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On the Periodic Solutions for the Perturbed Spatial Quantized Hill Problem

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Abstract: In this work, we investigated the differences and similarities among some perturbation approaches, such as the classical perturbation theory, Poincaré–Lindstedt technique, multiple scales method, the KB averaging method, and averaging theory. The necessary conditions to construct the periodic solutions for the spatial quantized Hill problem—in this context, the periodic solutions emerging from the equilibrium points for the spatial Hill problem—were evaluated by using the averaging theory, under the perturbation effect of quantum corrections. This model can be used to develop a Lunar theory and the families of periodic orbits in the frame work for the spatial quantized Hill problem. Thereby, these applications serve to reinforce the obtained results on these periodic solutions and gain its own significance.

Keywords: quantized Hill problem; averaging theory; periodic solution

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1. Introduction

A three-body problem plays a vital role in space science; in particular, the related fields of solar system motions, stars, planets, and their moons. The model of this problem can be used to characterize the dynamical behavior of the most stellar and planetary systems and to give accurate pictures on the motion. The three-body problem acquisition is important due to its application, not only in space science, but also in many fields, such as applied mathematics and mathematical physics. In addition, it is considered the simplest non-integrable dynamical system in space dynamics, and it can be approximated to more simple systems, such as the perturbed two-body problem [1], Robe's restricted problem [2,3], and Hill's problem [4,5].

Periodic solutions of a dynamical system are solutions that characterize some repeated phenomena identically at regular intervals. These solutions play a vital role in many branches of science, such as physics and engineering, but appear in celestial mechanics, to study the dynamical structures of the two-body problem [6,7]; an analysis the infinitesimal body motion within the frame of the three-body problem [8,9] or N-body problem [10–12].

Considerable contributions have been made to the periodic solutions of either the unperturbed Kepler problem or the perturbed problem. For example, in [6], the authors

have explored the existence of two periodic orbits at each energy level, in the framework of the anisotropic Kepler system, which emerges from elliptic orbits of the Kepler motion with high eccentricity, when the parameter of the anisotropy is small. Some interesting works have been constructed to analyze the periodic solutions [13–15].

The periodic solutions have particular significance in a three-body problem, regarding its extended applications in both space dynamics and celestial mechanics. Thereby, many dynamical systems that could be studied by considering the problem of the restricted three-body. Some of these systems have applications in space missions for spacecrafts in the planet–Moons systems (such as the Earth–Moon system). Further, this problem has application in stellar systems, to study the behavior of exoplanets in the proximity of one or both objects of a binary star system [16–18].

In fact, periodic solutions have considerable significance, because most of the natural phenomena in physical and engineering sciences, as well as celestial mechanics, could be characterized by periodic solutions of dynamical systems, characterized by ordinary or partial differential equations systems. These systems have a wide variety of applications, i.e., in physical, mathematical, and engineering sciences, as well as in the fields of biology, chemistry, and neural networks [19–22].

Many methods have been developed to analyze periodic solutions, such as the averaging method, Poincaré–Lindstedt technique, Krylov–Bogoliubov–Mitropolsky (KBM), and multiple scale methods, [23,24]. Periodic solutions are everywhere in the analysis of dynamical systems. Each field of science, in particularly, celestial mechanics, has its own oscillatory phenomena, which can be described by periodic solutions.

In this work, we aimed to find the periodic solutions of the dynamical system of the spatial quantized Hill problem. Thus, we evaluated the equilibria points of the linear system, and the necessary conditions were used to calculate the periodic solutions arising from the equilibria points for the spatial quantized Hill problem, by using the averaging theory. This system was constructed for the first time by Abouelmagd et al. (2020) [5]; this motivated us to study the dynamical structures of this system, through finding its own periodic solutions.

2. Perturbation Techniques

Perturbation techniques play substantial roles in analyzing non-linear dynamics; they have become a driving force, "pushing" mathematical researchers to extreme efforts, to explore and characterize the features of dynamical systems. These methods are considered excellent tools, and are designed to be applied in many fields.

In this chapter, we shed light on techniques that could be used to construct appropriate analytical periodic solutions of perturbed dynamical systems. Moreover, these techniques could be applied to some problems, where closed form solutions do not exist or where the exact solutions are either impossible or unrealistic of their physical meanings. In general, these systems are, often, neither linear nor autonomous in nature.

Now, we assume that $\mathcal{H}$ is an $n-$dimensional Hamiltonian system defined in terms of conjugate variables $(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} \in \Gamma$, Γ is an open set of $\mathbb{R}^n$ and $\mathbf{Y} \in T^n$, here T denotes the standard one-dimensional torus. Then, a nearly integrable Hamiltonian system $\mathcal{H}$ can be read as

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}) + \varepsilon f(\mathbf{X}, \mathbf{Y}) \tag{1}$$

where h and f are analytical functions called integrable (or unperturbed) Hamiltonian and the perturbing function, respectively. However, ε is a small parameter, which measures the size of the perturbation force.

In the case of $\varepsilon = 0$, the Hamiltonian function is given by

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}), \tag{2}$$

using Hamiltonian relation (2), the associated equations of motion are given as

$$\begin{aligned} \dot{\mathbf{X}} &= 0 \\ \dot{\mathbf{Y}} &= \mathbf{w}(\mathbf{X}), \end{aligned} \tag{3}$$

where, denoted with a dot, it is used for the derivatives, with respect to time, $\mathbf{w}$ is the frequency vector, defined as $\mathbf{w} = \partial h(\mathbf{X})/\partial \mathbf{X}$.

The integration of System (3) is

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{X}_0 \\ \mathbf{Y} &= \mathbf{w}_0 t + \mathbf{X}_0, \end{aligned} \tag{4}$$

where $\mathbf{X}_0 = \mathbf{X}(0)$ and $\mathbf{w}_0 = \mathbf{w}(\mathbf{X}(0))$, Solution (4) shows that the variable $\mathbf{X}$ is constant, while its conjugate varies linearly with time.

