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On Indeterminacy of Interval Multiplicative Pairwise Comparison Matrix

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Abstract: The interval multiplicative pairwise comparison matrix (IMPCM) is widely used to model human judgments affected by uncertainty and/or ambiguity. To improve the quality of an IMPCM, consistency is not sufficient. The indeterminacy should also be within an acceptable threshold because a consistent IMPCM may be deemed unacceptable due to high indeterminacy. Regarding indeterminacy, two metrics have been proposed in the literature: the *indeterminacy ratio* and the *indeterminacy index*. The former is from a local view, and the latter is from a global view. We have proposed an acceptable IMPCM model, which guarantees that an inconsistent IMPCM can be transformed into a consistent IMPCM, and the maximal *indeterminacy ratio* can be reduced. However, there is still a research gap. That is, a concomitant question naturally arises: can the *indeterminacy index* be reduced as well? In this paper, we further prove that the *indeterminacy index* of an originally inconsistent IMPCM can be reduced under the proposed model. Three numerical examples are presented to illustrate the feasibility and superiority of the proposed model. We also flowcharted the proposed model from a pragmatic view such that we can judiciously reduce the *indeterminacy index* of the IMPCM to a certain satisfactory level. That is, by applying the proposed model once, the original inconsistent IMPCM can be transformed into a consistent IMPCM that will possess less indeterminacy than the original one has. Consequently, by successively applying the proposed model, we can reduce or even eventually eliminate the indeterminacy of the IMPCM. In other words, we can/may obtain an MPCM rather than an IMPCM. In addition to mathematical proofs, we present experimental results of computer simulations to corroborate our argument. In summary, this model is not only effective but also efficient because it only requires arithmetic operations without solving complex optimization problems.

Keywords: analytic hierarchy process; consistent transformation; interval multiplicative pairwise comparison matrix; subjective judgment; uncertainty and/or ambiguity



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1. Introduction

Pairwise Comparison (PC), a long-established and powerful method, has been successfully applied in multiple criteria decision making (MCDM) to manifest human judgment [1]. MCDM can be roughly divided into two subfields: MODM (Multi-Objective Decision Making) and MADM (Multi-Attribute Decision Making) [2,3]. The former includes goal programming (GP) [4,5] and multiple objective programming (MOP) [6]. The latter can be further categorized into three types according to its applications: *structure modelling methods* (e.g., Interpretive Structural Modeling (ISM) [7], Decision Making Trial, and Evaluation Laboratory (DEMATEL) [8]), *priority weighting methods* (e.g., Analytic Hierarchy/Network Process (AHP/ANP) [9,10], Criteria Importance Through Intercriteria Correlation (CRITIC) [11]), and *ranking methods* (e.g., Simple Additive Weight (SAW) [12], Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) [13], Grey Relational Analysis (GRA) [14], Elimination Et Choice Translating Reality (ELECTRE) [15], and Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE) [16,17]).

The concept of PC method is used in structure modelling methods (such as DEMATEL), in priority weighting methods (such as AHP), and in ranking methods (such as ELECTRE). In this study, we focus on using PC in the AHP.

Undoubtedly, prioritization is a crucial stage for MCDM that will significantly affect the decision. In AHP, the source information (preference and/or expertise) of this stage is collected through PC that manifests human subjective judgment. On the other hand, there are methods for the determination of objective weights, such as the Entropy [18], the CRITIC [11], and the MEREC (Method based on the Removal Effects of Criteria) [19].

As an extensive framework that deals with problem decomposition in a systematic way, the AHP enables us simultaneously to deal with the intuitive, the rational, and the irrational when we are confronted with real-life decision problems [20]. It is well known that an important building block of the AHP is PC. By using PC, we inquire with experts about comparing two entities at a time and expressing the degree of intensity of one entity over the other. Unfortunately, human judgments sometimes may be inconsistent, especially in a situation with many entities involved. That is, the main challenge of using pairwise comparison is its lack of consistency that originates from the human judgments, which in practice, is very often the case; in other words, most PC matrices are inconsistent.

Since Saaty's [9,20] development of the AHP, many applications in practical decision making have been reported [21,22]. By using PC, the kernel of AHP, the decision maker collects human judgments to derive estimated priority vector of $w = (w_1, w_2, \dots, w_n)$ [20]. For this, let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of n entities that can be criteria or alternatives. A pairwise comparison matrix (PCM) A on X can be presented as a square matrix $A = [a_{ij}]_{n \times n}$, where a_{ij} and for every $i, j = 1, \dots, n$. This can present human assessment based on the relative importance, preference, or estimation of an entity i to another entity j . There are several reviews and survey papers on the AHP and the PC [21–24].

To present human judgments, various measurement scales have been proposed [9,25] (note that this study is independent of what measurement scale is adopted). For example, Saaty [9,21] suggested the widely used bipolar ratio scale $S = [\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9]$ with a neutral value of 1. Here, "1" means *equal* importance, "3" *moderate* importance, "5" *strong* importance, "7" *very strong* importance, and "9" *extreme* importance; intermediate values of "2", "4", "6", and "8" stand for judgments between the two adjacent judgments. This scale requires that $a_{ij} \in S$, and $a_{ii} = 1$ for all $i, j = 1, 2, \dots, n$. A crisp value of a_{ij} can be interpreted as the entity x_i is a_{ij} times preferred to another entity x_j . The greater the value of the a_{ij} , the more strongly x_i is preferred to x_j .

However, it is difficult to assign precise values in a PCM for real-life decisions since they are usually characterized by uncertainty and/or ambiguity [26]. How to deal with the uncertainty and/or ambiguity confronted in pairwise comparisons has received increasing research attention in the past decades [27,28]. Various models have been proposed, such as interval fuzzy preference relations (IFPRs) [29,30], interval multiplicative preference relations (IMPRs) [31–38] (there is no unified name in the literature; we adopt the interval multiplicative pairwise comparison matrix (IMPCM)), triangular fuzzy preference relations [39–41], linguistic preference relations (LPRs) [42,43], intuitionistic fuzzy preference relations [44,45], hesitant fuzzy preference relations [46,47], and Fermatean fuzzy sets (FFSs) [48,49]. Some review and survey papers can be found [50,51]. Since the IMPCM introduced by Saaty and Vargas [10] is widely used to manifest human judgments with uncertainty and/or ambiguity, we here focus on it.

Three key issues should be considered to cope with an IMPCM: consistency, indeterminacy, and normality [52]. Various definitions of consistency have been proposed [53,54]. However, those definitions only can be used to check whether an IMPCM is consistent or not and not directly used to construct a consistent IMPCM.

However, a consistent IMPCM is still not sufficient to be acceptable, and a highly indeterminate IMPCM is considered useless [55]. Therefore, two metrics of indeterminacy (indeterminacy ratio and indeterminacy index) have been proposed [55]. Besides, nor-

malized interval priority weights are better derived from an IMPCM [53]. For this, two definitions of normalization (additive and multiplicative) have been proposed [30,55].

