mathematics

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Besicovitch Almost Periodic Solutions of Abstract Semi-Linear Differential Equations with Delay

Yongkun Li [1,*], Mei Huang [1] and Bing Li [2]

1 Department of Mathematics, Yunnan University, Kunming 650091, China; huangmei9614@163.com
2 School of Mathematics and Computer Science, Yunnan Minzu University, Kunming 650500, China; bli123@126.com
* Correspondence: yklie@ynu.edu.cn

Abstract: In this paper, first, we give a definition of Besicovitch almost periodic functions by using the Bohr property and the Bochner property, respectively; study some basic properties of Besicovitch almost periodic functions, including composition theorem; and prove the equivalence of the Bohr definition and the Bochner definition. Then, using the contraction fixed point theorem, we study the existence and uniqueness of Besicovitch almost periodic solutions for a class of abstract semi-linear delay differential equations. Even if the equation we consider degenerates into ordinary differential equations, our result is new.

Keywords: Besicovitch almost periodic function; semi-linear differential equation; time delay

MSC: 43A60; 47B37; 47D06

Citation: Li, Y.; Huang, M.; Li, B. Besicovitch Almost Periodic Solutions of Abstract Semi-Linear Differential Equations with Delay. *Mathematics* **2022**, *10*, 639. https://doi.org/10.3390/math10040639

Academic Editor: Christopher Goodrich

Received: 6 February 2022
Accepted: 16 February 2022
Published: 18 February 2022

1. Introduction

The concept of almost periodicity was first proposed by the Danish mathematician, H. Bohr [1–3]. Once this concept was put forward, it immediately attracted the attention of many famous mathematicians, such as W. Stepanov, H. Weyl, N. Wiener, S. Bochner, and so on. They put forward many important generalizations and variants of Bohr almost periodic concept. It is worth mentioning that the generalization of the concept of almost periodic functions can be considered in two different ways. One is to use Bohr property or Bochner property to define almost periodic functions in more general function spaces, and the other is to define almost periodic functions as the elements in the closure of the set composed of trigonometric polynomials in a more general function space according to a certain norm or seminorm. A. S. Besicovitch adopted the second approach [4–12].

On the one hand, Besicovitch's almost periodicity is a natural extension of Bohr's almost periodicity [4,5,7]. The space of Besicovitch almost periodic functions is the completion of trigonometric polynomials in the form of

$$a_1e^{i\lambda_1 t}+a_2e^{i\lambda_2 t}+\ldots+a_ne^{i\lambda_n t}$$

with respect to the seminorm

$$\|f\|_M=\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}},$$

where $1\leq\flat<\infty$ (see in [5,7]). The Besicovitch almost periodic functions defined in this way (denote by $B^\flat$ the space of all such functions) possess both the Bohr and the Bochner properties [7]. C. Corduneanu and A.S. Besicovitch have given some basic properties of Besicovitch almost periodic functions in [5,7]. Besicovitch almost periodic functions can also be defined in Marcinkiewicz space using the Bohr property or the Bochner property.

However, Besicovitch almost periodic functions defined in the above ways are equivalent classes of functions rather than usual functions. This brings great difficulties to the study of Besicovitch almost periodic solutions of differential equations. Therefore, there are few results on Besicovitch almost periodic solutions of differential equations.

On the other hand, it is well known that the existence of periodic solutions and almost periodic solutions is one of the important research contents of the qualitative theory of differential equations, see in [13–22] and the references therein. Nevertheless, there are still few results on the existence of Besicovitch almost periodic solutions for differential equations [23].

Based on the above discussion, the main purpose of this paper is to study some basic properties of Besicovitch almost periodic functions defined in the first way, that is, to give the definition of Besicovitch almost periodic functions by using the Bohr property and the Bochner property, respectively, and to study some basic properties of this kind of Besicovitch almost periodic functions, including composition theorem and proves the equivalence of Bohr's definition and Bochner's definition. As an application, the existence and uniqueness of Besicovitch almost periodic solutions for a class of abstract semi-linear delay differential equations are studied by using the contraction fixed point theorem.

The remainder of this paper is arranged as follows. In Section 2, we introduce some basic definitions and lemmas. In Section 3, we first give the definition of Besicovitch almost periodic functions by using the Bohr property, deduce some basic properties of Besicovitch almost periodic functions in Bohr's sense, including translation invariance and composition theorem, and then prove the equivalence between the concept of Besicovitch almost periodic functions defined by the Bohr property and the concept of Besicovitch almost periodic functions defined by the Bochner property. In Section 4, we study the existence and uniqueness of Besicovitch almost periodic solutions for a class of abstract semi-linear delay differential equations. In Section 5, we give a brief conclusion.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space. We denote by $C(\mathbb{R}, \mathbb{X})$ the collection of continuous functions from $\mathbb{R}$ to $\mathbb{X}$ and by $BC(\mathbb{R}, \mathbb{X})$ be the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$.

For $1 \leq \flat < \infty$, we denote by $L^{\flat}_{loc}(\mathbb{R}, \mathbb{X})$ the set of measurable functions from $\mathbb{R}$ into $\mathbb{X}$ that are $\flat$-th power locally integrable and by $M_{\flat}(\mathbb{R}, \mathbb{X})$ the collection of functions $f \in L^{\flat}_{loc}(\mathbb{R}, \mathbb{X})$ satisfying

$$\|f\|_{M_{\flat}} := \left(\limsup_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|f(t)\|^{\flat}_{\mathbb{X}} dt\right)^{\frac{1}{\flat}} < +\infty.$$

Let $L^{\infty}(\mathbb{R}, \mathbb{X})$ be the space of functions $f : \mathbb{R} \to \mathbb{X}$ that are measurable and essentially bounded. Then, space $L^{\infty}(\mathbb{R}, \mathbb{X})$ is a Banach space when it is endowed with the norm

$$\|f\|_{\infty} := \inf\{D \geq 0 : \|f(t)\|_{\mathbb{X}} \leq D \ a.e.\ t \in \mathbb{R}\}.$$

Definition 1 ([7])**.** *Let $f \in C(\mathbb{R}, \mathbb{X})$. A function f is said to be (Bohr) almost periodic if for any $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that in each interval of length l of $\mathbb{R}$, there is a number $\tau \in (a, a + l)$ with the property (Bohr's property)*

$$\sup_{t\in\mathbb{R}} \|f(t + \tau) - f(t)\|_{\mathbb{X}} < \varepsilon. \tag{1}$$

The set of such functions will be denoted by $AP(\mathbb{R}, \mathbb{X})$.

