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On a Fractional Stochastic Risk Model with a Random Initial Surplus and a Multi-Layer Strategy

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Abstract: The paper deals with a fractional time-changed stochastic risk model, including stochastic premiums, dividends and also a stochastic initial surplus as a capital derived from a previous investment. The inverse of a ν-stable subordinator is used for the time-change. The submartingale property is assumed to guarantee the net-profit condition. The long-range dependence behavior is proven. The infinite-horizon ruin probability, a specialized version of the Gerber–Shiu function, is considered and investigated. In particular, we prove that the distribution function of the infinite-horizon ruin time satisfies an integral-differential equation. The case of the dividends paid according to a multi-layer dividend strategy is also considered.

Keywords: stochastic premiums and claims; fractional Poisson process; multi–layer dividend strategy; ruin probability; piecewise integro-differential equation

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1. Introduction

The motivation of such a contribution relies on the need to specialize (in the fractional context) the Cramer–Lundberg-type risk model with a random initial surplus (cf. [1–3]). The choice of the time change by means of the inverse of a ν-stable subordinator ([4]) applied to the classical (integer) risk model is related to the possibility to make the last one more flexible for applications: indeed, the fractional model evolves on a stochastic time scale, and this aspect is revealed to be optimal in the financial application context in which the changes in the capital value evolve on a time scale strictly linked with the occurrence of other stochastic events ([5–8]). Moreover, the insertion of a random variable as the initial surplus is made in view to start from capital derived from previous investments.

The paper focuses on the study of the stochastic process denoted by $S^f_{\nu,t}$ (which stands for $S^f_\nu(t)$) as defined in (1)). It can be useful to model a surplus process of an insurance company with an initial capital, with probability density function f, subject to the dividend payment in a random time, regulated by a ν-stable subordinator, and also subject to further random variations due to the premiums and claims occurring in random times and with random sizes, respectively. The process $S^f_{\nu,t}$ can assume positive real values; in the case where it assumes zero value or negative values, the insurance company is ruined, and the time of this occurrence is called the ruin time. In the stochastic modeling of such financial phenomena, it is of interest to describe the ruin time probability (see, for instance, ref. [9]).

The real life examples useful to understand why such kinds of time-changed models are particularly advantageous include financial (as well as biological and other nature) dynamics subject to random changes in random times ([5,9]). More specifically, this means that random variations are applied in correspondence of the occurrence of other phenomenological random events affecting the evolution of the focused process. For such types of dynamics, the use of the inverse of a stable subordinator is suitable because, in particular, an ν-stable subordinator is a pure jump Lévy process, and its inverse (the new time) shows random jumps with random amplitude as well as plateau periods (freezing times or times of constancy) in correspondence with the jumps of the subordinator.

A further feature of time-changed stochastic processes is to show a long-range dependence in the correlation function. This is often used to construct models with the so-called long-memory properties (see, for instance, in the financial context, [10] and the references therein). Indeed, such processes are particularly suitable to describe dynamics, including memory effects. See, for instance, theoretical settings and applications of this type of process in different contexts, such as in neuronal modeling ([11,12]), in diffusion dynamics ([13,14]), in population dynamics and birth–death processes ([15,16]), and in service systems modeling and queuing theory ([17,18]).

By keeping in mind all these advantageous properties, here, we introduce a fractional time-changed risk model, and in order to provide a further generalization, we also consider a random initial capital and a multi-layer dividend strategy. Hence, in the next subsections, we define the proposed model, providing all details about the involved processes, such as the fractional compound Poisson processes and the inverse of the subordinators. In Section 2, we prove the submartigale property, and in Section 3, we provide the mean and covariance of the fractional risk process. In Section 4, we prove the long-range dependence property. In Section 5, we address the problem of the ruin probability. In Section 6, we provide the integral-differential equation for the distribution function of the infinite-horizon ruin time, and we also consider the multi-layer dividend payment strategy.