We put forward a new definition of *acceptable* IMPCM that simultaneously considers *consistency*, *indeterminacy*, and *normality* [52]. Although the maximal *indeterminacy* ratio of an IMPCM can be reduced under the proposed model, it is necessary to investigate whether this model can reduce the *indeterminacy index* of an IMPCM. This is the research question of this study. The answer is supported by conclusive mathematical proofs and experimental results from a series of computer simulations. In Section 2, two metrics of *indeterminacy* are reviewed. In Section 3, we give a brief overview for the proposed model. In Section 4, we prove that the proposed model can reduce the *indeterminacy index*. In Section 5, we first present three numerical examples to illustrate the feasibility and superiority of the proposed model, then flowchart the proposed model from a pragmatic view. In Section 6, the experimental results of computer simulations are presented. Finally, discussion and conclusions are given.

2. Literature Review

This section first briefly introduces the concepts of a multiplicative PC matrix (MPCM). For an MPCM, Saaty [9,20] presented the following two definitions as follows:

Definition 1. An MPCM $\mathbf{A} = [a_{ij}]_{n \times n}$ is reciprocal if

$$a_{ij} = \frac{1}{a_{ji}}, \forall i, j = 1, 2, \dots, n.$$

Thus, if entity x_i is a_{ij} times preferred to entity x_j , then entity x_j is $\frac{1}{a_{ij}}$ times preferred to entity x_i . It is worth noting that the entry a_{ij} can be a preference ratio, i.e., a multiplicative case, or a preference difference, i.e., an additive case [56,57], or a value belonging to $[0, 1]$ that measures the distance from the indifference (0.5), i.e., a fuzzy case [58].

What follows here are two fundamental issues of using a PC matrix: consistency and prioritization. First, the issue of consistency has attracted increasing attention from inception to recent years and resulted in the proposal of various indices for the measurement of inconsistency [57,59–61]. However, there is no consensually accepted index because every proposed index is itself a different definition [59]. Second, the issue of prioritization has mainly focused on a debate about which is better: the *eigenvector method* [62,63] or the *geometric mean method* [64]. Some support the former [65], whereas some prefer the latter [66,67]. Although sometimes, we have to deal with an *incomplete* PCM [68–70], hereinafter, a PCM means a *complete* PCM. On the other hand, for a complex decision problem, it usually needs group decision making [71–75]. However, to give a detailed exposition of these mentioned-above topics is beyond the scope of this article. Interested readers may refer to some retrospective literature [23,24,61].

Definition 2. An MPCM $\mathbf{A} = [a_{ij}]_{n \times n}$ is consistent if

$$a_{ij} = a_{ik}a_{kj}, \forall i, j, k = 1, 2, \dots, n.$$

Consistency is a normative property requesting that, for all triple entities i , j , and k , if entity x_i is conceived as being a_{ik} times preferable to entity x_k , and entity x_k is conceived as being a_{kj} times preferable to entity x_j , then entity x_i should be conceived as being $a_{ik}a_{kj}$ times preferable to entity x_j .

All PC matrices discussed here are *reciprocal* since they are natural and rational; however, they are not necessarily *consistent*, which is the property to be achieved from both the academic perspective and the decision-maker perspective.

Based on the Definition 2, a prevalent and normative definition of consistency, we introduce a new but logically equivalent definition as follows [52];

Definition 3. An MPCM $\mathbf{A} = [a_{ij}]_{n \times n}$ is consistent if

$$a_{ij} = \sqrt[n]{\left(\prod_{k=1}^n a_{ik}\right)\left(\prod_{k=1}^n a_{kj}\right)} = \sqrt[n]{R_i C_j}, \forall i, j = 1, 2, \dots, n.$$

Here, R_i stands for $\prod_{k=1}^n a_{ik}$, and C_j stands for $\prod_{k=1}^n a_{kj}$. Obviously, $R_i C_i = 1$ for all $i = 1, 2, \dots, n$.

Since Definition 3 begins with the prevailing Definition 2 of consistency, this new definition itself is clearly a consistent transformation [52]. Note that this consistent transformation utilizes the MPCM $\mathbf{A}$ holistically, and it makes the transformation endogenously.

However, it is usually challenging to assign crisp values to making pairwise comparisons in real-life decision-making problems since such problems generally include uncertainty and/or ambiguity. To model the uncertainty and/or ambiguity experienced in making pairwise comparisons, Saaty and Vargas introduced the IMPCM [31]. Current research has increasingly investigated IMPCM and obtained extensive theoretical results [32–39,53–55].

Definition 4. An IMPCM $\bar{\mathbf{A}}$ on $\mathbf{X}$ is characterized by an interval judgment matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n}$ with $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ and

$$\frac{1}{s} \leq a_{ij}^- \leq a_{ij}^+ \leq s, a_{ij}^- a_{ji}^+ = 1, a_{ii}^- = a_{ii}^+ = 1, \forall i, j = 1, 2, \dots, n. \quad (1)$$

where $\bar{a}_{ij}$ is an interval preference ratio that indicates a compared entity, and x_i is conceived as being between a_{ij}^- and a_{ij}^+ times as important as another compared entity x_j .

Various definitions for the consistency of an IMPCM have been proposed [35–37,39,53,54]. However, those definitions cannot be directly used to construct a consistent IMPCM, and even a consistent IMPCM is still not sufficient to be acceptable because a highly indeterminate IMPCM usually is considered worthless in practice [55]. Therefore, the interval width is often adopted to measure the indeterminacy level of an interval judgment.

To measure the degree of uncertainty and/or ambiguity of IMPCMs, Li et al. [55] introduced the following two measurements of indeterminacy: the *indeterminacy ratio* and the *indeterminacy index*.

Definition 5. Let $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ be an interval comparison judgment on a bounded scale $[1/s, s]$; then, its indeterminacy ratio, denoted by $\text{IR}(\bar{a}_{ij})$, is defined by $\text{IR}(\bar{a}_{ij}) = \frac{a_{ij}^+}{a_{ij}^-}$.

Obviously, $1 \leq \text{IR}(\bar{a}_{ij}) \leq s^2$ holds; $\text{IR}(\bar{a}_{ij}) = 1$ implies $\bar{a}_{ij}$ is a crisp value, and larger values of the $\text{IR}(\bar{a}_{ij})$ indicate more indeterminacy for the judgment $\bar{a}_{ij}$.

Definition 6. Letting $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$ be an IMPCM, a geometric-mean-based indeterminacy index of $\bar{\mathbf{A}}$ is defined as

$$\Pi(\bar{\mathbf{A}}) = \left(\prod_{i < j} \left(\frac{a_{ij}^+}{a_{ij}^-} \right) \right)^{\frac{2}{n^2 - n}}$$

It is obvious that $\Pi(\bar{\mathbf{A}}) \geq 1$. If $\Pi(\bar{\mathbf{A}}) = 1$, then $a_{ij}^+ = a_{ij}^-$ for all $i, j = 1, 2, \dots, n$. That is, all $\bar{a}_{ij}$ values become crisp, and $\bar{\mathbf{A}}$ is thus reduced to an MPCM; otherwise, $\bar{\mathbf{A}}$

contains indeterminacy, and the larger the $\Pi(\bar{\mathbf{A}})$, the higher the indeterminacy of this $\bar{\mathbf{A}}$ [55]. Moreover, within a hierarchical structure, the global priority weights that aggregate the local priority weights will become meaningless if the local priority weights are not normalized [53]. In order to bridge these three gaps (consistency, indeterminacy, and normality) mentioned above, we have proposed a model of *acceptable* IMPCM [52].

