Differential Geometry of Submanifolds in Complex Space Forms Involving δ -Invariants [†]

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† In memory of Professor Oscar Jesus Garay (26 January 1959–4 May 2021).

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Abstract: One of the fundamental problems in the theory of submanifolds is to establish optimal relationships between intrinsic and extrinsic invariants for submanifolds. In order to establish such relations, the first author introduced in the 1990s the notion of δ -invariants for Riemannian manifolds, which are different in nature from the classical curvature invariants. The earlier results on δ -invariants and their applications have been summarized in the first author's book published in 2011 *Pseudo-Riemannian Geometry, δ -Invariants and Applications* (ISBN: 978-981-4329-63-7). In this survey, we present a comprehensive account of the development of the differential geometry of submanifolds in complex space forms involving the δ -invariants done mostly after the publication of the book.

Keywords: δ -invariants; Chen invariants; complex space form; inequality; squared mean curvature; ideal immersions; δ -Casorati curvatures.

MSC: 53C40; 53C42; 53B25

Citation: Chen, B.-Y.; Blaga, A.M.; Vilcu, G.-E. Differential Geometry of Submanifolds in Complex Space Forms Involving δ -Invariants. *Mathematics* **2022**, *10*, 591. <https://doi.org/10.3390/math10040591>

Academic Editor: Sorin V. Sabau

Received: 24 January 2022

Accepted: 9 February 2022

Published: 14 February 2022

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1. Introduction

One of the most fundamental problems in submanifold theory is the immersibility of a Riemannian manifold into a Euclidean space. In 1956, John F. Nash proved the following well-known theorem.

Theorem 1 ([1]). *Every n -dimensional Riemannian manifold can be isometrically embedded in a Euclidean space with sufficiently high codimension.*

Nash's embedding theorem was aimed for in the hope that if a Riemannian manifold could be regarded as isometrically embedded submanifold, this would then yield the opportunity to use help from extrinsic geometry. However, this hope was not materialized until the publication of M. Gromov's article [2] in 1985.

There were several reasons why it is so difficult to apply Nash's embedding theorem. One reason is that it requires in general very large codimension for a Riemannian manifold to admit an isometric embedding into a Euclidean space. On the other hand, submanifolds of higher codimension are very difficult to be understood. Another reason is that at that time there did not exist general optimal relationships between the known intrinsic invariants and the main extrinsic invariants for arbitrary submanifolds of Euclidean spaces except the

three fundamental equations (see [3,4]). This leads to another fundamental problem in the theory of submanifolds (see, e.g., [5,6]).

Problem 1. Find optimal relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold.

In order to establish such relationships, the first author introduced in the 1990s the notion of δ -invariants (also known as Chen invariants in literature) for Riemannian manifolds. The earlier results on δ -invariants and applications have been summarized in the first author's book [7] published in 2011.

The purpose of this survey article is to present a comprehensive account of the development on the differential geometry of submanifolds in complex space forms involving δ -invariants done mostly after the publication of the book [7].

2. Preliminaries

2.1. Basic Formulas and Equations of Submanifolds

Suppose that M is an n -dimensional submanifold of a Riemannian m -manifold $\tilde{M}^m$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. Let R and $\tilde{R}$ be the Riemann curvature tensors of M and $\tilde{M}^m$, respectively.

The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (1)$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (2)$$

where X and Y are vector fields tangent to M , ξ normal vector field, h the second fundamental form, D the normal connection, and A the shape operator of M .

Let $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ be an orthonormal frame field of $\tilde{M}^m$ such that, restricted to M , $e_1, \dots, e_n$ are tangent to M and hence $\xi_{n+1}, \dots, \xi_m$ are normal to M . Then the coefficients of the second fundamental form h , denoted by $\{h_{ij}^r\}$, $i, j = 1, \dots, n$; $r = n+1, \dots, m$ with respect to $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ are defined by

$$h_{ij}^r = \langle h(e_i, e_j), \xi_r \rangle = \langle A_r e_i, e_j \rangle,$$

where $A_r = A_{\xi_r}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\tilde{M}^m$.

The mean curvature vector $\vec{H}$ of M is defined by

$$\vec{H} = \frac{1}{n} \text{trace } h = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$$

and the squared mean curvature is given by $H^2 = \langle \vec{H}, \vec{H} \rangle$. The submanifold M is called *minimal* (resp., *totally geodesic*) if its mean curvature vector (resp., its second fundamental form) vanishes identically.

If $\tilde{M}^m$ is of constant curvature c , the three fundamental equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle A_{h(Y, Z)}X, W \rangle - \langle A_{h(X, Z)}Y, W \rangle \\ &\quad + c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned} \quad (3)$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \quad (4)$$

$$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle \quad (5)$$

for vector fields X, Y, Z, W tangent to M and ξ, η normal to M , where $R^\perp$ is the normal curvature tensor associated with the normal connection D and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (6)$$

2.2. δ -Invariants

Let $K(\pi)$ denote the sectional curvature of a Riemannian n -manifold M associated with a plane section $\pi \subset T_x M$ at a point $x \in M$. For an orthonormal basis $e_1, \dots, e_n$ of $T_x M$, the scalar curvature M at x is given by

$$\tau(x) = \sum_{i < j} K(e_i \wedge e_j).$$

Suppose that L is a r -dimensional subspace of $T_x M$ with $r \geq 2$ and that $\{e_1, \dots, e_r\}$ is an orthonormal basis of L . Then the scalar curvature $\tau(L)$ of L is given by

$$\tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For a unit tangent vector X of M at x , the Ricci curvature $\text{Ric}(X)$ at X is defined by

$$\text{Ric}(X) = \sum_{j=2}^n K(X, e_j)$$

where $X = e_1, \dots, e_n$ is an orthonormal basis of $T_x M$.

For a given integer $k \geq 0$, we denote by $\mathcal{S}(n, k)$ the finite set of unordered k -tuples $(n_1, \dots, n_k)$ of integers ≥ 2 satisfying

$$n_1 < n \quad \text{and} \quad n_1 + \dots + n_k \leq n.$$

Let $\mathcal{S}(n)$ denote the set of unordered k -tuples with $k \geq 0$ for a fixed n . For a k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ the invariant $\delta(n_1, \dots, n_k)$ is defined by

$$\delta(n_1, \dots, n_k)(x) = \tau(x) - \inf\{\tau(L_1) + \dots + \tau(L_k)\}, \quad (7)$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of $T_x M$ such that $\dim L_j = n_j$, $j = 1, \dots, k$ (see [8,9]). Clearly, the δ -invariants $\delta(\emptyset), \delta(2), \delta(n-1)$ and $\delta(2, 2)$ are given respectively by

$$\begin{aligned} \delta(\emptyset) &= \tau \quad (k=0, \text{ the trivial } \delta\text{-invariant}), \\ \delta(2) &= \tau - \inf K, \text{ where } K \text{ is the sectional curvature,} \\ \delta(n-1) &= \max \text{Ric, and} \\ \delta(2, 2)(x) &= \tau(x) - \inf\{\tau(L_1) + \tau(L_2)\}, \end{aligned}$$

where L_1 and L_2 run over all mutually orthogonal 2-planes of $T_x M$. We remark that the invariant $\delta(2)$ was first introduced by the first author in [10].

2.3. Horizontal Lift of Lagrangian Submanifolds

There exists a link between Legendrian submanifolds and Lagrangian submanifolds via horizontal lift given by [11].

Case (a): $\mathbb{C}P^n(4)$. Consider the Hopf fibration $\pi : S^{2n+1}(1) \rightarrow \mathbb{C}P^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at u is the orthogonal complement of iu , $i = \sqrt{-1}$, with respect to the metric on $S^{2n+1}(1)$ induced from the metric on $\mathbb{C}^{n+1}$. Let $\iota : M \rightarrow \mathbb{C}P^n(4)$ be a Lagrangian isometric immersion. Then there exist a covering map $\tau : \hat{M} \rightarrow M$ and a horizontal immersion $\hat{\iota} : \hat{M} \rightarrow S^{2n+1}(1)$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Therefore, each Lagrangian immersion can be lifted locally (or globally if M is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $\mathbb{C}P^n(4)$ is lifted to a minimal Legendrian submanifold of

the Sasakian sphere $S^{2n+1}(1)$. Conversely, if $f : \hat{M} \rightarrow S^{2n+1}$ is a Legendrian isometric immersion, then $\iota = \pi \circ f : M \rightarrow \mathbb{C}P^n(4)$ is a Lagrangian isometric immersion.

Case (b): $\mathbb{C}H^n(-4)$. Consider the complex number space $\mathbb{C}_1^{n+1}$ endowed with the pseudo-Euclidean metric: $g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j$. Let $\langle \cdot, \cdot \rangle$ denote the induced inner product on $\mathbb{C}_1^{n+1}$. Let

$$H_1^{2n+1}(-1) = \{z \in \mathbb{C}_1^{2n+1} : \langle z, z \rangle = -1\}$$

equipped with the canonical Sasakian structure. Put

$$T'_z = \{u \in \mathbb{C}^{n+1} : \langle u, z \rangle = 0\}, \quad H_1^1 = \{\lambda \in \mathbb{C} : \lambda \bar{\lambda} = 1\}.$$

Then there exists an H_1^1 -action on $H_1^{2n+1}(-1)$ given by $z \mapsto \lambda z$ and the vector $\xi = -iz$ is tangent to the flow of the action. Since g_0 is Hermitian, we have $\langle \xi, \xi \rangle = -1$. The quotient space $H_1^{2n+1}(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $\mathbb{C}H^n(-4)$ with constant holomorphic sectional curvature -4 .

Just like case (a), if $\iota : M \rightarrow \mathbb{C}H^n(-4)$ is a Lagrangian immersion, then there exists an isometric covering map $\tau : \hat{M} \rightarrow M$ and a Legendrian immersion $f : \hat{M} \rightarrow H_1^{2n+1}(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus, each Lagrangian immersion into $\mathbb{C}H^n(-4)$ can be lifted locally (or globally if M is simply-connected) to a Legendrian immersion into $H_1^{2n+1}(-1)$. Conversely, if $f : \hat{M} \rightarrow H_1^{2n+1}(-1)$ is Legendrian, then $\iota = \pi \circ f : M \rightarrow \mathbb{C}H^n(-4)$ is a Lagrangian immersion.

2.4. H-Umbilical Submanifolds of Kaehler Manifolds

Since there do not exist non-totally geodesic, totally umbilical Lagrangian submanifolds in a non-flat complex space form (see [12]), the first author introduced the following notion in [13,14].

Definition 1. Let $\tilde{M}$ be a Kaehler manifold of complex dimension n . A non-totally geodesic Lagrangian submanifold of $\tilde{M}$ is called *H-umbilical* if its second fundamental form satisfies

$$\begin{aligned} h(e_j, e_j) &= \lambda J e_n, \quad h(e_j, e_n) = \lambda J e_j, \quad j = 1, \dots, n-1, \\ h(e_n, e_n) &= \mu J e_n, \quad h(e_j, e_k) = 0, \quad 1 \leq j \neq k \leq n-1, \end{aligned} \quad (8)$$

for some functions λ, μ with respect to an orthonormal frame $\{e_1, \dots, e_n\}$. If the ratio of $\mu : \lambda$ is a constant r , the *H-umbilical submanifold* is said to be of ratio r .

The *H-umbilical submanifolds* in complex space forms were classified by the first author in a series of his papers [13–15]. It was also proved in [16] that there always exist *H-umbilical submanifolds* of ratio r in $\mathbb{C}P^n(4)$ and in $\mathbb{C}H^n(-4)$ for any real number r .

A unit speed curve $z : I \rightarrow S^3(1) \subset \mathbb{C}^2$ (resp., $z : I \rightarrow H_1^3(-1) \subset \mathbb{C}_1^2$) is said to be *Legendre* if it satisfies $\langle z', iz \rangle = 0$. It was proved in [14] that a unit speed curve z in $S^3(1)$ (resp., in $H_1^3(-1)$) is Legendre if and only if it satisfies

$$z'' = i\lambda z' - z \quad (\text{resp., } z'' = i\lambda z' + z) \quad (9)$$

for a real-valued function λ . It is also known from [14] that λ is the curvature function of the curve z in $S^3(1)$ (resp., in $H_1^3(-1)$) (see also [17] Lemmas 3.1 and 3.2).

The following links between Legendre curves and *H-umbilical submanifolds* in $\mathbb{C}P^n$ and in $\mathbb{C}H^n$ were established in [14,18].

Proposition 1. An H -umbilical submanifold M of ratio 4 in $\mathbb{C}P^n(4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{2n+1}(1) \rightarrow \mathbb{C}P^n(4)$ is the Hopf fibration, and $\psi : M \rightarrow S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ is given by

$$\psi(t, y_1, \dots, y_n) = (z_1(t), z_2(t)y), \quad \{y \in \mathbb{R}^n : \langle y, y \rangle = 1\}, \quad (10)$$

where z is a unit speed Legendre curve in $S^3(1)$ satisfying $z'' = 4i\mu z' - z$, and μ is a non-trivial zero solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 + 1) = 0$.

Proposition 2. An H -umbilical submanifold M of ratio 4 in $\mathbb{C}H^n(-4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : H_1^{2n+1}(-1) \rightarrow \mathbb{C}H^n(-4)$ is the Hopf fibration and $\psi : M \rightarrow H_1^{2n+1}(-1) \subset \mathbb{C}^{n+1}$ is either one of

$$\begin{aligned} \psi(t, y_1, \dots, y_n) &= (z_1(t), z_2(t)y), \quad \{y \in \mathbb{R}^n : \langle y, y \rangle = 1\}, \\ \psi(t, y_1, \dots, y_n) &= (z_1(t)y, z_2(t)), \quad \{y \in \mathbb{R}_1^n : \langle y, y \rangle = -1\}, \end{aligned}$$

where z is a unit speed Legendre curve in $H_1^3(-1)$ satisfying $z'' = 4i\mu z' + z$, and μ is a non-trivial solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 - 1) = 0$; or ψ is

$$\begin{aligned} \psi(t, u_1, \dots, u_{n-1}) &= \frac{\sqrt{\mu(t)}}{\sqrt{\mu(0)}} e^{i \int_0^t \mu(t) dt} \left(\frac{1}{2} + \frac{1}{2} \sum_{j=1}^{n-1} u_j^2 - it\mu(0) + \frac{\mu(0)}{2\mu(t)}, \right. \\ &\quad \left. \left(i\mu(0) - \frac{\mu'(0)}{2\mu(0)} \right) \left(\frac{1}{2} \sum_{j=1}^{n-1} u_j^2 - it\mu(0) + \frac{\mu(0)}{2\mu(t)} - \frac{1}{2} \right), u_1, \dots, u_{n-1} \right), \end{aligned}$$

where z is a unit speed Legendre curve in $H_1^3(-1)$ and μ is a non-trivial solution of $\mu'^2 = 4\mu^2(1 - \mu^2)$.