Lemma 1 ([8]). *The space $(AP(\mathbb{R},\mathbb{X}),\|\cdot\|_{AP})$ is a Banach space, where $\|f\|_{AP}=\sup_{t\in\mathbb{R}}\|f(t)\|_{\mathbb{X}}$ for $f\in AP(\mathbb{R},\mathbb{X})$. If $f\in AP(\mathbb{R},\mathbb{X})$, then f is bounded and uniformly continuous on $\mathbb{R}$ with respect to $\|\cdot\|_{AP}$.*

Lemma 2 ([7], Bochner property). *If $f\in BC(\mathbb{R},\mathbb{X})$, then $f\in AP(\mathbb{R},\mathbb{X})$ if only if the family $\mathcal{F}=\{f(t+h);h\in\mathbb{R}\}$ is relatively compact in $BC(\mathbb{R},\mathbb{X})$.*

Lemma 3 ([7]). *The space $(M_\flat(\mathbb{R},\mathbb{X}),\|\cdot\|_{M_\flat})$ is a complete seminorm space.*

3. Besicovitch Almost Periodic Functions

Denote

$$L_\flat=\left\{f:f\in M_\flat(\mathbb{R},\mathbb{X}),\ \limsup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t)\|_{\mathbb{X}}^{\flat}dt=0\right\}.$$

Lemma 4 ([7]). *The set $L_\flat$ is a closed linear manifold in $M_\flat(\mathbb{R},\mathbb{X})$.*

Define a relation in $M_\flat(\mathbb{R},\mathbb{X})$ as follows:

$$u\simeq v,\ u,v\in M_\flat(\mathbb{R},\mathbb{X}),\ if\ u-v\in L_\flat.$$

One can easily verify that the relation $\simeq$ is indeed an equivalence relation.

The quotient space $M_\flat(\mathbb{R},\mathbb{X})/L_\flat$ is the set of equivalence classes with respect to the relation $\simeq$, organized in accordance with the operations $[u]+[v]=[u+v]$, $u,v\in M_\flat(\mathbb{R},\mathbb{X})$, and $\lambda[u]=[\lambda u]$ for $u\in M_\flat(\mathbb{R},\mathbb{X})$ and for $\lambda\in\mathbb{R}$.

Definition 2 ([7]). *The quotient space $M_\flat(\mathbb{R},\mathbb{X})/L_\flat$ is called Marcinkiewicz space and denoted by $\mathcal{M}_\flat$.*

Lemma 5 ([7]). *The Marcinkiewicz function space $\mathcal{M}_\flat(\mathbb{R},\mathbb{X})$ is a Banach space with the norm defined by*

$$\|[x]\|_{\mathcal{M}_\flat}=\|x+L_\flat\|_{\mathcal{M}_\flat}:=\inf\{\|y\|_{M_\flat}:y\in x+L_\flat\},$$

where $[x]\in\mathcal{M}_\flat$.

Remark 1. *One can easily check that $\|x+L_\flat\|_{\mathcal{M}_\flat}=\|x\|_{M_\flat}$ for every $x\in M_\flat(\mathbb{R},\mathbb{X})$.*

Remark 2. *If $f,g\in\mathcal{M}_\flat(\mathbb{R},\mathbb{X})$, then $(f+L_\flat)-(g+L_\flat)=f-g+L_\flat$.*

Definition 3 (Bohr's definition). *Let $f\in\mathcal{M}_\flat(\mathbb{R},\mathbb{X})$, then f is said to be Besicovitch almost periodic if f posses the following two properties:*

(1) *f is uniformly continuous in the $\mathcal{M}_\flat$-norm, and*
(2) *for every $\varepsilon>0$, there is an $l=l(\varepsilon)>0$ such that each interval with length l contains a number $\tau=\tau(\varepsilon)$ satisfying*

$$\|f(t+\tau)-f(t)\|_{\mathcal{M}_\flat}<\varepsilon.$$

The τ is called a ε-translation number of f. The set of such functions will be denoted by $B^\flat_{AP}(\mathbb{R},\mathbb{X})$. For $f\in B^\flat_{AP}(\mathbb{R},\mathbb{X})$, we denote $\|f\|_{B^\flat}=\|f\|_{\mathcal{M}_\flat}$.

Remark 3. *R. Doss has used the Bohr property to define Besicovitch almost periodic functions in space $L^\flat_{loc}(\mathbb{R},\mathbb{R})$ [4,24,25]. Here, we use the Bohr property to define Besicovitch almost periodic functions in Marcinkiewicz space space $M_\flat(\mathbb{R},\mathbb{X})/L_\flat$. Therefore, Doss's Besicovitch almost periodic functions are ordinary functions, while the Besicovitch almost periodic functions defined by Definition 3 are not ordinary functions but the equivalence classes of functions.*

In the sequel, let us denote by $UC\mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$ the set of all functions $f \in \mathcal{M}_\flat$ that satisfy the condition (1) in Definition 3.

Remark 4. *It should be noted that, in general, when we write that a function $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ we do not have in mind the function f itself, it does represent a whole class that is equivalent to f.*

Lemma 6. *Let $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ and $\alpha \in \mathbb{R}$, then $f(\cdot - \alpha) \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$.*

Proof. As $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, for every $\varepsilon > 0$, there is an $l = l(\varepsilon) > 0$ such that each interval with length l contains a number $\tau = \tau(\varepsilon)$ satisfying

$$\|f(t+\tau) - f(t)\|_{B^\flat} < \varepsilon.$$

Noting that

$$\begin{aligned}
\|f(t+\tau+\alpha) - f(t+\alpha)\|_{B^\flat} &= \left(\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T+\alpha}^{T+\alpha} \|f(t+\tau) - f(t)\|^\flat_{\mathbb{X}} dt \right)^{\frac{1}{\flat}} \\
&\le \left(\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T-\alpha}^{T+\alpha} \|f(t+\tau) - f(t)\|^\flat_{\mathbb{X}} dt \right)^{\frac{1}{\flat}} < \varepsilon.
\end{aligned}$$

One concludes that $f(\cdot - \alpha) \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. This completes the proof. □

Remark 5. *In the proof of Lemma 6, the f's in the left side of the formula are equivalence classes and the f's in the right side are representative elements of the equivalence classes. We will not make an explanation later, it is clear from the context.*

Lemma 7. *If $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ and $\lambda \in \mathbb{R}$, then $\lambda f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$.*

Proof. For every $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that each interval of length l contains a τ satisfying

$$\|f(t+\tau) - f(t)\|_{B^\flat} < \varepsilon.$$

Noting the fact that

$$\|\lambda f(t+\tau) - \lambda f(t)\|_{B^\flat} = |\lambda| \cdot \|f(t+\tau) - f(t)\|_{B^\flat} < |\lambda|\varepsilon.$$

We have $\lambda f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. The proof is completed. □

Lemma 8. *If $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, then we have that the family $\mathcal{F} = \{f(t+h); h \in \mathbb{R}\}$ is relatively compact in the topology of $\mathcal{M}_\flat$.*