3. An Acceptable IMPCM

As mentioned above, even a consistent IMPCM is still not sufficient to be acceptable, and a highly indeterminate IMPCM is considered useless [55]. To take *indeterminacy* into consideration, Li et al. introduced the following definition of an *acceptable* IMPCM [55]:

Definition 7. Let $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$ be an IMPCM and t_{ur} ($t_{ur} > 1$) be an acceptable indeterminacy ratio threshold; if $IR(\bar{a}_{ij}) \leq t_{ur}$ for all $i, j = 1, 2, \dots, n$, and $\bar{\mathbf{A}}$ is acceptably consistent, then $\bar{\mathbf{A}}$ is called acceptable; otherwise, $\bar{\mathbf{A}}$ is unacceptable.

However, in Definition 7, Li et al. used the *indeterminacy ratio* and not the *indeterminacy index*; moreover, about consistency, they adopted a looser definition of consistency—so-called acceptably consistent.

Therefore, we proposed a definition of an *acceptable* IMPCM as follows [52]:

Definition 8. For an IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$, let the acceptable indeterminacy ratio threshold be denoted by $a^* = \max_{i,j} IR(\bar{a}_{ij})$. Then, a constructed IMPCM $\bar{\mathbf{B}} = (\bar{b}_{ij})_{n \times n} = ([b_{ij}^-, b_{ij}^+])_{n \times n}$ is called an acceptable IMPCM if it satisfies the following three conditions: (1) $\bar{\mathbf{B}}$ is consistent, (2) $IR(\bar{b}_{ij}) < a^*$, for all $i, j = 1, 2, \dots, n$, and (3) interval priority weights $\mathbf{W}$ of $\bar{\mathbf{B}}$ are (additively) normalized.

As mentioned above, an IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n}$ must satisfy reciprocity of $\bar{a}_{ji} = \frac{1}{\bar{a}_{ij}}$. That is, $\bar{a}_{ji} = [\frac{1}{a_{ij}^+}, \frac{1}{a_{ij}^-}]$. Thus, for an IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n}$ with $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$, we denote, hereinafter,

$$R_i^- = \left(\prod_{k=1}^{i-1} \frac{1}{a_{ki}^+} \right) \left(\prod_{k>i}^n a_{ik}^- \right), \quad (2)$$

$$R_i^+ = \left(\prod_{k=1}^{i-1} \frac{1}{a_{ki}^-} \right) \left(\prod_{k>i}^n a_{ik}^+ \right). \quad (3)$$

Obviously, $R_i^- \leq R_i^+$ for all $i = 1, 2, \dots, n$. Based on Definition 3, it is a consistent transformation; we then propose the following definition [52]:

Definition 9. An IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$ is consistent if and only if

$$a_{ij}^- a_{ij}^+ = \sqrt[n]{\frac{R_i^- R_i^+}{R_j^- R_j^+}}, \forall i, j = 1, 2, \dots, n. \quad (4)$$

To guarantee that $b_{ij}^- \leq b_{ij}^+$ will hold, we further suggest the following transformation method (it is worth noting that Equation (5) is an endogenous transformation):

$$b_{ij}^- = \min \left(\sqrt[n]{\frac{R_i^-}{R_j^-}}, \sqrt[n]{\frac{R_i^+}{R_j^+}} \right), b_{ij}^+ = \max \left(\sqrt[n]{\frac{R_i^-}{R_j^-}}, \sqrt[n]{\frac{R_i^+}{R_j^+}} \right). \quad (5)$$

For this constructed IMPCM $\bar{\mathbf{B}} = (\bar{b}_{ij})_{n \times n} = ([b_{ij}^-, b_{ij}^+])_{n \times n}$, we can obtain a normalized interval priority weights by following three steps. First, according to Equations (2) and (3), we have

$$R_i'^- = \left(\prod_{k=1}^{i-1} \frac{1}{b_{ki}^+} \right) \left(\prod_{k>i}^n b_{ik}^- \right), \text{ and } R_i'^+ = \left(\prod_{k=1}^{i-1} \frac{1}{b_{ki}^-} \right) \left(\prod_{k>i}^n b_{ik}^+ \right). \quad (6)$$

Then, we normalize the geometric means of rows of $\bar{\mathbf{B}}$ to have

$$R_i''^- = \frac{\sqrt[n]{R_i'^-}}{\sum_{i=1}^n \sqrt[n]{R_i'^-}} \text{ and } R_i''^+ = \frac{\sqrt[n]{R_i'^+}}{\sum_{i=1}^n \sqrt[n]{R_i'^+}}, \quad (7)$$

and obtain the interval priority weights

$$w_i^- = \min(R_i''^-, R_i''^+) \text{ and } w_i^+ = \max(R_i''^-, R_i''^+). \quad (8)$$

Finally, we proved the following three theorems to guarantee that an *acceptable* IMPCM will be obtained [52] (since Theorems 1–3 are the main parts of our previous paper ([52]), we omit their proof for the sake of space and to avoid self-plagiarism).

Theorem 1. For an IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n}$ with $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$, the constructed IMPCM $\bar{\mathbf{B}} = (\bar{b}_{ij})_{n \times n} = ([b_{ij}^-, b_{ij}^+])_{n \times n}$, under the transformation by Equation (5), is consistent.

Theorem 2. By Equation (5), we will have $\text{IR}(\bar{b}_{ij}) < a^* = \max_{i,j} \text{IR}(\bar{a}_{ij})$ for all i, j .

Theorem 3. The interval priority weights, by Equation (8), of the constructed IMPCM $\bar{\mathbf{B}}$ are additively normalized.

In other words, the *indeterminacy ratio* of an originally inconsistent IMPCM can be reduced under the transformation by Equation (5). Additionally, the concomitant question then arose: can the *indeterminacy index* also be reduced? That is the research question of this study. To answer it, we used mathematical proofs (in Section 4) and experimental results of a series of computer simulations (in Section 5).

