n_0$ with m is odd, n is even. Hence, we have that $\{\varkappa_{2n}\}$ converges to $(0, 0) \in D$ w.r.t. τ_θ and

$$\lim_{n,k \to \infty} \theta(\varkappa_{2n}, \varkappa_{2k}) = \theta((0, 0), (0, 0) = 0,$$

that is, the sequence $\{\varkappa_{2n}\}$ has a subsequence satisfying (2). However, neither D nor E is 0-complete. Indeed, if we take a sequence $\{\varkappa_n\} = \left\{\left(0, -1 + \frac{1}{n}\right)\right\}$ in D, then we have $\lim_{n,m \to \infty} \theta(\varkappa_n, \varkappa_m) = 0$. Hence, the sequence $\{\varkappa_n\}$ is a 0-Cauchy sequence in D, but it is not convergent in D. Hence, D is not 0-complete. Similarly, we can show that E is not 0-complete by considering the sequence $\{x_n\} = \left\{\left(0, 2 - \frac{1}{n}\right)\right\}$ in E.

Now, we give a new definition in partial metric spaces.

Definition 9. *Let $\emptyset \neq D$ be a subset of a partial metric space (Λ, θ). Then, D is called 0-boundedly compact if every bounded sequence $\{\varkappa_n\}$ has a convergent subsequence $\{\varkappa_{n_k}\}$ to a point $\varkappa^* \in D$ w.r.t. τ_θ such that*

$$\lim_{k,l \to \infty} \theta(\varkappa_{n_k}, \varkappa_{n_l}) = \theta(\varkappa^*, \varkappa^*) = 0.$$

Remark 2. *Note that if either D or E is a 0-boundedly compact, then the pair (D, E) is a cyclically 0-complete pair. However, the converse may not be true. Example 2 can be given to show this fact.*

Proposition 1. *Let (Λ, θ) be a partial metric space and $\emptyset \neq D, E \subseteq \Lambda$. Suppose that $\Upsilon : D \cup E \to D \cup E$ is a p-cyclic contraction mapping. If for any sequence $\{\varkappa_n\}$ defined by $\varkappa_{n+1} = \Upsilon\varkappa_n$ with the initial point $\varkappa_0 \in D$, there is $n_0 \in \mathbb{N}$ such that*

$$\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) \leq \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}),$$

then Υ has a best proximity point in $D \cup E$.

Proof. Assume that $\{\varkappa_n\}$ is an arbitrary sequence defined by $\varkappa_{n+1} = \Upsilon\varkappa_n$ with the initial point $x_0 \in D \cup E$. Since Υ is a p-cyclic contraction mapping, there is q in $[0, 1)$ such that

$$\begin{aligned} \theta(\varkappa_n, \varkappa_{n+1}) &= \theta(\Upsilon\varkappa_{n-1}, \Upsilon\varkappa_n) \\ &\leq q\{\theta(\varkappa_{n-1}, \varkappa_n) + |\theta(\varkappa_{n-1}, \Upsilon\varkappa_{n-1}) - \theta(\varkappa_n, \Upsilon\varkappa_n)|\} \\ &\quad +(1-q)\theta(D, E) \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$. Now, if there is $n_0 \in \mathbb{N}$ such that

$$\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) \leq \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}),$$

then from (4) we get

$$\begin{aligned} \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) &= \theta(\Upsilon\varkappa_{n_0}, \Upsilon\varkappa_{n_0+1}) \\ &\leq q\big\{\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) + \big|\theta(\varkappa_{n_0}, \Upsilon\varkappa_{n_0}) - \theta(\varkappa_{n_0+1}, \Upsilon\varkappa_{n_0+1})\big|\big\} \\ &\quad +(1-q)\theta(D, E) \\ &= q\big\{\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) + \big|\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) - \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2})\big|\big\} \\ &\quad +(1-q)\theta(D, E) \\ &= q\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) - q\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) + q\theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) \\ &\quad +(1-q)\theta(D, E) \\ &= q\theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) + (1-q)\theta(D, E). \end{aligned}$$

Hence, we have

$$\theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) \leq \theta(D, E).$$

Additionally, since $\theta(D, E) \leq \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2})$, we get

$$\theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) = \theta(D, E).$$

Then, we obtain

$$\theta(D, E) \leq \theta(\varkappa_{n_0}, \Upsilon\varkappa_{n_0}) \leq \theta(\varkappa_{n_0+1}, \Upsilon\varkappa_{n_0+1}) = \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}) = \theta(D, E)$$