In the case of the perturbing force, it has its own effect ($\varepsilon \neq 0$), the Hamiltonian function is given in Relation (1); the associated equations of motion are

$$\begin{aligned} \dot{\mathbf{X}} &= -\varepsilon \frac{\partial f}{\partial \mathbf{Y}} \\ \dot{\mathbf{Y}} &= \mathbf{w}(\mathbf{X}) + \varepsilon \frac{\partial f}{\partial \mathbf{X}}. \end{aligned} \tag{5}$$

System (5) may not be integrable and chaotic motion could appear.

2.1. On Perturbation Techniques

2.1.1. Importance of Perturbation Techniques

Physical phenomena in nature are mostly nonlinear and non-autonomous in their structures. The descriptions of these phenomena, within a linear sense, are not realistic, and present inaccurate information about their behaviour. Thus, it is necessary to preform the nonlinear dynamical systems, which describe these phenomena, but there are more difficulties in treating these systems by direct methods, and the perturbation techniques are considered the best choices in most cases. The perturbation techniques are employed for the dynamical systems, which consist of ordinary or partial differential equations.

Numerous methods can be used to obtain periodic solutions. For example the averaging method, for details, see [25–29]. The Liouville–Green method, which is known as the LG or WKB method, Lyapunov's theorem, Poincaré–Lindstedt technique, and KBM method [30,31]. There are more methods, such as the straightforward expansion technique (classical perturbation theory), but this method may fail in removing secular terms. Moreover, the multiple scales method is considered one of the strongest techniques in obtaining periodic solutions [32].

2.1.2. Advantages and Disadvantages of Perturbation Techniques

Exact solution of the dynamical system is rare, not only in celestial mechanics, but also in branches of applied mathematics: quantum mechanics, fluid mechanics, solid mechanics, and theoretical physics. This concerns nonlinearity behavior of a physical phenomenon. Therefore, engineers, physicists, and mathematicians are forced to find approximate solutions for mathematical models. These solutions may be purely analytical, purely numerical, or a combination of analytical and numerical techniques.

Perturbation techniques provide the most multilateral tools obtainable in non-linear sets of differential equations, which can be applied and employed to even more complex models. However, perturbation techniques have their own limitations, which mainly depend on presumptions, and very small parameters must appear in the prevailing equations.

2.2. Validity of Perturbation Techniques

Many applications of perturbation methods are not available without the existence of a small parameter. An overwhelming number of non-linear dynamical systems, in particular, have "strong non-linearity" and no small parameters. In some estimation cases, this parameter is more the result of a technical procedure than scientific methodology. The convenient selection of small parameters could lead to intelligible results. On the contrary, incorrect choice for these parameters create inaccurate or even unrealistic solutions. Even if there exist appropriate small parameters that the perturbation methods provide us with by analytical solutions, they are adequate in cases of bounding the parameters.

The structures and analyses of perturbation approaches, such as *classical perturbation theory*, *Poincaré–Lindstedt technique*, *multiple scales method*, *Krylov–Bogoliubov (KB) averaging method* and *averaging theory* are familiar in the literature, as we have mentioned in the previous subsections. However, we will show the differences and similarities among the methods and rationales in choosing the averaging theory to find the periodic solution for the spatial quantized Hill problem.

The aforementioned techniques demand that the dynamical system be weakly non-linear or weakly non-autonomous, meaning that those terms in the equation, including non-linearity or non-autonomy, are small. Alternatively systems of this structure can be thought of as almost linear or quasi–linear. One consequence of almost linear systems is that differential equations will have linear terms and small non-linear or non-autonomous terms separated from each other. These techniques can be applied to nearly integrable Hamiltonian systems, as in System (5).

The classical perturbation theory is called the straightforward expansion technique. It is used to find analytically solutions in the power series of the following form:

$$\mathbf{X}(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{X}_k(t). \tag{6}$$

In general, this method has secular terms that will provide unbounded solutions in the case of long intervals. Straightforward application of the classical perturbation theory to periodic nonlinear motion gives a result with secular terms, which will be proportional with time, in spite of the fact that the behavior of motion is known to be bounded. One of the common approaches used to remove those unwanted secular terms is the Poincaré–Lindstedt technique (also called the continuation method).

The multiple-scales technique is a generalized formula for the Poincaré–Lindstedt method. The latter method is dependant on the angular velocity of the non–linear oscillation and it may depend on its amplitude. The angular velocity is expanded in an asymptotic series, and the coefficients of each term in the series are evaluated in such a way that the solution has no secular terms.

In a similar manner, in the multiple-scales technique, the solution varies on fast and slow time separation scales. The first variable, "fast time scale", with respect to the first linear order in Poincaré–Lindstedt expansion, and every slow scale, matches with the second and higher terms in this expansion. The major difference is that multiple scales suppose that the coefficients of each scale are fixed, equal, and employ a variation-of-parameters approach, considering the integration constants that appear in the linear solution be functions in the slow time scales variables. This process results in the dynamical system of partial differential equations at each order that has to be solved, to obtain a uniform solution of the dynamical system. The multiple-scales technique is a much more powerful tool than the Poincaré–Lindstedt method, because the former "admits" the constants compared to the slow time scale variables, while the latter only introduces one free variable at each order in the perturbed parameter. However, this method may result in a system, where the solution for obtaining periodicity conditions is more difficult than the main system.

The Krylov–Bogolyubov averaging method is a technique used to find the periodic solution of a non-linear system, based on the averaging principle, where the exact system is replaced by an averaged one. To obtain the solution of the perturbed system or non-linear motion, using this method, the constants of linear motion vary slowly with time t and the perturbation parameter ε. The significance of this method is that a general averaging approach is developed and proves that the solutions of the averaged systems give precise approximation to the original system [7].