4. Mathematical Proofs

In this section, we prove that the *indeterminacy index* of an original inconsistent IMPCM can be reduced under the proposed model. In the proposed model, when transformed by Equation (5), the maximal *indeterminacy ratio* of an IMPCM can be reduced [52]. Recall that the *indeterminacy ratio* is from a local view, and the *indeterminacy index* is from a global view, leading to the concomitant question mentioned above: whether the *indeterminacy index* can also be reduced. In contrast to Definition 6, which explores the upper triangle elements of an IMPCM, we defined the following *indeterminacy index* by considering the entire matrix [52]:

Definition 10. Letting $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$ be an IMPCM, a geometric-mean-based *indeterminacy index* of $\bar{\mathbf{A}}$ is defined as

$$\Pi(\bar{\mathbf{A}}) = \sqrt[n^2]{\prod_{i=1}^n \prod_{j=1}^n \left(\frac{a_{ij}^+}{a_{ij}^-} \right)} = \left(\prod_{i=1}^n \sqrt[n]{\frac{R_i^+}{R_i^-}} \right)^{\frac{1}{n}}. \quad (9)$$

Note that the value of an *indeterminacy index* under Definition 10 is the square of the value of Definition 6. For the constructed IMPCM $\mathbf{B} = (b_{ij})_{n \times n} = ([b_{ij}^-, b_{ij}^+])_{n \times n}$, according to Definition 10, we will have

$$\Pi(\mathbf{B}) = \left(\prod_{i=1}^n \sqrt[n]{\frac{R_i'^+}{R_i'^-}} \right)^{\frac{1}{n}}. \quad (10)$$

Here, $R_i'^- = \left(\prod_{k=1}^{i-1} \frac{1}{b_{ki}^+} \right) \left(\prod_{k>i}^n b_{ik}^- \right)$, and $R_i'^+ = \left(\prod_{k=1}^{i-1} \frac{1}{b_{ki}^-} \right) \left(\prod_{k>i}^n b_{ik}^+ \right)$. Obviously, by Equations (2) and (3), for an IMPCM, the following two properties should hold:

$$(i) \left(\prod_{i=1}^n \sqrt[n]{R_i'^-} \right) \left(\prod_{i=1}^n \sqrt[n]{R_i'^+} \right) = 1, \quad (11)$$

and

$$(ii) \prod_{i=1}^n \sqrt[n]{R_i'^-} \leq 1 \leq \prod_{i=1}^n \sqrt[n]{R_i'^+}. \quad (12)$$

Therefore, if we can prove that $\prod_{i=1}^n \sqrt[n]{R_i'^-} \geq \prod_{i=1}^n \sqrt[n]{R_i^-}$, then we will have $\prod_{i=1}^n \sqrt[n]{R_i'^+} \leq \prod_{i=1}^n \sqrt[n]{R_i^+}$. Consequently, we will also have

$$\Pi(\mathbf{B}) = \left(\prod_{i=1}^n \sqrt[n]{\frac{R_i'^+}{R_i'^-}} \right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^n \sqrt[n]{\frac{R_i^+}{R_i^-}} \right)^{\frac{1}{n}} = \Pi(\mathbf{A}).$$

Let $N = \{1, 2, \dots, n\}$, and N_1 and N_2 be two subsets of N that satisfy $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$. According to properties (i) and (ii), it is simple to derive the following two properties:

$$(iii) \left(\prod_{i \in N_1} R_i^- \right) \left(\prod_{j \in N_2} R_j^+ \right) \geq \left(\prod_{i \in N_1} \frac{R_i^-}{R_i^+} \right), \quad (13)$$

and

$$(iv) \left(\prod_{i \in N_1} R_i^- \right) \left(\prod_{j \in N_2} R_j^+ \right) \leq \left(\prod_{j \in N_2} \frac{R_j^+}{R_j^-} \right). \quad (14)$$

To argue that the *indeterminacy index* of an IMPCM can be reduced, we first introduce the formula of $\sqrt[n]{\prod_{i=1}^n \sqrt[n]{R_i'^-}}$.

It is well known that a linear ordering of the elements of the set $\{1, 2, \dots, n\}$ is called an n -permutation, and the number of n -permutation is $n!$. We denote an n -permutation by $p = p_1 p_2 \dots p_n$, with p_i being the i th entry in the linear order given by p . In other words, an n -permutation p can be described by a function α , defined as: $\alpha(i) = p_i$, and $\alpha^{-1}(p_i) = i$. According to Equations (2), (3), and (5), we will have a general form of $R_i'^-$

$$R_i'^- = \left(\prod_{\alpha^{-1}(j) > \alpha^{-1}(i)}^n \frac{R_j^-}{R_j^+} \right) \left(\prod_{\alpha^{-1}(j) < \alpha^{-1}(i)}^n \frac{R_i^+}{R_j^+} \right). \quad (15)$$

Using Equation (15), we can investigate the formula of $\sqrt[n]{\prod_{i=1}^n \sqrt[n]{R_i'^-}}$ as follows (Recall that $R_i^- \leq R_i^+$ for all $i = 1, 2, \dots, n$).

When n is even, for example, $n = 4$, assume that the following inequalities hold (this is only one of 24 cases of permutations for $n = 4$):

$$\frac{R_3^-}{R_3^+} \leq \frac{R_2^-}{R_2^+} \leq \frac{R_4^-}{R_4^+} \leq \frac{R_1^-}{R_1^+} \leq 1 \leq \frac{R_1^+}{R_1^-} \leq \frac{R_4^+}{R_4^-} \leq \frac{R_2^+}{R_2^-} \leq \frac{R_3^+}{R_3^-}.$$

That is, $p = p_1 p_2 p_3 p_4 = 3241 = \alpha(1)\alpha(2)\alpha(3)\alpha(4)$. In other words, $\alpha^{-1}(1) = 4$, $\alpha^{-1}(2) = 2$, $\alpha^{-1}(3) = 1$, and $\alpha^{-1}(4) = 3$. Thus, according to Equation (15), we will have

$$R_1'^- = \frac{R_1^+}{R_2^+} \frac{R_1^+}{R_3^+} \frac{R_1^+}{R_4^+}, R_2'^- = \frac{R_2^-}{R_1^-} \frac{R_2^-}{R_3^+} \frac{R_2^-}{R_4^-}, R_3'^- = \frac{R_3^-}{R_1^-} \frac{R_3^-}{R_2^-} \frac{R_3^-}{R_4^-}, \text{ and } R_4'^- = \frac{R_4^-}{R_1^-} \frac{R_4^+}{R_2^+} \frac{R_4^+}{R_3^+}.$$

Thus, we have

$$\sqrt[4]{\prod_{i=1}^4 \sqrt[4]{R_i'^-}} = \sqrt[4]{\sqrt[4]{\left(\frac{R_3^-}{R_3^+}\right)^3 \left(\frac{R_2^-}{R_2^+}\right) \left(\frac{R_4^+}{R_4^-}\right) \left(\frac{R_1^+}{R_1^-}\right)^3}}.$$

When n is odd, for example, $n = 5$, assume that the following inequalities hold (this is only one of 120 cases of permutations for $n = 5$):

$$\frac{R_2^-}{R_2^+} \leq \frac{R_1^-}{R_1^+} \leq \frac{R_3^-}{R_3^+} \leq \frac{R_4^-}{R_4^+} \leq \frac{R_5^-}{R_5^+} \leq 1 \leq \frac{R_5^+}{R_5^-} \leq \frac{R_4^+}{R_4^-} \leq \frac{R_3^+}{R_3^-} \leq \frac{R_1^+}{R_1^-} \leq \frac{R_2^+}{R_2^-}.$$

That is, $p = p_1 p_2 p_3 p_4 p_5 = 21345 = \alpha(1)\alpha(2)\alpha(3)\alpha(4)\alpha(5)$.