1.1. The Fractional Time-Changed Risk Model

Here, we consider a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a natural filtration $(\mathcal{F}_t)_{t\geq 0}$ with respect to all stochastic processes and the random variables here considered. We specifically consider the ν-fractional surplus process $S^f_{\nu,t}$ described by the following equation:

$$S^f_{\nu,t} = X - c\mathcal{L}_\nu(t) + \sum_{i=1}^{N^A_{\nu,t}} A_i - \sum_{i=1}^{N^R_{\nu,t}} R_i, \qquad t > 0,\ \nu \in (0,1) \tag{1}$$

with $S^f_{\nu,0} = X \quad (a.s.)$. In this risk model, we consider an initial non-negative absolute continuous random surplus X from which the dynamics starts, where $f(x)$ is the probability density function (pdf) of X. Then, $\mathcal{L}_\nu(t)$ is the stochastic time process defined as the right-continuous inverse of an ν-stable subordinator σ_ν ([19]), i.e.,

$$\mathcal{L}_\nu(t) = \inf\{s \geq 0 : \sigma_\nu(s) > t\} \qquad t \geq 0, \quad \nu \in (0,1). \tag{2}$$

We refer to the ν-stable subordinator $\{\sigma_\nu(s), s \geq 0\}$ defined as an increasing Lévy process such that, for $\theta > 0 : \mathbb{E}[e^{-\theta\sigma_\nu(s)}] = e^{-s\theta^\nu}$. (See also [15] for details and other examples of subordinators).

Hence, a stochastic dividend payment as time t and $\mathcal{L}_\nu(t)$ increase, with the rate $c > 0$, is subtracted from the surplus value. Then, the value of the surplus $S^f_{\nu,t}$ can be augmented by stochastic premiums, whose sizes is described by the sequence $(A_i)_{i\geq 1}$ of non-negative independent and identically distributed (i.i.d.) random variables (r.v.s) with a cumulative distribution function (c.d.f.) $F_A(a) = \mathbb{P}[A_i \leq a]$ and $\mu_A > 0$ finite expectation. The random number of premiums in the time interval $[0,t]$ is a fractional time-changed Poisson process $(N^A_{\nu,t})_{t\geq 0}$ obtained as

$$N^A_{\nu,t} = N^A_\nu(t) = N^A(\mathcal{L}_\nu(t)), \quad t \geq 0,\ \nu \in (0,1), \tag{3}$$

where $N^A(t)$ is the classical Poisson process, with constant intensity $\lambda_A > 0$, independent on $\mathcal{L}_\nu(t)$ (cf. [2,4]). Hence, the fractional compound Poisson process $\sum_{i=1}^{N^A_{\nu,t}} A_i$ models the additive premium process.

Instead, the value of the surplus $S^f_{\nu,t}$ can be reduced by stochastic claims, whose sizes is described by the sequence $(R_i)_{i\geq 1}$ of i.i.d. r.v.s with c.d.f. $F_R(r) = \mathbb{P}[R_i \leq r]$ and $\mu_R > 0$ finite expectation. The number of claims in the time interval $[0, t]$ is a fractional time-changed Poisson process $(N^R_{\nu,t})_{t\geq 0}$ defined in analogy to (3) as

$$N^R_{\nu,t} = N^R_\nu(t) = N^R(\mathcal{L}_\nu(t)), \quad t \geq 0,\ \nu \in (0,1), \tag{4}$$

where $N^R(t)$ is the classical Poisson process, with constant intensity $\lambda_R > 0$, independent on $\mathcal{L}_\nu(t)$. Thus, the total claims in $[0, t]$ are modeled by the fractional compound Poisson process $\sum_{i=1}^{N^R_{\nu,t}} R_i$. Moreover, we assume that $\sum_{i=1}^{N^A_{\nu,t}} A_i = 0$ if $N^A_{\nu,t} = 0$ and $\sum_{i=1}^{N^R_{\nu,t}} R_i = 0$ if $N^R_{\nu,t} = 0$. Take into account that the r.v.s $(R_i)_{i\geq 1}$, $(A_i)_{i\geq 1}$, $(N^R_{\nu,t})_{t\geq 0}$ and $(N^A_{\nu,t})_{t\geq 0}$ are mutually independent.