Averaging is a mathematical method to replace a given field by its own average over a specified variable, such as time or an angular variable to get asymptotic approximation to the original system, aiming to obtain a periodic solution. In dynamical systems, the averaging method or the averaging theory utilizes systems, including time scale separation: fast oscillation versus a slow drift. We propose averaging over a certain interval of time to iron out the fast oscillation and monitor the qualitative behaviour from the resulting dynamics. It is a familiar problem where a trade-off exists between how perfect the approximate solution is balanced by how much time it holds to be similar to the original solution [27].

In this work, we used the averaging theory of dynamical systems, because it is especially worthy in the case of systems that can have isolated periodic orbits, such as in this case. In the next section, we will apply this method to Hill's version of the quantized three-body problem to find periodic solutions.

3. Mathematical Model

Hill's version of the quantized three-body problem was derived and analyzed for first time in [5], and the equations of motion are given by

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= 3\xi + 2(\alpha_1 - \alpha_{11}) - \frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\xi, \\ \ddot{\eta} + 2\dot{\xi} &= -\frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\eta, \\ \ddot{\zeta} &= -\zeta - \frac{1}{r^3}\left[1 + \frac{2\alpha_{21}}{r} + \frac{3\alpha_{22}}{r^2}\right]\zeta. \end{aligned} \tag{7}$$

System (7) characterizes the perturbed spatial Hill problem, the system is perturbed by quantum corrections; hence, this system is called the spatial quantized Hill problem (SQHP). Furthermore, this system is considered a "limiting case" from the spatial quantized restricted three-bodies problem; it was developed in [33]. In addition, this system can be described by the style of writing of the restricted three-body problem,

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Psi_{\xi}(\xi, \eta, \zeta), \\ \ddot{\eta} + 2\dot{\xi} &= \Psi_{\eta}(\xi, \eta, \zeta), \\ \ddot{\zeta} &= \Psi_{\zeta}(\xi, \eta, \zeta), \end{aligned} \tag{8}$$

where

$$\Psi(\xi, \eta, \zeta) = \frac{1}{2}\left[3\,\xi^2 + 4(\alpha_1 - \alpha_{11})\xi - \zeta^2\right] + \frac{1}{r}\left[1 + \frac{\alpha_{21}}{r} + \frac{\alpha_{22}}{r^2}\right], \tag{9}$$

and Ψ_{ξ}, Ψ_{β}, Ψ_{ζ} refer to the partial derivatives of the potential function, with respect to the variables ξ, β, and ξ, while the separation distance r is given by $r = \sqrt{\xi^2 + \beta^2 + \zeta^2}$.

Here, α_1, α_{11}, and α_{21} are very small quantities with order $\mathcal{O}(1/c^2)$, while α_{22} is more smaller with order $\mathcal{O}(1/c^3)$, where c is the speed of light; for comprehensive details,

see [33]. Since α_1, α_{11}, and α_{21} have the same order, then $\alpha_1 - \alpha_{11} \cong 0$, and System (8) and the potential function can be simplified and rewritten

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \bar{\Psi}_{\xi}(\xi,\, \eta,\, \zeta), \\ \ddot{\eta} + 2\dot{\xi} &= \bar{\Psi}_{\eta}(\xi,\, \eta,\, \zeta), \\ \ddot{\zeta} &= \bar{\Psi}_{\zeta}(\xi,\, \eta,\, \zeta), \end{aligned} \tag{10}$$

where

$$\Psi(\xi, \eta, \zeta) = \frac{1}{2}\left[3\,\xi^2 - \zeta^2\right] + \frac{1}{r}\left[1 + \frac{\alpha_{21}}{r} + \frac{\alpha_{22}}{r^2}\right]. \tag{11}$$

Although many different analyses in celestial mechanics have been accomplished in the framework of the three-body problem, but there are also many dynamical concepts can be carried out within frame of Hill problem [34] without loosing the required accuracy underlying of using simple model. Furthermore, the perturbed model of this problem can be used to study the effect of some perturbed forces on the dynamical properties, such as the emerging periodic solutions from the equilibria points, which will be analyzed in the next sections.

4. Periodic Solutions

Periodic solutions or periodic orbits are considered one the major reasons of stability and are continuous in life; for example, the periodicity motion of the Sun and Moon. When the Hill model was constructed, the researchers devoted their work toward calculating the families of periodic orbits. More work was developed to analyze the lunar theory dependent on Hill's problem. For the importance of Hill's problem, we to explore the presence of periodic solutions emerged from equilibria points by underlying SQHP.

By taking $\alpha_1 = \alpha_{11} = \alpha_{21} = 1.5 \times 10^{-3}, \alpha_{22} = 1.5 \times 10^{-5}$ of the differential System (7), we get the following equilibrium points $E_1 = (\delta, 0, 0)$ and $E_2 = (-\delta, 0, 0)$ where $\delta \approx 0.694035$.

Thus, we study the presence of periodic solutions, which emerge from equilibria points E_1 and the same ones are valid for E_2. To study the motion around or in the proximity of the equilibria points E_1 and E_2, we first have to linearize System (8). Thus, we impose that $\xi = x_1 - \delta, \eta = y_1, \zeta = z_1$, where x_1, y_1 z_1 are very small displacements from the equilibria points; thereby, the associated linear system to non-linear System (8) are given by

$$\begin{aligned} \frac{d^2x_1}{dt^2} - 2\frac{dy_1}{dt} - \alpha x_1 &= 0, \\ \frac{d^2y_1}{dt^2} + 2\frac{dx_1}{dt} + \beta y_1 &= 0, \\ \frac{d^2z_1}{dt^2} + \gamma z_1 &= 0, \end{aligned} \tag{12}$$

with $\alpha \approx 9.03$, $\beta \approx 3$ and $\gamma \approx 3$.