Hence, $\varkappa_{n_0}$ and $\varkappa_{n_0+1}$ are best proximity points of Υ. □

Remark 3. *If for the sequence $\{\varkappa_n\}$ mentioned in Proposition 1 there is $n_0 \in \mathbb{N}$ such that $\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) \leq \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2})$, then Υ has a best proximity point in $D \cup E$. Therefore, we will investigate the condition $\theta(\varkappa_{n+1}, \varkappa_{n+2}) \leq \theta(\varkappa_n, \varkappa_{n+1})$ for all $n \in \mathbb{N}$ in the rest of paper.*

Proposition 2. *Let (Λ, θ) be a partial metric space and $\emptyset \neq D, E \subseteq \Lambda$. Assume that $\Upsilon : D \cup E \to D \cup E$ is a p-cyclic contraction mapping. Then, for every sequence $\{\varkappa_n\} \subseteq D \cup E$ created as in Proposition 1, we have $\theta(\varkappa_n, \varkappa_{n+1}) \to \theta(D, E)$ as $n \to \infty$.*

Proof. Let $\{\varkappa_n\} \subseteq D \cup E$ be a sequence constructed as in Proposition 1. Since Υ is a p-cyclic contraction mapping, considering Remark 3 we have

$$\begin{aligned}
\theta(\varkappa_n, \varkappa_{n+1}) &= \theta(\Upsilon\varkappa_{n-1}, \Upsilon\varkappa_n) \\
&\leq q\{\theta(\varkappa_{n-1}, \varkappa_n) + |\theta(\varkappa_{n-1}, \Upsilon\varkappa_{n-1}) - \theta(\varkappa_n, \Upsilon\varkappa_n)|\} + (1-q)\theta(D,E) \\
&= q\{\theta(\varkappa_{n-1}, \varkappa_n) + |\theta(\varkappa_{n-1}, \varkappa_n) - \theta(\varkappa_n, \varkappa_{n+1})|\} + (1-q)\theta(D,E) \\
&= q\{\theta(\varkappa_{n-1}, \varkappa_n) + \theta(\varkappa_{n-1}, \varkappa_n) - \theta(\varkappa_n, \varkappa_{n+1})\} + (1-q)\theta(D,E) \\
&= 2q\theta(\varkappa_{n-1}, \varkappa_n) - q\theta(\varkappa_n, \varkappa_{n+1}) + (1-q)\theta(D,E),
\end{aligned}$$

and so we get

$$\theta(\varkappa_n, \varkappa_{n+1}) \leq \alpha\theta(\varkappa_{n-1}, \varkappa_n) + \beta\theta(D,E)$$

for all $n \in \mathbb{N}$, where $\alpha = \frac{2q}{1+q} < 1$ and $\beta = \frac{1-q}{1+q}$. By using the last inequality, we have

$$\begin{aligned}
\theta(D,E) &\leq \theta(\varkappa_n, \varkappa_{n+1}) \\
&\leq \alpha\theta(\varkappa_{n-1}, \varkappa_n) + \beta\theta(D,E) \\
&\leq \alpha(\alpha\theta(\varkappa_{n-2}, \varkappa_{n-1}) + \beta\theta(D,E)) + \beta\theta(D,E) \\
&= \alpha^2\theta(\varkappa_{n-2}, \varkappa_{n-1}) + \beta\theta(D,E)(1+\alpha) \\
&\vdots \\
&\leq \alpha^n\theta(\varkappa_0, \varkappa_1) + \beta\theta(D,E)\left(1 + \alpha + \cdots + \alpha^{n-1}\right) \\
&= \alpha^n\theta(\varkappa_0, \varkappa_1) + \beta\theta(D,E)\left(\frac{1-\alpha^n}{1-\alpha}\right) \\
&= \alpha^n\theta(\varkappa_0, \varkappa_1) + \theta(D,E)(1-\alpha^n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we get

$$\lim_{n\to\infty} \theta(\varkappa_n, \varkappa_{n+1}) = \theta(D,E). \tag{5}$$

□

The following proposition is crucial for our main result.

