Article

Applications of (M, N) -Lucas Polynomials on a Certain Family of Bi-Univalent Functions

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Abstract: In the current article, making use of certain operator, we initiate and explore a certain family $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$ of holomorphic and bi-univalent functions in the open unit disk $\mathbb{D}$. We establish upper bounds for the initial Taylor–Maclaurin coefficients and the Fekete–Szegő type inequality for functions in this family.

Keywords: (M, N) -Lucas polynomials; bi-univalent function; upper bounds; holomorphic function; Fekete–Szegő problem

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1. Introduction

We indicate by $\mathcal{A}$ the family of holomorphic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

By $\mathcal{S}$ we denote the subfamily of $\mathcal{A}$ consisting of all functions which are also univalent in $\mathbb{D}$.

The famous Koebe one-quarter theorem [1] ensures that the image of $\mathbb{D}$ under each univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Each function $f \in \mathcal{S}$ has an inverse f^{-1} and the inverse is defined by $f^{-1}(f(z)) = z$ and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (2)$$

We state that a function $f \in \mathcal{A}$ is bi-univalent in the open unit disk $\mathbb{D}$ if the functions f and f^{-1} are univalent in $\mathbb{D}$. The family of all bi-univalent functions in $\mathbb{D}$ is denoted by Σ .

There have been many papers in recent years on analytic and bi-univalent functions, e.g., [2–16].

From the paper in [10], we mention some examples of functions in the family Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

We know that some familiar functions $f \in \mathcal{S}$, such as the Koebe function $k(z) = \frac{z}{(1-z)^2}$, its rotation function $k_\zeta(z) = \frac{z}{(1-e^{i\zeta}z)^2}$, $f(z) = z - \frac{z^2}{2}$ and $f(z) = \frac{z}{1-z^2}$, are not members of Σ .

The problem lies in obtaining the general coefficient bounds on the Taylor–Maclaurin coefficients

$$|a_n| \quad (n \in \mathbb{N}; n \geq 3)$$

for function $f \in \Sigma$ is still not completely addressed for many of the subfamilies of Σ . The Fekete–Szegő problem $|a_3 - \eta a_2^2|$ for $f \in \mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin is in the disproof by Fekete and Szegő in [1] of the Littlewood–Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The Fekete–Szegő problem has been studied in recent years for many classes of univalent functions, see, for example: [2–6,8–10,17–34].

In [35], the principle of subordination between holomorphic functions is presented: if the functions f and g are holomorphic in $\mathbb{D}$, we state that the function f is subordinate to g , if there exists a Schwarz function ω , which is analytic in $\mathbb{D}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = g(\omega(z)).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{D}).$$

It is well known that if the function g is univalent in $\mathbb{D}$, then

$$f \prec g \quad (z \in D) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

For obtaining the original results in this paper, some elements of the (p, q) -calculus must be used. For more details on the concepts of (p, q) -calculus, see the following papers: [23,31,36–39].

For a holomorphic function f , the (p, q) -derivative operator is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z} \quad (z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}), 0 < q < p \leq 1,$$

and

$$D_{p,q}f(0) = f'(0).$$

For function $f \in \mathcal{A}$, we deduce that

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1},$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} \quad (p \neq q),$$

(see [40,41])

$$\lim_{p \rightarrow 1^-} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}.$$

We see that the notation $[n]_{p,q}$ is symmetric,

$$[n]_{p,q} = [n]_{q,p}.$$

Wanas and Cotîrlă [42] introduced the operator $W_{\alpha,\beta,p,q}^{\sigma,\delta} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$W_{\alpha,\beta,p,q}^{\sigma,\delta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[\Psi_n(\sigma, \alpha, \beta)]_{p,q}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}} \right)^{\delta} a_n z^n,$$

where

$$\Psi_n(\sigma, \alpha, \beta) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\alpha^{\tau} + n\beta^{\tau}), \quad \Psi_1(\sigma, \alpha, \beta) = \sum_{\tau=1}^{\sigma} \binom{\sigma}{\tau} (-1)^{\tau+1} (\alpha^{\tau} + \beta^{\tau}),$$

$$n-1, \sigma \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in \mathbb{R}_0^+ \text{ with } \alpha + \beta > 0, \delta \in \mathbb{N}_0, 0 < q < p \leq 1, z \in \mathbb{D}.$$

Remark 1. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ is a generalization of several known operators studied in earlier investigations, which are recalled below.

1. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ reduces to $J_{q,\alpha}^{\nu}$, the q -Srivastava-Attiya operator, see [43], when $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-, p = \beta = \sigma = 1, \Re(\nu) > 1, \delta = -\nu$.
2. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the q -Bernardi operator, see [44], when $\delta = -1, \alpha > -1, p = \sigma = \beta = 1$.
3. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the q -Libera operator, see [44], when $\delta = -1, p = \alpha = \sigma = \beta = 1$.
4. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the q -Sălăgean operator, see [45], when $p = \beta = \sigma = 1, \alpha = 0$.
5. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ reduces to the operator $I_{\alpha,\beta}^{\delta}$, introduced and studied by Wanas in [46], when $p = 1, q \longrightarrow 1^-$.
6. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator $I_{\alpha,\beta}^{\delta}$ studied by Swamy in the paper [47], when $\sigma = p = 1, q \longrightarrow 1^-$.
7. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ reduces to the Srivastava-Attiya operator J_{α}^{ν} which was studied in [48], when $\Re(\nu) > 1, \delta = -\nu, q \longrightarrow 1^-, p = \beta = \sigma = 1$, and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$.
8. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator I_{α}^{δ} , which was studied by Cho and Srivastava in [49], when $\alpha > -1, q \longrightarrow 1^-, p = \beta = \sigma = 1$.
9. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator I^{δ} , which was studied by Uralegaddi and Somanatha in [50], when $\sigma = p = \alpha = \beta = 1, q \longrightarrow 1^-$.
10. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator I^{ξ} introduced by Jung et al. in [51], when $\xi > 0, \delta = -\xi, \sigma = \alpha = \beta = p = 1, q \longrightarrow 1^-$. The operator I^{ξ} is the Jung-Kim-Srivastava integral operator.
11. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ reduces to the Bernardi operator in [52] when $\sigma = \beta = 1 = p, q \longrightarrow 1^-, \alpha > -1, \delta = -1$.
12. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ reduces to the Alexander operator in [53] when $p = \beta = \sigma = 1, q \longrightarrow 1^-, \alpha = 0, \delta = -1$.
13. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator D_{β}^{δ} , which was studied by Al-Oboudi in the paper [54], when $\sigma = 1 = p, q \longrightarrow 1^-, \alpha = \beta - 1$ and $\beta > 0$.
14. The operator $W_{\alpha,\beta,p,q}^{\sigma,\delta}$ becomes the operator S^{δ} , which was studied by Sălăgean in the paper [55], when $\sigma = 1 = p, q \longrightarrow 1^-, \alpha = 0, \beta = 1$.

In [56], the (M, N) -Lucas Polynomials, $L_{M,N,k}(x)$, for the polynomials with real coefficients $M(x)$ and $N(x)$ are defined by the recurrence relation:

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x), \quad (k \geq 2),$$

and

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \text{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x). \quad (3)$$

The Lucas Polynomials play an important role in a range of disciplines in mathematics, statistics, engineering sciences and physics (see, for example [57–59]). The generating function of the (M, N) -Lucas Polynomial $L_{M,N,k}(x)$ (see [58]) is given by

$$T_{M(x),N(x)}(z) = \sum_{k=2}^{\infty} L_{M,N,k}(x)z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}. \quad (4)$$

Remark 2. If we choose particular values for $M(x)$ and $N(x)$, then the (M, N) -Lucas Polynomial $L_{M,N,k}(x)$ leads to the following polynomials:

1. $L_{x,1,k}(x) =: L_k(x)$, the Lucas polynomials;
2. $L_{2x,1,k}(x) =: P_k(x)$, the Pell–Lucas polynomials;
3. $L_{1,2x,k}(x) =: J_k(x)$, the Jacobsthal polynomials;
4. $L_{3x,-2,k}(x) =: F_k(x)$, the Fermat–Lucas polynomials;
5. $L_{2x,-1,k}(x) =: T_k(x)$, the first-kind Chebyshev polynomials.

The (M, N) -Lucas Polynomial has been presented and investigated analogously by various classes of functions (see, for example [32,60–65]).