Since the means of the averaging theory is one of the powerful tools for finding periodic solutions, then we will apply this method to study the existence of periodic solutions of the following system

$$\begin{aligned} \frac{d^2x_1}{dt^2} - 2\frac{dy_1}{dt} - \alpha x_1 &= \varepsilon \mathcal{F}_1\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, , z_1, \frac{dz_1}{dt}\right), \\ \frac{d^2y_1}{dt^2} + 2\frac{dx_1}{dt} + \beta y_1 &= \varepsilon \mathcal{F}_2\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right), \\ \frac{d^2z_1}{dt^2} + \gamma z_1 &= \varepsilon \mathcal{F}_3\left(t, x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right), \end{aligned} \tag{13}$$

where ε is the perturbation parameter; it is a very small quantity and the functions $\mathcal{F}_1, \mathcal{F}_2$, $\mathcal{F}_3$ represent the non-linear terms, which will be ignored when $\varepsilon = 0$, but this function satisfies the following properties:

- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are smooth functions;
- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are periodic functions in variable t;
- $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are resonance in $\iota_1 : \iota_2$ with periodic solutions for System (12);

where ι_1 and ι_2 are primes numbers.

There is a unique singular point for the unperturbed System (12) at the origin with eigenvalues $\pm\Omega, \pm\omega_1 i, \pm\omega_2 i$, where $\Omega \approx 2.51$, $\omega_1 \approx 2.07$ and $\omega_2 \approx 1.73$. In the phase space

$$\left(x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right)$$

the aforementioned system (the unperturbed system) has two planes filed of periodic solutions with the exception of the origin, where the periods of solutions are

$$T_1 = 2\pi/\omega_1 \text{ or } 2\pi/\omega_2,$$

here, periods T_1 and T_2 are related to the eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We will explore which of the periodic solutions continue for the perturbed System (13), where the parameter of perturbation ε is small enough and there are two periods, either $\iota_1 T_1/\iota_2$ or $\iota_1 T_2/\iota_2$, for the perturbed function $\mathcal{F}_i$ for $i \in \{1, 2, 3\}$.

Consider $Z^0 = (Z_1^0, Z_2^0)$, and $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$ for the System (13), with

$$\begin{aligned}
\mathcal{H}_1\left(Z^0\right) &= \frac{1}{\iota_1 T_2}\int_0^{\iota_1 T_2} \langle(\cos(\omega_2 t), -\sin(\omega_2 t)), (\mathcal{F}_5^*(t), \mathcal{F}_6^*(t))\rangle dt \\
&= \frac{1}{\iota_1 T_2}\int_0^{\iota_1 T_2} -\sin(\omega_2 t)\mathcal{F}_6^*(t)dt, \\
\mathcal{H}_2\left(Z^0\right) &= \frac{1}{pT_2}\int_0^{pT_2} \langle(\sin(\omega_2 t), \cos(\omega_2 t)), (\mathcal{F}_5^*(t), \mathcal{F}_6^*(t))\rangle dt \\
&= \frac{1}{pT_2}\int_0^{pT_2} \cos(\omega_2 t)\mathcal{F}_6^*(t)dt,
\end{aligned} \tag{14}$$

where $\langle , \rangle$ is the scalar product and $\mathcal{F}_5^*(t) = 0$, $\mathcal{F}_6^* = 1.12\,\mathcal{F}_3$, $\mathcal{F}_3 = \mathcal{F}_3\left(\eta_1^2(t), \ldots, \eta_6^2(t)\right)$ and $\eta_j^2(t) = 0, j = 1, \ldots, 4$, while

$$\begin{aligned}
\eta_5^2(t) &= 0.50\left(Z_1^0\cos(\omega_2 t) + Z_2^0\sin(\omega_2 t)\right), \\
\eta_6^2(t) &= 0.86\left(Z_2^0\cos(\omega_2 t) - Z_1^0\sin(\omega_2 t)\right).
\end{aligned}$$

Now, we impose that $Z^{0*} = (Z_1^{0*}, Z_2^{0*})$ is the zero of non-linear system $\mathcal{H}\left(Z^0\right) = 0$ where

$$\left|\frac{\partial\mathcal{H}}{\partial Z^0}\right| \neq 0 \text{ when } Z^0 = Z^{0*},$$

Then, we can state that the system has a simple zero [35]. The expression of simple zero or pole is used to describe the zero or pole of order one, and sometimes the term of "degree" is used instead of "order". The property of this zero or pole leading to this zero can be isolated and its neighbourhood has no other zero.

We emphasize that if the Malkin bifurcation function $\mathcal{H}$ has a simple zero Z^{0*} and the solution of the unperturbed system has a period T_2 by using initial value Z^{0*}, then the perturbed system will has also T_2-periodic solution.

The periodic solution of the dynamical System (13) is considered the main result in this work, where this solution will bifurcate from the T_2-periodic solution of the unperturbed system; hence, we present the following theorem:

Theorem 1. *We impose that ι_1 and ι_2 are prime numbers, and*

- *$\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ are smooth functions of System (13)*
- *$\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ are periodic, with period $\iota_1 T_2/\iota_2$ in variable t*

For each simple zero, $Z^{0*} \neq 0$ of the non-linear system $\mathcal{H}(Z^0) = 0$ when $\varepsilon \neq 0$ and enough small, we can find a periodic solution for the perturbed System (13) takes the form $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ and tends to the periodic solution $(x_1(t), y_1(t), z_1(t)) = (\eta_1^2(t), \eta_3^2(t), \eta_5^2(t))|_{Z^0 Z^{0*}}$ of the unperturbed System (12) traveled ι_1 times.

We will presented the proof of Theorem 1 in Section 5. Further, the following corollary is considered the application of this theorem and its proof will be presented in Section 6.