Summing up, in the model based on Equation (1), we specify that $\nu, c, \lambda_A, \lambda_R, \mu_A, \mu_R$ are parameters, whereas X, A_i, R_i are random variables, t is the time, $\mathcal{L}_\nu(t)$ is the stochastic process used to the time-change, $N^A_{\nu,t}$ and $N^R_{\nu,t}$ are the fractional stochastic counting processes ([20–22]).

Note that if $\nu = 1$, the considered model is the corresponding *integer* risk model:

$$S^f_t = X - ct + \sum_{i=1}^{N^A_t} A_i - \sum_{i=1}^{N^R_t} R_i, \quad t > 0, \tag{5}$$

with $S^f_0 = X\ (a.s.)$, where N^A_t and N^R_t are classical Poisson processes.

1.2. The Fractional Counting Processes for Premiums and Claims

Referring to the premiums, by using the probability density $f_\nu(s,t)$ of $\mathcal{L}_\nu(t)$ such that $f_\nu(s,t) = \frac{\partial}{\partial s}\mathbb{P}(\mathcal{L}_\nu(t) \leq s)$ (cf. [23]), the one-dimensional distribution of $N^A_{\nu,t}$ can be obtained with a subordinator operation, such as (cf. [2]):

$$\mathbb{P}(N^A_\nu(t) = k) = \int_0^\infty \frac{(\lambda_A s)^k}{k!} e^{-\lambda_A s} f_\nu(s,t)ds = \frac{(\lambda_A t^\nu)^k}{k!} E^{(k)}_\nu(-\lambda_A t^\nu)$$

where $E^{(k)}_\nu(-\lambda_A t^\nu)$ is the k-th derivative of the following Mittag–Leffler function ([24,25])

$$E_\nu(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\nu + 1)}, \quad \nu > 0,\ z \in \mathbb{C}, \tag{6}$$

evaluated in $z = -\lambda_A t^\nu$.

Furthermore, the mean and the covariance of the process $N^A_{\nu,t}$ are, respectively,:

$$\mathbb{E}[N^A_\nu(t)] = \frac{\lambda_A t^\nu}{\Gamma(\nu+1)}$$

and

$$\mathrm{Cov}(N^A_\nu(t), N^A_\nu(s)) = \frac{\lambda_A(\min(t,s))^\nu}{\Gamma(\nu+1)} + \lambda_A^2 \mathrm{Cov}(\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)), \tag{7}$$

where the covariance of the inverse of the ν-stable subordinator $\mathcal{L}_\nu(t)$ is ([26])

$$\mathrm{Cov}(\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)) = \frac{t^{2\nu}B(s/t;\nu,\nu+1) + s^{2\nu}B(\nu,\nu+1)}{\Gamma(\nu)\Gamma(\nu+1)} - \frac{(st)^\nu}{\Gamma^2(\nu+1)} \tag{8}$$

with $B(a,b)$ and $B(z;a,b)$ are the Beta function and the incomplete Beta function, respectively. Then, its variance is

$$\mathrm{Var}(N^A_\nu(t)) = \frac{\lambda_A t^\nu}{\Gamma(\nu+1)} + \frac{\lambda_A^2 t^{2\nu}}{\Gamma^2(\nu+1)}\left(\frac{\nu\Gamma(\nu)}{\Gamma(2\nu)} - 1\right).$$

All the same above specifications can be obtained similarly for the counting process $N_\nu^R(t)$ of claims by substituting λ_A with λ_R.

1.3. The Fractional Compound Poisson Process

For the two fractional compound Poisson processes $\sum_{i=1}^{N_{\nu,t}^A} A_i$ and $\sum_{i=1}^{N_{\nu,t}^R} R_i$, we give the same details in the following Lemma by referring to the general counting Poisson process denoted by $N_{\nu,t}$ and the general fractional compound Poisson process denoted by $\sum_{i=1}^{N_{\nu,t}} Y_i$, with Y_i i.i.d. random variables.