Proof. If $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, then $\{f(t+h_n);\ n \ge 1\} \subset \mathcal{F}$ is a sequence of translates of f. By $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, there exists an $l = l(\varepsilon) > 0$ such that every interval $[-h_n, -h_n + l]$ contains a number $\tau_n = \tau_n(\varepsilon)$ in this interval with the property that

$$\|f(t+\tau_n) - f(t)\|_{B^\flat} < \varepsilon. \tag{2}$$

From Definition 3, one has that f is uniformly continuous in the norm $\|\cdot\|_{B^\flat}$, that is, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\|f(t+h) - f(t)\|_{B^\flat} < \varepsilon, \tag{3}$$

for all $h \in \mathbb{R}$ with $|h| < \delta$. Moreover, for every $h_n \in \mathbb{R}$, one has $\tau_n + h_n \in [0, l]$, which implies that there exists a subsequence $\{h_{n_k}; k \geq 1\} \subset \{h_n\}_{n \in \mathbb{N}}$ such that $\{\tau_n + h_{n_k}; k \geq 1\}$ is convergent. Consequently, there is an integer $N = N(\varepsilon) > 0$ such that

$$|\tau_n + h_{n_k} - \tau_n + h_{n_{k'}}| < \delta,$$

for $k,\ k' > N$. It follows from (2) and (3) that

$$\begin{aligned}
\|f(t + h_{n_k}) - f(t + h_{n_{k'}})\|_{B^\flat} &\leq \|f(t + h_{n_k}) - f(t + \tau_n + h_{n_k})\|_{B^\flat} \\
&\quad + \|f(t + \tau_n + h_{n_k}) - f(t + \tau_n + h_{n_{k'}})\|_{B^\flat} \\
&\quad + \|f(t + \tau_n + h_{n_{k'}}) - f(t + h_{n_{k'}})\|_{B^\flat} \\
&< 3\varepsilon,
\end{aligned}$$

which implies that $\{f(t + h_{n_k}); k \geq 1\}$ is uniformly convergent in the $B^\flat$-norm, thus, the family $\mathcal{F} = \{f(t+h); h \in \mathbb{R}\}$ is relatively compact. This completes the proof. □

Definition 4 (Bochner's definition)**.** *Function $f \in UC\mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$ is called Besicovitch almost periodic if for every sequence $\{h'_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, there exists a subsequence $\{h_n\}_{n \in \mathbb{N}} \subset \{h'_n\}_{n \in \mathbb{N}}$ such that $\{f(t + h_n); n \in \mathbb{N}\}$ converges in the norm $\|\cdot\|_{\mathcal{M}_\flat}$.*

Lemma 9. *The Bohr definition and the Bochner definition of Besicovitch almost periodicity are equivalent.*

Proof. According to Lemma 8, one can easily see that Bohr's definition implies the Bochner definition. To finish the proof, we only need to prove that the Bochner definition implies the Bohr definition, that is, for $f \in UC\mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$, we need to show that if the set $\mathcal{F} = \{f(t+h); h \in \mathbb{R}\}$ is relatively compact set of $\mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$, then f posses Bohr's property, that is, for each $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that in every interval $(a, a + l) \subset \mathbb{R}$, there exists a τ such that

$$\|f(t+\tau) - f(t)\|_{\mathcal{M}_\flat} = \left(\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t+\tau) - f(t)\|_{\mathbb{X}}^\flat dt \right)^{\frac{1}{\flat}} < \varepsilon. \tag{4}$$

Otherwise, if f does not possess Bohr's property, then one can find a $\varepsilon_0 > 0$ such that $l = l(\varepsilon_0)$ does not exist. That is to say, for every $l > 0$, we can find an interval of length l that does not contain any τ such that (4) holds. Now, choose an arbitrary $h_1 \in \mathbb{R}$ and an interval $(a_1, b_1) \subset \mathbb{R}$ of length larger than $2|h_1|$ such that (a_1, b_1) does not contain any τ satisfying (4). Set $h_2 = \frac{a_1 + b_1}{2}$, then $h_2 - h_1 \in (a_1, b_1)$, and thus $h_2 - h_1$ cannot be chosen as τ, which implies that

$$\|f(t + h_2 - h_1) - f(t)\|_{\mathcal{M}_\flat} = \left(\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t + h_2 - h_1) - f(t)\|_{\mathbb{X}}^\flat dt \right)^{\frac{1}{\flat}} \geq \varepsilon_0.$$

Let us choose an interval $(a_2, b_2) \subset \mathbb{R}$ of length larger than $2(|h_1| + |h_2|)$ that does not contain any τ such that (4) holds. Continue this process, letting $h_3 = \frac{a_2 + b_2}{2}$, then one has $h_3 - h_2, h_3 - h_1 \in (a_2, b_2)$, and this implies that $h_3 - h_2$ and $h_3 - h_1$ cannot be chosen as τ, that is,

$$\|f(t + h_3 - h_1) - f(t)\|_{\mathcal{M}_\flat} = \left(\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t + h_3 - h_1) - f(t)\|_{\mathbb{X}}^\flat dt \right)^{\frac{1}{\flat}} \geq \varepsilon_0$$

and

$$\|f(t+h_3-h_2)-f(t)\|_{\mathcal{M}_\flat}=\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t+h_3-h_2)-f(t)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\geq\varepsilon_0.$$

Proceeding similarly, one constructs the numbers $h_4,\ h_5,\ldots,$ with the property that none of the differences h_i-h_j, $i>j$, could be chosen as number τ in (4). Therefore, for $i>j$, one gets

$$\begin{aligned}\|f(t+h_i-h_j)-f(t)\|_{\mathcal{M}_\flat}&=\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t+h_i-h_j)-f(t)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\\&=\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T-h_j}^{T-h_j}\|f(t+h_i)-f(t+h_j)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\geq\varepsilon_0.\end{aligned}$$

Therefore, one obtains

$$\begin{aligned}\|f(t+h_i)-f(t+h_j)\|_{\mathcal{M}_\flat}&=\left(\limsup_{T\to+\infty}\frac{1}{2(T+h_j)}\int_{-T-h_j}^{T+h_j}\|f(t+h_i)-f(t+h_j)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\\&=\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T-h_j}^{T+h_j}\|f(t+h_i)-f(t+h_j)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\\&\geq\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T-h_j}^{T-h_j}\|f(t+h_i)-f(t+h_j)\|_{\mathbb{X}}^{\flat}dt\right)^{\frac{1}{\flat}}\geq\varepsilon_0,\end{aligned}$$

which contradicts the property of relative compactness of the family $\mathcal{F}=\{f(t+h);h\in\mathbb{R}\}$. Consequently, f does posses Bohr's property. The proof is completed. □

Lemma 10. *Let $\{f_n\}_{n\in\mathbb{N}}\subset B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$ satisfy that there exists $f\in\mathcal{M}_\flat(\mathbb{R},\mathbb{X})$ with $\|f_n-f\|_{B^\flat}\to 0$ as $n\to\infty$, then $f\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$.*