Corollary 1. *Considering that $\mathcal{F}_1(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$, $\mathcal{F}_2(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$, $\mathcal{F}_3(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = z^5 + \sin(\omega_2 t)\dot{z}^2$. Thus, the System (13) with $\varepsilon \neq 0$ is enough small, and has one periodic solution $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ approximating to the periodic solutions $(x_1(t), y_1(t), z_1(t)) = (\eta_1^2(t), \eta_3^2(t), \eta_5^2(t))|_{Z^0 = Z^{0*}}$ of (2) when $\varepsilon \to 0$, given by $Z^{0*} = (0, -2.11)$.*

Now, we impose that $Y^0 = (Y_1^0, Y_2^0)$, and considering the Malkin bifurcation function $\bar{\mathcal{H}} = (\mathcal{H}_3, \mathcal{H}_4)$ for the System (13) controlled by

$$\begin{aligned}
\mathcal{H}_3\left(Y^0\right) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\cos(\omega_1 t), -\sin(\omega_1 t)), (\mathcal{F}_3^*(t), \mathcal{F}_4^*(t))\rangle dt \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)\mathcal{F}_3^*(t) - \sin(\omega_1 t)\mathcal{F}_4^*(t))dt, \\
\mathcal{H}_4\left(Y^0\right) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\sin(\omega_1 t), \cos(\omega_1 t)), (\mathcal{F}_3^*(t), \mathcal{F}_4^*(t))\rangle dt \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)\mathcal{F}_4^*(t) + \sin(\omega_1 t)\mathcal{F}_3^*(t))dt,
\end{aligned}$$

where

$$\mathcal{F}_3^* = -0.45\,\mathcal{F}_1, \quad \mathcal{F}_4^* = -1.46\,\mathcal{F}_2,$$

with $\mathcal{F}_i = \mathcal{F}_i(\eta_1^1(t), \ldots, \eta_6^1(t)), i \in \{1,2\}$ and $\eta_j^2(t) = 0, j = 5, 6$,

$$\eta_1^1(t) = 0.12\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right),$$

$$\eta_2^1(t) = -0.26\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right),$$

$$\eta_3^1(t) = -0.41\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right),$$

$$\eta_4^1(t) = -0.85\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right),$$

As we aforementioned with the existence of a simple zero Y^{0*} of the Malkin bifurcation function $\bar{\mathcal{H}}$, one will obtain from T_1-periodic solution of the unperturbed system, with initial value Y^{0*}, emerges the solution of a perturbed system with T_1-periodic solution, because the simple zero can be isolated and its neighbourhood has no other zero.

The second result related to the periodic solutions, which is associated to unperturbed System (13), will be stated in the following theorem:

Theorem 2. *We impose that ι_1 and ι_2 are primes numbers and*

- *$\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ are smooth functions of System (13)*
- *$\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ are periodic with period $\iota_1 T_1/\iota_2$ in variable t*

For each simple zero $Y^{0*} \neq 0$ of the non-linear system $\bar{\mathcal{H}}(Y^0) = 0$, when $\varepsilon \neq 0$ is enough small, then the perturbed System (13) has a periodic solution $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ going to the periodic solution $(x_1(t), y_1(t), z_1(t)) = (\eta_1^1(t), \eta_3^1(t), \eta_5^1(t))|_{Y^0=Y^{0*}}$ of the unperturbed System (12) traveled ι_1 times.

The application of Theorem 2 can be stated in the following corollary:

Corollary 2. *Considering that $\mathcal{F}_1(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = \sin(\omega_1 t) + x_1 + 3x_1^2 y_1 + y_1$, $\mathcal{F}_2(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = \cos(\omega_1 t) - (\dot{y}_1)^2 + x_1$, $\mathcal{F}_3(t, x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1) = 0$. Then the System (13) for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x_1(t,\varepsilon), y_1(t,\varepsilon), z_1(t,\varepsilon))$ tending to the periodic solutions $(x_1(t), y_1(t), z_1(t)) = (\eta_1^1(t), \eta_3^1(t), \eta_5^1(t))|_{Y^0=Y^{0*}}$ of System (12) when $\varepsilon \to 0$, given by $Y^{0*} = (2.59\,,\, 9.21)$.*

The proof of Corollary 2 will be given in Section 6.

5. Proof of the Theorems 1 and 2

To accomplish the proofs of Theorems 1 and 2, we will use the following variables:

$$(x_1, x_2, y_1, y_2, z_1, z_2) = \left(x_1, \frac{dx_1}{dt}, y_1, \frac{dy_1}{dt}, z_1, \frac{dz_1}{dt}\right)$$

Thus, the dynamical System (13) can be rewritten in the form of a first order in $\mathbb{R}^6$

$$\begin{aligned} \frac{dx_1}{dt} &= x_2\,, & \frac{dx_2}{dt} &= \alpha x_1 + 2y_2 + \varepsilon \mathcal{F}_1(x_1, x_2, y_1, y_2, z_1, z_2), \\ \frac{dy_1}{dt} &= y_2\,, & \frac{dy_2}{dt} &= -2x_2 - \beta y_1 + \varepsilon \mathcal{F}_2(x_1, x_2, y_1, y_2, z_1, z_2), \\ \frac{dz_1}{dt} &= z_2\,, & \frac{dz_2}{dt} &= -\gamma z_1 + \varepsilon \mathcal{F}_3(x_1, x_2, y_1, y_2, z_1, z_2). \end{aligned} \tag{15}$$