Lemma 1. *For a fractional compound Poisson process $\sum_{i=1}^{N_{\nu,t}} Y_i$, with Y_i i.i.d. random variables, and $N_{\nu,t} = N(\mathcal{L}_\nu(t))$ with $\lambda-$intensity Poisson process $N(t)$ independent on $\mathcal{L}_\nu(t)$, it holds*

(i)

$$\mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}} Y_i\right] = \frac{\mathbb{E}[Y_i]\lambda t^\nu}{\Gamma(\nu+1)}, \ \mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}} Y_i, \sum_{i=1}^{N_{\nu,s}} Y_i\right] = \frac{\mathbb{E}[Y_i^2]\lambda s^\nu}{\Gamma(\nu+1)} + \lambda^2(\mathbb{E}[Y_i])^2\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)],$$

(ii)

$$\mathrm{Cov}\left[\mathcal{L}_\nu(t), \sum_{i=1}^{N_{\nu,s}} Y_i\right] = \mathrm{Cov}\left[\mathcal{L}_\nu(s), \sum_{i=1}^{N_{\nu,t}} Y_i\right] = \mathbb{E}[Y_i]\lambda\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)].$$

Proof. (i)

At first, by applying the freezing lemma, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}} Y_i\right] &= \mathbb{E}\left[\mathbb{E}\left[Y_1 + Y_2 + \ldots + Y_{N_{\nu,t}} | N_{\nu,t}\right]\right] \\ &= \mathbb{E}[Y_i]\mathbb{E}[N_{\nu,t}] = \mathbb{E}[Y_i]\lambda\mathbb{E}[\mathcal{L}_\nu(t)] = \mathbb{E}[Y_i]\frac{\lambda t^\nu}{\Gamma(\nu+1)}.\end{aligned}$$

Then, by taking into account the mutual independence of Y_i, similarly,

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=1}^{N_{\nu,s}} Y_i\right)^2\right] &= \mathbb{E}\left[\sum_{i=1}^{N_{\nu,s}} Y_i^2 + \sum_{i,j=1, i\neq j}^{N_{\nu,s}} Y_i Y_j\right] \\ &= \mathbb{E}[Y_i^2]\mathbb{E}[N_{\nu,s}] + \mathbb{E}[Y_i Y_j](\mathbb{E}[N_{\nu,s}^2] - \mathbb{E}[N_{\nu,s}]) \\ &= \mathbb{E}[Y_i^2]\frac{\lambda s^\nu}{\Gamma(\nu+1)} + (\mathbb{E}[Y_i])^2\frac{\lambda^2 s^{2\nu}}{\Gamma^2(\nu+1)}\frac{\nu\Gamma(\nu)}{\Gamma(2\nu)}.\end{aligned}$$

For $s < t$, we also have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N_{\nu,t}} Y_i\right)\left(\sum_{i=1}^{N_{\nu,s}} Y_i\right)\right] = \mathbb{E}\left[\left(\sum_{i=1}^{N_{\nu,s}} Y_i\right)^2 + \left(\sum_{i=N_{\nu,s}+1}^{N_{\nu,t}} Y_i\right)\left(\sum_{j=1}^{N_{\nu,s}} Y_j\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{N_{\nu,s}} Y_i\right)^2\right] + \mathbb{E}\left[\sum_{i=N_{\nu,s}+1}^{N_{\nu,t}} \sum_{j=1}^{N_{\nu,s}} Y_i Y_j\right] \\
&= \mathbb{E}[Y_i^2]\frac{\lambda s^{\nu}}{\Gamma(\nu+1)} + \mathbb{E}[Y_iY_j](\mathbb{E}[N_{\nu,s}^2] - \mathbb{E}[N_{\nu,s}]) + \mathbb{E}[Y_iY_j]\mathbb{E}[N_{\nu,s}(N_{\nu,t} - N_{\nu,s})] \\
&= \mathbb{E}[Y_i^2]\frac{\lambda t^{\nu}}{\Gamma(\nu+1)} + (\mathbb{E}[Y_i])^2\big(\mathbb{E}[N_{\nu,s}N_{\nu,t}] - \mathbb{E}[N_{\nu,s}]\big) \\
&= \mathbb{E}[Y_i^2]\frac{\lambda t^{\nu}}{\Gamma(\nu+1)} + (\mathbb{E}[Y_i])^2\big(\mathrm{Cov}(N_{\nu,s}, N_{\nu,t}) + \mathbb{E}[N_{\nu,s}]\mathbb{E}[N_{\nu,t}] - \mathbb{E}[N_{\nu,s}]\big) \\
&= \mathbb{E}[Y_i^2]\frac{\lambda t^{\nu}}{\Gamma(\nu+1)} + (\mathbb{E}[Y_i])^2\left(\lambda^2\mathrm{Cov}(\mathcal{L}_{\nu}(t), \mathcal{L}_{\nu}(s)) + \frac{\lambda^2 s^{\nu} t^{\nu}}{(\Gamma(\nu+1))^2}\right).
\end{aligned}
$$