2. Main Results

We define the family $\mathcal{W}_{\Sigma}(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$ in this section as follows:

Definition 1. Assume that $\gamma \geq 0$, $0 \leq \lambda \leq 1$ and h is analytic in $\mathbb{D}$, $h(0) = 1$. The family $\mathcal{W}_{\Sigma}(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$ contains all the functions $f \in \Sigma$ that satisfy the subordinations

$$(1 - \gamma) \left[(1 - \lambda) \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} + \lambda \left(1 + \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)''}{\left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'} \right) \right] + \gamma \frac{\lambda z^2 \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'' + z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{\lambda z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)' + (1 - \lambda) W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} \prec h(z)$$

and

$$(1 - \gamma) \left[(1 - \lambda) \frac{w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'}{W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w)} + \lambda \left(1 + \frac{w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)''}{\left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'} \right) \right] + \gamma \frac{\lambda w^2 \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'' + w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'}{\lambda w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)' + (1 - \lambda) W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w)} \prec h(w),$$

where the function f^{-1} is of the form (2).

Theorem 1. Assume that $0 \leq \lambda \leq 1$ and $\gamma \geq 0$. If the family $\mathcal{W}_{\Sigma}(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$, where $h(z) = 1 + e_1 z + e_2 z^2 + \dots$, contains the functions $f \in \Sigma$ given by Relation (1), then

$$|a_2| \leq \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta} e_1}{(\lambda + 1) [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{\delta}} \quad (5)$$

and

$$|a_3| \leq \min \left\{ \max \left\{ \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right|, \right. \right. \\ \left. \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_2}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} + \frac{(\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1^2}{2(2\lambda + 1)(\lambda + 1)^2[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right| \right\}, \\ \max \left\{ \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right|, \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_2}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right| \right. \\ \left. \left. - \frac{(4(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{2\delta}[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta - (\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta})e_1^2}{2(2\lambda + 1)(\lambda + 1)^2[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\}. \quad (6)$$

Proof. Suppose that $f \in \mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; e_1; e_2)$. It follows that there exist $\varepsilon, \zeta : \mathbb{D} \rightarrow \mathbb{D}$ holomorphic functions, of the form

$$\varepsilon(z) = d_1 z + d_2 z^2 + d_3 z^3 + \dots \quad (z \in \mathbb{D}) \quad (7)$$

and

$$\zeta(w) = l_1 w + l_2 w^2 + l_3 w^3 + \dots \quad (w \in \mathbb{D}), \quad (8)$$

with $\varepsilon(0) = \zeta(0) = 0$, $|\varepsilon(z)| < 1$, $|\zeta(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(1 - \gamma) \left[(1 - \lambda) \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} + \lambda \left(1 + \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)''}{\left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'} \right) \right] \\ + \gamma \frac{\lambda z^2 \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'' + z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{\lambda z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)' + (1 - \lambda) W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} = 1 + e_1 \varepsilon(z) + e_2 \varepsilon^2(z) + \dots \quad (9)$$

and

$$(1 - \gamma) \left[(1 - \lambda) \frac{w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'}{W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w)} + \lambda \left(1 + \frac{w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)''}{\left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'} \right) \right] \\ + \gamma \frac{\lambda w^2 \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'' + w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)'}{\lambda w \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w) \right)' + (1 - \lambda) W_{\alpha, \beta, p, q}^{\sigma, \delta} f^{-1}(w)} = 1 + e_1 \zeta(w) + e_2 \zeta^2(w) + \dots \quad (10)$$

Combining (7)–(10) yields

$$(1 - \gamma) \left[(1 - \lambda) \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} + \lambda \left(1 + \frac{z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)''}{\left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'} \right) \right] \\ + \gamma \frac{\lambda z^2 \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'' + z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)'}{\lambda z \left(W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z) \right)' + (1 - \lambda) W_{\alpha, \beta, p, q}^{\sigma, \delta} f(z)} = 1 + e_1 d_1 z + [e_1 d_2 + e_2(x) d_1^2] z^2 + \dots \quad (11)$$

and

$$(1-\gamma) \left[(1-\lambda) \frac{w \left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)'}{W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w)} + \lambda \left(1 + \frac{w \left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)''}{\left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)'} \right) \right] \\ + \gamma \frac{\lambda w^2 \left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)'' + w \left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)'}{\lambda w \left(W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w) \right)' + (1-\lambda) W_{\alpha,\beta,p,q}^{\sigma,\delta} f^{-1}(w)} = 1 + e_1 l_1 w + [e_1 l_2 + e_2 l_1^2] w^2 + \dots \quad (12)$$

If for $z, w \in \mathbb{D}$, it is already known that $|\varepsilon(z)| < 1$ and $|\zeta(w)| < 1$, then

$$|d_j| \leq 1, \quad |l_j| \leq 1, \quad j \in \mathbb{N}, \quad (13)$$

please see [66] for more details.