3. Universal Inequalities for Riemannian Submanifolds

By a *real space form* we mean a Riemannian n -manifold of constant sectional curvature with $n \geq 2$.

For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ as defined in [8,19] we put

$$\begin{aligned} a(n_1, \dots, n_k) &= \frac{1}{2}n(n-1) - \frac{1}{2} \sum_{j=1}^k n_j(n_j-1), \\ b(n_1, \dots, n_k) &= \frac{n^2(n+k-1 - \sum_{j=1}^k n_j)}{2(n+k - \sum_{j=1}^k n_j)}, \end{aligned} \quad (11)$$

where $a(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ are positive numbers.

3.1. Optimal Inequalities for Submanifolds

For submanifolds in real space forms, we have the following optimal universal inequalities involving δ -invariants obtained in [8,19].

Theorem 2. If $\phi : M \rightarrow R^m(c)$ is an isometric immersion from a Riemannian n -manifold M into a real space form $R^m(c)$ of constant curvature c , then for any $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have

$$\delta(n_1, \dots, n_k) \leq b(n_1, \dots, n_k)H^2 + a(n_1, \dots, n_k)c, \quad (12)$$

where H^2 is the squared mean curvature of M .

The equality case of (12) holds at a point $x \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ at x such that the shape operators at x take the forms:

$$A_r = \begin{pmatrix} B_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & B_k^r & \\ & & 0 & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, m, \quad (13)$$

where I is an identity matrix and B_j^r is a symmetric $(n_j \times n_j)$ -submatrix such that

$$\text{trace}(B_1^r) = \dots = \text{trace}(B_k^r) = \mu_r. \quad (14)$$

Remark 1. If $k = 1$ and $n_1 = 2$, then inequality (12) reduces to

$$\delta(2) \leq \frac{n^2(n-2)}{n-1} H^2 + (n+1)(n-2)c, \quad (15)$$

which is known as the Chen first inequality in [20–22] among others.

Remark 2. An isometric immersion of a Riemannian n -manifold into another Riemannian manifold is called $\delta(n_1, \dots, n_k)$ -ideal if it satisfies the equality case of the inequality (12) identically. Roughly speaking, an ideal submanifold is a submanifold which receives the least amount of tension from its ambient space at each point (see [3,7,8]). There are many articles studying ideal submanifolds by various authors. In particular, $\delta(2)$ -ideal submanifolds in space forms have been studied extensively by many authors (see, e.g., the 2011 book [7] of the first author).

Remark 3. For an affine version of Theorem 2, we refer to [23].

Theorem 2 was extended to arbitrary Riemannian ambient spaces as follows.

Theorem 3 ([24]). Let $\phi : M \rightarrow \tilde{M}^m$ be an isometric immersion of a Riemannian n -manifold into a Riemannian m -manifold. Then, for each $x \in M$ and each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have

$$\delta(n_1, \dots, n_k)(x) \leq b(n_1, \dots, n_k)H^2(x) + a(n_1, \dots, n_k) \max \tilde{K}(x), \quad (16)$$

where $\max \tilde{K}(x)$ denotes the maximum of the sectional curvature function of $\tilde{M}^m$ restricted to 2-plane sections of the tangent space $T_x M$ of M at x .

The equality case of (16) holds at $x \in M$ if and only if the following conditions hold:

- (a) There exists an orthonormal basis $e_1, \dots, e_n, \xi_{n+1}, \dots, \xi_m$ at x such that the shape operators of M in $\tilde{M}^m$ at x take the following forms of (13) and (14).
- (b) For mutual orthogonal subspaces $L_1, \dots, L_k \subset T_x M$ satisfying

$$\delta(n_1, \dots, n_k) = \tau - \sum_{j=1}^k \tau(L_j)$$

at x , we have $\tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(x)$ for $\alpha_i \in \Gamma_i, \alpha_j \in \Gamma_j, 0 \leq i \neq j \leq k$, where

$$\begin{aligned} \Gamma_0 &= \{1, \dots, n_1\}, \dots, \Gamma_{k-1} = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\}, \\ \Gamma_k &= \{n_1 + \dots + n_k + 1, \dots, n\}. \end{aligned}$$

Remark 4. Since $\delta(\emptyset) = \tau$, the inequality (16) with $k = 0$ is related to inequality (1.7) of [25] obtained by B. D. Suceavă and M. B. Vajiac.

From [9], we also have the following.

If the ambient space is $\mathbb{C}H^m(-4)$, we have the following result from [26].

Proposition 3. *If M is an n -dimensional submanifold of a complex space form $\mathbb{C}H^m(-4)$, then we have*

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)}H^2 + \frac{1}{2}(n+1)(n-2). \quad (17)$$

The equality case of (17) holds at a point $x \in M$ if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x M$ and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}\}$ of $T_x^\perp M$ such that the following three conditions hold:

- (a) The subspace spanned by $\{e_3, \dots, e_n\}$ is totally real;
- (b) $K(e_1 \wedge e_2) = \inf K$ at x ;
- (c) The shape operator A satisfies

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & h_{22}^r & 0 & \dots & 0 \\ 0 & 0 & \mu_r & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_r \end{pmatrix}, \quad r = n+2, \dots, m,$$

where $\mu_r = h_{11}^r + h_{22}^r$.

3.2. Some Immediate Applications of Theorem 3

Theorem 3 implies the following.

Theorem 4. *Let M be an n -dimensional submanifold of a complex space form $\mathbb{C}P^m(4)$ with an arbitrary m . Then for any k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \dots, n_k) \leq b(n_1, \dots, n_k)H^2 + \frac{1}{2}\left\{n^2 + 2n - \sum_{j=1}^k n_j(n_j - 1)\right\}. \quad (18)$$

The equality case of (18) holds identically if and only if M is a holomorphic, totally geodesic submanifold.

Theorem 5. *If M is an n -dimensional submanifold of the complex hyperbolic m -space $\mathbb{C}H^m(-4)$ with an arbitrary m , then for any k -tuple $(n_1, \dots, n_k)$, we have*

$$\delta(n_1, \dots, n_k) \leq b(n_1, \dots, n_k)H^2 - \frac{1}{2}\left\{n(n-1) - \sum_{j=1}^k n_j(n_j - 1)\right\}. \quad (19)$$

Theorem 6. *Let $\tilde{M}^m$ be a Riemannian manifold whose sectional curvature function is bounded above by c . If M is any Riemannian n -manifold such that*

$$\delta(n_1, \dots, n_k)(x) > \frac{1}{2}\left\{n(n-1) - \sum_{j=1}^k n_j(n_j - 1)\right\}c$$

for some k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ at some point $x \in M$, then M admits no minimal isometric immersion into $\tilde{M}^m$.

In particular, Theorem 6 implies the following non-existence result.

Corollary 1. Let M be a Riemannian n -manifold which satisfies $\delta(n_1, \dots, n_k) > 0$ at some point in M for some k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$. Then M admits no minimal isometric immersion into any Riemannian m -manifold with non-positive sectional curvature.

Since $\delta(n-1) = \max \text{Ric}$, inequality (16) implies immediately the following.

Corollary 2. Let M be an n -dimensional submanifold of a Riemannian m -manifold $\tilde{M}^m$. Then, for any unit vector X tangent to M , we have

$$\text{Ric}(X) \leq \frac{n^2}{4} H^2 + (n-1) \max \tilde{K}. \quad (20)$$

This corollary extends Theorem 4(1) of [15] (see also Theorem 1.2(a) of [27]).

4. Some Further Applications of δ -Invariants

δ -invariants have many applications to several areas in mathematics. For instance, it is proved in [28] that every $\delta(2)$ -ideal null 2-type hypersurface of a Euclidean space is a spherical cylinder. In this section, we provide some further applications of δ -invariants.

4.1. Applications to Spectral Theory

By applying Theorem 3 and Nash's embedding theorem, we discovered the following sharp estimate of the first nonzero eigenvalue of Laplacian in terms of δ -invariants (see [7–9]).

Theorem 7. Let M be a compact irreducible homogeneous Riemannian n -manifold. Then the first nonzero eigenvalue λ_1 of the Laplacian Δ satisfies

$$\lambda_1 \geq n \Delta(n_1, \dots, n_k) \quad (21)$$

for any k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

If $k = 0$, then (21) reduces to the following well-known result of T. Nagano obtained in [29]:

$$\lambda_1 \geq n\rho, \quad (22)$$

where $\rho = \frac{2\tau}{n(n-1)}$ is the normalized scalar curvature.

4.2. Applications to Warped Products

Another application of δ -invariants is to obtain the following sharp result for warped products.

Theorem 8 ([30]). If $\phi : N_1 \times_f N_2 \rightarrow \tilde{M}$ is an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold $\tilde{M}$, then the warping function f satisfies

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 \max \tilde{K}, \quad n_i = \dim N_i, \quad i = 1, 2,$$

where Δ is the Laplacian on N_1 .

Theorem 8 has many immediate consequences as well. For instance, it yields the following.

Corollary 3. Let $N_1 \times_f N_2$ be a warped product of Riemannian manifolds. If the warping function f is harmonic, then we have

- (a) $N_1 \times_f N_2$ never admits minimal immersion into any Riemannian manifold of negative sectional curvature;
- (b) Every minimal immersion of $N_1 \times_f N_2$ into any Euclidean space is a warped product immersion regardless of codimension.

Corollary 4. Let f be an eigenfunction of Δ with eigenvalue $\lambda > 0$ on N_1 . Then every warped product $N_1 \times_f N_2$ never admits a minimal immersion into any Riemannian manifold with non-positive sectional curvature.

Corollary 5. If N_1 is a compact Riemannian manifold, then we have

- (a) Every warped product $N_1 \times_f N_2$ never admits a minimal immersion into any Riemannian manifold of negative sectional curvature;
- (b) Every warped product $N_1 \times_f N_2$ never admits a minimal immersion into any Euclidean space.

For further results in this direction, see [31–36] among many others.

4.3. Applications to Submersions

Let M and B be Riemannian manifolds with $n = \dim M > \dim B = b > 0$. A surjective map $\pi : M \rightarrow B$ is called a *Riemannian submersion* if it satisfies the following two axioms:

(S1) π has maximal rank;

(S2) the differential π_* preserves lengths of horizontal vectors.

Let $\pi : M \rightarrow B$ be a Riemannian submersion. Then, for each $x \in B$, $\pi^{-1}(x)$ is a submanifold of M . The submanifolds $\pi^{-1}(x)$, $x \in B$, are called *fibers*. A vector field on M is called *vertical* if it is always tangent to fibers, and *horizontal* if it is always orthogonal to fibers. We also use corresponding terminology for individual tangent vectors.

The simplest type of Riemannian submersion is the projection of a Riemannian product manifold onto one of its factors. For such a Riemannian submersion, horizontal and vertical distributions are totally geodesic distributions.

A Riemannian manifold M is said to admit a *non-trivial Riemannian submersion* if there exists a Riemannian submersion $\pi : M \rightarrow B$ such that the horizontal and vertical distributions of the submersion are not both totally geodesic distributions.

For a Riemannian submersion $\pi : M \rightarrow B$, let $\{e_1, \dots, e_b, e_{b+1}, \dots, e_n\}$ be an orthonormal basis of $T_x M$ ($x \in M$) such that $e_1, \dots, e_b$ are horizontal and $e_{b+1}, \dots, e_n$ are vertical. The following submersion δ -invariant of π was introduced in [37] as

$$\check{A}_\pi(p) = \sum_{i=1}^b \sum_{r=b+1}^n \langle A_{e_i} e_r, A_{e_i} e_r \rangle, \quad (23)$$

where A is the O'Neill integrability tensor. By applying O'Neill tensor, the first author proved the following non-immersion theorem.

Theorem 9 ([37]). If a Riemannian manifold admits a non-trivial Riemannian submersion with totally geodesic fibers, then it can not be isometrically immersed into any Riemannian manifold with non-positive sectional curvature as a minimal submanifold.

Let $\pi : M \rightarrow B$ be a Riemannian submersion with totally geodesic fibers and let N be a Riemannian submanifold of B . We denote by $\hat{N}$ the pre-image $\pi^{-1}(N)$ of N in M . Then $\hat{\pi} : \hat{N} \rightarrow N$ is also a Riemannian submersion with totally geodesic fibers, where $\hat{\pi}$ is the restriction $\pi|_{\hat{N}}$. For a horizontal 2-plane $H_x \subset T_x \hat{N}$, we denote by $\hat{H}_x$ the $(n - b + 2)$ -subspace spanned by H_x , where $n = \dim M$ and $b = \dim B$.

The following *submersion δ -invariant* on $\hat{N}$ was defined in [38] as

$$\delta^H(x) = \tau_{\hat{N}}(x) - \inf_{\hat{H}_x} \tau_{\hat{N}}(\hat{H}_x), \quad (24)$$

where $\hat{H}_x$ runs over $(n - b + 2)$ -subspaces associated with all horizontal 2-planes H_x at $x \in \hat{N}$.

Let M be a submanifold of a Kaehler manifold $(\tilde{M}, \tilde{g}, J)$. For each $X \in TM$ we put

$$JX = PX + FX, \quad (25)$$

where PX and FX are the tangential and normal components of JX , respectively. Let $\|P\|^2$ denote the squared norm of the P defined in (25).

The following sharp inequality involving δ^H was obtained in [38].

Theorem 10. *Let $\pi : S^{2m+1} \rightarrow \mathbb{C}P^m(4)$ be the Hopf fibration and let M be an n -dimensional submanifold of $\mathbb{C}P^m(4)$. Then we have*

$$\delta^H \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \|P\|^2 + \frac{1}{2}(n^2 - n - 2). \quad (26)$$

The equality case of (26) holds identically if and only if there exists an orthonormal frame $e_1, \dots, e_m$ such that the following two conditions are satisfied:

(a) The shape operator A of M in $\mathbb{C}P^m(4)$ satisfies

$$A_{e_s} = \begin{pmatrix} B_s & 0 \\ 0 & \mu_s I \end{pmatrix}, \quad s = n+1, \dots, 2m, \quad (27)$$

where I is an identity $(n-2) \times (n-2)$ matrix and B_s are symmetric 2×2 submatrices satisfying $\mu_s = \text{trace } B_s$, $s = n+1, \dots, 2m$, and

(b) $Pe_1 = Pe_2 = 0$.

It is also proved in [38] that there exist many non-totally geodesic, totally real submanifolds of $\mathbb{C}P^m(4)$ which satisfy the equality case of the inequality (26) identically.

4.4. Applications to Symplectic Geometry

An application of δ -invariants to Lagrangian submanifolds is provided by the following sharp result.