Proposition 3. *Let $\emptyset \neq D, E$ be subsets of a partial metric space (Λ, θ). Assume that $\Upsilon : D \cup E \to D \cup E$ is a p-cyclic contraction mapping. Then, every sequence $\{\varkappa_n\} \subseteq D \cup E$ created as in Proposition 1 is bounded.*

Proof. Let $\{\varkappa_n\} \subseteq D \cup E$ be a sequence constructed as in Proposition 1. Hence, from Proposition 2 the sequence $\{\theta(\varkappa_{2n}, \varkappa_{2n+1})\}$ converges to $\theta(D,E)$ as $n \to \infty$, and so the sequence $\{\theta(\varkappa_{2n}, \varkappa_{2n+1})\}$ is bounded. Then, there exists $L > 0$ such that

$$\theta(\varkappa_{2n}, \varkappa_{2n+1}) \leq L$$

for all $n \in \mathbb{N}$. Since Υ is a p-cyclic contraction mapping, considering Remark 3, we have

$$\begin{aligned}
\theta(\varkappa_{2n+1}, \varkappa_1) &\leq \theta(\varkappa_{2n+1}, \varkappa_{2n+2}) + \theta(\varkappa_{2n+2}, \varkappa_1) \\
&\leq L + \theta(\Upsilon\varkappa_{2n+1}, \Upsilon\varkappa_0) \\
&\leq L + q\{\theta(\varkappa_{2n+1}, \varkappa_0) + |\theta(\varkappa_{2n+1}, \Upsilon\varkappa_{2n+1}) - \theta(\varkappa_0, \Upsilon\varkappa_0)|\} + (1-q)\theta(D,E) \\
&= L + q\{\theta(\varkappa_{2n+1}, \varkappa_0) + |\theta(\varkappa_{2n+1}, \varkappa_{2n+2}) - \theta(\varkappa_0, \varkappa_1)|\} + (1-q)\theta(D,E) \\
&= L + q\theta(\varkappa_{2n+1}, \varkappa_0) - q\theta(\varkappa_{2n+1}, \varkappa_{2n+2}) + q\theta(\varkappa_0, \varkappa_1) + (1-q)\theta(D,E) \\
&\leq L + q\theta(\varkappa_{2n+1}, \varkappa_1) + 2q\theta(\varkappa_0, \varkappa_1) + (1-q)\theta(D,E)
\end{aligned}$$

for each $n \in \mathbb{N}$ which implies that

$$\theta(\varkappa_{2n+1}, \varkappa_1) \leq \frac{L}{1-q} + \frac{2q}{1-q}\theta(\varkappa_0, \varkappa_1) + \theta(D, E).$$

Let

$$M = \frac{L}{1-q} + \frac{2q}{1-q}\theta(\varkappa_0, \varkappa_1) + \theta(D, E).$$

Hence, $\{\varkappa_{2n+1}\}$ is bounded. Additionally, we get

$$\theta(\varkappa_{2n}, \varkappa_1) \leq \theta(\varkappa_{2n}, \varkappa_{2n+1}) + \theta(\varkappa_{2n+1}, \varkappa_1) \leq L + M.$$

Hence,$\{\varkappa_{2n}\}$ is bounded. Therefore, $\{\varkappa_n\}$ is bounded. □

Theorem 3. *Let $\emptyset \neq D, E$ be subsets of a partial metric space (Λ, θ) where (D, E) is a cyclically 0-complete pair. If $\Upsilon : D \cup E \to D \cup E$ is a p-cyclic contraction mapping, Υ has a best proximity point.*

Proof. Let $\{\varkappa_n\} \subseteq D \cup E$ be a sequence constructed as in Proposition 1 with the initial point $x_0 \in D$. If there is $n_0 \in \mathbb{N}$ such that

$$\theta(\varkappa_{n_0}, \varkappa_{n_0+1}) \leq \theta(\varkappa_{n_0+1}, \varkappa_{n_0+2}),$$

then, from Proposition 1 Υ has a best proximity point. Now assume

$$\theta(\varkappa_{n+1}, \varkappa_{n+2}) \leq \theta(\varkappa_n, \varkappa_{n+1})$$

for all $n \in \mathbb{N}$. In this case, using Proposition 2, we have

$$\lim_{n\to\infty} \theta(\varkappa_n, \varkappa_{n+1}) = \theta(D, E). \tag{6}$$