Consequentially, we obtain the second of (i) as

$$
\begin{aligned}
&\mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}} Y_i, \sum_{i=1}^{N_{\nu,s}} Y_i\right] = \mathbb{E}\left[\left(\sum_{i=1}^{N_{\nu,t}} Y_i\right)\left(\sum_{i=1}^{N_{\nu,s}} Y_i\right)\right] - \mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}} Y_i\right]\mathbb{E}\left[\sum_{i=1}^{N_{\nu,s}} Y_i\right] \\
&= \mathbb{E}[Y_i^2]\frac{\lambda t^{\nu}}{\Gamma(\nu+1)} + (\mathbb{E}[Y_i])^2\left(\lambda^2\mathrm{Cov}(\mathcal{L}_{\nu}(t), \mathcal{L}_{\nu}(s)) + \frac{\lambda^2 s^{\nu} t^{\nu}}{(\Gamma(\nu+1))^2} - \mathbb{E}[N_{\nu,t}]\mathbb{E}[N_{\nu,s}]\right).
\end{aligned}
$$

(ii) Furthermore, we also obtain (ii) by recalling that

$$
\begin{aligned}
&\mathrm{Cov}\left[\mathcal{L}_{\nu}(t), \sum_{i=1}^{N_{\nu,s}} Y_i\right] \\
&= \mathbb{E}\left[\mathcal{L}_{\nu}(t)\sum_{i=1}^{N_{\nu,s}} Y_i\right] - \mathbb{E}[\mathcal{L}_{\nu}(t)]\mathbb{E}\left[\sum_{i=1}^{N_{\nu,s}} Y_i\right] \\
&= \mathbb{E}\left[\mathcal{L}_{\nu}(t)\mathbb{E}\left[\sum_{i=1}^{N_{\nu,s}} Y_i \middle| N_{\nu,s}\right]\right] - \mathbb{E}[\mathcal{L}_{\nu}(t)]\mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_{\nu,s}} Y_i \middle| N_{\nu,s}\right]\right] \\
&= \mathbb{E}[Y_i]\mathbb{E}[\mathcal{L}_{\nu}(t)\lambda\mathcal{L}_{\nu}(s)] - \mathbb{E}[\mathcal{L}_{\nu}(t)]\mathbb{E}[Y_i]\lambda\mathbb{E}[\mathcal{L}_{\nu}(s)] \\
&= \mathbb{E}[Y_i]\lambda\mathrm{Cov}[\mathcal{L}_{\nu}(t)\mathcal{L}_{\nu}(s)].
\end{aligned}
$$

□

2. The Submartingale Property

Assumption 1. *To guarantee the net profit condition, we assume that the process*

$$N_{\nu,t}^{A}\mu_A - N_{\nu,t}^{R}\mu_R - c\mathcal{L}_{\nu}(t) \tag{9}$$

is a submartingale.

Hence, the direct consequence of such an assumption is the following proposition.