Proof. It is easy to see that f is uniformly continuous in the norm $\|\cdot\|_{B^\flat}$. As $\|f_n-f\|_{B^\flat}\to 0$ as $n\to\infty$, for any $\varepsilon>0$, there is a large enough integer $N_1=N_1(\varepsilon)>0$ such that

$$\|f_{N_1}-f\|_{B^\flat}<\varepsilon.$$

Because $f_{N_1}\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$, then there is an $l=l(\varepsilon)$ such that every interval of length l contains a number τ satisfying

$$\|f_{N_1}(t+\tau)-f_{N_1}(t)\|_{B^\flat}<\varepsilon.$$

Consequently, we have

$$\begin{aligned}&\|f(t+\tau)-f(t)\|_{B^\flat}\\&\leq\|f(t+\tau)-f_{N_1}(t+\tau)\|_{B^\flat}+\|f_{N_1}(t+\tau)-f_{N_1}(t)\|_{B^\flat}+\|f_{N_1}(t)-f(t)\|_{B^\flat}\\&<\varepsilon+\varepsilon+\varepsilon=3\varepsilon,\end{aligned}$$

which implies that $f\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$. The proof is completed. □

Using the proof method of Proposition 3.21 in [7], one can easily show that

Lemma 11. *If $f_k\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X}),k=1,2,\ldots,n$, then for each $\varepsilon>0$, there are common ε-translation numbers for these functions.*

Lemma 12. *Let $f, g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, then $f + g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$.*

Proof. According to Lemma 11, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a τ with the property

$$\|f(t+\tau) - f(t)\|_{B^\flat} < \varepsilon \text{ and } \|g(t+\tau) - g(t)\|_{B^\flat} < \varepsilon.$$

Hence, we have

$$\begin{aligned}
&\|f(t+\tau)g(t+\tau) - f(t)g(t)\|_{B^\flat} \\
\leq{}& \|f(t+\tau)g(t+\tau) - f(t)g(t+\tau)\|_{B^\flat} + \|f(t)g(t+\tau) - f(t)g(t)\|_{B^\flat} \\
\leq{}& \|f(t+\tau) - f(t)\|_{B^\flat}\|g(t+\tau)\|_{B^\flat} + \|f(t)\|_{B^\flat}\|g(t+\tau) - g(t)\|_{B^\flat} \\
\leq{}& \varepsilon\|g(t+\tau)\|_{B^\flat} + \varepsilon\|f(t)\|_{B^\flat},
\end{aligned}$$

therefore, $f + g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. The proof is complete. □

Lemma 13. *The space $B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ is a Banach space with the norm $\|\cdot\|_{B^\flat}$.*

Proof. Clearly, $B^\flat_{AP}(\mathbb{R}, \mathbb{X}) \subset \mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$ and by Lemma 10, the space $B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ is closed. Therefore, $B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ is also a Banach space. The proof is complete. □

Remark 6. *Unlike in the case of Bohr almost periodic functions, the product $f \cdot g$ does not necessarily belong to $B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. However, we replace a Besicovitch almost periodic function with a Bohr almost periodic function, the result is still valid.*

Lemma 14. *If $f \in AP(\mathbb{R}, \mathbb{X})$ and $g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, then we have $f \cdot g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$.*

Proof. Clearly, we have $f \cdot g \in \mathcal{M}_\flat$ and $f \cdot g$ is uniformly continuous in the norm $\|\cdot\|_{B^\flat}$. As $f \in AP(\mathbb{R}, \mathbb{X})$, for any sequence $\{h_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\{h_{1n}\}_{n\in\mathbb{N}}$ of $\{h_n\}_{n\in\mathbb{N}}$ such that $\{f(t + h_{1n}); n \in \mathbb{N}\}$ converges uniformly on $\mathbb{R}$. As $g \in B^\flat(\mathbb{R}, \mathbb{X})$ it follows that for the sequence $\{g(t + h_{1n}); n \in \mathbb{N}\}$, there exists a subsequence $\{h_{2n}\}_{n\in\mathbb{N}} \subset \{h_{1n}\}_{n\in\mathbb{N}}$ such that the sequence of functions $\{g(t + h_{2n}); n \in \mathbb{N}\}$ converges in the norm $\|\cdot\|_{B^\flat}$. It is then clear that $\{f(t + h_{2n}); n \in \mathbb{N}\}$ converges uniformly in $t \in \mathbb{R}$.

Now, for any $\varepsilon > 0$, there exists a large enough integer $N = N(\varepsilon)$ such that

$$\begin{aligned}
&\|f(t+h_{2n})g(t+h_{2n}) - f(t)g(t)\|_{B^\flat} \\
\leq{}& \|f(t+h_{2n})g(t+h_{2n}) - f(t+h_{2n})g(t)\|_{B^\flat} + \|f(t+h_{2n})g(t) - f(t)g(t)\|_{B^\flat} \\
\leq{}& \|f(t+h_{2n})\|_{B^\flat}\|g(t+h_{2n}) - g(t)\|_{B^\flat} + \|f(t+h_{2n}) - f(t)\|_{B^\flat}\|g(t)\|_{B^\flat} \\
\leq{}& \|f(t+h_{2n})\|_{AP}\|g(t+h_{2n}) - g(t)\|_{B^\flat} + \|f(t+h_{2n}) - f(t)\|_{AP}\|g(t)\|_{B^\flat} \\
<{}& \|f\|_{AP}\varepsilon + \varepsilon\|g(t)\|_{B^\flat},
\end{aligned}$$

for $n \geq N$, which means that $\{f(t+h_{2n})g(t+h_{2n}); n \in \mathbb{N}\}$ converges in the norm $\|\cdot\|_{B^\flat}$. Hence, $f \cdot g$ satisfies the Bochner definition, namely, $f \cdot g \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. The proof is complete. □

Lemma 15. *If $x \in AP(\mathbb{R}, \mathbb{R})$ and $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, then $f(\cdot - x(\cdot)) \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$.*

Proof. Step 1. We will show that $f(\cdot - x(\cdot)) \in UC\mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$. Since $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$ and $x \in AP(\mathbb{R}, \mathbb{R})$, f and x are bounded and uniformly continuous in their corresponding norms, and for every $\varepsilon > 0$, there exists $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that

$$\|f(t+h) - f(t)\|_{B^\flat} < \varepsilon \text{ and } \|x(t+h) - x(t)\|_{AP} < \varepsilon, \tag{5}$$

for $|h| < \delta, h \in \mathbb{R}$. By the fact that the set $\{t - x(t) : t \in \mathbb{R}\} \subset \mathbb{R}$ and the boundedness of f and x, one can easily get that $f(\cdot - x(\cdot))$ is bounded in $\|\cdot\|_{B^\flat}$. Thus, $f(\cdot - x(\cdot)) \in \mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$. From (5), we have

$$\begin{aligned}
&\|f(t+h-x(t+h)) - f(t-x(t))\|_{B^\flat}\\
\le &\|f(t+h-x(t+h)) - f(t+h-x(t))\|_{B^\flat}\\
&+ \|f(t+h-x(t) - f(t-x(t)\|_{B^\flat}\\
< &2\varepsilon.
\end{aligned}$$

Therefore, $f(\cdot - x(\cdot))$ is uniformly continuous.