In the relations (11) and (12), after simplifying, we find that

$$\frac{(\lambda+1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta}} a_2 = e_1 d_1, \quad (14)$$

$$\frac{2(2\lambda+1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta}} a_3 - \frac{(\lambda\gamma(\lambda-1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} a_2^2 = e_1 d_2 + e_2 d_1^2, \quad (15)$$

$$- \frac{(\lambda+1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta}} a_2 = e_1 l_1 \quad (16)$$

and

$$\frac{2(2\lambda+1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta}} (2a_2^2 - a_3) - \frac{(\lambda\gamma(\lambda-1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} a_2^2 = e_1 l_2 + e_2 l_1^2. \quad (17)$$

The Inequality (5) follows from (14) and (16). In view of (14) and (15), we conclude that

$$\frac{2(2\lambda+1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta} e_1} a_3 = d_2 + \left(\frac{e_2}{e_1} + \frac{(\lambda\gamma(\lambda-1) + 3\lambda + 1)e_1}{(\lambda+1)^2} \right) d_1^2, \quad (18)$$

and on the basis of the well-known sharp result please see [67], (p. 10):

$$|d_2 - \mu d_1^2| \leq \max\{1, |\mu|\} \quad (19)$$

for all $\mu \in \mathbb{C}$, we obtain

$$\left| \frac{2(2\lambda+1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta} e_1} a_3 \right| \leq \max \left\{ 1, \left| \frac{e_2}{e_1} + \frac{(\lambda\gamma(\lambda-1) + 3\lambda + 1)e_1}{(\lambda+1)^2} \right| \right\}. \quad (20)$$

Following (16) and (17), we deduce that

$$- \frac{2(2\lambda+1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta}}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta} e_1} a_3 \\ = l_2 + \left(\frac{e_2}{e_1} - \frac{(4(2\lambda+1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{\delta} [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^{\delta} - (\lambda\gamma(\lambda-1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}) e_1}{(\lambda+1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right) l_1^2. \quad (21)$$

Applying (19), we obtain

$$\left| \frac{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta}{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1} \right| |a_3| \leq \max \left\{ 1, \left| \frac{e_2}{e_1} - \frac{(4(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta - (\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\}. \quad (22)$$

Inequality (23) follows from (20) and (22). $\square$

From Relation (3), we have $e_1 = M(x)$ and $e_2 = M^2(x) + 2N(x)$ and if we consider the generating function (4) of the (M, N) -Lucas polynomials, $L_{M,N,k}(x)$ as $h(z) + 1$, Theorem 1 provides the following corollary.

Corollary 1. If $f \in \Sigma$ of the form (1) is in the class $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; T_{M(x), N(x)} - 1)$, then

$$|a_2| \leq \frac{|M(x)| [\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta}{[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^\delta (\lambda + 1)}$$

and

$$|a_3| \leq \min \left\{ \max \left\{ \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta M(x)}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right|, \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta (M^2(x) + 2N(x))}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} + \frac{(\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta M^2(x)}{2(2\lambda + 1)(\lambda + 1)^2 [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right| \right\}, \right. \\ \left. \max \left\{ \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta M(x)}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right|, \left| \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta (M^2(x) + 2N(x))}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \right|, \right. \\ \left. \left| \frac{(4(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^{2\delta} [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta - (\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} M^2(x)}{2(2\lambda + 1)(\lambda + 1)^2 [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\} \right\}, \quad (23)$$

for all $\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q, x$ such that $\sigma \in \mathbb{N}$, $\gamma \geq 0$, $\alpha \in \mathbb{R}$, $0 \leq \lambda \leq 1$, $\alpha + \beta > 0$, $\beta \in \mathbb{R}_0^+$, $\delta \in \mathbb{N}_0$, $0 < q < p \leq 1$ and $x \in \mathbb{R}$, where $T_{M(x), N(x)}$ is given by (4).