Proposition 1. *Under Assumption 1, the fractional risk model* (1) *is a submartigale.*

Proof. Consider, for $t > s$

$$
\begin{aligned}
&\mathbb{E}\left[S_\nu^f(t) - S_\nu^f(s)|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}^A} A_i - \sum_{i=1}^{N_{\nu,t}^R} R_i - c\mathcal{L}_\nu(t) - \sum_{i=1}^{N_{\nu,s}^A} A_i + \sum_{i=1}^{N_{\nu,s}^R} R_i + c\mathcal{L}_\nu(s)|\mathcal{F}_s\right] \\
&= -c\mathbb{E}[\mathcal{L}_\nu(t) - \mathcal{L}_\nu(s)|\mathcal{F}_s] + \mathbb{E}\left[\sum_{i=N_{\nu,s}^A+1}^{N_{\nu,t}^A} A_i - \sum_{i=N_{\nu,s}^R+1}^{N_{\nu,t}^R} R_i|\mathcal{F}_s\right] \\
&= -c\mathbb{E}[\mathcal{L}_\nu(t) - \mathcal{L}_\nu(s)|\mathcal{F}_s] + \mathbb{E}\left[\mathbb{E}\left[\sum_{i=N_{\nu,s}^A+1}^{N_{\nu,t}^A} A_i|\mathcal{F}_t\right]\Big|\mathcal{F}_s\right] - \mathbb{E}\left[\mathbb{E}\left[\sum_{i=N_{\nu,s}^R+1}^{N_{\nu,t}^R} R_i|\mathcal{F}_t\right]\Big|\mathcal{F}_s\right] \\
&= -c\mathbb{E}[\mathcal{L}_\nu(t) - \mathcal{L}_\nu(s)|\mathcal{F}_s] + \mathbb{E}\left[(N_{\nu,t}^A - N_{\nu,s}^A)\mu_A|\mathcal{F}_s\right] - \mathbb{E}\left[(N_{\nu,t}^R - N_{\nu,s}^R)\mu_R|\mathcal{F}_s\right] \\
&= \mathbb{E}[N_{\nu,t}^A\mu_A - N_{\nu,t}^R\mu_R - c\mathcal{L}_\nu(t)|\mathcal{F}_s] - \mathbb{E}[N_{\nu,s}^A\mu_A - N_{\nu,s}^R\mu_R - c\mathcal{L}_\nu(s)|\mathcal{F}_s].
\end{aligned}
$$

Hence, $\mathbb{E}\left[S_\nu^f(t) - S_\nu^f(s)|\mathcal{F}_s\right] \geq 0$ if and only if

$$
\mathbb{E}[N_{\nu,t}^A\mu_A - N_{\nu,t}^R\mu_R - c\mathcal{L}_\nu(t)|\mathcal{F}_s] \geq N_{\nu,s}^A\mu_A - N_{\nu,s}^R\mu_R - c\mathcal{L}_\nu(s) \quad (a.s.),
$$

which holds under Assumption 1 of the submartingale property for the process (9). □

3. Moments

Proposition 2. *The expectation of the fractional risk model* (1) *is:*

$$
\mathbb{E}[S_\nu^f(t)] = \mathbb{E}[X] + (\lambda_A\mu_A - \lambda_R\mu_R - c)\frac{t^\nu}{\Gamma(\nu+1)} \tag{10}
$$

with $(\lambda_A\mu_A - \lambda_R\mu_R - c) \geq 0$ *under Assumption 1.*

Proof. In order to evaluate the mean of the fractional risk model (1), we first write the expectation of the fractional counting processes for premiums and claims. Indeed, we have

$$
\mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}^A} A_i\right] = \frac{\lambda_A t^\nu}{\Gamma(\nu+1)}\mu_A, \quad \mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}^R} R_i\right] = \frac{\lambda_R t^\nu}{\Gamma(\nu+1)}\mu_R.
$$