Now, let us show that $\{\varkappa_n\}$ is a cyclically Cauchy sequence. Assume $n, m \in \mathbb{N}$ with $n \geq m$. Since Υ is a p-cyclic contraction mapping, we get

$$\begin{aligned}
\theta(\varkappa_n, \varkappa_m) &= \theta(\Upsilon\varkappa_{n-1}, \Upsilon\varkappa_{m-1}) \\
&\leq q\{\theta(\varkappa_{n-1}, \varkappa_{m-1}) + |\theta(\varkappa_{m-1}, \varkappa_m) - \theta(\varkappa_{n-1}, \varkappa_n)|\} + (1-q)\theta(D, E) \\
&= q\{\theta(\varkappa_{n-1}, \varkappa_{m-1}) + \theta(\varkappa_{m-1}, \varkappa_m) - \theta(\varkappa_{n-1}, \varkappa_n)\} + (1-q)\theta(D, E) \\
&\leq q\{\theta(\varkappa_{n-1}, \varkappa_{m-1}) + \theta(\varkappa_{m-1}, \varkappa_m) - \theta(D, E)\} + (1-q)\theta(D, E) \\
&= q\{\theta(\varkappa_{n-1}, \varkappa_{m-1}) + \theta(\varkappa_{m-1}, \varkappa_m)\} + (1-2q)\theta(D, E)
\end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n \geq m$. Additionally, from Proposition 2 we obtain

$$\theta(\varkappa_m, \varkappa_{m+1}) \leq \alpha\theta(\varkappa_{m-1}, \varkappa_m) + \beta\theta(D, E)$$

for all $m \in \mathbb{N}$ where $\alpha = \frac{2q}{1+q} < 1$ and $\beta = \frac{1-q}{1+q}$. Therefore, we have

$$
\begin{aligned}
\theta(D,E) &\leq \theta(\varkappa_n,\varkappa_m)\\
&= \theta(\Upsilon\varkappa_{n-1},\Upsilon\varkappa_{m-1})\\
&\leq q\theta(\varkappa_{n-1},\varkappa_{m-1}) + q\theta(\varkappa_{m-1},\varkappa_m) + (1-2q)\theta(D,E)\\
&\leq q\{q\theta(\varkappa_{n-2},\varkappa_{m-2}) + q\theta(\varkappa_{m-2},\varkappa_{m-1}) + (1-2q)\theta(D,E)\}\\
&\quad +q\{\alpha\theta(\varkappa_{m-2},\varkappa_{m-1}) + \beta\theta(D,E)\} + (1-2q)\theta(D,E)\\
&= q^2\theta(\varkappa_{n-2},\varkappa_{m-2}) + \left(q^2+\alpha q\right)\theta(\varkappa_{m-2},\varkappa_{m-1}) + \beta q\theta(D,E)\\
&\quad +(1+q)(1-2q)\theta(D,E)\\
&\quad \vdots\\
&\leq q^m\theta(\varkappa_{n-m},\varkappa_0)\\
&\quad +q^m\left\{1+\frac{\alpha}{q}+\left(\frac{\alpha}{q}\right)^2+\left(\frac{\alpha}{q}\right)^3+\cdots+\left(\frac{\alpha}{q}\right)^{m-1}\right\}\theta(\varkappa_0,\varkappa_1)\\
&\quad +\beta\left\{\begin{array}{c}\{q+q^2+q^3+\cdots+q^{m-2}\}\\ +\alpha\{q+q^2+q^3+\cdots+q^{m-1}\}\\ +\alpha^2\{q+q^2+q^3+\cdots+q^{m-2}\}\\ \vdots\\ +\alpha^{m-1}q\end{array}\right\}\theta(D,E)\\
&\quad +(1-2q)\left\{1+q+q^2+\cdots+q^{m-1}\right\}\theta(D,E)\\
&\leq q^m\theta(\varkappa_0,\varkappa_{n-m}) + q^m\sum_{i=0}^{m-1}\left(\frac{\alpha}{q}\right)^i\theta(\varkappa_0,\varkappa_1)\\
&\quad +\beta\left\{\sum_{i=1}^{\infty}q^i+\alpha\sum_{i=1}^{\infty}q^i+\alpha^2\sum_{i=1}^{\infty}q^i+\cdots+\alpha^{m-1}\sum_{i=1}^{\infty}q^i\right\}\theta(D,E)\\
&\quad +(1-2q)\sum_{i=0}^{m-1}q^i\theta(D,E)\\
&= q^m\theta(\varkappa_{n-m},\varkappa_0) + q^m\frac{1-\left(\frac{\alpha}{q}\right)^m}{1-\frac{\alpha}{q}}\theta(\varkappa_0,\varkappa_1)\\
&\quad +\beta\frac{q}{1-q}\left\{1+\alpha+\alpha^2+\cdots+\alpha^{m-1}\right\}\theta(D,E) + (1-2q)\frac{1-q^m}{1-q}\theta(D,E)
\end{aligned}
$$