Theorem 11 ([8,9]). *Assume that M is a compact Riemannian n -manifold with null first Betti number, that is, $b_1(M) = 0$. If there exists a k -tuple $(n_1, \dots, n_k)$ such that $\delta(n_1, \dots, n_k) > 0$, then M admits no Lagrangian isometric immersion into the complex Euclidean n -space $\mathbb{C}^n$.*

Remark 5. *If the fundamental group of M is finite, then $b_1(M) = 0$ holds automatically. We note that the condition $b_1(M) = 0$ is necessary in Theorem 11 (see [3,7]).*

5. A Link between Ideal Immersions and Covering Maps

The following notion of ideal immersions was introduced in [8,9].

Definition 2. *An n -dimensional submanifold M of a real space form $R^m(c)$ is called ideal if it is $\delta(n_1, \dots, n_k)$ -ideal for some $(n_1, \dots, n_k) \in \mathcal{S}(n)$.*

By a covering map $\pi : M \rightarrow N$ between two Riemannian manifolds we mean a covering map which is a locally isometry.

For a compact Riemannian manifold M , we denote by $\lambda_1(M)$ the first positive eigenvalue of the Laplacian Δ on M . For the first nonzero eigenvalue λ_1 , we recall the following result of K. Yoshiji.

Theorem 12 ([39]). *Every non-orientable compact manifold M , excepting the real projective plane, admits a Riemannian metric g for which the first eigenvalue coincides with that of its Riemannian double cover $\hat{M}$, that is, one has $\lambda_1(M) = \lambda_1(\hat{M})$ with respect to the metric g and its covering metric $\hat{g}$ on $\hat{M}$.*

A Riemannian manifold is called *homogeneous* if its group of isometries acts transitively on the manifold (see, e.g., [40] Theorem 4.6). A compact homogeneous manifold with irreducible isotropy action is called an *irreducible compact homogeneous space*.

The next theorem provides a simple link between covering maps and ideal embeddings.

Theorem 13 ([41]). *Let $\pi : M \rightarrow N$ be a covering map between two irreducible compact homogeneous spaces. If $\lambda_1(M) \neq \lambda_1(N)$ holds, then N does not admit an ideal embedding into any Euclidean space regardless of codimension.*

Example 1. Let $S^n(1)$ and $\mathbb{R}P^n(1)$ denote the n -sphere and the real projective n -space of constant sectional curvature 1. Then there exists a twofold covering map $\pi : S^n(1) \rightarrow \mathbb{R}P^n(1)$ which carries each pair of antipodal points on $S^n(1)$ to a single point in $\mathbb{R}P^n(1)$.

It is well-known that

$$n = \lambda_1(S^n(1)) \neq \lambda_1(\mathbb{R}P^n(1)) = 2(n+1).$$

Thus Theorem 13 implies that $\mathbb{R}P^n(1)$ never admits an ideal embedding into any Euclidean space regardless of codimension. On the other hand, the standard embedding $S^n(1) \subset \mathbb{E}^{n+1}$ is an ideal embedding of $S^n(1)$ in $\mathbb{E}^{n+1}$.

6. Real Hypersurfaces of Complex Space Forms

For general references to real hypersurfaces of complex space forms, we refer to [42]. A real hypersurface M in a complex space form is called a *Hopf hypersurface* if $J\xi$ is a principal curvature vector, where ξ is a unit normal vector of M (see, e.g., [42,43]).

6.1. Real Hypersurfaces of Complex Space Forms Involving $\delta(2, \dots, 2)$

For a real hypersurface of the complex projective m -space $\mathbb{C}P^m(4)$, we have the two classification results from [9].

Theorem 14. *If M is a real hypersurface of $\mathbb{C}P^m(4)$ ($m \geq 2$), then we have*

$$\delta(2, \dots, 2) \leq \frac{(2m-1)^2(2m-k-2)}{2(2m-k-1)}H^2 + 2m^2 - k - 2 \quad (28)$$

for any natural number $k \leq m-1$, where 2 appears k -times in $\delta(2, \dots, 2)$.

If M is a Hopf hypersurface, then the equality case of (28) holds for some k if and only if one of the following three cases occurs:

- (a) $k = m-1$ and M is an open portion of a geodesic sphere with radius $\frac{\pi}{4}$;
- (b) m is odd, $k = m-1$, and M is an open portion of a tubular hypersurface over a totally geodesic $\mathbb{C}P^{(m-1)/2}(4c)$ with radius $r \in (0, \frac{\pi}{2})$;
- (c) $m = 2$, $k = 1$, and M is an open portion of a tubular hypersurface over a complex quadric curve Q_1 with radius

$$r = \arctan\left(\frac{1}{2}\left(1 + \sqrt{5} - \sqrt{2 + 2\sqrt{5}}\right)\right) = 0.33311971 \dots$$

For a real hypersurface of the complex hyperbolic m -space $\mathbb{C}P^m(4)$, we have

Theorem 15. Let M be a real hypersurface of a complex hyperbolic m -space $\mathbb{C}H^m(-4)$. Then we have

$$\delta(2, \dots, 2) \leq \frac{(2m-1)^2(2m-k-2)}{2(2m-k-1)} H^2 - (2m^2 - 4k - 2) \quad (29)$$

for any natural number $k \leq m-1$, where 2 appears k -times in $\delta(2, \dots, 2)$.

The equality case of (29) holds for some k if and only if one of the following two cases occurs:

- (a) m is odd, $k = m-1$, and M is an open portion of a tubular hypersurface over a totally geodesic $\mathbb{C}H^{(m-1)/2}(-4)$ with radius $r \in \mathbb{R}_+$;
- (b) M is an open portion of a horosphere in $\mathbb{C}H^m(-4)$.

Remark 6. Theorems 14 and 15 extend Theorems 6 and 8 of [26] on $\delta(2)$, respectively.

Remark 7. Since there do not exist totally umbilical hypersurfaces in a non-flat complex space form (see, e.g., [12]), there are no hypersurfaces in a non-flat complex space form which satisfy either the equality case of (28) or the equality case of (29) with $k = 0$.

6.2. Classification of Non-Hopf Hypersurfaces in $\mathbb{C}P^2(4)$

In view of Theorems 14 and 15, T. Sasahara studies non-Hopf real hypersurfaces of the complex projective plane $\mathbb{C}P^2(4)$ in [44,45]. In [44], he proved the following.

Theorem 16. Let M be a real hypersurface of $\mathbb{C}P^2(4)$ which is non-Hopf at every point. If M has constant mean curvature, then M satisfies the equality case of

$$\delta(2) \leq \frac{9}{4}H^2 + 5 \quad (30)$$

identically if and only if it is a minimal ruled real hypersurface which is given by $\psi \circ z$, where $\psi : S^5 \rightarrow \mathbb{C}P^2(4)$ is the Hopf fibration and

$$z(u, v, \theta, w) = e^{iw}(\cos u \cos v, \cos u \sin v, (\sin u)e^{i\theta})$$

for $u \in (-\pi/2, \pi/2)$ and $v, \theta, w \in [0, 2\pi)$.

The following example of $\delta(2)$ -ideal hypersurface in $\mathbb{C}P^2(4)$ was constructed in [46].

Example 2. Let $\alpha(s), \beta(s), \gamma(s)$ and $\mu(s)$ be functions on an open interval $I \subset \mathbb{R}$ which satisfy

$$\begin{aligned} \alpha' &= \beta(\alpha + \gamma - 3\mu), \quad \beta' = \beta^2 + \gamma^2 + \mu(\alpha - 2\gamma) + 1, \\ \gamma' &= \frac{(\gamma - \mu)(\gamma^2 - \alpha\gamma - 1)}{\beta} + \beta(2\gamma + \mu), \quad \beta \neq 0. \end{aligned} \quad (31)$$

If $\alpha + \gamma = \mu$ on I , then, by applying Theorem 5 of [47], there exists a smooth immersion $\Phi : I \times \mathbb{E}^2 \rightarrow \mathbb{C}P^2$ defined on some subinterval of I which gives rise to a non-Hopf, non-minimal $\delta(2)$ -ideal hypersurface in $\mathbb{C}P^2$.

The next classification theorem was obtained in [46].

Theorem 17. Let M be a $\delta(2)$ -ideal non-Hopf real hypersurface in $\mathbb{C}P^2(4)$. If the mean curvature is constant along each integral curve of the Reeb vector field, then M is locally obtained by the construction described in Example 2.

7. Lagrangian Submanifolds of Complex Space Forms

We recall the following definition (see, e.g., [48,49]).

Definition 3. A totally real submanifold M of a Kaehler manifold $(\tilde{M}, \tilde{g}, J)$ is a Riemannian submanifold such that the complex structure J of $\tilde{M}$ carries each tangent space of M into the corresponding normal space, that is, $J(T_x M) \subset T_x^\perp M$, $x \in M$. A totally real submanifold M of $\tilde{M}$ is called Lagrangian if we have $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \tilde{M}$.

Totally real and Lagrangian submanifolds of Kaehler manifolds have been studied extensively from differential geometric point of view by many geometers during the last half century (see, e.g., [48]).

By a complex space form we mean a Kaehler manifold of constant holomorphic sectional curvature. A complete simply-connected complex space form $\tilde{M}^m(c)$ is holomorphically isometric to a complex projective m -space $\mathbb{C}P^m$, a complex Euclidean m -space $\mathbb{C}^m$, or a complex hyperbolic m -space $\mathbb{C}H^m$ depending on the holomorphic sectional curvature c is positive, zero, or negative, respectively.

Since the proof of Theorem 2 was based only on the Gauss equation and that Gauss equation for a totally real submanifold in a complex space form $\tilde{M}^m(4c)$ is the same as the Gauss equation for submanifolds of a real space form $R^m(c)$, we have immediately the following.

Theorem 18. Let M be an n -dimensional totally real submanifold of a complex space form $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$. Then for any k -tuple $(n_1, \dots, n_k)$ we have

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)H^2 + b(n_1, \dots, n_k)c. \quad (32)$$

The equality case of (32) holds at a point $x \in M$ if and only if (13) and (14) hold at x with respect to a suitable orthonormal basis $e_1, \dots, e_n, \dots, e_{2m}$ at x .

An m -dimensional Lagrangian submanifold of a complex space form $\tilde{M}^m(4c)$ is called an ideal Lagrangian submanifold if it satisfies the equality case of inequality (32) identically.

The following theorem from [19] extends a result on $\delta(2)$ from [50] (see also [51]).

Theorem 19. Every ideal Lagrangian submanifold of a complex space form is a minimal submanifold.

Remark 8. Theorem 19 is false if the “Lagrangian” condition is replaced by the “totally real” condition. For instance, if $S^n(r) \subset S^{n+1}(1)$ is a small hypersphere with radius $r \in (0, 1)$, then the following composition map from $S^n(r)$ into $\mathbb{C}P^{n+1}(4)$:

$$S^n(r) \subset S^{n+1}(1) \xrightarrow{\pi} \mathbb{R}P^{n+1}(1) \xrightarrow{\phi} \mathbb{C}P^{n+1}(4)$$

is a non-minimal, ideal, totally real isometric immersion of $S^n(r)$ into $\mathbb{C}P^{n+1}(4)$, where $S^{n+1}(1) \xrightarrow{\pi} \mathbb{R}P^{n+1}(1)$ is the canonical double covering and $\mathbb{R}P^{n+1}(1) \xrightarrow{\phi} \mathbb{C}P^{n+1}(4)$ is a totally geodesic Lagrangian embedding.

8. Improved Inequalities for Lagrangian Submanifolds

We need to fix some notations. For a given δ -invariant $\delta(n_1, \dots, n_k)$ on a Riemannian n -manifold M and a point $x \in M$, we consider mutually orthogonal subspaces $L_1, \dots, L_k$ of $T_x M$ with $\dim L_i = n_i$, minimizing the quantity $\tau(L_1) + \dots + \tau(L_k)$. We shall choose an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_x M$ such that

$$e_1, \dots, e_{n_1} \in L_1, e_{n_1+1}, \dots, e_{n_1+n_2} \in L_2, \dots, e_{n_1+\dots+n_{k-1}+1}, \dots, e_{n_1+\dots+n_k} \in L_k,$$

and we put

$$\Delta_1 := \{1, \dots, n_1\}, \quad \Delta_2 := \{n_1 + 1, \dots, n_1 + n_2\}, \quad \dots, \\ \Delta_k := \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\}, \quad \Delta_{k+1} := \{n_1 + \dots + n_k + 1, \dots, n\}.$$

We will use the following conventions for the ranges of summation indices:

$$A, B, C \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, k\}, \quad \alpha_i, \beta_i \in \Delta_i, \quad r, s \in \Delta_{k+1}.$$

we also put $n_{k+1} = n - n_1 - \dots - n_k$. Note that this may be zero, in which case Δ_{k+1} is empty. As before, we denote the components of the second fundamental form by $h_{AB}^C = \langle h(e_A, e_B), e_C \rangle$.

In view of Theorem 19, it is very natural to look for optimal inequalities for non-minimal Lagrangian submanifolds in complex space forms. In this respect, the first author, F. Dillen, J. Van der Veken and L. Vrancken improved in [52] the inequality (32) of Theorem 32 to the following.

Theorem 20. *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then for any $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $\sum_{i=1}^k n_i < n$ we have*

$$\delta(n_1, \dots, n_k) \leq \frac{n^2 \{n - \sum_{i=1}^k n_i + 3k - 1 - 6 \sum_{i=1}^k (2 + n_i)^{-1}\}}{2 \{n - \sum_{i=1}^k n_i + 3k + 2 - 6 \sum_{i=1}^k (2 + n_i)^{-1}\}} H^2 \\ + \frac{1}{2} \left\{ n(n-1) - \sum_{i=1}^k n_i(n_i - 1) \right\} c. \quad (33)$$

If the equality case of (33) holds at a point $x \in M$, then with the choice of basis and the notations introduced above, we have

- (a) $h_{BC}^A = 0$ if A, B, C are mutually different and not all in the same Δ_i with $i \in \{1, \dots, k\}$,
- (b) $h_{\alpha_j \alpha_j}^{\alpha_i} = h_{rr}^{\alpha_i} = \sum_{\beta_i \in \Delta_i} h_{\beta_i \beta_i}^{\alpha_i} = 0$ for $i \neq j$,
- (c) $h_{rr}^r = 3h_{ss}^r = (n_i + 2)h_{\alpha_i \alpha_i}^r$ for $r \neq s$.

It was shown in [53] and also in [52] that the inequality (33) is optimal, that is, it is always possible to construct a non-minimal Lagrangian immersion which realizes equality at least in one point. More precise, we have the following.

Theorem 21. *For each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, there exists a non-minimal Lagrangian submanifold satisfying the equality case of (33).*

This theorem implies that the inequality (33) in Theorem 20 can not be improved further.

Remark 9. When $k = 1$ and $n_1 = 2$, inequality (33) is due to T. Oprea [54].

Another consequence of Theorem 20 is the following result of T. Oprea.

Corollary 6 ([55]). *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then we have*

$$\delta(n-1) \leq \frac{n(n-1)}{4} H^2 + (n-1)c. \quad (34)$$

The next proposition is due to S. Deng [56].