It is clear that the perturbed System (15) ($\varepsilon \neq 0$) can be reduced to the unperturbed System (12) when $\varepsilon = 0$. Now we write the perturbed System (15) with the style that the linear part at the origin point will take the real Jordan expression, after that, we can change the variables to the following form:

$$(x_1, x_2, y_1, y_2, z_1, z_2) \to (X_1, X_2, Y_1, Y_2, Z_1, Z_2)$$

given by

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -0.32 & 0.32 & 0.0 & 0.12 & 0.0 & 0.0 \\ -0.81 & -0.81 & -0.26 & 0 & 0.0 & 0.0 \\ 0.17 & 0.17 & -0.41 & 0 & 0.0 & 0.0 \\ 0.44 & -0.44 & 0.0 & -0.85 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.86 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix}, \tag{16}$$

and

$$\begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -1.92 & -0.53 & 0.34 & -0.28 & 0.0 & 0.0 \\ 1.92 & -0.53 & 0.34 & 0.28 & 0.0 & 0.0 \\ 0.0 & -0.45 & -2.11 & 0.0 & 0.0 & 0.0 \\ -1.98 & 0.0 & 0.0 & -1.46 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix},$$

the differential System (15) becomes

$$\begin{aligned} \dot{X}_1 &= \Omega X_1 + \varepsilon \mathcal{F}_1^*, \\ \dot{X}_2 &= -\Omega X_2 + \varepsilon \mathcal{F}_2^*, \\ \dot{Y}_1 &= \omega_1 Y_2 + \varepsilon \mathcal{F}_3^*, \\ \dot{Y}_2 &= -\omega_1 Y_1 + \varepsilon \mathcal{F}_4^*, \\ \dot{Z}_1 &= \omega_2 Z_2 + \varepsilon \mathcal{F}_5^*, \\ \dot{Z}_2 &= -\omega_2 Z_1 + \varepsilon \mathcal{F}_6^*, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mathcal{F}_1^* &= -0.53\mathcal{F}_1 - 0.28\mathcal{F}_2, \\ \mathcal{F}_2^* &= -0.53\mathcal{F}_1 + 0.28\mathcal{F}_2, \\ \mathcal{F}_3^* &= -0.45\mathcal{F}_1, \\ \mathcal{F}_4^* &= -1.46\mathcal{F}_2, \\ \mathcal{F}_5^* &= 0, \\ \mathcal{F}_6^* &= 1.15\mathcal{F}_3, \end{aligned}$$

with $\mathcal{F}_i = \mathcal{F}_i(\eta_1, \ldots, \eta_6), i \in \{1, 2, 3\}$ and

$$\begin{aligned} \eta_1 &= -0.32X_1 + 0.32X_2 + 0.12Y_2, \\ \eta_2 &= -0.81X_1 - 0.81X_2 - 0.26Y_1, \\ \eta_3 &= 0.17X_1 + 0.17X_2 - 0.41Y_1, \\ \eta_4 &= 0.44X_1 - 0.44X_2 - 0.85Y_2, \\ \eta_5 &= 0.5Z_1, \\ \eta_6 &= 0.86Z_2, \end{aligned}$$

To prove Theorems 1 and 2, we first depict the periodic solution of the unperturbed system through the following *Lemma*

Lemma 1. *The periodic solutions* $(X_1(t), X_2(t), Y_1(t), Y_2(t), Z_1(t), Z_2(t))$ *of System* (17) *when* $\varepsilon = 0$ *are*

$$\left(0, 0, \mathrm{Y}_1^0 \cos(\omega_1 t) + \mathrm{Y}_2^0 \sin(\omega_1 t), \mathrm{Y}_2^0 \cos(\omega_1 t) - \mathrm{Y}_1^0 \sin(\omega_1 t), 0, 0\right), \tag{18}$$

where T_1 *is the period of motion, and*

$$\left(0, 0, 0, 0, \mathrm{Z}_1^0 \cos(\omega_2 t) + \mathrm{Z}_2^0 \sin(\omega_2 t), \mathrm{Z}_2^0 \cos(\omega_2 t) - \mathrm{Z}_1^0 \sin(\omega_2 t)\right), \tag{19}$$

here, T_2 *is the period of motion.*

Proof of Lemma 1. Since System (17) is linear when $\varepsilon = 0$ thereby, the proof can be easily established. □

Proof of Theorem 1. We impose that $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of (13) are periodic functions in t with period $\iota_1 T_1/\iota_2$ where ι_1 and ι_2 are primes numbers. Hence, the same periodicity features are the same for System (17) and the periodic Solutions (19) with period $\iota_1 T_2$. By applying Theorem A1 in [35] and using the same notation and terminology to the System (17), then we can write the System (17) in the following form

$$\dot{X}(t) = H_0(t, X) + \varepsilon\, H_1(t, X) + \varepsilon^2\, H_2(t, X, \varepsilon)$$

Then, we can consider

$$X = \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Z_1 \\ Z_2 \end{pmatrix}, \quad H_0(t,X) = \begin{pmatrix} \Omega X_1 \\ -\Omega X_2 \\ \omega_1 Y_2 \\ -\omega_1 Y_1 \\ \omega_2 Z_2 \\ -\omega_2 Z_1 \end{pmatrix}, \quad H_1(t,X) = \begin{pmatrix} \mathcal{F}_1^* \\ \mathcal{F}_2^* \\ \mathcal{F}_3^* \\ \mathcal{F}_4^* \\ \mathcal{F}_5^* \\ \mathcal{F}_6^* \end{pmatrix}, \quad H_2(t,X,\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this context, the periodic solution of the unperturbed System (17) ($\varepsilon = 0$) will be studied within the Type (19) to continue as a periodic solution for the perturbed system (when $\varepsilon \neq 0$ is enough small). First, we characterize the different parameters, which are stated in Theorem A1 (see the Appendix in [35] for details) due to the certain case of the System (17). Now, we assume that $\rho_1 > 0$ and $\rho_2 > 0$, where $\rho_1 > 0$ is chosen to be small, while $\rho_2 > 0$ is chosen to be large. We also assume that V is bounded and open subset of the plane $X_1 = X_2 = Y_1 = Y_2 = 0$ of the form

$$V = \left\{ \left(0,0,0,0,Z_1^0,Z_2^0\right) \in \mathbb{R}^6 : \rho_1 < \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} < \rho_2 \right\}.$$

Since V is bounded and open subset of $\mathbb{R}^2$, we can choose two numbers $\rho_1 > 0, \rho_2 > 0$, such that

$$V = \left\{ (\beta(\alpha),\alpha) \in \mathbb{R}^2 : \rho_1 < \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} < \rho_2 \right\}.$$

where $\alpha \in R^2$ and $\beta(\alpha) \in R^4$.