Hence, we have

$$\prod_{i=1}^5 \sqrt[5]{R_i'^-} = \sqrt[5]{\sqrt[5]{\left(\frac{R_2^-}{R_2^+}\right)^4 \left(\frac{R_1^-}{R_1^+}\right)^2 \left(\frac{R_4^+}{R_4^-}\right)^2 \left(\frac{R_5^+}{R_5^-}\right)^4}}.$$

Then, after a detailed analysis, we have the following two propositions with respect to the formula of $\sqrt[n]{\prod_{i=1}^n \sqrt[n]{R_i'^-}}$.

Proposition 1. When n is even, $\sqrt[n]{\prod_{i=1}^n \sqrt[n]{R_i'^-}}$ is

$$\sqrt[n]{\sqrt[n]{\left(\frac{R_{\alpha(1)}^-}{R_{\alpha(1)}^+}\right)^{n-1} \left(\frac{R_{\alpha(2)}^-}{R_{\alpha(2)}^+}\right)^{n-3} \cdots \left(\frac{R_{\alpha(\frac{n}{2})}^-}{R_{\alpha(\frac{n}{2})}^+}\right)^1} \times \sqrt[n]{\sqrt[n]{\left(\frac{R_{\alpha(\frac{n}{2}+1)}^+}{R_{\alpha(\frac{n}{2}+1)}^-}\right)^1 \cdots \left(\frac{R_{\alpha(n-1)}^+}{R_{\alpha(n-1)}^-}\right)^{n-3} \left(\frac{R_{\alpha(n)}^+}{R_{\alpha(n)}^-}\right)^{n-1}}}.$$

Proposition 2: When n is odd, $\sqrt[n]{\prod_{i=1}^n \sqrt[n]{R_i'^-}}$ is

$$\sqrt[n]{\sqrt[n]{\left(\frac{R_{\alpha(1)}^-}{R_{\alpha(1)}^+}\right)^{n-1} \left(\frac{R_{\alpha(2)}^-}{R_{\alpha(2)}^+}\right)^{n-3} \cdots \left(\frac{R_{\alpha(\frac{n+1}{2}-1)}^-}{R_{\alpha(\frac{n+1}{2}-1)}^+}\right)^2} \times \sqrt[n]{\sqrt[n]{\left(\frac{R_{\alpha(\frac{n+1}{2}+1)}^+}{R_{\alpha(\frac{n+1}{2}+1)}^-}\right)^2 \cdots \left(\frac{R_{\alpha(n-1)}^+}{R_{\alpha(n-1)}^-}\right)^{n-3} \left(\frac{R_{\alpha(n)}^+}{R_{\alpha(n)}^-}\right)^{n-1}}}.$$

The following theorem reveals that the *indeterminacy index* of the original inconsistent IMPCM can be reduced by the consistent transformation, i.e., Equation (5).

Theorem 4. For an IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{n \times n} = ([a_{ij}^-, a_{ij}^+])_{n \times n}$, the indeterminacy index of the constructed, by Equation (5), IMPCM $\bar{\mathbf{B}} = (\bar{b}_{ij})_{n \times n} = ([b_{ij}^-, b_{ij}^+])_{n \times n}$ will be less than or equal to the indeterminacy index of IMPCM $\bar{\mathbf{A}}$. That is, $\Pi(\bar{\mathbf{B}}) \leq \Pi(\bar{\mathbf{A}})$.

Proof. Part 1, when n is even.

Without loss of generality, for example, $n = 4$, we can assume that

$$\frac{R_3^-}{R_3^+} \leq \frac{R_2^-}{R_2^+} \leq \frac{R_4^-}{R_4^+} \leq \frac{R_1^-}{R_1^+} \leq 1 \leq \frac{R_1^+}{R_1^-} \leq \frac{R_4^+}{R_4^-} \leq \frac{R_2^+}{R_2^-} \leq \frac{R_3^+}{R_3^-}.$$

Thus, according to Proposition 1, we need to prove that

$$\prod_{i=1}^4 \sqrt[4]{R_i'^-} = \sqrt[4]{\sqrt[4]{\left(\frac{R_3^-}{R_3^+}\right)^3 \left(\frac{R_2^-}{R_2^+}\right) \left(\frac{R_4^-}{R_4^+}\right) \left(\frac{R_1^-}{R_1^+}\right)^3}} \geq \prod_{i=1}^4 \sqrt[4]{R_i^-} = \sqrt[4]{R_1^- R_2^- R_3^- R_4^-}.$$

That is, we need to prove that $\sqrt[4]{\sqrt[4]{\frac{(R_3^-)^3 (R_2^-) (R_4^-) (R_1^-)^3}{(R_3^+ R_2^+ R_4^+ R_1^+) (R_3^+)^2 (R_1^+)^2}}} \geq \sqrt[4]{R_1^- R_2^- R_3^- R_4^-}$.

According to Equation (11), $R_1^- R_2^- R_3^- R_4^- R_1^+ R_2^+ R_3^+ R_4^+ = 1$; thus, we have

$$\sqrt[4]{\sqrt[4]{\frac{(R_3^-)^3 (R_2^-) (R_4^-) (R_1^-)^3}{\frac{1}{R_3^- R_2^- R_4^- R_1^-} (R_3^+)^2 (R_1^+)^2}}} \geq \sqrt[4]{R_1^- R_2^- R_3^- R_4^-}.$$

This can be reformulated as follows:

$$\sqrt[4]{\left(\frac{R_1^+}{R_1^-}\right)^4 \frac{1}{(R_1^- R_2^- R_3^- R_4^-)^2} \left(\frac{R_4^+}{R_4^-}\right)^2} \geq 1.$$

Since for all i , we have $\frac{R_i^+}{R_i^-} \geq 1$, we only need to prove that

$$R_1^- R_2^- R_3^+ R_4^- \leq 1.$$

According to property (iii), we have (iii-1) $R_1^+ R_2^+ R_3^- R_4^+ \geq \frac{R_3^-}{R_3^+}$. Now, if $R_1^- R_2^- R_3^+ R_4^- > 1$ holds, then we will have (v) $R_1^+ R_2^+ R_3^- R_4^+ < 1$, and (vi) $R_1^- R_2^- R_3^- R_4^- > \frac{R_3^-}{R_3^+}$.

Clearly, (v) is not in concordance with (iii-1), and (vi) is not in concordance with left side of (ii). On the other hand, since we want $R_1^- R_2^- R_3^+ R_4^- \leq 1$ to hold, then we will have (vii) $R_1^+ R_2^+ R_3^- R_4^+ \geq 1$, and (viii) $R_1^- R_2^- R_3^- R_4^- \leq \frac{R_3^-}{R_3^+}$. It is clear that (vii) is in accord with (iii-1), and (viii) is in accord with the left side of (ii).

Part 2, when n is odd.