Proposition 4. Let N be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then the Ricci curvature of N satisfies

$$\text{Ric}(u) = \frac{n(n-1)}{4}H^2 + (n-1)c \quad (35)$$

for each unit tangent vector u to M if and only if either M is totally geodesic or M is an H -umbilical submanifold with ratio 3.

For the case: $\sum_{i=1}^k n_i = n$, we have the following inequality with a different coefficient in [52].

Theorem 22. Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then for any $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $\sum_{i=1}^k n_i = n$ we have

$$\begin{aligned} \delta(n_1, \dots, n_k) \leq & \frac{n^2 \{k-1-2\sum_{i=2}^k (2+n_i)^{-1}\}}{2\{k-2\sum_{i=2}^k (2+n_i)^{-1}\}} H^2 \\ & + \frac{1}{2} \left\{ n(n-1) - \sum_{i=1}^k n_i(n_i-1) \right\} c, \end{aligned} \quad (36)$$

where we assume that $n_1 = \min\{n_1, \dots, n_k\}$.

If the equality case of (36) holds at a point $x \in M$, then the components of the second fundamental form h with respect to a suitable orthonormal basis $\{e_1, \dots, e_n\}$ for $T_x M$ satisfy the following conditions:

- (a) $h_{\alpha_i \alpha_j}^A = 0$ for $i \neq j$ and $A \neq \alpha_i, \alpha_j$;
- (b) If $n_j \neq \min\{n_1, \dots, n_k\}$, we have $h_{\alpha_i \alpha_i}^{\beta_j} = 0$, $i \neq j$, and $\sum_{\alpha_j \in \Delta_j} h_{\alpha_j \alpha_j}^{\beta_j} = 0$;
- (c) If $n_j = \min\{n_1, \dots, n_k\}$, we have $\sum_{\alpha_j \in \Delta_j} h_{\alpha_j \alpha_j}^{\beta_j} = (n_i + 2)h_{\alpha_i \alpha_i}^{\beta_j}$ for $i \neq j$ and $\alpha_i \in \Delta_i$.

The following result of [52] implies that inequality (36) is optimal as well.

Theorem 23. For each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $\sum_{i=1}^k n_i = n$, there exists a Lagrangian submanifold in $\tilde{M}^n(4c)$ satisfying the equality of the improved inequality (36) identically.

This result shows that the inequality (36) in Theorem 22 can not be improved further as well.

Remark 10. For minimal Lagrangian submanifolds of a complex space form, inequality (32) of Theorem 18 and the improved inequalities (33) of Theorem 20 and (36) of Theorem 22 are the same.

9. Special Cases of Ideal Lagrangian Submanifolds for Improved Inequalities

Lagrangian submanifolds in complex space forms satisfying the improved inequalities (33) for $\delta(2)$ have been studied extensively in [7,52–54,57–63] among others. For $\delta(2,2)$ -ideal Lagrangian submanifolds, we refer to [16,18,64].

In this section, we pay our attention to Lagrangian submanifolds in complex space forms which are either $\delta(n_1)$ -ideal, $\delta(n-1)$ -ideal, $\delta(2, n-2)$ -ideal, or $\delta(p, \dots, p)$ -ideal (with k times p).

9.1. $\delta(n_1)$ -Ideal Lagrangian Submanifolds

The following result was proved by the first author and F. Dillen in [53].

Theorem 24. A $\delta(n_1)$ -ideal Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ with $n_1 < n-1$ is minimal.

For $\delta(2)$ -ideal Lagrangian submanifolds in $\mathbb{C}P^n$, L. Vrancken proved the following.

Theorem 25 ([63]). *Let M be a $\delta(2)$ -ideal complete Lagrangian submanifold of $\mathbb{C}P^n(4)$. Then either M is totally geodesic or $n = 3$.*

In dimension $n = 3$, an example of a compact $\delta(2)$ -ideal non-totally geodesic Lagrangian submanifold was constructed earlier in [51].

9.2. $\delta(n-1)$ -Ideal Lagrangian Submanifolds

The complete classification of non-minimal $\delta(n-1)$ -ideal Lagrangian submanifolds of complex space forms was proved by the first author, F. Dillen and L. Vrancken. Here, we present the results only for the cases whose ambient space is either $\mathbb{C}^n$ or $\mathbb{C}P^n(4)$.

Theorem 26 ([60]). *Let M be a non-minimal Lagrangian submanifold of the complex Euclidean n -space $\mathbb{C}^n$. Then we have*

$$\delta(n-1) \leq \frac{1}{4}n(n-1)H^2. \quad (37)$$

The equality case of (37) holds identically if and only if, up to dilations and rigid motions, M is given by

$$L(\lambda, u_2, \dots, u_n) = \frac{(n+1)e^{-i\varphi}}{(n+1)\mu + i\lambda} \phi(u_2, \dots, u_n),$$

where

$$\varphi(\lambda) = -\frac{n+1}{n} \csc^{-1}\left((n+1)c\lambda^{\frac{n}{1-n}}\right), \quad \mu(\lambda) = \sqrt{c^2\lambda^{\frac{2}{1-n}} - (n+1)^{-2}\lambda^2}$$

for some real number $c > 0$, and $\phi(u_2, \dots, u_n)$ is a minimal Legendre submanifold of $S^{2n-1}(1)$.

Theorem 27 ([60]). *Let M be a non-minimal Lagrangian submanifold of the complex projective n -space $\mathbb{C}P^n(4)$. Then we have*

$$\delta(n-1) \leq \frac{n-1}{4}(nH^2 + 4). \quad (38)$$

The equality case holds identically if and only if M is congruent to the Lagrangian submanifold defined by the composition $\pi \circ L$, where $\pi : S^{2n+1}(1) \rightarrow \mathbb{C}P^n(4)$ is the Hopf fibration and

$$L(t, u_2, \dots, u_n) = \left(\frac{e^{-i\theta}\phi}{\sqrt{1+\mu^2+(n+1)^{-2}\lambda^2}}, \frac{(i(n+1)^{-1}\lambda - \mu)e^{-ni\theta}}{\sqrt{1+\mu^2+(n+1)^{-2}\lambda^2}} \right), \quad (39)$$

where $\lambda(t)$, $\mu(t)$ and $\theta(t)$ satisfy

$$\frac{d\theta}{dt} = -\frac{\lambda}{n+1}, \quad \frac{d\lambda}{dt} = (n-1)\lambda\mu, \quad \frac{d\mu}{dt} = -1 - \mu^2 - \frac{n\lambda^2}{(n+1)^2}, \quad \lambda \neq 0, \quad (40)$$

and ϕ is a minimal Legendre immersion in $S^{2n-1}(1)$.

9.3. $\delta(2, n-2)$ -Ideal Lagrangian Submanifolds

This is the simplest case of ideal Lagrangian submanifold with $\sum_{i=1}^k n_i = n$. The $\delta(2, n-2)$ -ideal Lagrangian submanifolds in a complex space form have been studied and completely classified in [65]. Here, we present the classification theorem for ideal Lagrangian submanifolds merely in $\mathbb{C}^n$.

Theorem 28. Let M be a Lagrangian submanifold of the complex Euclidean space $\mathbb{C}^n$ with $n \geq 5$. Then we have

$$\delta(2, n-2) \leq \frac{n^2(n-2)}{4(n-1)} H^2. \quad (41)$$

Assume that M is non-minimal. Then M is $\delta(2, n-2)$ -ideal, that is, it satisfies the equality case of (41) identically, if and only if M is locally congruent to the image of one of the following two immersions:

(a)

$$L(x, u_2, \dots, u_n) = \frac{e^{i\theta(x)}}{\varphi(x) + ix} \Phi(u_2, \dots, u_n),$$

with

$$\theta(x) = \frac{n-1}{2-n} \arcsin\left(cx^{\frac{n-2}{n-3}}\right), \quad \varphi(x) = \sqrt{\frac{1}{c^2 x^{\frac{2}{n-3}}} - x^2},$$

where c is a positive constant and Φ is a minimal Legendre submanifold of $S^{2n-1}(1) \subseteq \mathbb{C}^n$ which is mapped to a $\delta(n-2)$ -ideal minimal Lagrangian submanifold of $\mathbb{C}P^{n-1}(4)$ by the Hopf fibration;

(b)

$$L(x, y, u_1, \dots, u_{n-2}) = (f(x, y)e^{ix}\Phi(u_1, \dots, u_{n-2}), z(x, y)),$$

where Φ defines a minimal Legendre immersion in $S^{2n-3}(1) \subset \mathbb{C}^{n-1}$ and (fe^{ix}, z) is a Lagrangian surface in $\mathbb{C}^2$, where f is determined by

$$\frac{f_{yy}}{f^{n-2}} - (n-2)\frac{f_y^2}{f^{n-1}} + (n-1)f^{n-1} + (n-2)f^{n-3}f_x^2 + f^{n-2}f_{xx} = 0$$

and z by

$$z_x = e^{i(n-1)x} \frac{f_y}{f^{n-2}}, \quad z_y = e^{i(n-1)x} f^{n-1} \left(i - \frac{f_x}{f}\right).$$

9.4. $\delta(2, \dots, 2)$ -Ideal Lagrangian Submanifolds

For $\delta(2, \dots, 2)$ -ideal Lagrangian submanifolds in complex space forms we have the following classification theorems from [18].

Theorem 29. Let M be a non-minimal $\delta(2, \dots, 2)$ -ideal Lagrangian submanifold in $\mathbb{C}^n$ with $n = 2k + 1$, where k is the multiplicity of 2 in $\delta(2, \dots, 2)$. Then M is one of the following Lagrangian submanifolds:

- (a) an H -umbilical submanifold of ratio 4;
- (b) a Lagrangian submanifold defined by

$$L(\mu, u_1, \dots, u_{2k}) = \frac{e^{\frac{4}{3}i \tan^{-1} \sqrt{\mu^3/(c^2 - \mu^3)}}}{\sqrt{c^2 \mu^{-1} - \mu^2} + i\mu} \phi(u_1, \dots, u_{2k}),$$

where c is a positive real number and $\phi(u_1, \dots, u_{2k})$ is a horizontal lift of a non-totally geodesic minimal $\delta(2, \dots, 2)$ -ideal Lagrangian immersion in $\mathbb{C}P^{n-1}(4)$, and the multiplicity of 2 is less than k .

Theorem 30. Let M be a non-minimal $\delta(2, \dots, 2)$ -ideal Lagrangian submanifold in $\mathbb{C}P^n(4)$ with $n = 2k + 1$, where k is the multiplicity of 2 in $\delta(2, \dots, 2)$. Then M is one of the following Lagrangian submanifolds:

- (a) an H -umbilical submanifold of ratio 4;

- (b) a Lagrangian submanifold defined by $\pi \circ L$, where $\pi : S^{2n+1}(1) \rightarrow \mathbb{CP}^n(4)$ is the Hopf fibration and $L : M \rightarrow S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ is given by

$$L(\mu, u_1, \dots, u_{2k}) = \frac{1}{c} \left(\sqrt{\mu} e^{i\theta} \phi, e^{3i\theta} (\sqrt{c^2 - \mu^3 - \mu} - i\mu^{\frac{3}{2}}) \right),$$

where c is a positive real number, $\phi : N^{2k} \rightarrow S^{2n-1}(1) \subset \mathbb{C}^n$ is a horizontal lift of a non-totally geodesic minimal $\delta(2, \dots, 2)$ -ideal Lagrangian immersion in $\mathbb{CP}^{n-1}(4)$, the multiplicity of 2 is less than k , and $\theta(\mu)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2\sqrt{c^2\mu^{-1} - \mu^2 - 1}}$.

Theorem 31. Let M be a non-minimal $\delta(2, \dots, 2)$ -ideal Lagrangian submanifold in $\mathbb{CH}^n(-4)$ with $n = 2k + 1$, where k is the multiplicity of 2 in $\delta(2, \dots, 2)$. Then M is either

- (a) an H -umbilical submanifold of ratio 4, or
 (b) a Lagrangian submanifold defined by $\pi \circ L$, where $\pi : H_1^{2n+1}(-1) \rightarrow \mathbb{CH}^n(-4)$ is the Hopf fibration and $L : M \rightarrow H_1^{2n+1}(-1) \subset \mathbb{C}_1^{n+1}$ is one of the following:
 (b.1) $L = \frac{1}{c} \left(\sqrt{\mu} e^{i\theta} \phi(u_1, \dots, u_{2k}), e^{-i\theta} (\sqrt{\mu - \mu^3 - c^2} - i\mu^{\frac{3}{2}}) \right)$, where c is a positive number, $\phi : N^{2k} \rightarrow H_1^{2n-1}(-1) \subset \mathbb{C}_1^n$ is a horizontal lift of a non-totally geodesic minimal $\delta(2, \dots, 2)$ -ideal Lagrangian immersion in $\mathbb{CH}^{n-1}(-4)$, the multiplicity of 2 is less than k and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2} \sqrt{1 - \mu^2 - c^2 \mu^{-1}}$;
 (b.2) $L = \frac{1}{c} \left(e^{i\theta} (\sqrt{\mu - \mu^3 + c^2} - i\mu^{\frac{3}{2}}), \sqrt{\mu} e^{i\theta} \phi(u_1, \dots, u_{2k}) \right)$, where c is a positive number, $\phi : N^{2k} \rightarrow S^{2n-1}(1) \subset \mathbb{C}^n$ is a horizontal lift of a non-totally geodesic minimal $\delta(2, \dots, 2)$ -ideal Lagrangian immersion in $\mathbb{CP}^{n-1}(4)$, the multiplicity of 2 is less than k , and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2} \sqrt{1 - \mu^2 + c^2 \mu^{-1}}$;
 (b.3) L is given by

$$L = \frac{1}{\cosh t - i \sinh t} \left(2t + w + i \left\{ \cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right\}, \psi, \right. \\ \left. 2t + w + i \left\{ \cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right\} \right),$$

where $\psi(u_1, \dots, u_{2k})$ is a non-totally geodesic minimal $\delta(2, \dots, 2)$ -ideal Lagrangian immersion in $\mathbb{C}^{n-1}$, the multiplicity of 2 is less than k and up to a constant $w(u_1, \dots, u_{2k})$ is the unique solution to the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, \dots, 2k$;

- (b.4) L is given by

$$L = \frac{1}{\cosh t - i \sinh t} \left(2t + w + i \left\{ \cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right\}, \psi, \right. \\ \left. 2t + w + i \left\{ \cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right\} \right),$$

where $\psi = (\psi_1, \dots, \psi_k)$ is the direct product immersion of k non-totally geodesic Lagrangian minimal immersions $\psi_\alpha : N_\alpha^2 \rightarrow \mathbb{C}^2$, $\alpha = 1, \dots, k$, and up to a constant $w(u_1, \dots, u_{2k})$ is the unique solution to the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, \dots, k$.

9.5. $\delta(p, \dots, p)$ -Ideal Lagrangian Submanifolds with $kp = n - 1$

This case can only occur if k is a divisor of $n - 1$. In this case, we have the following classification result from [66].