Step 2. We will show that $f(\cdot - x(\cdot)) \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. Thanks to Lemma 9, to complete the proof, we only need to show the family $\mathcal{F} = \{f(t+h-x(t+h)) : h \in \mathbb{R}\}$ is relatively compact.

As $f \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$, $x \in AP(\mathbb{R}, \mathbb{R})$, by Lemmas 2 and 9, for any $\varepsilon > 0$ and $\{h_q\}_{q\in\mathbb{N}} \subset \mathbb{R}$, we can have

$$\begin{aligned}
&\|f(t+h_{1q}) - f(t+h_{1q'})\|_{B^\flat} < \varepsilon, \quad \text{for} \quad q, q' \ge N_1(\varepsilon),\\
&|x(t+h_{1q}) - x(t+h_{1q'})| < \delta(\varepsilon), \quad \text{for} \quad q, q' \ge N_2(\varepsilon),
\end{aligned}$$

where $\{h_{1q}\}_{q\in\mathbb{N}}, \{h_{1q'}\}_{q\in\mathbb{N}} \subset \{h_q\}_{q\in\mathbb{N}}$ and $\delta(\varepsilon)$ is mentioned in Step 1. By the uniform continuity of f in the norm $\|\cdot\|_{B^\flat}$, denote $N = \max\{N_1, N_2\}$, one gets

$$\begin{aligned}
&\|f(t+h_{1q} - x(t+h_{1q})) - f(t+h_{1q'} - x(t+h_{1q'}))\|_{B^\flat}\\
\le &\|f(t+h_{1q} - x(t+h_{1q})) - f(t+h_{1q'} - x(t+h_{1q}))\|_{B^\flat}\\
&+ \|f(t+h_{1q'} - x(t+h_{1q}) - f(t+h_{1q'} - x(t+h_{1q'})\|_{B^\flat}\\
< &\varepsilon + \varepsilon = 2\varepsilon,
\end{aligned}$$

for q, $q' \ge N$, which means that the function $f(\cdot - x(\cdot))$ satisfies the Bochner definition. Therefore, $f(\cdot - x(\cdot)) \in B^\flat_{AP}(\mathbb{R}, \mathbb{X})$. The proof is complete. □

Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be a Banach space.

Definition 5. *A function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$, $(t, x) \mapsto f(t, x)$ with $f(\cdot, x) \in \mathcal{M}_\flat$ for each $x \in \mathbb{X}$, is said to be Besicovitch almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{X}$ if the following properties hold true:*

(1) *For every bounded subset $\mathbb{B} \subset \mathbb{X}$, $f(t, x)$ is uniformly continuous in $x \in \mathbb{B}$ with respect to the norm $\|\cdot\|_{B^\flat}$ uniformly for $t \in \mathbb{R}$.*

(2) *For each $\varepsilon > 0$ and each bounded subset $\mathbb{B} \subset \mathbb{X}$, there exists $l = l(\varepsilon) > 0$ such that every interval of length l contains a number τ with the property*

$$\left(\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t+\tau, x) - f(t, x)\|^\flat_{\mathbb{Y}} dt\right)^{\frac{1}{\flat}} < \varepsilon,$$

uniformly in $x \in \mathbb{B}$.

The collection of these functions will be denoted by $B^\flat_{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$.

4. Besicovitch Almost Periodic Solutions

Consider the following semi-linear differential equation with delay:

$$x'(t) = Ax(t) + F(t, x(t), x(t - \tau(t))), \tag{6}$$

in which A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \ge 0\}$ on a Banach space $\mathbb{X}$, $F : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a measurable function.

A function $x : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution of (6), if for $x(\theta) = \varphi(\theta), \theta \in [t_0 - \bar{\tau}, t_0]$, where $\varphi \in C(t_0 - \bar{\tau}, t_0]$ and $\bar{\tau} = \sup\limits_{t \in \mathbb{R}} \tau(t)$, it satisfies the integral equation

$$x(t) = T(t - t_0)x(t_0) + \int_{t_0}^{t} T(t - s)F(s, x(s), x(s - \tau(s)))ds, \; t \geq t_0.$$

In the following, in order to distinguish, let

$$L_{\flat_1} = \left\{ f : f \in M_\flat(\mathbb{R}, \mathbb{X}), \; \limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\|_{\mathbb{X}}^{\flat} dt = 0 \right\},$$

and $L_{\flat_2}$ be the set of functions $f \in M_\flat(\mathbb{R} \times \mathbb{X}^2, \mathbb{X})$ with

$$\limsup_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t, x)\|_{\mathbb{X}}^{\flat} dt = 0$$

uniformly in $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}^2$. The quotient spaces corresponding to $L_{\flat_1}$ and $L_{\flat_2}$ are denoted by $\mathcal{M}_{\flat_1}$ and $\mathcal{M}_{\flat_2}$, respectively.

We assume that the following conditions hold throughout the rest of the paper:

(H_1) $\tau \in AP(\mathbb{R}, \mathbb{R}^+)$ and F satisfies $F + L_{\flat_2} \in B^{\flat}_{AP}(\mathbb{R} \times \mathbb{X}^2, \mathbb{X})$.

(H_2) There exist positive constants $\mathfrak{L}_1$ and $\mathfrak{L}_2$ such that for all $x_1, y_1, x_2, y_2 \in \mathbb{X}$ and $t \in \mathbb{R}$,

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\|_{\mathbb{X}} \leq \mathfrak{L}_1 \|x_1 - x_2\|_{\mathbb{X}} + \mathfrak{L}_2 \|y_1 - y_2\|_{\mathbb{X}}$$

and $F(t, 0, 0) = 0$.

(H_3) A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t) : t \geq 0\}$, that is, there exist numbers $N, \lambda > 0$ such that $\|T(t)\| \leq Ne^{-\lambda t}, t \geq 0$.

(H_4) $\kappa := \frac{N(\mathfrak{L}_1 + \mathfrak{L}_2)}{\lambda} < 1$, where N is defined in (H_3).

Let

$$\mathbb{W} = \{y \in L^\infty(\mathbb{R}, \mathbb{X}) : \; y + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})\}$$

with the norm $\| \cdot \|_{\mathbb{W}} := \| \cdot \|_\infty$.