Now, we assume that $\alpha = Z^0 = (Z_1^0, Z_2^0)$, then we characterize V with the set $\{\alpha \in \mathbb{R}^2 : \rho_1 < ||\alpha|| < \rho_2\}$, being $||\cdot||$ the Euclidean norm in $\mathbb{R}^2$, while the function $\beta(\alpha)$ is defined as $\beta : C1(V) \to \mathbb{R}^4$ such that $\beta(\alpha) = (0,0,0,0)$, here $C1(V)$ refers to the closure of V. Hence, for the proposed system, one obtains

$$\begin{aligned} \mathcal{Z} &= \{\mathbf{z}_\alpha = (\beta(\alpha),\alpha), \alpha \in C1(V)\} \\ &= \left\{ \left(0,0,0,0,Z_1^0,Z_2^0\right) \in \mathbb{R}^6 : \rho_1 \leq \sqrt{\left(Z_1^0\right)^2 + \left(Z_2^0\right)^2} \leq \rho_2 \right\}. \end{aligned}$$

We take for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the periodic solution

$$\mathbf{x}(t,\mathbf{z}_\alpha) = (0,0,0,0,Z_1(t),Z_2(t)),$$

controlled by System (19) of period $\iota_1 T_2$. Calculating the matrix $M_{\mathbf{z}_\alpha}(t)$ of the linear System (17), which is called the fundamental matrix where $\varepsilon = 0$ related to the $\iota_1 T_2$ periodic solution $\mathbf{z}_\alpha = (0,0,0,0,Z_1^0,Z_2^0)$ and also $M_{\mathbf{z}_\alpha}(0)$ is the identity element in space $\mathbb{R}^6$; thus, one obtains

$$M_{\mathbf{z}_\alpha}(t) = M(t) = \begin{pmatrix} e^{\Omega t} & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\Omega t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ 0 & 0 & -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}.$$

There is no correlation between the matrix $M_{\mathbf{z}_\alpha}(t)$ and particular periodic solution $\mathbf{x}(t,\mathbf{z}_\alpha,0)$.

Now, we impose that the matrix $\hbar$ is defined by

$$\hbar = M^{-1}(0) - M^{-1}(\iota_1 T_2)$$

then the matrix $\hbar$ is satisfied the stated assumptions (ii) in Theorem A1 (see the Appendix in [35] for details), where the matrix $\hbar$ and its determinant are given by

$$\hbar = \begin{pmatrix} 1-e^{-2\pi\Omega\iota_1/\omega_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-e^{2\pi\Omega\iota_1/\omega_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2(\pi\iota_1\omega_1/\omega_2) & \sin(2\pi\iota_1\omega_1/\omega_2) & 0 & 0 \\ 0 & 0 & -\sin(2\pi\iota_1\omega_1/\omega_2) & 2\sin^2(\pi\iota_1\omega_1/\omega_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sin^2(\pi\iota_1) & \sin(2\pi\iota_1) \\ 0 & 0 & 0 & 0 & -\sin(2\pi\iota_1) & 2\sin^2(\pi\iota_1) \end{pmatrix},$$

$$|\hbar| = \begin{vmatrix} 1-e^{-2\pi\Omega\iota_1/\omega_2} & 0 & 0 & 0 \\ 0 & 1-e^{2\pi\Omega\iota_1/\omega_2} & 0 & 0 \\ 0 & 0 & 2\sin^2(\pi\iota_1\omega_1/\omega_2) & \sin(2\pi\iota_1\omega_1/\omega_2) \\ 0 & 0 & -\sin(2\pi\iota_1\omega_1/\omega_2) & 2\sin^2(\pi\iota_1\omega_1/\omega_2) \end{vmatrix}$$

hence,

$$|\hbar| = -16\sinh^2(\pi\Omega\iota_1/\omega_2)\sin^2(\pi\iota_1\omega_1/\omega_2) \neq 0,$$

because the ratio of the frequencies is non-resonant with π. Ti summarize, all of the stated assumptions in Theorem A1 are satisfied by the System (17).

In the proposed system, the map $\xi : \mathbb{R}^6 \longrightarrow \mathbb{R}^2$ can be written as

$$\xi(X_1, X_2, Y_1, Y_2, Z_1, Z_2) = (Z_1, Z_2),$$

by evaluating the function

$$\mathcal{H}\left(\mathcal{Z}_1^0, \mathcal{Z}_2^0\right) = \mathcal{H}(\alpha) = \xi\left(\frac{1}{pT_2}\int_0^{pT_2} M_{\mathbf{z}_\alpha}^{-1}(t)H_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))dt\right),$$

we get $\mathcal{H}(\mathcal{Z}^0) = (\mathcal{H}_1(\mathcal{Z}^0), \mathcal{H}_2(\mathcal{Z}^0))$, where the functions $\mathcal{H}_k$ for $k = 1, 2$ are the ones given in (14). Then, by Theorem A1, we have that, for every simple zero $\mathcal{Z}^{0*} \in V$ of the system of non-linear functions $\mathcal{H}(Z^0) = 0$, we have a periodic solution

$$(X_1, X_2, Y_1, Y_2, Z_1, Z_2)(t, \varepsilon)$$

of System (17), such that

$$(X_1, X_2, Y_1, Y_2, Z_1, Z_2)(0, \varepsilon) \longrightarrow \left(0, 0, 0, 0, Z_1^{0*}, Z_2^{0*}\right) \text{ when } \varepsilon \longrightarrow 0$$