Theorem 32. Let M be a non-minimal $\delta(p, \dots, p)$ -ideal (k times p) Lagrangian submanifold of $\mathbb{C}^n$, with $kp = n - 1$. Then M is defined by

$$L(x, y_2, \dots, y_n) = \frac{e^{i\varphi(x)}}{\mu(x) + ix} \Phi(y_2, \dots, y_n),$$

where

$$\varphi(x) = \frac{p+2}{p+1} \arcsin\left(cx^{\frac{m+1}{m}}\right), \quad \mu(x) = \sqrt{\frac{1}{c^2 x^{\frac{2}{m}}} - x^2}$$

for some positive real number c and Φ is a minimal Legendrian submanifold of the unit sphere $S^{2n-1}(1) \subset \mathbb{C}^n$ which is mapped to a $\delta(p, \dots, p)$ -ideal minimal Lagrangian submanifold of $\mathbb{C}P^{n-1}(4)$ by the Hopf fibration.

10. CR-Submanifolds of Complex Space Forms

10.1. Basic Properties of CR-Submanifolds

A submanifold M in an Hermitian manifold $\tilde{M}$ is called a CR-submanifold if there exists on M a totally real distribution $\mathcal{D}^\perp$ whose orthogonal complement $\mathcal{D}$ is a complex distribution, that is, $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, $J(\mathcal{D}^\perp) \subset T^\perp M$, and $J(\mathcal{D}) = \mathcal{D}$. A CR-submanifold is called *anti-holomorphic* if it satisfies $J(\mathcal{D}^\perp) = T^\perp M$. The two distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ of a CR-submanifold are called the *holomorphic distribution* and the *totally real distribution* of the CR-submanifold, respectively (see [67]).

A CR-submanifold is called *proper* if it is neither a holomorphic submanifold nor a totally real submanifold. Further, a CR-submanifold N is called a *CR-product* if it is a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^\perp$ of $\tilde{M}$.

Let M be an n -manifold and $T^\mathbb{C}M$ its complexified tangent bundle, that is,

$$T_x^\mathbb{C}M = T_x M \otimes_{\mathbb{R}} \mathbb{C} \equiv T_x M \oplus i(T_x M).$$

Let $\mathcal{H} \subset T^\mathbb{C}M$ be a complex subbundle of complex rank ℓ . A CR-manifold (in the sense of Greenfield) of real dimension n and CR-dimension ℓ is a pair $(M, \mathcal{H})$ such that $\mathcal{H}$ is involutive and $\mathcal{H}_x \cap \bar{\mathcal{H}}_x = \{0\}$ (see [68]).

Let $\mathcal{K}$ be a distribution of a Riemannian manifold M . Denote by $\mathcal{K}^\perp$ the orthogonal complementary distribution of $\mathcal{K}$. For any vector fields X, Y in $\mathcal{K}$ we put

$$\mathring{h}(X, Y) = (\nabla_X Y)^\perp, \quad (42)$$

where $(\nabla_X Y)^\perp$ is the $\mathcal{K}^\perp$ -component of $\nabla_X Y$. It follows from Frobenius' theorem that $\mathcal{K}$ is integrable if and only if $\mathring{h}$ is symmetric.

Let us define

$$\mathring{H} = \frac{1}{k} \text{trace } \mathring{h}, \quad (43)$$

where k is the rank of $\mathcal{K}$. Then, up to a sign, $\mathring{H}$ is a well-defined vector field, called the *mean curvature vector* of $\mathcal{K}$. The distribution $\mathcal{K}$ is called a *minimal distribution* if $\mathring{H} = 0$. In particular, if $\mathring{h} = 0$ identically, $\mathcal{K}$ is called a *totally geodesic distribution*.

CR-submanifolds have the following three fundamental properties.

Theorem 33 ([69]). Every CR-submanifold of a Hermitian manifold is a CR-manifold (in the sense of Greenfield [68]).

Theorem 34 ([69–71]). The totally real distribution of any CR-submanifold of a locally conformal Kaehler manifold is an integrable distribution.

Theorem 35 ([70,72]). *The holomorphic distribution of any CR-submanifold of a Kaehler manifold is a minimal distribution.*

By applying Theorems 34 and 35, the first author proved the following.

Theorem 36 ([71,73]). *For any compact CR-submanifold of a Kaehler manifold, there exists a canonical deRham cohomology class:*

$$c(M) \in H^{2\ell}(M, \mathbb{R}), \quad \ell = \dim_{\mathbb{C}} \mathcal{D}. \quad (44)$$

Moreover, this cohomology class is non-trivial if $\mathcal{D}$ is integrable and $\mathcal{D}^{\perp}$ is minimal.

By applying Theorem 36, he obtained the following.

Corollary 7 ([73]). *If M is a $(2m - 1)$ -dimensional compact manifold with $H^{2k}(N, \mathbb{R}) = \{0\}$ for some $k < m$, then any immersion of M into a Kaehler manifold $\tilde{M}^m$ of complex dimension m is a CR-hypersurface such that either its holomorphic distribution is non-integrable or its totally real distribution is non-minimal.*

10.2. CR-Submanifolds Involving δ -Invariants

By using Proposition 3, the first author and L. Vrancken proved the following two results for CR-submanifold in $\mathbb{C}H^m(-4)$.

Theorem 37 ([74]). *Let M be a CR-submanifold of dimension $n \geq 3$ of $\mathbb{C}H^m(-4)$. If M satisfies the equality (18) identically, then one of the following three cases must occur:*

- (a) $n = 3$;
- (b) M is a minimal proper CR-submanifold;
- (c) M is totally real.

Theorem 38 ([74]). *Let U be a domain of $\mathbb{C}$ and let $\Psi : U \rightarrow \mathbb{C}^{m-1}$ be a non-constant holomorphic curve in $\mathbb{C}^{m-1}$. Define $z : \mathbb{E}^2 \times U \rightarrow \mathbb{C}_1^{m+1}$ by*

$$z(u, t, w) = e^{it} \left(i - 1 - \frac{1}{2} \Psi(w) \bar{\Psi}(w), iu - \frac{1}{2} \Psi(w) \bar{\Psi}(w), \Psi(w) \right).$$

Then $\langle z, z \rangle = -1$ and $z(\mathbb{E}^2 \times U)$ in $H_1^{2m+1}(-1)$ is invariant under the action of $H_1^1 = \{\lambda \in \mathbb{C} : \lambda \bar{\lambda} = 1\}$. Away from points where $\Psi'(w) = 0$, the image $\pi(\mathbb{E}^2 \times U)$ via Hopf's fibration $\pi : H_1^{2m+1}(-1) \rightarrow \mathbb{C}H^m(-4)$ is a proper CR-submanifold of $\mathbb{C}H^m(-4)$ which satisfies (18).

Conversely, up to rigid motions, every proper CR-submanifold of $\mathbb{C}H^m(-4)$ satisfying (18) is obtained in such a way.

A submanifold is said to be *linearly full* in $\mathbb{C}H^m(-4)$ if it does not lie in any totally geodesic complex submanifold of $\mathbb{C}H^m(-4)$.

The next two results are due to T. Sasahara.

Proposition 5 ([75]). *Let M be an n -dimensional submanifold of a complex space form $\tilde{M}^m(4c)$. Then*

$$\delta(n-1) \leq \frac{n^2}{4} H^2 + (n+2)c, \quad \text{if } c \geq 0; \quad (45)$$

$$\delta(n-1) \leq \frac{n^2}{4} H^2 + (n-1)c, \quad \text{if } c < 0. \quad (46)$$

Theorem 39 ([75]). Let U be a domain of a Cartesian $2n$ -space $\mathbb{R}^{2n}$. Define $z : \mathbb{R}^2 \times U \rightarrow \mathbb{C}_1^{m+1}$ by

$$z(s, t, x_1, x_2, \dots, y_1, y_2) = \left(\psi(x_1, \dots, y_2) e^{is}, \frac{e^{it}}{\sqrt{2n-2}} \right) \quad (47)$$

where $|\psi|^2 = \frac{1-2n}{2n-2}$ and $\psi(x_1, \dots, y_2) e^{is}$ is a CR-submanifold of $\mathbb{C}_1^m$ such that the unit vector field η in the totally real distribution $\mathcal{D}^\perp$ is $\sqrt{\frac{2n-2}{2n-1}} \frac{\partial}{\partial s}$. Then $\langle z, z \rangle = -1$ and $z(\mathbb{R}^2 \times U)$ is H_1^1 -invariant. Moreover, $z(\mathbb{R}^2 \times U) / \sim$ is a $(2n+1)$ -dimensional CR-submanifold with $\dim \mathcal{D}^\perp = 1$ satisfying the equality case of (46) under the condition that the shape operator A_η has constant principal curvatures.

Conversely, in case $n > 1$ and $m > n+1$, up to rigid motions of $\mathbb{C}H^m(-4)$, every linearly full $(2n+1)$ -dimensional CR-submanifold with $\dim \mathcal{D}^\perp = 1$ satisfying the equality case of (46) under the condition that A_η has constant principal curvatures is obtained in such a way.

11. CR-Warped Products Involving δ -Invariants

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and let f be a positive function defined on B . Then the warped product $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric

$$g = g_B + f^2 g_F.$$

The function f is called the *warping function* of the warped product (cf. [76]).

In [77], the first author proved that if $N^\perp \times_f N^T$ is a warped product in a Kaehler manifold $\tilde{M}$ such that $N^\perp$ is totally real and N^T is a complex submanifold in $\tilde{M}$, then the warping function f must be constant. Further, he showed in [77] that there exist abundant warped product submanifolds of the form $N^T \times_f N^\perp$ in Kaehler manifolds, which are called *CR-warped products*.

For a CR-warped product $M = N^T \times_f N^\perp$ with $\ell = \dim_{\mathbb{C}} N^T$ and $p = \dim N^\perp$ in a Kaehler manifold $\tilde{M}^{\ell+p}$, we shall choose a local orthonormal frame $\{e_1, \dots, e_{2\ell+p}\}$ on M such that $e_1, \dots, e_\ell, e_{\ell+1} = J e_1, \dots, e_{2\ell} = J e_\ell$ are in $\mathcal{D}$ and $e_{2\ell+1}, \dots, e_{2\ell+p}$ in $\mathcal{D}^\perp$. In this section, we use the following convention on the range of indices:

$$\begin{aligned} i, j, k &= 1, \dots, 2\ell; \quad \alpha, \beta, \gamma = 1, \dots, \ell \\ r, s, t &= 2\ell+1, \dots, 2\ell+p; \quad A, B, C = 1, \dots, 2\ell+p. \end{aligned}$$

For a CR-submanifold M of a Kaehler manifold $\tilde{M}$, we put

$$h_{AB}^r = \langle h(e_A, e_B), J e_r \rangle,$$

where h is the second fundamental form of M . The two partial mean curvature vectors $\vec{H}_{\mathcal{D}}$ and $\vec{H}_{\mathcal{D}^\perp}$ of M are defined by

$$\vec{H}_{\mathcal{D}} = \frac{1}{2\ell} \sum_{i=1}^{2\ell} h(e_i, e_i), \quad \vec{H}_{\mathcal{D}^\perp} = \frac{1}{p} \sum_{r=2\ell+1}^{2\ell+p} h(e_r, e_r).$$

A CR-submanifold M of a Kaehler manifold is called $\mathcal{D}$ -minimal (resp., $\mathcal{D}^\perp$ -minimal) if $\vec{H}_{\mathcal{D}} = 0$ (resp., $\vec{H}_{\mathcal{D}^\perp} = 0$) holds identically.

11.1. CR-Warped Products in Kaehler Manifolds

For any CR-warped product in a Kaehler manifold, we have the next result from [77].

Theorem 40. If $M = N^T \times_f N^\perp$ is a CR-warped product in a Kaehler manifold $\tilde{M}$, then we have:

- (a) The squared norm of the second fundamental form h of M satisfies

$$\|h\|^2 \geq 2q \|\nabla(\ln f)\|^2, \quad q = \dim N^\perp, \quad (48)$$

where $\nabla(\ln f)$ is the gradient of $\ln f$;

- (b) If the equality case of (48) holds identically, then N^T is totally geodesic and $N^\perp$ is totally umbilical in $\tilde{M}$. Further, M is a minimal in $\tilde{M}$;
- (c) If M is anti-holomorphic and $q > 1$, then the equality case of (48) holds identically if and only if $N^\perp$ is totally umbilical in $\tilde{M}$;
- (d) If M is anti-holomorphic and $q = 1$, then the equality case of (48) holds identically if the characteristic vector field $J\xi$ of M is a principal vector field with principal curvature zero. Conversely, if the equality case of (48) holds, then $J\xi$ is a principal vector field with principal curvature zero only when $M = N^T \times_f N^\perp$ is a trivial CR-warped product in $\tilde{M}$ as a totally geodesic hypersurface. When M is anti-holomorphic and $q = 1$, the equality case of (48) holds if and only if M is a minimal hypersurface in $\tilde{M}$.

For the classification of CR-warped products in complex space forms see [77–79].

11.2. Further Inequalities for CR-Warped Products in Complex Space Forms

The first author also proved in [80] that CR-warped products of a complex space form also satisfy the following general optimal inequality.

Theorem 41. Let $M = N^T \times_f N^\perp$ be a CR-warped product in a complex space form $\tilde{M}^m(4c)$. Then we have:

$$\|h\|^2 \geq 2p \{ \|\nabla(\ln f)\|^2 + \Delta(\ln f) + 2\ell c \}. \quad (49)$$

If the equality case of (49) holds identically, then N^T is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold. Moreover, M is a minimal submanifold in $\tilde{M}^m(4c)$.

The complete classification of CR-submanifolds in complex space forms satisfying the equality case of (49) was obtained in [79,80]. In addition, he established the following results in [81] for CR-submanifolds with compact holomorphic factor.

Theorem 42. Let $N^T \times_f N^\perp$ be a CR-warped product in the complex projective m -space $\mathbb{CP}^m(4)$. If N^T is compact, then $m \geq \ell + p + \ell p$. Moreover, if $N^T \times_f N^\perp$ is a CR-warped product in $\mathbb{CP}^{\ell+p+\ell p}(4)$ with compact N^T , then N^T is holomorphically isometric to $\mathbb{CP}^\ell$.

Theorem 43. For any CR-warped product $N^T \times_f N^\perp$ in $\mathbb{CP}^m(4)$ with compact N^T and for any $x \in N^\perp$, we have

$$\int_{N^T \times \{x\}} \|h\|^2 dV_T \geq 4\ell p \operatorname{vol}(N^T), \quad (50)$$

where dV_T is the volume element of N^T , and $\operatorname{vol}(N^T)$ is the volume of N^T .