Lemma 16. *The space $\mathbb{W}$ is a Banach space when it is endowed with the norm $\| \cdot \|_{\mathbb{W}}$.*

Proof. Let $\{f_n; n \geq 1\}$ be a Cauchy sequence in $\mathbb{W}$. Then, for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$\|f_n - f_m\|_{\mathbb{W}} < \varepsilon, \; n, m > N.$$

Because $(L^\infty(\mathbb{R}, \mathbb{X}), \| \cdot \|_\infty)$ is a Banach space and $\mathbb{W} \subset L^\infty(\mathbb{R}, \mathbb{X})$, so there exists $f \in L^\infty(\mathbb{R}, \mathbb{X})$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. As $\|f_n - f\|_{M_\flat} \leq \|f_n - f\|_\infty$, we have

$$\|f_n - f\|_{M_\flat} \to 0 \quad \text{as } n \to \infty, \tag{7}$$

which implies that $f \in M_\flat(\mathbb{R}, \mathbb{X})$ by the completeness of $M_\flat(\mathbb{R}, \mathbb{X})$. Moreover, we have $f + L_{\flat_1} \in \mathcal{M}_\flat(\mathbb{R}, \mathbb{X})$. In view of (7), for every $\varepsilon > 0$, there is a large enough $N_1 = N_1(\varepsilon)$ such that $\|f_{N_1} - f\|_{M_\flat} < \varepsilon$. Due to the fact that $f_{N_1} + L_{\flat_1} \in B^{\flat_1}_{AP}(\mathbb{R}, \mathbb{X})$ is uniformly continuous with respect to the norm $\| \cdot \|_{B^\flat}$, there is a $\delta = \delta(\varepsilon)$ such that

$$\|f_{N_1}(t) + L_{\flat_1} - (f_{N_1}(t + h) + L_{\flat_1})\|_{B^\flat} < \varepsilon,$$

for $|h| < \delta$. Therefore, we have

$$\begin{aligned}\|f(t+h)+L_{\flat_1}-(f(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \leq & \|f(t+h)+L_{\flat_1}-(f_{N_1}(t+h)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ & +\|f_{N_1}(t+h)+L_{\flat_1}-(f_{N_1}(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ & +\|f_{N_1}(t)+L_{\flat_1}-(f(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ < & \, 3\varepsilon,\end{aligned}$$

which means that $f+L_{\flat_1}$ is uniformly continuous in the norm $\|\cdot\|_{\mathcal{M}_\flat}$. Denote by τ the ε-translation number of $f_{N_1}+L_{\flat_1}$, then

$$\begin{aligned}\|f(t+\tau)+L_{\flat_1}-(f(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \leq & \|f(t+\tau)+L_{\flat_1}-(f_{N_1}(t+\tau)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ & +\|f_{N_1}(t+\tau)+L_{\flat_1}-(f_{N_1}(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ & +\|f_{N_1}(t)+L_{\flat_1}-(f(t)+L_{\flat_1})\|_{\mathcal{M}_\flat} \\ < & \varepsilon+\varepsilon+\varepsilon=3\varepsilon,\end{aligned}$$

which yields $f+L_{\flat_1}\in B^{\flat_1}_{AP}(\mathbb{R},\mathbb{X})$. Therefore, $f\in\mathbb{W}$. The proof is completed. □

Lemma 17. *Let* (H_1) *and* (H_2) *hold. If* $x\in\mathbb{W}$*, then* $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))\in L^\infty(\mathbb{R},\mathbb{X})$ *and* $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))+L_{\flat_1}\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$.

Proof. Step 1, we will prove that $x(\cdot-\tau(\cdot))\in\mathbb{W}$. As $\tau\in AP(\mathbb{R},\mathbb{R}^+)$, so $\{t-\tau(t);\ t\in\mathbb{R}\}\subset\mathbb{R}$, then

$$\|x(t-\tau(t))\|_{\mathbb{X}}\leq\|x\|_\infty,$$

which means $x(\cdot-\tau(\cdot))\in L^\infty(\mathbb{R},\mathbb{X})$. In addition, in view of Lemma 15, we see that $x(\cdot-\tau(\cdot))+L_{\flat_1}\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$. Thus, $x(\cdot-\tau(\cdot))\in\mathbb{W}$.

Step 2, we will show that $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))\in L^\infty(\mathbb{R},\mathbb{H})\cap M_\flat$. Based on (H_2), for all $t\in\mathbb{R}$, one has

$$\|F(t,x(t),x(t-\tau(t)))\|_{\mathbb{X}}\leq\mathfrak{L}_1\|x(t)\|_{\mathbb{X}}+\mathfrak{L}_2\|x(t-\tau(t))\|_{\mathbb{X}},$$

which implies $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))\in L^\infty(\mathbb{R},\mathbb{H})$. Moreover, by virtue of the Minkowski inequality, one get

$$\begin{aligned}&\left(\frac{1}{2T}\int_{-T}^{T}\|F(t,x(t),x(t-\tau(t)))\|^\flat_{\mathbb{X}}dt\right)^{\frac{1}{\flat}}\\ \leq&\left(\frac{1}{2T}\int_{-T}^{T}(\mathfrak{L}_1\|x(t)\|_{\mathbb{X}}+\mathfrak{L}_2\|x(t-\tau(t))\|_{\mathbb{X}})^\flat dt\right)^{\frac{1}{\flat}}\\ \leq&\,\mathfrak{L}_1\left(\frac{1}{2T}\int_{-T}^{T}\|x(t)\|^\flat_{\mathbb{X}}dt\right)^{\frac{1}{\flat}}+\mathfrak{L}_2\left(\frac{1}{2T}\int_{-T}^{T}\|x(t-\tau_q(t))\|^\flat_{\mathbb{X}}dt\right)^{\frac{1}{\flat}},\end{aligned}$$

then

$$\|F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))\|_{M_\flat}\leq\mathfrak{L}_1\|x\|_{M_\flat}+\mathfrak{L}_2\|x\|_{M_\flat},$$

which implies $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))\in M_\flat$.

Step 3, we will prove that $F(\cdot,x(\cdot),x(\cdot-\tau(\cdot)))+L_{\flat_1}\in B^{\flat}_{AP}(\mathbb{R},\mathbb{X})$. As $x,x(\cdot-\tau(\cdot))\in L^\infty(\mathbb{R},\mathbb{X})$, there exists a bounded subset $\mathbb{B}\subset\mathbb{X}^2$ such that

$$(x(t),x(t-\tau(t)))\in\mathbb{B},$$

for all $t\in\mathbb{R}$. From Lemma 11 it follows that for each $\varepsilon>0$, there exists an $\ell=\ell(\varepsilon)>0$ such that every interval of length ℓ contains an h satisfying

$$\left(\limsup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|F(t+h,u,v)-F(t,u,v)\|^\flat_{\mathbb{X}}dt\right)^{\frac{1}{\flat}}<\varepsilon,\ \ u,v\in\mathbb{X},$$

$$\begin{aligned}
&\|x(t+h) + L_{\flat_1} - (x(t) + L_{\flat_1})\|_{B^\flat} \\
=&\|x(t+h) - x(t)\|_{M_\flat} < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\|x(t+h-\tau(t+h)) + L_{\flat_1} - (x(t-\tau(t)) + L_{\flat_1})\|_{B^\flat} \\
=&\|x(t+h-\tau(t+h)) - x(t-\tau(t))\|_{M_\flat} < \varepsilon.
\end{aligned}$$