Let us changes the variables in System (16), then, one obtains a periodic solution

$$(x_1, x_2, y_1, y_2, z_1, z_2)(t, \varepsilon)$$

of System (17), where

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ y_1(t,\varepsilon) \\ y_2(t,\varepsilon) \\ z_1(t,\varepsilon) \\ z_2(t,\varepsilon) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5\left(Z_1^{0*}\cos(\omega_2 t) + Z_2^{0*}\sin(\omega_2 t)\right) \\ 0.86\left(Z_2^{0}\cos(\omega_2 t) - Z_1^{0}\sin(\omega_2 t)\right) \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0$$

Thus, periodic solution of System (17) $(x_1(t,\varepsilon), y_1(t,\varepsilon)$, and $z_1(t,\varepsilon)$ can be written as

$$(x_1, y_1, z_1)(t,\varepsilon) \to \begin{pmatrix} 0 \\ 0 \\ 0.5\left(Z_1^{0*}\cos(\omega_2 t) + Z_2^{0*}\sin(\omega_2 t)\right) \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0$$

The previous steps give the complete proof of Theorem 1. □

Proof of Theorem 2. To prove this theorem, we will follow the same steps of proving Theorem 1. Thus, the periodic solution can be written in the following form:

$$\begin{pmatrix} 0.12\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right) \\ -0.26\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right) \\ -0.41\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right) \\ -0.85\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right) \\ 0 \\ 0 \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0.$$

Hence, we get a periodic solution $(x_1, y_1, z_1)(t,\varepsilon)$ of System (13), such that

$$(x_1, y_1, z_1)(t,\varepsilon) \to \begin{pmatrix} 0.12\left(Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t)\right) \\ -0.41\left(Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t)\right) \\ 0 \end{pmatrix} \text{ when } \varepsilon \longrightarrow 0.$$

□

6. Proof of the Corollaries 1 and 2

Proof of Corollary 1. Under the aforementioned assumptions in Corollary 1, the nonlinear System (14) can be written as

$$\begin{aligned} \mathcal{H}_1\left(Z_1^0, Z_2^0\right) = & -0.005468749997\left(Z_1^0\right)^4 Z_2^0 - 0.05177199997\left(Z_2^0\right)^2 - 0.01093750000\left(Z_1^0\right)^2\left(Z_2^0\right)^3 \\ & - 0.1553160000\left(Z_1^0\right)^2 - 0.005468749997\left(Z_2^0\right)^5 \end{aligned}$$

$$\mathcal{H}_2\left(Z_1^0, Z_2^0\right) = 3.183098861 \times 10^{-12}\, Z_1^0\, (6872233931.0 \left(Z_1^0\right)^2 \left(Z_2^0\right)^2 + 3436116965.0 \left(Z_1^0\right)^4 + 3436116965.0 \left(Z_2^0\right)^4 - 65058613960.0\, Z_2^0$$

Then, the solution of the above system is

$$Z^{0*} = (0, -2.11).$$

Since

$$\left|\frac{\partial \mathcal{H}}{\partial Z^0}\right| = 0.21 \neq 0, \text{ when } Z^{0*} = (0\,,\, -2.11)$$

this solution is simple. Finally, by Theorem 1., we only have one periodic solution for this system and the proof is over. □

Proof of Corollary 2. Again, under the aforementioned assumptions in Corollary 2, the non-linear system $\bar{\mathcal{H}}\left(Y^0\right) = 0$ can be written as

$$\mathcal{H}_3\left(Y_1^0, Y_2^0\right) = -\,0.02699999999\, Y_2^0 + 0.004649999999\, Y_1^0 + 0.0009962999998 \left(Y_1^0\right)^3 + 0.0009962999998 \left(Y_2^0\right)^2 Y_1^0,$$

$$\mathcal{H}_4\left(Y_1^0, Y_2^0\right) = -\,0.9549999997 + 0.004649999999\, Y_2^0 + 0.0009962999998 \left(Y_1^0\right)^2 Y_2^0 + 0.0009962999998 \left(Y_2^0\right)^3 + 0.02699999999\, Y_1^0,$$

The above system satisfies the following solution:

$$Y^{0*} = (2.590\,,\, 9.210),$$

Moreover, since

$$\left|\frac{\partial \bar{\mathcal{H}}}{\partial Y^0}\right| = 0.020 \neq 0, \text{ when } Y^{0*} = (2.590\,,\, 9.210)$$

the obtained solution is simple, using Theorem 2, one can obtain only one periodic solution for this system, which gives a complete proof. □

7. Conclusions

The averaging theory is one of the most important perturbation methods that can be used to study the existence and stability of periodic solutions for the ordinary differential equation systems. It is a powerful tool and has proven its effectiveness many times in the literature by examining the existence and stability of the periodicity of dynamical systems, in both physical and engineering sciences.

In this work, the dynamical system of the perturbed spatial Hill problem by quantum corrections, which is called the spatial quantized Hill problem, was analyzed to find the possible periodic solutions. The importance of this problem was stated in the introduction. The differences and similarities among some perturbation approaches, such as the classical perturbation theory, Poincaré–Lindstedt technique, the multiple-scales method, the KB averaging method, and the averaging theory were investigated in the second section. Then, the equilibrium points of the linear system were evaluated. The necessary conditions were analyzed to calculate the periodic solutions emerging from the equilibrium points of the SQHP by using the averaging theory. The application of this theory on the quantized Hill problem has provided interesting and important results about the periodic solution through the proof of Theorems 1 and 2 and their associated corollaries.

We applied a known theory on a new model to state the new results on such a model. The difficulty of our proofs involved showing all hypotheses of the averaging theory of the dynamical systems for the perturbed spatial quantized Hill problem, in order to apply the theorems of this theory. Changes of variables, obtaining the normal form of this theory, and many technical tricks were needed in this aim, which allowed us to state dynamical information on the perturbed spatial quantized Hill model. Furthermore, this model can be used to develop a lunar theory and the families of periodic orbits in the framework for the spatial quantized Hill problem. Thereby, these applications serve to reinforce the results obtained about these periodic solutions.

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