The equality case of (50) holds identically if and only if we have:

- (a) The warping function f is constant;
- (b) (N^T, g_T) is holomorphically isometric to $\mathbb{CP}^\ell(4)$ and it is isometrically immersed into $\mathbb{CP}^m$ as a totally geodesic holomorphic submanifold;
- (c) $(N^\perp, f^2 g_\perp)$ is isometric to an open portion of the real projective p -space $\mathbb{RP}^p(1)$ of constant sectional curvature 1 and it is isometrically immersed into $\mathbb{CP}^m$ as a totally geodesic totally real submanifold;
- (d) $N^T \times_f N^\perp$ is immersed linearly fully into a complex subspace $\mathbb{CP}^{\ell+p+\ell p}(4)$ of $\mathbb{CP}^m(4)$; and moreover, the immersion is rigid.

Theorem 44. Let $N^T \times_f N^\perp$ be a CR-warped product with compact N^T in $\mathbb{C}P^m(4)$. If the warping function f is a non-constant function, then for each $x \in N^\perp$ we have

$$\int_{N^T \times \{x\}} \|h\|^2 dV_T \geq 2p\lambda_1 \int_{N^T} (\ln f)^2 dV_T + 4\ell p \operatorname{vol}(N^T), \quad (51)$$

where λ_1 is the first positive eigenvalue of the Laplacian Δ on N^T . Moreover, the equality case of (51) holds identically if and only if

- (a) $\Delta(\ln f) = \lambda_1 \ln f$;
- (b) The CR-warped product is N^T -totally geodesic and $N^\perp$ -totally geodesic.

Remark 11. After these results, there are many research articles published by various authors for different types of warped product submanifolds in different ambient manifolds (see, e.g., [33,82,83] and recent survey articles published in the two volumes [84,85]).

11.3. CR-Warped Products and δ -Invariants

For a CR-submanifold M of a Kaehler manifold, the CR δ -invariant $\delta(\mathcal{D})$ is defined in [86] as:

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x), \quad x \in M, \quad (52)$$

where $\tau(x)$ and $\tau(\mathcal{D}_x)$ are the scalar curvature of M and of $\mathcal{D}_x \subset T_x M$ at x , respectively.

The next result provides an optimal inequality for CR-warped submanifolds in complex space forms involving the CR δ -invariant.

Theorem 45 ([86]). If $M = N^T \times_f N^\perp$ is a CR-warped product in a complex space form $\tilde{M}^{\ell+p}(4c)$ with $\ell = \dim_{\mathbb{C}} N^T \geq 1$ and $p = \dim N^\perp \geq 2$, then we have

$$H^2 \geq \frac{2(p+2)}{(2\ell+p)^2(p-1)} \left\{ \delta(\mathcal{D}) - \frac{p\Delta f}{f} - \frac{p(p-1)c}{2} \right\}. \quad (53)$$

The equality case of (53) holds at a point $x \in M$ if and only if there exists an orthonormal basis $\{e_{2\ell+1}, \dots, e_n\}$ of $\mathcal{D}_x^\perp$ such that the coefficients of the second fundamental form h with respect to $\{e_{2\ell+1}, \dots, e_n\}$ satisfy

$$\begin{aligned} h_{rr}^r &= 3h_{ss}^r, & \text{for } 2\ell+1 \leq r \neq s \leq 2\ell+p; \\ h_{st}^r &= 0, & \text{for distinct } r, s, t \in \{2\ell+1, \dots, 2\ell+p\}. \end{aligned} \quad (54)$$

Let $w : S^p(1) \rightarrow \mathbb{C}^p$ be the map of the unit p -sphere into $\mathbb{C}^p$ defined by

$$w(y_0, y_1, \dots, y_p) = \frac{1+iy_0}{1+y_0^2} (y_1, \dots, y_p), \quad y_0^2 + y_1^2 + \dots + y_p^2 = 1.$$

The map w is a (non-isometric) Lagrangian immersion with one self-intersection point which is called the *Whitney p -sphere*.

All CR-warped products in $\mathbb{C}^{\ell+p}$ satisfying the equality case of inequality (53) are classified in [86] as the following.

Theorem 46. If $\psi : N^T \times_f N^\perp \rightarrow \mathbb{C}^{\ell+p}$ is a CR-warped product in $\mathbb{C}^{\ell+p}$ with $\ell = \dim_{\mathbb{C}} N^T \geq 1$ and $p = \dim N^\perp \geq 2$, then we have

$$H^2 \geq \frac{2(p+2)}{(2\ell+p)^2(p-1)} \left\{ \delta(\mathcal{D}) - \frac{p\Delta f}{f} \right\}. \quad (55)$$

The equality case of (55) holds identically if and only if, up to dilations and rigid motions of $\mathbb{C}^{\ell+p}$, one of the following three cases occur:

- (a) The CR-warped product is an open portion of the CR-product $\mathbb{C}^\ell \times W^p \subset \mathbb{C}^\ell \times \mathbb{C}^p$, where W^p is the Whitney p -sphere in $\mathbb{C}^p$;
- (b) N^T is an open portion of $\mathbb{C}^\ell$, $N^\perp$ is an open portion of the unit p -sphere S^p , $f = |z_1|$ and ψ is the minimal immersion defined by

$$(z_1 w_0, \dots, z_1 w_p, z_2, \dots, z_\ell),$$

where $z = (z_1, \dots, z_\ell) \in \mathbb{C}^\ell$ and $w = (w_0, \dots, w_p) \in S^p \subset \mathbb{E}^{p+1}$;

- (c) N^T is an open portion of $\mathbb{C}^\ell$, $N^\perp$ is the warped product of a curve and an open part of S^{p-1} with the warping function $\varphi = (\frac{\sqrt{c^2-1}}{\sqrt{2}})\text{cn}(ct, \frac{\sqrt{c^2-1}}{\sqrt{2c}})$, $c > 1$, $f = |z_1|$, and ψ is the non-minimal immersion given by

$$\left(z_1 e^{\int \frac{\varphi(\varphi' + ik\varphi^2)}{\varphi^2 - 1} dt}, z_1 \varphi e^{ik \int \varphi dt} w_1, \dots, z_1 \varphi e^{ik \int \varphi dt} w_p, z_2, \dots, z_\ell \right),$$

where $z = (z_1, \dots, z_\ell) \in \mathbb{C}^\ell$, $(w_1, \dots, w_p) \in S^{p-1}(1) \subset \mathbb{E}^p$, $k = \sqrt{c^4 - 1}/2$, and cn is a Jacobi's elliptic function.

12. Anti-Holomorphic Submanifolds and δ -Invariants

For anti-holomorphic submanifolds with $p = \text{rank } \mathcal{D}^\perp \geq 2$ in complex space forms, we have the following optimal inequality.

Theorem 47 ([87]). *If M is an anti-holomorphic submanifold of a complex space form $\tilde{M}^{\ell+p}(4c)$ with $\ell = \text{rank}_{\mathbb{C}} \mathcal{D} \geq 1$ and $p = \text{rank } \mathcal{D}^\perp \geq 2$, then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2\ell+p)^2}{2(p+2)} H^2 + \frac{p}{2}(4\ell+p-1)c. \quad (56)$$

The equality case of (56) holds identically if and only if the following three conditions are satisfied:

- (a) M is $\mathcal{D}$ -minimal;
- (b) M is mixed totally geodesic, that is, $h(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$;
- (c) There exists an orthonormal frame $\{e_{2\ell+1}, \dots, e_n\}$ of $\mathcal{D}^\perp$ such that the second fundamental form h of M satisfies:

$$\begin{cases} h_{rr}^r = 3h_{ss}^r, & \text{for } 2\ell+1 \leq r \neq s \leq 2\ell+p; \\ h_{st}^r = 0, & \text{for distinct } r, s, t \in \{2\ell+1, \dots, 2\ell+p\}. \end{cases}$$

Anti-holomorphic submanifolds satisfying the equality case of inequality (56) are classified in [87] as follows.

Theorem 48. *Let M be an anti-holomorphic submanifold of $\tilde{M}^{\ell+p}(4c)$ with $\ell = \text{rank}_{\mathbb{C}} \mathcal{D} \geq 1$ and $p = \text{rank } \mathcal{D}^\perp \geq 2$. If M satisfies the equality case of (56) and if $\mathcal{D}$ is integrable, then $c = 0$. Moreover, in this case we have either:*

- (a) M is a totally geodesic anti-holomorphic submanifold of $\mathbb{C}^{\ell+p}$ or,
- (b) up to dilations and rigid motions of $\mathbb{C}^{\ell+p}$, M is given by an open portion of the product immersion:

$$\varphi : \mathbb{C}^\ell \times S^p(1) \rightarrow \mathbb{C}^{\ell+p}; \quad (z, x) \mapsto (z, w(x)), \quad z \in \mathbb{C}^\ell, \quad x \in S^p(1),$$

where $w : S^p(1) \rightarrow \mathbb{C}^p$ is the Whitney p -sphere.

Theorem 49. Let M be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$ with $\text{rank}_{\mathbb{C}} \mathcal{D} = 1$ and $p = \text{rank } \mathcal{D}^{\perp} \geq 2$. Then we have

$$\delta(\mathcal{D}) \leq \frac{(p-1)(p+2)^2}{2(p+2)} H^2 + \frac{p}{2}(p+3)c. \quad (57)$$

The equality case of (57) holds identically if and only if $c = 0$ and either:

- (a) M is a totally geodesic anti-holomorphic submanifold of $\mathbb{C}^{1+p}$; or
- (b) Up to dilations and rigid motions, M is given by an open portion of the product immersion:

$$\phi : \mathbb{C} \times S^p(1) \rightarrow \mathbb{C}^{1+p}; \quad (z, x) \mapsto (z, w(x)), \quad z \in \mathbb{C}, \quad x \in S^p(1),$$

where $w : S^p(1) \rightarrow \mathbb{C}^p$ is the Whitney p -sphere.

13. Complex Submanifolds of Complex Space Forms and δ -Invariants

Assume that M is a Kaehler manifold of complex dimension $n \geq 2$. A 2-plane section $\pi \subset T_x M$, $x \in M$, is called *totally real* if $J\pi$ is orthogonal to π . Let us denote by $K(\pi^r)$ the sectional curvature of a totally real plane section π^r . We put

$$(\inf K^r)(x) = \inf_{\pi^r \subset T_x M} K(\pi^r),$$

where $K(\pi^r)$ runs over all totally real 2-plane sections at p .

From [88], we have:

Proposition 6. If M is a complex submanifold of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^m(4c)$, then $\inf K^r \leq c$, with the equality holding identically if and only if M is a totally geodesic complex submanifold.

For a real number k , the first author defined in [88] the notion of *totally real δ -invariant* δ_k^r by

$$\delta_k^r(x) = \tau(x) - k \inf K^r(x), \quad x \in M, \quad (58)$$

where $\inf K^r(p) = \inf_{\pi^r} \{K(\pi^r)\}$ and π^r runs over all totally real 2-planes in $T_x M$.

13.1. Inequality for Complex Submanifolds in Complex Space Forms

For complex submanifolds of complex space forms, we have the following result from [88].

Theorem 50. Let M be a complex submanifold of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^m(4c)$. Then we have

- (a) For each real number $k \leq 4$, δ_k^r satisfies:

$$\delta_k^r \leq (2n^2 + 2n - k)c; \quad (59)$$

- (b) Inequality (59) fails for every $k > 4$;
- (c) $\delta_k^r = (2n^2 + 2n - k)c$ holds identically for some $k < 4$ if and only if M is totally geodesic;

- (d) $\delta_4^r = (2n^2 + 2n - 4)c$ holds at a point $x \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_n, e_1^* = Je_1, \dots, e_n^* = Je_n, \xi_1, \dots, \xi_{m-n}, \xi_1^* = J\xi_1, \dots, \xi_{m-n}^* = J\xi_{m-n}$ of $T_p\tilde{M}$ such that the shape operator of M satisfies

$$\begin{aligned} A_\alpha &= \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix}, \quad A_\alpha = A_{\xi_\alpha}, \quad A_{\alpha^*} = A_{\xi_\alpha^*}, \\ A'_\alpha &= \begin{pmatrix} a_\alpha & b_\alpha & 0 \\ b_\alpha & -a_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} a_\alpha^* & b_\alpha^* & 0 \\ b_\alpha^* & -a_\alpha^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (60)$$

for some $n \times n$ matrices A'_α, A''_α , with respect to this basis.

Definition 4. A complex submanifold M of a Kaehler manifold $\tilde{M}^m$ is called strongly minimal if A'_α and A''_α in (60) satisfy $\text{trace } A'_\alpha = \text{trace } A''_\alpha = 0$ for $\alpha = 1, \dots, m - n$.

In [88] we also have the following two results.

Theorem 51. A complete complex submanifold M of complex dimension $n \geq 2$ in $\mathbb{CP}^m(4)$ satisfies $\delta_4^r = 2(n^2 + n - 2)$ identically if and only if either

- (a) M is a totally geodesic Kaehler submanifold, or
- (b) $n = 2$ and M is a strongly minimal Kaehler surface in $\mathbb{CP}^m(4)$.

Theorem 52. A complete complex submanifold M of complex dimension $n \geq 2$ of $\mathbb{C}^m$ satisfies $\delta_4^r = 0$ identically if and only if either

- (a) M is a complex n -plane of $\mathbb{C}^m$, or
- (b) M is a complex cylinder over a strongly minimal Kaehler surface.

For a real number k and an integer $\ell \in [2, \frac{n}{2}]$, B. D. Suceavă [89] defined the following

$$\delta_{\ell,k}^r(x) = \tau(x) - \frac{k}{\ell - 1} \inf_{\pi_\ell^r \subset T_x M} [\tau(\pi_\ell^r(x))], \quad (61)$$

where π_ℓ^r runs over all totally real ℓ -subspaces in $T_x\tilde{M}$. Clearly, we have $\delta_{2,k}^r = \delta_k^r$.

In [89,90], Suceavă proved the following.

Theorem 53. Let M be a complex submanifold of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+p}(4c)$. Then we have

- (a) For any $2 \leq \ell \leq \frac{n}{2}$, the following inequality holds

$$\delta_{4,\ell}^r \leq \left[2n^2 + 2n - \binom{\ell}{2} \right] c. \quad (62)$$

The equality case for $\ell = 2$ has been described in Theorem 50. Equality holds at every point for a fixed $\ell \geq 3$ if and only if M is a totally geodesic submanifold.

- (b) For any $k \in [0, 4]$, the following inequality holds

$$\delta_{k,\ell}^r \leq \left[2n^2 + 2n - \frac{k}{4} \binom{\ell}{2} \right] c. \quad (63)$$

Equality holds at every point for a fixed $\ell \geq 3$ if and only if M is a totally geodesic submanifold.

13.2. Examples of Strongly Minimal Surfaces

Now, we provide some examples of strongly minimal surfaces.

Example 3. Every totally geodesic complex submanifold of a complex space form is strongly minimal.