Since $\{\varkappa_n\}$ is a bounded sequence, considering the last inequality we obtain

$$
\begin{aligned}
\theta(D,E) &\leq \theta(\varkappa_n,\varkappa_m)\\
&\leq q^m M + \frac{q^m-\alpha^m}{1-\frac{\alpha}{q}}\theta(\varkappa_0,\varkappa_1) + \frac{q}{1-q}\theta(D,E)\\
&\quad +(1-2q)\frac{1-q^m}{1-q}\theta(D,E)
\end{aligned}
$$

for all $n,m\in\mathbb{N}$ with $n\geq m$ and for some $M>0$. Hence, we have

$$\lim_{n,m\to\infty}\theta(\varkappa_n,\varkappa_m) = \theta(D,E).$$

Now, since (D,E) is a cyclically 0-complete pair, without loss of the generality we can assume that $\{\varkappa_{2n}\}$ has a subsequence $\{\varkappa_{2n_i}\}$ such that

$$\lim_{i,j\to\infty}\theta(\varkappa_{2n_i},\varkappa_{2n_j}) = \lim_{i\to\infty}\theta(\varkappa_{2n_i},\varkappa^*) = \theta(\varkappa^*,\varkappa^*) = 0 \tag{7}$$

for some $\varkappa^* \in D$. Moreover, we have

$$\begin{aligned} \theta(D,E) &\leq \theta(\varkappa^*, \varkappa_{2n_i-1}) \\ &\leq \theta(\varkappa^*, \varkappa_{2n_i}) + \theta(\varkappa_{2n_i-1}, \varkappa_{2n_i}). \end{aligned}$$

Taking limit $i \to \infty$ in last inequality and using the equality (6) we have

$$\lim_{i\to\infty} \theta(\varkappa^*, \varkappa_{2n_i-1}) = \theta(D,E)$$

Additionally, we get

$$\begin{aligned} \theta(\varkappa_{2n_i}, \Upsilon\varkappa^*) &= \theta(\Upsilon\varkappa_{2n_i-1}, \Upsilon\varkappa^*) \\ &\leq q\{\theta(\varkappa_{2n_i-1}, \varkappa^*) + |\theta(\varkappa^*, \Upsilon\varkappa^*) - \theta(\varkappa_{2n_i-1}, \Upsilon\varkappa_{2n_i-1})|\} \\ &\quad +(1-q)\theta(D,E) \\ &= q\{\theta(\varkappa_{2n_i-1}, \varkappa^*) + |\theta(\varkappa^*, \Upsilon\varkappa^*) - \theta(\varkappa_{2n_i-1}, \varkappa_{2n_i})|\} \\ &\quad +(1-q)\theta(D,E). \end{aligned}$$

Taking limit $i \to \infty$ in last inequality, from (7) we get

$$\begin{aligned} \theta(\varkappa^*, \Upsilon\varkappa^*) &= \lim_{i\to\infty} \theta(\varkappa_{2n_i}, \Upsilon\varkappa^*) \\ &\leq q\theta(D,E) + q|\theta(\varkappa^*, \Upsilon\varkappa^*) - \theta(D,E)| + (1-q)\theta(D,E) \\ &= q\theta(D,E) + q\theta(\varkappa^*, \Upsilon\varkappa^*) - q\theta(D,E) + (1-q)\theta(D,E). \end{aligned}$$

Therefore, we have $\theta(\varkappa^*, \Upsilon\varkappa^*) \leq \theta(D,E)$, and so $\theta(\varkappa^*, \Upsilon\varkappa^*) = \theta(D,E)$. Hence, $\varkappa^*$ is a best proximity point of Υ in D. If $\{\varkappa_{2n+1}\}$ has a subsequence $\{\varkappa_{2n_i+1}\}$ such that

$$\lim_{i,j\to\infty} \theta(\varkappa_{2n_i+1}, \varkappa_{2n_j+1}) = \lim_{i\to\infty} \theta(\varkappa_{2n_i+1}, \zeta^*) = \theta(\zeta^*, \zeta^*) = 0$$

for some $\zeta^* \in E$. Then, by the similar way, it can be shown that ζ^* is a best proximity point of Υ in E. □