Without loss of generality, we can assume that, for example, $n = 5$

$$\frac{R_2^-}{R_2^+} \leq \frac{R_1^-}{R_1^+} \leq \frac{R_3^-}{R_3^+} \leq \frac{R_4^-}{R_4^+} \leq \frac{R_5^-}{R_5^+} \leq 1 \leq \frac{R_5^+}{R_5^-} \leq \frac{R_4^+}{R_4^-} \leq \frac{R_3^+}{R_3^-} \leq \frac{R_1^+}{R_1^-} \leq \frac{R_2^+}{R_2^-}.$$

According to Proposition 2, we need to prove that

$$\begin{aligned}\prod_{i=1}^5 \sqrt[5]{R_i^-} &= \sqrt[5]{\sqrt[5]{\left(\frac{R_2^-}{R_2^+}\right)^4 \left(\frac{R_1^-}{R_1^+}\right)^2 \left(\frac{R_4^+}{R_4^-}\right)^2 \left(\frac{R_5^+}{R_5^-}\right)^4}} \\ &\geq \prod_{i=1}^5 \sqrt[5]{R_i^-} = \sqrt[5]{R_1^- R_2^- R_3^- R_4^- R_5^-}.\end{aligned}$$

That is,

$$\sqrt[5]{\sqrt[5]{\frac{(R_2^-)^4 (R_1^-)^2 (R_4^+)^2 (R_5^+)^4}{(R_2^+ R_1^+ R_4^- R_5^-)^2 (R_2^+)^2 (R_5^-)^2}}} \geq \sqrt[5]{R_1^- R_2^- R_3^- R_4^- R_5^-}.$$

According to Equation (11), $R_1^- R_2^- R_3^- R_4^- R_5^- R_1^+ R_2^+ R_3^+ R_4^+ R_5^+ = 1$, we have

$$\sqrt[5]{\sqrt[5]{\frac{(R_2^-)^4 (R_1^-)^2 (R_4^+)^2 (R_5^+)^4}{\left(\frac{1}{R_2^- R_1^+ R_4^+ R_5^+ R_3^+ R_3^-}\right)^2 (R_2^+)^2 (R_5^-)^2}}} \geq \sqrt[5]{R_1^- R_2^- R_3^- R_4^- R_5^-}.$$

This can be reformulated as follows:

$$\sqrt[5]{\left(\frac{R_2^-}{R_2^+}\right)^2 \frac{1}{R_1^- R_2^- R_3^- R_4^- R_5^-} \left(\frac{R_3^+}{R_3^-}\right)^2 \left(\frac{R_4^+}{R_4^-}\right)^4 \left(\frac{R_5^+}{R_5^-}\right)^6} \geq 1.$$

Since $\frac{R_i^+}{R_i^-} \geq 1$, we only need to prove that

$$\left(\frac{R_2^-}{R_2^+}\right)^2 \frac{1}{R_1^- R_2^- R_3^- R_4^- R_5^-} \geq 1.$$

According to property (iii), we have (iii-2)

$$R_1^- R_2^+ R_3^- R_4^- R_5^- \geq \frac{R_1^- R_3^- R_4^- R_5^-}{R_1^+ R_3^+ R_4^+ R_5^+}.$$

Now, if $\left(\frac{R_2^-}{R_2^+}\right)^2 \frac{1}{R_1^- R_2^- R_3^- R_4^- R_5^-} \leq 1$ holds, we will have $R_1^- R_2^+ R_3^- R_4^- R_5^- \geq \frac{R_2^-}{R_2^+}$. Obviously, it would be a contradiction to (iii-2) if $\frac{R_2^-}{R_2^+} < \frac{R_1^- R_3^- R_4^- R_5^-}{R_1^+ R_3^+ R_4^+ R_5^+}$ holds. Unfortunately, that is possible in an IMPCM under the assumption that $\frac{R_2^-}{R_2^+} \leq \frac{R_1^-}{R_1^+} \leq \frac{R_3^-}{R_3^+} \leq \frac{R_4^-}{R_4^+} \leq \frac{R_5^-}{R_5^+} \leq 1$. On the other hand, since we want $\left(\frac{R_2^-}{R_2^+}\right)^2 \frac{1}{R_1^- R_2^- R_3^- R_4^- R_5^-} \geq 1$ to hold, then we will have $R_1^- R_2^+ R_3^- R_4^- R_5^- \leq \frac{R_2^-}{R_2^+}$. Since $\frac{R_2^-}{R_2^+} \leq \frac{R_2^+}{R_2^-}$, it will thus obey $R_1^- R_2^+ R_3^- R_4^- R_5^- \leq \frac{R_2^+}{R_2^-}$ according to property (iv). The proof is completed. $\square$

5. Illustrated Examples and Flowchart of the Proposed Model

What follows here is first three numerical examples that illustrate the feasibility and superiority of the proposed methods and then the flowchart of the proposed model that can transform an inconsistent IMPCM to a consistent IMPCM, which possesses less indeterminacy than the original one has and obtains additively normalized interval weights.

Example 1. Consider the following IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{4 \times 4} = ([a_{ij}^-, a_{ij}^+])_{4 \times 4}$ in Table 1, which was examined by [33,37], respectively (for simplicity, we use a table to represent an IMPCM).

Table 1. The original inconsistent IMPCM [37] (M1).

1	1	1	2	1	2	2	3
0.5	1	1	1	3	5	4	5
0.5	1	0.2	0.333333	1	1	6	8
0.333333	0.5	0.2	0.25	0.125	0.166667	1	1

By Definition 6, the indeterminacy index of M1 is 1.598238. By using an iterative algorithm [37] to improve the consistency of the inconsistent IMPCM (M1), the resulting IMPCM is presented in Table 2.

Table 2. The resulting IMPCM [37] (M2).

1	1	0.839	1.745	0.9764	2.0393	2.246	3.6655
0.573066	1.191895	1	1	2.3416	4.1827	4.0078	5.5942
0.490364	1.02417	0.23908	0.427058	1	1	4.6087	6.7708
0.272814	0.445236	0.178757	0.249513	0.147693	0.216981	1	1

By utilizing a convex combination method [33], the resulting IMPCM is presented in Table 3.

Table 3. The resulting IMPCM [33] (M3).

1	1	0.9428	1.4696	0.9943	1.934	2.1061	3.9754
0.680457	1.06067	1	1	2.7733	3.4565	4.1113	5.5707
0.517063	1.005733	0.28931	0.360581	1	1	5.5695	5.6122
0.251547	0.474811	0.179511	0.243232	0.178183	0.179549	1	1

By using the Equation (5), the following consistent IMPCM is induced from M1.

According to the IMPCMs (in Tables 1–4), we can obtain the following comparative results in Table 5.

Table 4. The resulting IMPCM (This paper).

1	1	0.759836	0.832358	1.3512	1.456475	3.935979	4.898979
1.201406	1.316074	1	1	1.749818	1.778279	5.18004	5.885662
0.686589	0.740083	0.562341	0.571488	1	1	2.912951	3.363586
0.204124	0.254066	0.169904	0.193049	0.297302	0.343295	1	1

Table 5. Comparative results for the original inconsistent IMPCM (M1).