For the family functions $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$, we prove the Fekete–Szegő inequality in the next theorem.

Theorem 2. If $f \in \Sigma$ of the form (1) is in the class $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1|}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \times \\ \times \min \left\{ \max \left\{ 1, \left| \frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2\eta(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\}, \right. \\ \left. \max \left\{ 1, \left| \frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2(\eta - 2)(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\} \right\} \quad (24)$$

for all $\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q$ such that $\alpha \in \mathbb{R}$, $\gamma \geq 0$, $\beta \in \mathbb{R}_0^+$, $0 \leq \lambda \leq 1$, $\sigma \in \mathbb{N}$, $\delta \in \mathbb{N}_0$, $\alpha + \beta > 0$, $0 < q < p \leq 1$, $\eta \in \mathbb{C}$.

Proof. We apply the notations from the proof of Theorem 1. From (14), (15) and (18), we have

$$a_3 - \eta a_2^2 = \frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \times \left(r_2 + \left(\frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2\eta(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right) r_1^2 \right) \quad (25)$$

and using the known sharp result $|r_2 - \mu r_1^2| \leq \max\{1, |\mu|\}$, we obtain

$$|a_3 - \eta a_2^2| \leq \frac{|[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1|}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \times \max \left\{ 1, \left| \frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2\eta(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\}. \quad (26)$$

In the same way, from (16), (17) and (21), we conclude that

$$a_3 - \eta a_2^2 = -\frac{[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \left(s_2 + \left(\frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2(\eta - 2)(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right) s_1^2 \right) \quad (27)$$

and using $|s_2 - \mu s_1^2| \leq \max\{1, |\mu|\}$, we obtain

$$|a_3 - \eta a_2^2| \leq \frac{|[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta e_1|}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \max \left\{ 1, \left| \frac{e_2}{e_1} + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2(\eta - 2)(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) e_1}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right| \right\}. \quad (28)$$

Inequality (29) follows from (26) and (28). $\square$

Corollary 2. If $f \in \Sigma$ of the form (1) is in the class $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; T_{M(x), N(x)} - 1)$, then

$$|a_3 - \eta a_2^2| \leq \frac{|[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta M(x)|}{2(2\lambda + 1)[\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta} \min \left\{ \max \left\{ 1, \left| \frac{M^2(x) + 2N(x)}{M(x)} \right| \right\}, \right. \\ \left. + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2\eta(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) M(x)}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right\}, \\ \max \left\{ 1, \left| \frac{M^2(x) + 2N(x)}{M(x)} \right| \right\} \\ \left. + \frac{((\lambda\gamma(\lambda - 1) + 3\lambda + 1)[\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta} + 2(\eta - 2)(2\lambda + 1)[\Psi_1(\sigma, \alpha, \beta)]_{p,q}^\delta [\Psi_3(\sigma, \alpha, \beta)]_{p,q}^\delta) M(x)}{(\lambda + 1)^2 [\Psi_2(\sigma, \alpha, \beta)]_{p,q}^{2\delta}} \right\}, \quad (29)$$

for all $\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q, x$ such that $\alpha \in \mathbb{R}$, $\gamma \geq 0$, $0 \leq \lambda \leq 1$, $\beta \in \mathbb{R}_0^+$, $\alpha + \beta > 0$, $\sigma \in \mathbb{N}$, $\delta \in \mathbb{N}_0$, $0 < q < p \leq 1$ and $x \in \mathbb{R}$, where $T_{M(x), N(x)}$ is given by (4).

3. Conclusions

We obtain in this paper a new family $\mathcal{W}_\Sigma(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$ of holomorphic and bi-univalent functions defined by a certain operator and also using the (M, N) -Lucas Polynomials $L_{M,N,k}(x)$, which are of the form (3) and generate the function $T_{M(x), N(x)}(z)$

in (4). We generate Taylor–Maclaurin coefficient inequalities for functions belonging to the family $\mathcal{W}_{\Sigma}(\lambda, \gamma, \sigma, \delta, \alpha, \beta, p, q; h)$ and consider the famous Fekete–Szegő problem.

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