Example 3. *Let $\Lambda = [0,\infty) \times [0,\infty)$ and $\theta : \Lambda \times \Lambda \to \mathbb{R}$ be a function defined by*

$$\theta(\varkappa, \zeta) = \max\{\varkappa_1, \zeta_1\} + |\varkappa_2 - \zeta_2|.$$

for $\varkappa = (\varkappa_1, \varkappa_2), \zeta = (\zeta_1, \zeta_2) \in \Lambda$. It is clear that (Λ, θ) is a partial metric space. Let

$$D = \{(a,1) : a \in [0,\infty)\},$$

and

$$E = \{(a,2) : a \in [0,\infty)\},$$

then $\theta(D,E) = 1$. Now, we show that the pair (D,E) is a cyclically 0-complete pair. Let $\{\varkappa_n\}$ be a cyclically Cauchy sequence in $D \cup E$ with $\{\varkappa_{2n}\} \subseteq D$ and $\{\varkappa_{2n+1}\} \subseteq E$. Then, we have

$$\lim_{n,m\to\infty} \theta(\varkappa_n, \varkappa_m) = \theta(D,E).$$

Hence, we get

$$\lim_{n,m\to\infty} \max\{\varkappa^1_{2n}, \varkappa^1_{2m+1}\} = 0$$

which implies that $\lim_{n\to\infty} \varkappa^1_{2n} = 0$ and $\lim_{m\to\infty} \varkappa^1_{2m+1} = 0$. Then, we have

$$\lim_{n,k\to\infty} \theta(\varkappa_{2n}, \varkappa_{2k}) = \lim_{n\to\infty} \theta(\varkappa_{2n}, (0,1)) = \theta((0,1),(0,1) = 0,$$

that is, the sequence $\{\varkappa_{2n}\}$ *has a subsequence satisfying* (2)*. If we define a mapping* $\Upsilon : D \cup E \to D \cup E$ *by*

$$\Upsilon\varkappa = \begin{cases} (\frac{a}{2}, 2) & , \quad \varkappa = (a,1) \in D \\ (\frac{a}{2}, 1) & , \quad \varkappa = (a,2) \in E \end{cases}$$

then, it is clear that Υ *is a p-cyclic contraction mapping for* $q = \frac{1}{2}$*. Hence, all conditions of Theorem* 3 *are satisfied, and so* Υ *has a best proximity point* $\varkappa^*$ *in* $D \cup E$*.*

Corollary 1. *Let* $\emptyset \neq D, E$ *be subsets of a partial metric space* (Λ, θ) *and* $\Upsilon : D \cup E \to D \cup E$ *is a p-cyclic contraction mapping. If D or E is 0-boundedly compact, then* Υ *has a best proximity point.*

Proof. From Remark 2, we know that if D or E is 0-boundedly compact, then the pair (D, E) is a cyclically 0-complete pair. Considering Theorem 3, we obtain that Υ has a best proximity point. □

Using Theorem 3 we obtain the following corollary which is a generalization of the main result of Popescu [5].

Corollary 2. *Let* (Λ, θ) *be a 0-complete partial metric space and* $\Upsilon : \Lambda \to \Lambda$ *be a mapping. If there exists q in* $[0, 1)$ *such that*

$$\theta(\Upsilon\varkappa, \Upsilon\zeta) \leq q\{\theta(\varkappa, \zeta) + |\theta(\varkappa, \Upsilon\varkappa) - \theta(\zeta, \Upsilon\zeta)|\} \tag{8}$$

then Υ *has a fixed point.*

Proof. Let (Λ, θ) be a 0-complete partial metric space. If we take $D = E = \Lambda$ and $\theta(D, E) = 0$, taking into Remark 1 we can say that (D, E) is a cyclically 0-complete pair. Additionally, from inequality (8) Υ is p-cyclic contraction mapping. Since all hypotheses of Theorem 3 are satisfied, we conclude that there exists a point $\varkappa^* \in \Lambda$ such that

$$\theta(\varkappa^*, \Upsilon\varkappa^*) = \theta(D, E) = 0$$

which implies that $\varkappa^* = \Upsilon\varkappa^*$. □

4. Application

In this section, we will consider the following nonlinear Fredholm integral equation