Example 4. Consider the complex quadric in $\mathbb{C}P^{n+1}(4)$ defined by

$$Q_n = \{(z_0, z_1, \dots, z_{n+1}) \in \mathbb{C}P^{n+1}(4) : z_0^2 + z_1^2 + \dots + z_{n+1}^2 = 0\},$$

where $\{z_0, z_1, \dots, z_{n+1}\}$ is a homogeneous coordinate system of $\mathbb{C}P^{n+1}(4)$.

For Q_2 we have $\tau = 8$, $\inf K^r = 0$ and $\delta_4^r = 8$. Thus Q_2 is a non-totally geodesic complex surface satisfying the equality case of (59). Thus Theorem 52 implies that Q_2 is a strongly minimal Kaehler surface in $\mathbb{C}P^3(4)$.

Example 5. The Kaehler surface defined by $\{z \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}$ is a strongly minimal surface in $\mathbb{C}^3$.

The next example is due to B. D. Suceavă [89].

Example 6. For any $k \in \mathbb{C}$, the Kaehler surface defined by $\{z \in \mathbb{C}^3 : z_1 + z_2 + z_3^2 = k\}$ is a strongly minimal Kaehler surface in $\mathbb{C}^3$.

14. Submanifolds of Space Forms and δ -Casorati Curvatures

The notion of Casorati curvature for surfaces in Euclidean 3-space was introduced and preferred by F. Casorati in [91] over the traditional Gauss curvature because it corresponds better with the common intuition of curvature. In fact, the Casorati curvature of a surface in Euclidean 3-space is zero at all points only for planes, which indeed are not curved at all in the sense of our common intuition. It was pointing out in [92] that Casorati curvature is useful in computer vision and image processing, where this curvature occurs as bending energy.

Since the 1990s, the study of δ -invariants introduced in [8–10] has attracted a lot of attention and many Chen-like inequalities were obtained by many authors for various classes of submanifolds in different ambient spaces (see e.g., [7]). Some other invariants of similar nature were also defined by several authors and also called δ -invariants.

14.1. Submanifolds Involving δ -Casorati Curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$

For an n -dimensional submanifold M of a Riemannian m -manifold $\tilde{M}$, the squared norm $\|h\|^2$ of the second fundamental form h over n is denoted by $\mathcal{C}$ and is called the *Casorati curvature* of the submanifold, that is,

$$\mathcal{C} = \frac{1}{n} \|h\|^2. \quad (64)$$

Similarly, for an ℓ -dimensional linear subspace $L \subset T_x M$ spanned by an orthonormal basis $\{e_1, \dots, e_\ell\}$, the *Casorati curvature* of L is defined by:

$$\mathcal{C}(L) = \frac{1}{\ell} \sum_{r=n+1}^m \left(\sum_{i,j=1}^{\ell} (h_{ij}^r)^2 \right). \quad (65)$$

In [93], S. Decu, S. Haesen and L. Verstraelen introduced the normalized δ -Casorati curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ as follows:

$$\begin{aligned} [\delta_C(n-1)]_x &= \frac{1}{2} \mathcal{C}_x + \frac{n+1}{2n} \inf\{\mathcal{C}(L) | L \text{ a hyperplane of } T_x M\}, \\ [\hat{\delta}_C(n-1)]_x &= 2\mathcal{C}_x - \frac{2n-1}{2n} \sup\{\mathcal{C}(L) | L \text{ a hyperplane of } T_x M\}. \end{aligned} \quad (66)$$

Note that in contrast to the first author's δ -invariants which are intrinsic, the δ -Casorati curvatures are extrinsic.

S. Decu, S. Haesen and L. Verstraelen proved the following result for normalized δ -Casorati curvatures.

Theorem 54 ([93]). *Let M be an n -dimensional submanifold of a real space form $R^m(c)$. If ρ denotes the normalized scalar curvature of M , then we have*

(a) *The normalized δ -Casorati curvature $\delta_C(n-1)$ satisfies*

$$\delta_C(n-1) \geq \rho - c; \quad (67)$$

(b) *The normalized δ -Casorati curvature $\hat{\delta}_C(n-1)$ satisfies*

$$\hat{\delta}_C(n-1) \geq \rho - c. \quad (68)$$

An alternate proof of Theorem 54 was given in [94] by P. Zhang and L. Zhang using T. Oprea's optimization method on Riemannian submanifolds (see [55]). Further, L. Zhang, X. Pan, P. Zhang [95] improved the inequality (68) in the case of Lagrangian submanifolds of complex space forms as the following.

Theorem 55. *Let M be an n -dimensional Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then we have:*

$$\hat{\delta}_C(n-1) \geq \rho - c + \frac{2n(2n-3)}{(n-1)(2n+3)} H^2. \quad (69)$$

Moreover, the equality case of (69) holds identically if and only if M is a totally geodesic Lagrangian submanifold.

Recently, the last author improved the inequality (69) to:

Theorem 56 ([96]). *Let M be an n -dimensional Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then we have:*

$$\delta_C(n-1) \geq \rho - c + \frac{n}{n+3} H^2. \quad (70)$$

Remark 12. *The last author also determined in [96] all Casorati $\delta_C(n-1)$ -ideal Lagrangian submanifolds of a complex space form $\tilde{M}^n(4c)$, that is, all the Lagrangian submanifolds of a complex space form $\tilde{M}^n(4c)$, which satisfy the equality case of (70) identically.*

Theorem 56 implies immediately the following obstruction to the existence of minimal Lagrangian isometric immersions of a Riemannian n -manifold into $\mathbb{C}^n$.

Corollary 8. *Let M be a Riemannian n -manifold. If there exists a point $x \in M$ such that*

$$\rho(x) > [\delta_C(n-1)]_x$$

holds, then M does not admit any minimal Lagrangian isometric immersion into $\mathbb{C}^n$.

14.2. Casorati $\delta_C(n-1)$ -Ideal Lagrangian Submanifold of a Complex Space Form

The last author determined in [96] all Casorati $\delta_C(n-1)$ -ideal Lagrangian submanifolds of a complex space form $\tilde{M}^n(4c)$. In particular, he proved the following results.

Theorem 57 ([96]). *If M is a Casorati $\delta_C(n-1)$ -ideal Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$, then it is one of the following submanifolds:*

- (a) A totally geodesic Lagrangian submanifold;
- (b) An H-umbilical Lagrangian submanifold with ratio 4.

Corollary 9 ([96]). Let M be a Casorati $\delta_C(n-1)$ -ideal Lagrangian submanifold without totally geodesic points in a complex space form $\tilde{M}^n(4c)$. Then M is an H-umbilical submanifold of ratio 4.

The last author also provides in [96] the following explicit example of H-umbilical Lagrangian submanifold of ratio 4 in $\mathbb{C}H^n(-4)$ which satisfies the equality case of (70) identically.

Example 7. Consider the following map:

$$\psi(t, u_1, \dots, u_{n-1}) = \frac{e^{i \arctan(\tanh t)}}{\cosh 2t} \left(\frac{1}{2} - it + \frac{1}{2} \sum_{j=1}^{n-1} u_j^2 + \frac{\cosh 2t}{2}, \right. \\ \left. t - \frac{i}{2} + \frac{i}{2} \sum_{j=1}^{n-1} u_j^2 + \frac{i \cosh 2t}{2}, u_1, \dots, u_{n-1} \right).$$

As $\langle \psi, \psi \rangle = -1$, it follows that the composition $\pi \circ \psi$, where $\pi : H_1^{2n+1}(-1) \rightarrow \mathbb{C}H^n(-4)$ is the Hopf fibration, gives rise to an H-umbilical Lagrangian submanifold of ratio 4 in $\mathbb{C}H^n(-4)$ attaining the equality case of (70) identically.

14.3. Submanifolds Involving δ -Casorati Curvatures $\delta_C(r, n-1)$ and $\hat{\delta}_C(r, -1)$

For a given positive number $r \neq n(n-1)$, we put

$$a(r) = \frac{(n-1)(r+n)(n^2-n-r)}{nr}. \quad (71)$$

If M is an n -dimensional submanifold of a Riemannian manifold, the generalized δ -Casorati curvatures $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$ are defined by S. Decu, S. Haesen and L. Verstraelen in [97] as follows:

$$[\delta_C(r; n-1)]_x = r\mathcal{C}_x + a(r) \inf\{\mathcal{C}(L) | L \text{ a hyperplane of } T_x M\} \text{ if } 0 < r < n(n-1), \\ [\hat{\delta}_C(r; n-1)]_x = r\mathcal{C}_x + a(r) \sup\{\mathcal{C}(L) | L \text{ a hyperplane of } T_x M\} \text{ if } r > n(n-1).$$

The generalized δ -Casorati curvatures $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$ are generalizations of $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ respectively, since we have (see [98])

$$\delta_C(n(n-1)/2; n-1) = n(n-1)\delta_C(n-1), \\ \hat{\delta}_C(2n(n-1); n-1) = n(n-1)\hat{\delta}_C(n-1). \quad (72)$$

S. Decu, S. Haesen and L. Verstraelen proved the following results in [97] for the generalized δ -Casorati curvatures.

Theorem 58. If M is an n -dimensional submanifold of a real space form $R^m(c)$, then for a real number r such that $0 < r < n(n-1)$, we have

$$2\tau \leq \delta_C(r; n-1) + n(n-1)c; \quad (73)$$

and for a real number $r > n(n-1)$ we have

$$2\tau \leq \hat{\delta}_C(r; n-1) + n(n-1)c. \quad (74)$$

In [99], M. Aquib, J. W. Lee, G.-E. Vilcu and D. W. Yoon improved Theorem 56 for Lagrangian submanifolds to the following.

Theorem 59. Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then we have

(a) For $r \in (0, n-1)$, the generalized normalized δ -Casorati curvature $\delta_C(r; n-1)$ satisfies

$$\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} + c - \frac{2r^2}{(n-1)(n^2 + n(r-1) + r)} H^2. \quad (75)$$

(b) For $r > n(n-1)$, the generalized normalized δ -Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies

$$\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + c - \frac{2n(n^2 + n(r-1) - 2r)}{(n-1)(n^2 + n(r-1) + r)} H^2. \quad (76)$$

Moreover, the equality case of (76) holds identically if and only if M is a totally geodesic Lagrangian submanifold.

For Lagrangian $\delta_C(r; n-1)$ -Casorati ideal submanifolds, M. Aquib, J. W. Lee, G.-E. Vilcu and D. W. Yoon obtained the following result.

Theorem 60 ([99]). If M is an n -dimensional Lagrangian Casorati $\delta_C(n-1)$ -ideal submanifold satisfying the equality case of (75), then it is one of the following submanifolds:

- (a) A totally geodesic Lagrangian submanifold;
- (b) An H -umbilical Lagrangian submanifold with ratio $(n^2 - n + 2r)/r$.

The authors of [99] also provide an example to show that the improved inequalities (75) and (76) are the best possible. Thus, the constants in (75) and (76) can not be improved.

Remark 13. For further results on δ -Casorati curvatures, we refer to the recent survey [100] on δ -Casorati curvature invariants and to those articles cited in [100]. We also note that the counterpart of Theorems 55, 56 and 59 in Sasakian geometry can be found in the recent articles [101,102], while the extension of the above theorems in quaternion setting is still an open problem.

15. Riemannian Maps and δ -Invariants

In 1991, A. E. Fischer defined the notion of Riemannian map as a very natural generalization of the concepts of isometric immersions and Riemannian submersions [103]. Geometrically speaking, a Riemannian map is a map whose derivative is nothing but a linear isometry between the domain tangent space modulo the kernel and its range. Explicitly, let us suppose that (M^m, g_M) and (N^n, g_N) are Riemannian manifolds of dimension m and n , respectively, and let $F : (M^m, g_M) \rightarrow (N^n, g_N)$ be a smooth map such that $0 \leq \text{rank } F = r \leq \min\{m, n\}$. Then we have the following orthogonal complement decomposition of the tangent bundle TM of M :

$$TM = \ker F_* \oplus (\ker F_*)^\perp = \ker F_* \oplus \mathcal{H},$$

where $\mathcal{H} = (\ker F_*)^\perp$. Moreover, if $y = F(x) \in N$, $x \in M$, then the tangent space $T_y N$ has the next orthogonal complement decomposition:

$$T_y N = (\text{Im } F_{*x}) \oplus (\text{Im } F_{*x})^\perp.$$

Then $F : (M^m, g_M) \rightarrow (N^n, g_N)$ with $0 \leq \text{rank } F = r \leq \min\{m, n\}$ is called a *Riemannian map at the point* $x \in M$ if the horizontal restriction $F_{*x}^h : (\ker F_{*x})^\perp \rightarrow \text{Im } F_{*x}$ is a linear isometry. In addition, if F is a Riemannian map at each point $x \in M$, then F is called a *Riemannian map*. In the particular case when $\ker F_* = \{0\}$, the Riemannian map F reduces to an isometric immersion, while when $(\text{Im } F_*)^\perp = \{0\}$, the Riemannian map F reduces to a Riemannian submersion.

Note that one of the most remarkable properties of Riemannian maps is that such maps satisfy the generalized eikonal equation and thus have constant rank on each connected

component [103]. As a consequence, these maps lead to a bridge between physics and geometrical optics [104].

15.1. Chen First Inequality for Riemannian Maps

Assume that (M, g_M) and (N, g_N) are two Riemannian manifolds and let $F : M \rightarrow N$ be a smooth map. Then it is known that the differential F_* of F can be seen as a section of the bundle $\text{Hom}(TM, F^{-1}TN)$ on M , where $F^{-1}TN$ is the pullback bundle, with fibre $(F^{-1}TN)_x = T_{F(x)}N$, $x \in M$. It is known that $\text{Hom}(TM, F^{-1}TN)$ has an induced connection ∇ from the Levi-Civita connection ∇^M on M . Recall now that the second fundamental form of the map F is given by:

$$(\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X^M Y),$$

for $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the space consisting of all smooth sections in TM . It is known that ∇F_* is symmetric. Moreover, if F is a Riemannian map, then $(\nabla F_*)(X, Y)$ has no components in $\text{Im } F_*$, provided that $X, Y \in \Gamma((\ker F_*)^\perp)$ (see [104]).

Recall now that if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then the map F is called totally geodesic. On the other hand, F is said to be harmonic if $\text{trace } F_* = 0$.

In the following, we will denote by ∇^N both the Levi-Civita connection on (N, g_N) and its pullback along F . Suppose $\nabla_X^{F\perp} V$ is the orthogonal projection of $\nabla_X^N V$ onto $(\text{Im } F_*)^\perp$, $X \in \Gamma(TM)$, $V \in \Gamma((\text{Im } F_*)^\perp)$. Due to the fact that $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_N = 0$, one can consider the operator S_V on $F_*(TM)$ defined by

$$\nabla_X^N V = -S_V F_* X + \nabla_X^{F\perp} V,$$

where $-S_V F_* X$ denotes the tangential component of $\nabla_X^N V$. We would like to point out that S_V is a symmetric linear transformation of $\text{Im } F_*$, called the *shape operator* of the Riemannian map F [104].