Models	Consistency	Indeterminacy Index
M1	inconsistent	1.598238
M2	inconsistent	1.720843
M3	inconsistent	1.46134
This paper	consistent	1.11865

Example 2. Consider the following IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{4 \times 4} = ([a_{ij}^-, a_{ij}^+])_{4 \times 4}$, which was examined by [32–34,52,55], respectively.

By Definition 6, the indeterminacy index of M4 is 2.376177.

After 25 iterations of his method [34], López-Morales obtained an acceptably consistent IMPCM as follows.

By using the Equation (5), the following consistent IMPCM is induced from M4.

According to the IMPCMs (in Tables 6–11), we can obtain the following comparative results in Table 12.

Table 6. The original inconsistent IMPCM [52,55] (M4).

1	1	2	5	2	4	1	3
0.2	0.5	1	1	1	3	1	2
0.25	0.5	0.333333	1	1	1	0.5	1
0.333333	1	0.5	1	1	2	1	1

Table 7. The resulting IMPCM [52,55] (M5).

1	1	0.9428	1.4696	0.9943	1.934	2.1061	3.9754
0.680457	1.06067	1	1	2.7733	3.4565	4.1113	5.5707
0.517063	1.005733	0.28931	0.360581	1	1	5.5695	5.6122
0.251547	0.474811	0.179511	0.243232	0.178183	0.179549	1	1

Table 8. The resulting IMPCM [32,55] (M6).

1	1	1.5143	3.2239	2	4.899	1.6818	3.5676
0.310183	0.660371	1	1	0.9036	2.2134	0.7598	1.6119
0.204123	0.5	0.451794	1.106684	1	1	0.5	1.2247
0.2803	0.594601	0.620386	1.316136	0.816526	2	1	1

Table 9. The resulting IMPCM [33,55] (M7).

1	1	1.3512	3.3098	2	5.18	1.1892	4.3562
0.3021	0.7401	1	1	1.8922	1.9479	0.7071	1.6381
0.1931	0.5	0.5285	0.5134	1	1	0.4518	1.1067
0.2296	0.8409	0.6105	1.4142	0.9036	2.2134	1	1

Table 10. The resulting IMPCM [34] (M8).

1	1	2	3.9339	2.1524	4	1.181	3
0.254201	0.5	1	1	1	2.6226	1	1.8
0.25	0.464598	0.381301	1	1	1	0.5	0.9408
0.333333	0.84674	0.555556	1	1.062925	2	1	1

Table 11. The resulting IMPCM (this paper).

1	1	2.114743	2.114743	3.130169	3.309751	2.213364	2.340347
0.472871	0.472871	1	1	1.480166	1.565085	1.046635	1.106682
0.302138	0.319472	0.638943	0.6756	1	1	0.707107	0.707107
0.427287	0.451801	0.903602	0.955443	1.414214	1.414214	1	1

Table 12. Comparative results for the original inconsistent IMPCM (M4).

Models	Consistency	Indeterminacy Index
M4	inconsistent	2.376177
M5	inconsistent	2.376305
M6	inconsistent	2.280894
M7	inconsistent	2.26706
M8	inconsistent	2.086358
This paper	consistent	1.037891

Example 3. Consider the following IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{5 \times 5} = ([a_{ij}^-, a_{ij}^+])_{5 \times 5}$, which was examined by [35].

By Definition 6, the indeterminacy index of M9 is 2.166491. By using a linear programming model [35], the following satisfactorily consistent IMPCM is induced from M9.

By using the Equation (5), the following consistent IMPCM is induced from M9.

According to the IMPCMs (in Tables 13–15), we can obtain the following comparative results in Table 16.

Table 13. The original inconsistent IMPCM [35] (M9).

1	1	0.5	1.5	1.5	2.5	1	3	1.6667	3
0.6667	2	1	1	5	6	1.2	3	2	4
0.4	0.6667	0.1667	0.2	1	1	2	3	0.8	2
0.3333	1	0.3333	0.8333	0.3333	0.5	1	1	0.8	3
0.3333	0.6	0.25	0.5	0.5	1.25	0.3333	1.25	1	1

Table 14. The resulting IMPCM [35] (M10).

1	1	0.5	1.5	1.5	2.5	1	3	1.6667	3
0.6667	2	1	1	1.897	6	1.2	3.795	2	4
0.4	0.6667	0.1667	0.5271	1	1	0.775	2.45	0.8	2
0.3333	1	0.2635	0.8333	0.4082	1.2903	1	1	0.8	3
0.3333	0.5999	0.25	0.5	0.5	1.25	0.3333	1.25	1	1

Table 15. The resulting IMPCM (this paper).

1	1	0.6899	0.7481	1.636	2.1137	1.9332	2.1137	2.3522	2.4595
1.3367	1.4496	1	1	2.3714	2.8252	2.584	3.0639	3.144	3.5652
0.4731	0.6113	0.354	0.4217	1	1	0.9146	1.292	1.1128	1.5034
0.4731	0.5173	0.3264	0.387	0.774	1.0934	1	1	1.1636	1.2167
0.4066	0.4251	0.2805	0.3181	0.6652	0.8986	0.8219	0.8594	1	1

Table 16. Comparative results for the original inconsistent IMPCM (M9).

Models	Consistency	Indeterminacy Index
M9	inconsistent	2.166491
M10	inconsistent	2.632847
This paper	consistent	1.177408

From the comparative results of above examples, we illustrated the feasibility and superiority of the proposed model. Afterward, in Figure 1, we show the flowchart of the proposed model for a pragmatic purpose.

According to Theorem 4 proved in previous section, we know that, by applying the proposed model once, the original inconsistent IMPCM can be transformed into a consistent IMPCM that will possess less indeterminacy than the original one has. Therefore, we can judiciously reduce the *indeterminacy index* of the IMPCM to a certain satisfactory level. Consequently, by applying the proposed model successively, we can reduce or even eliminate the indeterminacy of the IMPCM. In other words, we can/may obtain an MPCM rather than an IMPCM.

For example, if the initial input is an IMPCM M1 (in Table 1), then we will reduce the values of *indeterminacy index* within three iterations from 1.592328 to 1.11865, 1.038917, and to 1.006607. If the initial input is an IMPCM M9 (in Table 13), then we will reduce the values of *indeterminacy index* within four iterations from 2.166495 to 1.177408, 1.055343, 1.023438, and to 1.012103. On the other hand, if the initial input is an IMPCM M4 (in Table 6), then we will reduce the values of *indeterminacy index* from 2.376177 to 1.037891 and finally to 1 and obtain an MPCM. Subsequently, in next section, we will show the experimental results to support the theorem proven in the Section 4.