Based on the inequalities above, for $t \in \mathbb{R}$ and $|h| < \delta = \min\{\delta_1, \delta_2\}$, one gets

$$\begin{aligned}
&\|F(t+h, x(t+h), x(t+h-\tau(t+h))) + L_{\flat_1} - (F(t, x(t), x(t-\tau(t))) + L_{\flat_1})\|_{\mathcal{M}_\flat} \\
=&\|F(t+h, x(t+h), x(t+h-\tau(t+h))) - F(t, x(t), x(t-\tau(t)))\|_{M_\flat} \\
\leq&\Bigg(\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|F(t+h, x(t+h), x(t+h-\tau(t+h))) \\
&- F(t, x(t+h), x(t+h-\tau(t+h)))\|_{\mathbb{X}}^{\flat} dt \Bigg)^{\frac{1}{\flat}} \\
&+ \left(\limsup_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|F(t, x(t+h), x(t+h-\tau(t+h))) - F(t, x(t), x(t-\tau(t)))\|_{\mathbb{X}}^{\flat} dt \right)^{\frac{1}{\flat}} \\
\leq&\varepsilon(1 + \mathcal{L}_1 + \mathcal{L}_2),
\end{aligned}$$

which means that $F(\cdot, x(\cdot), x(\cdot - \tau(\cdot))) + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})$. This completes the proof. □

Lemma 18. *Let (H_1)–(H_3) hold. If $x \in \mathbb{W}$, then the function $U : \mathbb{R} \to \mathbb{X}$ defined by*

$$U(t) = \int_{-\infty}^{t} T(t-s) F(s, x(s), x(s-\tau(s))) ds \tag{8}$$

belongs to $\mathbb{W}$.

Proof. By Lemma 17, one sees that $F(\cdot, x(\cdot), x(\cdot - \tau(\cdot))) + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})$. First of all, let us to show that U is well defined. According to (H_3), one has

$$\begin{aligned}
\|U(t)\|_{\mathbb{X}} =& \left\| \int_{-\infty}^{t} T(t-s) F(s, x(s), x(s-\tau(s))) ds \right\|_{\mathbb{X}} \\
\leq& \int_{-\infty}^{t} \|T(t-s)\|_{\mathbb{X}} \|F(s, x(s), x(s-\tau(s)))\|_{\mathbb{X}} ds \\
\leq& \int_{-\infty}^{t} N e^{-\lambda(t-s)} (\mathcal{L}_1 \|x(s)\|_{\mathbb{X}} + \mathcal{L}_2 \|x(s-\tau(s))\|_{\mathbb{X}}) ds \\
\leq& \frac{(\mathfrak{L}_1 + \mathcal{L}_2)\|x\|_{\mathbb{W}}}{\lambda},
\end{aligned}$$

which implies that

$$\|U\|_{\infty} \leq \frac{(\mathfrak{L}_1 + \mathcal{L}_2)}{\lambda} \|x\|_{\mathbb{W}}.$$

Therefore, U is well defined, in addition, one has gained that $U \in L^{\infty}(\mathbb{R}, \mathbb{H})$.

Next, we will prove $U + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})$. As $x \in AP(\mathbb{R}, \mathbb{X})$, from Lemma 17, we have $F(\cdot, x(\cdot), x(\cdot - \tau(\cdot))) + L_{\flat} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})$. Based on this fact, for every $\varepsilon > 0$, let $\sigma = \sigma(\varepsilon) > 0$ be the ε-translation number of $F(\cdot, x(\cdot), x(\cdot - \tau(\cdot))) + L_{\flat_1}$, then we have

$$\begin{aligned}
&\|F(t+\sigma, x(t+\sigma), x(t+\sigma-\tau(t+\sigma))) + L_{\flat_1} - (F(t, x(t), x(t-\tau(t))) + L_{\flat_1})\|_{B^\flat} \\
=& \|F(t+\sigma, x(t+\sigma), x(t+\sigma-\tau(t+\sigma))) - F(t, x(t), x(t-\tau(t)))\|_{M_\flat} < \varepsilon.
\end{aligned}$$

In addition, by using the Höld inequality, we have

$$
\begin{aligned}
&\|U(t+\sigma)-U(t)\|_{\mathbb{X}}^{p}\\
=&\left\|\int_{-\infty}^{t+\sigma} T(t+\sigma-s)F(s,x(s),x(s-\tau(s)))ds-\int_{-\infty}^{t} T(t-s)F(s,x(s),x(s-\tau(s)))\right\|_{\mathbb{X}}^{p}\\
=&\left\|\int_{-\infty}^{t} T(t-s)F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))ds\right.\\
&\left.-\int_{-\infty}^{t} T(t-s)F(s,x(s),x(s-\tau(s)))ds\right\|_{\mathbb{X}}^{p}\\
\leq&\left\|\int_{-\infty}^{t} T(t-s)[F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))ds-F(s,x(s),x(s-\tau(s)))]ds\right\|_{\mathbb{X}}^{p}\\
\leq&\left[\int_{-\infty}^{t}\|T(t-s)\|^{\frac{q}{p}}ds\right]^{\frac{p}{q}}\int_{-\infty}^{t}\|T(t-s)\|^{\frac{p}{q}}\|F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))ds\\
&-F(s,x(s),x(s-\tau(s)))\|_{\mathbb{X}}^{p}ds\\
\leq& N^{p}\left[\int_{-\infty}^{t} e^{-\frac{q}{p}\lambda(t-s)}ds\right]^{\frac{p}{q}}\int_{-\infty}^{t} e^{-\frac{p}{q}\lambda(t-s)}\|F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))ds\\
&-F(s,x(s),x(s-\tau(s)))\|_{\mathbb{X}}^{p}ds\\
\leq& N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\int_{-\infty}^{t} e^{-\frac{p}{q}\lambda(t-s)}\|F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))ds\\
&-F(s,x(s),x(s-\tau(s)))\|_{\mathbb{X}}^{p}ds,
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