Recall that the Riemannian map F is said to be *umbilical* at $x \in M$ if

$$S_V F_{*x}(X_x) = \lambda F_{*x}(X_x),$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{Im } F_*)^\perp)$, where λ is a smooth function on M . Moreover, F is called an *umbilical Riemannian map* if F is umbilical at each point $x \in M$.

In [105], Şahin obtained the following basic Chen inequality for a Riemannian map with target space a real space form.

Theorem 61 ([105]). *Let (M, g_M) be a Riemannian manifold and $(R^n(c), g_{R^n(c)})$ be a real space form of constant curvature c . If $F : M \rightarrow R^n(c)$ is a Riemannian map with $\text{rank } F \geq 3$, then the following inequality holds true for each $x \in M$ and for each plane section π of $T_x M$:*

$$K(\pi) \geq \rho_{\mathcal{H}} - \frac{\text{rank } F - 2}{2} \left[(\text{rank } F + 1)c + \frac{1}{\text{rank } F - 1} \|\tau^{\mathcal{H}}\|^2 \right], \quad (77)$$

where $\rho_{\mathcal{H}}$ denotes the scalar curvature on $\mathcal{H}$ and $\tau^{\mathcal{H}}$ is given by

$$\tau^{\mathcal{H}} = \sum_{i=1}^{\text{rank } F} g_{R(c)}((\nabla F_*)(e_i, e_i), (\nabla F_*)(e_i, e_i)),$$

for an orthonormal basis $\{e_i\}_{i=1, \dots, \text{rank } F}$ of $\mathcal{H}$.

Moreover, the equality case of (77) holds identically in (77) if there exist an orthonormal basis $\{e_i\}_{i=1,\dots,\text{rank } F}$ of $\mathcal{H}$ and an orthonormal basis $\{V_j\}_{j=\text{rank } F+1,\dots,\text{rank } F+d}$ of $(\text{Im } F_{*x})^\perp$, where $d = \dim(\text{Im } F_{*x})$, such that the shape operator takes the form

$$A_{\text{rank } F+1} = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ 0 & 0 & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C \end{pmatrix}, \quad A + B = C$$

and

$$A_\alpha = \begin{pmatrix} D_{11}^\alpha & D_{12}^\alpha & 0 & \cdots & 0 \\ D_{12}^\alpha & -D_{11}^\alpha & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \alpha = \text{rank } F + 1, \dots, \text{rank } F + d.$$

Obviously, the inequality stated in Theorem 61 is a generalization of Chen first inequality for both isometric immersions [10] and Riemannian submersions [37,106]. Moreover, as an immediate consequence of Theorem 61, we derive the next result.

Corollary 10 ([105]). Let (M, g_M) be a Riemannian manifold and suppose that $F : M \rightarrow \mathbb{E}^n$ is a Riemannian map from (M, g_M) onto the Euclidean n -space $\mathbb{E}^n$, with $\text{rank } F \geq 3$. If F is harmonic, then the following inequality holds true for each $x \in M$ and for each plane section π of $T_x M$:

$$K(\pi) \geq \rho_{\mathcal{H}}.$$

Example 8. As shown in [105], if a Riemannian map $F : M \rightarrow \mathbb{R}^n(c)$ as in the hypotheses of Theorem 61 is umbilical, then (77) is satisfied with equality. For explicit examples of such maps, see ([104] Chapter 8).

15.2. Chen–Ricci Inequality for Riemannian Maps

Suppose that (M, g_M) and (N, g_N) are Riemannian manifolds and let $F : M \rightarrow N$ be a Riemannian map. Then we denote by $\text{Ric}^{\mathcal{H}}$ the Ricci curvature on $\mathcal{H}$ and if $\{e_i\}_i$ is an orthonormal basis of $\mathcal{H}$ and $\{V_\alpha\}_\alpha$ is an orthonormal basis of $(\text{Im } F_{*x})^\perp$, then we use the following notation

$$\zeta_{ij}^{\mathcal{H}} = g_N((\nabla F_*)(e_i, e_j), V_\alpha).$$

Moreover, we have

$$\|\zeta^{\mathcal{H}}\|^2 = \sum_{i,j=1}^{\text{rank } F} g_N((\nabla F_*)(e_i, e_j), (\nabla F_*)(e_i, e_j)),$$

$$\text{trace } \zeta^{\mathcal{H}} = \sum_{i=1}^{\text{rank } F} (\nabla F_*)(e_i, e_i)$$

and

$$\|\text{trace } \zeta^{\mathcal{H}}\|^2 = g_N(\text{trace } \zeta^{\mathcal{H}}, \text{trace } \zeta^{\mathcal{H}}).$$

Then we have the following Chen–Ricci inequality for Riemannian maps having as target space a real space form.

Theorem 62 ([107]). Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) onto a real space form $(R^n(c), g_{R^n(c)})$ with $\text{rank } F < n$ and let X be a unit horizontal vector field on M . Then we have:

$$\text{Ric}^{\mathcal{H}}(X) \leq (r-1)c + \frac{1}{4} \|\text{trace } \zeta^{\mathcal{H}}\|^2. \quad (78)$$

Moreover:

(a) The equality case of (11) holds for $X \in \mathcal{H}_x$ if and only if:

$$\begin{cases} (\nabla F_*)(X, Y) = 0 & \text{for all } Y \in \mathcal{H}_x \text{ orthogonal to } X, \\ (\nabla F_*)(X, X) = \frac{1}{2} \text{trace } \zeta^{\mathcal{H}}, \end{cases} \quad (79)$$

(b) The equality case of (11) holds identically for all $X \in \mathcal{H}_x$ if and only if either $\zeta^{\mathcal{H}} = 0$ or $\text{rank } F = 2$ and $\zeta^{\mathcal{H}\alpha}_{11} = \zeta^{\mathcal{H}\alpha}_{22}$, for all α .

Example 9. Suppose that $\pi : S^7 \rightarrow S^4$ is the Hopf fibration and let $f : S^4 \rightarrow \mathbb{R}^5$ be the canonical embedding of the 4-sphere S^4 as a hypersurface of $\mathbb{R}^5$. Then the composition $F = f \circ \pi : S^7 \rightarrow \mathbb{R}^5$ is a totally umbilical Riemannian map which satisfies (78).

Example 10. Any totally geodesic Riemannian map with target space a real space form satisfies the equality of (11) at all points. For more results and examples of such remarkable Riemannian maps see [104,108].

We remark now that Theorem 62 leads to the next estimate for the squared norm of $\text{trace } \zeta^{\mathcal{H}}$ when the target space of a Riemannian map is the Euclidean n -space $\mathbb{E}^n$.

Corollary 11 ([107]). Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) into an Euclidean space $\mathbb{E}^n$ such that $\text{rank } F < n$. Then

$$\|\text{trace } \zeta^{\mathcal{H}}\|^2 \geq 4 \max_X \text{Ric}^{\mathcal{H}}(X), \quad (80)$$

where X runs over all unit horizontal vectors at $x \in M$.

Recall now that a Riemannian map $F : (M^m, g_M) \rightarrow (N^n, J_N, g_N)$ from a Riemannian manifold (M^m, g_M) onto an almost Hermitian manifold (N^n, J_N, g_N) is said to be an *anti-invariant Riemannian map* at $x \in M$ if $J_N(\text{Im } F_{*x}) \subset (\text{Im } F_{*x})^\perp$. Moreover, if the Riemannian map F is anti-invariant for every $x \in M$, then F is said to be an *anti-invariant Riemannian map*. Non-trivial examples of such maps can be found in [105]. With similar arguments as in the proof of Theorem 62, one can establish a Chen–Ricci inequality for anti-invariant Riemannian maps with target manifold a complex space form, as follows.

Theorem 63 ([107]). Let F be an anti-invariant Riemannian map from a Riemannian manifold (M^m, g_M) onto a complex space form $(N^n(4c), g_N)$ and let X be a unit horizontal vector on M . Then,

$$\text{Ric}^{\mathcal{H}}(X) \leq (\text{rank } F - 1)c + \frac{1}{4} \|\text{trace } \zeta^{\mathcal{H}}\|^2. \quad (81)$$

Moreover:

(a) The equality case of (81) holds for $X \in \mathcal{H}_x$ if and only if:

$$\begin{cases} (\nabla F_*)(X, Y) = 0 & \text{for all } Y \in \mathcal{H}_x \text{ orthogonal to } X, \\ (\nabla F_*)(X, X) = \frac{1}{2} \text{trace } \zeta^{\mathcal{H}}; \end{cases} \quad (82)$$

- (b) The equality case of (81) holds identically for all $X \in \mathcal{H}_x$ if and only if either $\zeta^{\mathcal{H}} = 0$ or $\text{rank } F = 2$ and $\zeta^{\mathcal{H}^\alpha}_{11} = \zeta^{\mathcal{H}^\alpha}_{22}$, for all α .

Remark 14. If the map F in Theorem 63 is Lagrangian, that is, $\text{rank } F = \dim(\text{Im } F_*)^\perp$, then inequality (81) is not optimal. In this case, an improved Chen–Ricci inequality can be stated in the following form.

Theorem 64 ([107]). Let $F : (M^m, g_M) \rightarrow (N^n(4c), g_N)$ be a Lagrangian Riemannian map from a Riemannian manifold (M^m, g_M) to a complex space form $(N^n(4c), J_N, g_N)$ of complex dimension $n > 1$. Then, for any unit horizontal vector X at $p \in M$, we have:

$$\text{Ric}^{\mathcal{H}}(X) \leq (n-1)c + \frac{n-1}{4n} \|\text{trace } \zeta^{\mathcal{H}}\|^2. \quad (83)$$

Moreover, the equality holds in the above inequality for any unit horizontal vector at p if and only if either

- (a) $\zeta^{\mathcal{H}} = 0$ or
 (b) $n = 2$ and

$$(\nabla F_*)(e_1, e_1) = 3\mu J e_1, \quad (\nabla F_*)(e_2, e_2) = \mu J e_1, \quad (\nabla F_*)(e_1, e_2) = \mu J e_2$$

for some function μ with respect to some orthonormal local frame field $\{e_1, e_2\}$.

Note that if the map F in Theorem 64 is an isometric Lagrangian immersion, then one recovers the main inequality stated in [56].

15.3. Casorati Inequalities for Riemannian Maps

Some optimal inequalities involving Casorati curvatures for Riemannian maps to real and complex space forms were recently stated in [109], the authors recovering in some particular cases several basic inequalities in the field. Therefore, in case of Riemannian maps having the target manifold a real space form, we have the next result.

Theorem 65 ([109]). Suppose that F is a Riemannian map defined on the Riemannian manifold (M^m, g_M) of dimension m and with target manifold a real space form $(R^n(c), g_{R^n(c)})$ of dimension n , such that $3 \leq r = \text{rank } F < \min\{m, n\}$. Then the following inequalities are valid:

$$\rho^{\mathcal{H}} \leq \delta_C^{\mathcal{H}}(r-1) + c, \quad \rho^{\mathcal{H}} \leq \widehat{\delta}_C^{\mathcal{H}}(r-1) + c, \quad (84)$$

where $\delta_C^{\mathcal{H}}(r-1)$ and $\widehat{\delta}_C^{\mathcal{H}}(r-1)$ denote the normalized Casorati curvatures on $\mathcal{H}$.

Moreover, the equality is attained in (84) at $x \in M$ if and only if with respect to some orthonormal bases $\{e_i\}_i$ on $\mathcal{H}_x = \ker F_{*x}^\perp$ and $\{V_\alpha\}_\alpha$ on $(\text{Im } F_{*x})^\perp$, the components of ζ satisfy

$$\zeta^{\mathcal{H}^\alpha}_{11} = \zeta^{\mathcal{H}^\alpha}_{22} = \cdots = \zeta^{\mathcal{H}^\alpha}_{r-1, r-1} = \frac{1}{2} \zeta^{\mathcal{H}^\alpha}_{rr}, \quad \text{for all } \alpha, \\ \zeta^{\mathcal{H}^\alpha}_{ij} = 0, \quad \text{for all } \alpha \text{ and } i \neq j.$$

Similarly, when the target manifold of a Riemannian map is a complex space form, one can state the next result.

Theorem 66 ([109]). Suppose that F is a Riemannian map defined on a Riemannian manifold (M^m, g_M) of dimension m , having the target manifold a complex space form $(N^n(c), J, g_N)$ of complex dimension n , such that $3 \leq r = \text{rank } F < \min\{m, 2n\}$. Then the following inequalities are valid:

$$\rho^{\mathcal{H}} \leq \delta_C^{\mathcal{H}}(r-1) + \frac{c}{4} + \frac{3c}{4r(r-1)} \|P^{\mathcal{H}}\|^2, \quad \rho^{\mathcal{H}} \leq \widehat{\delta}_C^{\mathcal{H}}(r-1) + \frac{c}{4} + \frac{3c}{4r(r-1)} \|P^{\mathcal{H}}\|^2, \quad (85)$$

where $\delta_C^{\mathcal{H}}(r-1)$ and $\widehat{\delta}_C^{\mathcal{H}}(r-1)$ are the normalized Casorati curvatures on $\mathcal{H}$.

Moreover, the equality is attained in (85) at $x \in M$ if and only if with respect to some orthonormal bases $\{e_i\}_i$ on $\mathcal{H}_x = \ker F_{*x}^\perp$ and $\{V_\alpha\}$ on $(\operatorname{Im} F_{*p})^\perp$, $x \in M$, the components of ζ satisfy

$$\zeta^{\mathcal{H}\alpha}_{11} = \zeta^{\mathcal{H}\alpha}_{22} = \cdots = \zeta^{\mathcal{H}\alpha}_{r-1\ r-1} = \frac{1}{2}\zeta^{\mathcal{H}\alpha}_{rr}, \quad \text{for all } \alpha, \\ \zeta^{\mathcal{H}\alpha}_{ij} = 0, \quad \text{for all } \alpha \text{ and } i \neq j.$$

The authors of [109] also provide examples to show that inequalities (84) and (85) are optimal. An open problem is to obtain optimal inequalities involving the generalized normalized Casorati curvatures for Riemannian maps to real and complex space forms.

Author Contributions: Conceptualization, B.-Y.C., A.M.B. and G.-E.V.; methodology, B.-Y.C., A.M.B. and G.-E.V.; software, A.M.B. and G.-E.V.; validation, B.-Y.C., A.M.B. and G.-E.V.; formal analysis, B.-Y.C., A.M.B. and G.-E.V.; investigation, B.-Y.C., A.M.B. and G.-E.V.; resources, B.-Y.C. and G.-E.V.; data curation, A.M.B. and G.-E.V.; writing—original draft preparation, B.-Y.C., A.M.B. and G.-E.V.; writing—review and editing, B.-Y.C., A.M.B. and G.-E.V.; visualization, A.M.B. and G.-E.V.; supervision, B.-Y.C. and G.-E.V.; project administration, A.M.B. and G.-E.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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