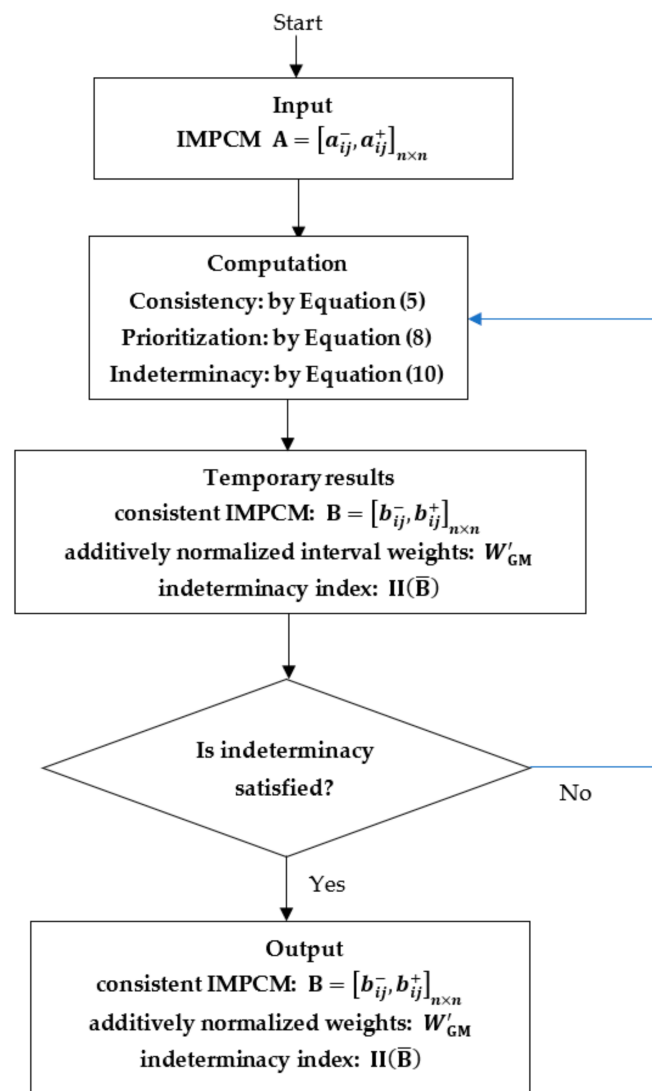


Figure 1. Flowchart of the proposed model.

6. Experimental Results

In addition to the theorem proven in the Section 4, we conducted a series of computer simulations to corroborate our argument (all simulations were executed under the Microsoft Excel environment). Based on the widely used bipolar ratio scale $\mathbf{S} = \left[\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9 \right]$ suggested by Saaty, we call a bipolar ratio scale $\left[\frac{1}{s}, \frac{1}{s-1}, \dots, \frac{1}{2}, 1, 2, \dots, s-1, s \right]$ an s -scale and use various s -scale from $s = 2$ to $s = 9$. By using a random number generator, we generated ten numbers (since there are ten entries located in the upper triangle of a square matrix with size of $n = 5$) in each single simulation for the value of a_{ij}^- . Then, according to these values, we further randomly generated their corresponding values as a_{ij}^+ such that $a_{ij}^- < a_{ij}^+$ in order to construct an IMPCM. In other words, for each single simulation, twenty random numbers were generated to construct an IMPCM. We conducted eight independent runs with 10,000 simulations per run and present the experimental results as follows.

Table 17, for $n = 5$, shows the results where the values are averages and standard deviations for the *indeterminacy indexes* following 10,000 simulations in eight independent runs, using various s -scales (from $s = 2$ to $s = 9$).

Table 17. Indeterminacy index by different scales ($n = 5$).

s	II(A)		II(B)		$1 - \frac{II(B)}{II(A)}$
	Average	Std.	Average	Std.	Reduced%
2	2.39	0.23	1.12	0.05	53.28%
3	2.74	0.51	1.25	0.1	54.19%
4	3.1	0.76	1.35	0.14	56.45%
5	3.44	1.03	1.43	0.18	58.29%
6	3.76	1.28	1.51	0.21	59.92%
7	4.11	1.58	1.57	0.24	61.93%
8	4.43	1.86	1.63	0.27	63.15%
9	4.47	2.15	1.69	0.3	64.56%

Obviously, the *indeterminacy index* of IMPCM is dramatically reduced by the proposed model, especially when we adopt a wider bipolar ratio scale, that is, when we use a bigger s . Moreover, for all eight s -scales, the number of simulations that have $II(B) < II(A)$ are exactly 10,000. In other words, there are no exceptions in all 80,000 independent simulations. This provides strong evidence for the argument of Theorem 4.

7. Discussion and Conclusions

To deal with uncertain and/or ambiguous information, human judgments can effectively be expressed in terms of intervals rather than exact numbers. The Interval Multiplicative Pairwise Comparison Matrix (IMPCM) is widely used to manifest human judgments with uncertainty and/or ambiguity. To improve the consistency of an IMPCM is one way to improve the quality of decision making with uncertainty and/or ambiguity. Thus, indeterminacy is an important issue to conquer. In the literature, two metrics have been proposed to measure uncertainty and/or ambiguity: the *indeterminacy ratio* and the *indeterminacy index*.

To take *indeterminacy* into consideration, Li et al. introduced an acceptable IMPCM model [45]. By setting a threshold for the *indeterminacy ratio*, they check whether an IMPCM is *acceptable*. However, three concomitant and interrelated problems arose. First, setting a threshold itself is heuristic. Second, without strong evidence of relationship between an *indeterminacy ratio* and reliability of derived priority weights, how can we decide a meaningful threshold as a criterion of acceptance for an IMPCM? Third, since the *indeterminacy ratio* is from a local view (i.e., from the entry of an IMPCM), a reduction of the *indeterminacy ratio* by one entry may result in increasing the *indeterminacy ratio* of another entry. Recently, by considering acceptable consistency and controlling uncertainty of priority weights, Wang, Z. J. [36] also introduced an *acceptable* IMPCM model. Note that he used two thresholds: one for consistency and the other for prioritization. Beyond the issues of using a threshold mentioned above, he adopted 0.1 (the most widely employed in checking acceptable consistency of MPCM, suggested by Saaty [20]) as the threshold for checking acceptability of consistency of IMPCM. This threshold has long been criticized for failing to account for the ordinal consistency, such as in [76].

We have proposed an acceptable IMPCM model, which guarantees that an inconsistent IMPCM can be transformed into a consistent IMPCM, and the maximal *indeterminacy ratio* can be reduced [42]. The contribution of this study is to fill a research gap by giving a proof that the model can also reduce the *indeterminacy index*. Therefore, by applying Equation (5) once, the original inconsistent IMPCM can be transformed into a consistent IMPCM that will possess less indeterminacy than the original one has. Consequently, by applying Equation (5) successively, we can reduce or even eventually eliminate the indeterminacy of the IMPCM. In other words, we can/may obtain an MPCM rather than an IMPCM.

The advantage of this method is that it is theoretically grounded on a prevalent and normative definition of consistency. On the other hand, the limitation of this method is that it may not be directly applicable to pairwise comparison matrices with qualitative entries.

In summary, the proposed model is both effective and efficient because it only requires some arithmetic operations without solving complex optimization problems. Finally, three points are also worth noting. First, the conclusion of this study is independent of what measurement scale is adopted. Second, the methodologies of this study may be applicable to cases of interval fuzzy preference relations (IFPRs), which is one of our ongoing research topics. Third, since the FFSs are capable of handling higher levels of uncertainties, we shall also try to extend it to the FFSs, and this is worth further investigating.

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