Then, by taking into account also that

$$
\mathbb{E}[\mathcal{L}_\nu(t)] = \frac{t^\nu}{\Gamma(\nu+1)},
$$

the (10) is obtained by substituting of the above results in the following formula

$$
\begin{aligned}
&\mathbb{E}[S_\nu^f(t)] = \mathbb{E}[X] - c\mathbb{E}[\mathcal{L}_\nu(t)] + \mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}^A} A_i\right] - \mathbb{E}\left[\sum_{i=1}^{N_{\nu,t}^R} R_i\right] \\
&= \mathbb{E}[X] + \mathbb{E}[\mathcal{L}_\nu(t)](\lambda_A\mu_A - \lambda_R\mu_R - c).
\end{aligned}
$$

□

Proposition 3. *Under the assumption of the mutual independence of all involved random variables, the covariance of the fractional risk model $S_\nu^f(t)$ is, for $s<t$,*

$$\begin{aligned}\mathrm{Cov}[S_\nu^f(t),S_\nu^f(s)] = &\; Var(X)\\ &+\; \mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)](c^2-2c\mu_A\lambda_A+2c\mu_R\lambda_R+\lambda_A^2\mu_A^2+\lambda_R^2\mu_R^2)\\ &+\; \left(\mathbb{E}[A_i^2]\lambda_A+\mathbb{E}[R_i^2]\lambda_R\right)\frac{s^\nu}{\Gamma(\nu+1)}.\end{aligned} \tag{11}$$

Proof. From the definition (1) of the process $S_\nu^f(t)$, we can write

$$\begin{aligned}\mathrm{Cov}[S_\nu^f(t),S_\nu^f(s)] = &\; Var(X)+c^2\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)]\\ &-\; c\,\mathrm{Cov}\left[\mathcal{L}_\nu(t),\sum_{i=1}^{N_{\nu,s}^A}A_i\right]+c\,\mathrm{Cov}\left[\mathcal{L}_\nu(t),\sum_{i=1}^{N_{\nu,s}^R}R_i\right]\\ &-\; c\,\mathrm{Cov}\left[\mathcal{L}_\nu(s),\sum_{i=1}^{N_{\nu,t}^A}A_i\right]+c\,\mathrm{Cov}\left[\mathcal{L}_\nu(s),\sum_{i=1}^{N_{\nu,t}^R}R_i\right]\\ &+\; \mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^A}A_i,\sum_{i=1}^{N_{\nu,s}^A}A_i\right]+\mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^R}R_i,\sum_{i=1}^{N_{\nu,s}^R}R_i\right]\end{aligned}$$

where

$$\mathrm{Cov}\left[\mathcal{L}_\nu(t),\sum_{i=1}^{N_{\nu,s}^A}A_i\right]=\mathrm{Cov}\left[\mathcal{L}_\nu(s),\sum_{i=1}^{N_{\nu,t}^A}A_i\right]=\mu_A\lambda_A\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)],$$

$$\mathrm{Cov}\left[\mathcal{L}_\nu(t),\sum_{i=1}^{N_{\nu,s}^R}R_i\right]=\mathrm{Cov}\left[\mathcal{L}_\nu(s),\sum_{i=1}^{N_{\nu,t}^R}R_i\right]=\mu_R\lambda_R\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)]$$

and

$$\mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^A}A_i,\sum_{i=1}^{N_{\nu,s}^A}A_i\right]=\frac{\mathbb{E}[A_i^2]\lambda_A s^\nu}{\Gamma(\nu+1)}+\lambda_A^2\mu_A^2\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)],$$

$$\mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^R}R_i,\sum_{i=1}^{N_{\nu,s}^R}R_i\right]=\frac{\mathbb{E}[R_i^2]\lambda_R s^\nu}{\Gamma(\nu+1)}+\lambda_R^2\mu_R^2\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)].$$