$$u(t) = \varphi(t) + \int_0^1 K(t, s, u(s))ds \tag{9}$$

where the functions $\varphi : [0, 1] \to [0, \infty)$ and $K : [0, 1]^2 \times [0, \infty) \to [0, \infty)$ are continuous. In mathematics and other sciences such as physics, chemistry, biology, etc., some problems can be modeled by this kind of integral equations. In general, to find an exact solution to these integral equations may not be possible. Hence, it can be used the iterative methods as an alternative way to approach the solution [29–31]. We investigate the existence a solution of nonlinear Fredholm integral equations by taking into account Corollary 2. Now, we consider the space Λ as the positive cone of $C[0, 1]$, that is,

$$\Lambda = \{u \in C[0, 1] : u(t) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Define a partial metric on Λ as

$$\theta(u, v) = \begin{cases} \sup_{t \in [0,1]} u(t) & , \quad u = v \\ \sup_{t \in [0,1]} \{u(t) + v(t)\} & , \quad u \neq v \end{cases}$$

Then, (Λ, θ) is a 0-complete partial metric space.

Theorem 4. *Assume the following conditions hold:*

(i) *the mapping* $\Upsilon : \Lambda \to \Lambda$ *defined by*

$$\Upsilon u(t) = \varphi(t) + \int_0^1 K(t, s, u(s))ds$$

for all $u \in \Lambda$ *and* $t \in [0, 1]$ *is continuous,*

(ii) *there is* q *in* $[0, 1)$ *such that*

$$K(t, s, u(s)) + K(t, s, v(s)) \le q \left\{ \begin{array}{c} u(s) + v(s) \\ + \left| \begin{array}{c} \sup_{s\in[0,1]}\{u(s) + \Upsilon u(s)\} \\ -\sup_{s\in[0,1]}\{v(s) + \Upsilon v(s)\} \end{array} \right| \end{array} \right\} - 2\varphi(t)$$

for all $t, s \in [0, 1]$ *and* $u, v \in \Lambda$.

Then, the integral Equation (9) has a positive solution.

Proof. If we prove that Υ has a fixed point, we show that Equation (9) has a solution. Now, for all $t \in [0, 1]$ and $u, v \in \Lambda$, we have

$$\begin{aligned}
\Upsilon u(t) + \Upsilon v(t) &= \varphi(t) + \int_0^1 K(t, s, u(s))ds + \varphi(t) + \int_0^1 K(t, s, v(s))ds \\
&= 2\varphi(t) + \int_0^1 \{K(t, s, u(s)) + K(t, s, v(s))\}ds \\
&\le 2\varphi(t) + \int_0^1 \left\{ q \left\{ \begin{array}{c} u(s) + v(s) \\ + \left| \begin{array}{c} \sup_{s\in[0,1]}\{u(s) + \Upsilon u(s)\} \\ -\sup_{s\in[0,1]}\{v(s) + \Upsilon v(s)\} \end{array} \right| \end{array} \right\} - 2\varphi(t) \right\} ds \\
&= 2\varphi(t) + q\int_0^1 \sup_{s\in[0,1]} \{u(s) + v(s)\}ds \\
&\quad + q\int_0^1 \left| \sup_{s\in[0,1]} \{u(s) + \Upsilon u(s)\} - \sup_{s\in[0,1]} \{v(s) + \Upsilon v(s)\} \right|)ds \\
&\quad -2\varphi(t)\int_0^1 ds \\
&= q \sup_{s\in[0,1]} \{u(s) + v(s)\} \\
&\quad + q \left| \begin{array}{c} \sup_{s\in[0,1]}\{u(s) + \Upsilon u(s)\} \\ -\sup_{s\in[0,1]}\{v(s) + \Upsilon v(s)\} \end{array} \right| \\
&= q\{\theta(u, v) + |\theta(u, \Upsilon u) - \theta(v, \Upsilon v)|\}.
\end{aligned}$$

This implies that

$$\sup_{t\in[0,1]} \{\Upsilon u(t) + \Upsilon v(t)\} \le q\{\theta(u, v) + |\theta(u, \Upsilon u) - \theta(v, \Upsilon v)|\},$$

and so we have

$$\theta(\Upsilon u, \Upsilon v) \le q\{\theta(u, v) + |\theta(u, \Upsilon u) - \theta(v, \Upsilon v)|\}.$$

Therefore, all conditions of Corollary 2 hold, and so the integral Equation (9) has a positive solution. □

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