By a change of variables, Fubini's theorem and Lebesgue's dominated convergence theorem, from the inequality above, we have

$$
\begin{aligned}
&\|U(t+\sigma)-U(t)\|_{M_{\flat}}^{p}\\
=&\limsup_{l\to\infty}\frac{1}{2l}\int_{-l}^{l}\|U(t+\sigma)-U(t)\|_{\mathbb{X}}^{p}dt\\
\leq& N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\limsup_{l\to\infty}\frac{1}{2l}\int_{-l}^{l}\left[\int_{-\infty}^{t} e^{-\frac{p}{q}\lambda(t-s)}\|F(s+\sigma,x(s+\sigma),x(s+\sigma-\tau(s+\sigma)))\right.\\
&\left.-F(s,x(s),x(s-\tau(s)))\|_{\mathbb{X}}^{p}ds\right]dt\\
=& N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\limsup_{l\to\infty}\frac{1}{2l}\int_{0}^{\infty} e^{-\frac{p}{q}\lambda s}\left[\int_{-l}^{l}\|F(t-s+\sigma,x(t-s+\sigma),x(t-s+\sigma-\tau(t-s+\sigma)))\right.\\
&\left.-F(t-s,x(t-s),x(t-s-\tau(t-s)))\|_{\mathbb{X}}^{p}dt\right]ds\\
\leq& N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\int_{0}^{\infty} e^{-\frac{p}{q}\lambda s}\|F(t-s+\sigma,x(t-s+\sigma),x(t-s+\sigma-\tau(t-s+\sigma)))\\
&-F(t-s,x(t-s),x(t-s-\tau(t-s)))\|_{M_{\flat}}ds\\
<& N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\frac{q}{\lambda p}\varepsilon,
\end{aligned}
$$

thus

$$
\|U(t+\sigma)+L_{\flat_1}-(U(t)+L_{\flat_1})\|_{\mathcal{M}_{\flat}}^{p}<N^{p}\left(\frac{p}{\lambda q}\right)^{\frac{p}{q}}\frac{q}{\lambda p}\varepsilon,
$$

which implies that $U + L_{\flat_1}$ meets Bohr's property. When σ is replaced by a small enough real number, we see that $U_p + L_{\flat_1}$ is uniformly continuous. Consequently, $U + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{H})$. The proof is completes. $\square$

Definition 6. *By a Besicovitch almost periodic mild solution $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} : \mathbb{R} \to \mathbb{X}$ of system (6), we mean that $x + L_{\flat_1} \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X})$ and x satisfies*

$$x(t) = \int_{-\infty}^{t} T(t-s)F(s, x(s), x(s-\tau(s)))ds, \ t \in \mathbb{R}.$$

Theorem 1. *If (H_1)–(H_4) hold, then system (6) posses a unique Besicovitch almost periodic mild solution.*

Proof. Consider the operator $\Psi : \mathbb{W} \to \mathbb{W}$ defined by

$$(\Psi x)(t) = \int_{-\infty}^{t} T(t-s)F(s, x(s), x(s-\tau(s)))ds, \ \ x \in \mathbb{W}, \ \ t \in \mathbb{R}.$$

By Lemma 18, we see that Ψ is well defined and maps $\mathbb{W}$ into $\mathbb{W}$. Therefore, we only need to prove that $\Psi : \mathbb{W} \to \mathbb{W}$ is a contraction mapping. Indeed, for every $x, y \in \mathbb{W}$, one has

$$\begin{aligned}\|(\Psi x)(t) - (\Psi y)(t)\|_{\mathbb{X}} \leq & \int_{-\infty}^{t} Ne^{-\lambda(t-s)} \big(\mathfrak{L}_1\|x(s) - y(s)\|_{\mathbb{X}} + \mathfrak{L}_2\|x(s-\tau(s)) - y(s-\tau(s))\|_{\mathbb{X}}\big)ds \\ \leq & \frac{N(\mathfrak{L}_1 + \mathfrak{L}_2)}{\lambda}\|x - y\|_{\mathbb{W}}, \ t \in \mathbb{R},\end{aligned}$$

which combined with (H_4) yields

$$\|\Psi x - \Psi y\|_{\mathbb{W}} \leq \kappa\|x - y\|_{\mathbb{W}}.$$

Thus, Ψ is a contraction mapping from $\mathbb{W}$ to $\mathbb{W}$. By the Banach fixed point theorem, Ψ posses a unique fixed point in $\mathbb{W}$. Consequently, system (6) posses a unique Besicovitch almost periodic mild solution. The proof is completed. $\square$

Finally, we present an example to illustrate the feasibility of our results obtained in this section.

Example 1.

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t, u(t,x), u(t-\tau(t), x)), \ t \geq 0, x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \ t \geq 0, \\ u(\theta, x) = \varphi(\theta, x), \ \theta \in [-\bar{\tau}, 0], \ x \in [0,\pi], \end{cases} \tag{9}$$

where $\varphi \in C([-\bar{\tau}, 0] \times [0,\pi]), \tau(t) = 3 - \sin(\sqrt{3}t)$ and

$$\begin{aligned}& f(t, u(t,x), u(t-\tau(t),x)) \\ =& \frac{1}{60}\left(\cos t + 2\cos\sqrt{5}t + 4e^{-|t|} - \frac{3}{1+t^2}\right)(\sin u(t,x) + 3\sin u(t-\tau(t),x)).\end{aligned}$$

Take $\mathbb{X} = L^2(0,\pi), (Au)x = u''(x)$ for $x \in [0,\pi]$ and $u \in D(A) = \{u \in C^1[0,\pi] : u'$ is absolutely continuous on $[0,\pi], u'' \in \mathbb{X}, u(0) = u(\pi) = 0\}$. It is well known that A generates a C_0 semigroup $T(t)$ with the property that $\|T(t)\| \leq e^{-t}$ for $t \geq 0$.

Clearly, $f \in B^{\flat}_{AP}(\mathbb{R}, \mathbb{X}^2), \tau \in AP(\mathbb{R}, \mathbb{R}^+)$, $N = \lambda = 1$, $\mathcal{L}_1 = \frac{1}{6}, \mathcal{L}_2 = \frac{1}{2}$. By a simple calculation, we have $\kappa = \frac{2}{3}$. Therefore, conditions (H_1)–(H_3) are verified. Therefore, by Theorem 1, system (9) posses a unique Besicovitch almost periodic solution.

5. Conclusions

In this paper, we have proposed a definition of Besicovitch almost periodic functions in Marcinkiewicz space by using the Bohr property and the Bochner property; studied some basic properties of Besicovitch almost periodic functions, including composition theorem; and proved the equivalence of the Bohr definition and the Bochner definition. On this basis, using the contraction fixed point theorem, we have obtained the existence and uniqueness of Besicovitch almost periodic solutions for a class of abstract semi-linear delay differential equations. Our results are new, and our results and methods can be used to study the existence of Besicovitch almost periodic solutions for other types of equations, such as abstract semi-linear evolution equations with or without delay, semi-linear neutral differential equations and so on.

Author Contributions: Conceptualization, Y.L.; formal analysis, Y.L., M.H. and B.L.; investigation, Y.L., M.H. and B.L.; writing¡—original draft preparation, Y.L., M.H.; writing¡—review and editing, Y.L. and B.L.; supervision, Y.L.; funding acquisition, Y.L. and B.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China under Grant No. 11861072 and the Applied Basic Research Foundation of Yunnan Province under Grant No. 2019FB003.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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