Hence,

$$\begin{aligned}\mathrm{Cov}[S_\nu^f(t),S_\nu^f(s)] = &\; \mathrm{Var}(X)+c^2\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)]\\ &-\; 2c\,\mu_A\lambda_A\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)]+2c\,\mu_R\lambda_R\mathrm{Cov}[\mathcal{L}_\nu(t),\mathcal{L}_\nu(s)]\\ &+\; \mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^A}A_i,\sum_{i=1}^{N_{\nu,s}^A}A_i\right]+\mathrm{Cov}\left[\sum_{i=1}^{N_{\nu,t}^R}R_i,\sum_{i=1}^{N_{\nu,s}^R}R_i\right]\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{Cov}[S_\nu^f(t), S_\nu^f(s)] = &\ \mathrm{Var}(X) + c\,\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)](c - 2\mu_A\lambda_A + 2\mu_R\lambda_R) \\ &+ \frac{\mathbb{E}[A_i^2]\lambda_A s^\nu}{\Gamma(\nu+1)} + \lambda_A^2\mu_A^2\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)] \\ &+ \frac{\mathbb{E}[R_i^2]\lambda_R s^\nu}{\Gamma(\nu+1)} + \lambda_R^2\mu_R^2\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)].\end{aligned}$$

Finally, we obtain (11). □

4. The Long-Range Dependence

We recall that, for a fixed $s > 0$ and $t > s$, a non-stationary process Z_t (related to a fractional order α) is said to show so-called *long-range dependence* behavior if the correlation function is such that

$$\mathrm{Corr}(Z(s), Z(t)) \sim c_\alpha(s)t^{-\alpha}, \quad \alpha \in (0,1), \quad as\, t \to \infty$$

with $c_\alpha(s)$ a constant depending on α and s.

(Note that the notation "$f(x) \sim g(x)$ for $x \to \infty$"means that the two functions show the same asymptotic behavior as x increases.)

4.1. The Long-Range Dependence of the Process $\mathcal{L}_\nu(t)$

From [2], the inverse of the ν-stable subordinator, the process $\mathcal{L}_\nu(t)$ has *long-range dependence* behavior. Indeed, it was proven that, for fixed s and large t, of the covariance of $\mathcal{L}_\nu(t)$ is the following:

$$\mathrm{Cov}(\mathcal{L}_\nu(s), \mathcal{L}_\nu(t)) = \left(\frac{-\nu s^{\nu+1}t^{\nu-1}}{\Gamma(\nu)\Gamma(2+\nu)} + \cdots + s^{2\nu}\right)B(\nu+1,\nu) \tag{12}$$

and the variance

$$\mathrm{Var}(\mathcal{L}_\nu(t)) = t^{2\nu}\left(\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\right) \tag{13}$$

in such a way that the correlation function is

$$\mathrm{Corr}(\mathcal{L}_\nu(s), \mathcal{L}_\nu(t)) = \frac{\left(\frac{-\nu s^{\nu+1}t^{\nu-1}}{\Gamma(\nu)\Gamma(2+\nu)} + \cdots + s^{2\nu}\right)B(\nu+1,\nu)}{\sqrt{s^{2\nu}\left(\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\right)}\sqrt{t^{2\nu}\left(\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\right)}}.$$

Hence, it shows long-range dependence behavior

$$\mathrm{Corr}(\mathcal{L}_\nu(s), \mathcal{L}_\nu(t)) \sim \frac{B(\nu+1,\nu)}{c_\nu(s)}t^{-\nu}, \quad t \to \infty, \tag{14}$$

with $c_\nu(s)$ as a constant depending on ν and s.

4.2. The Long-Range Dependence of the Fractional Risk Process $S_\nu^f(t)$

Proposition 4. *The fractional risk process $S_\nu^f(t)$ has the following long-range dependence behavior*

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] \sim K(s,\nu)t^{-\nu}, \quad t \to \infty$$

with $\nu \in (0,1)$ and $K(s,\nu)$ is a constant depending on s and ν.

Proof. From (11), we have

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] = \frac{\mathrm{Var}(X) + h\,\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)] + k\,\frac{s^\nu}{\Gamma(\nu+1)}}{\sqrt{\mathrm{Var}(X) + h\,\mathrm{Var}[\mathcal{L}_\nu(s)] + k\,\frac{s^\nu}{\Gamma(\nu+1)}}\sqrt{\mathrm{Var}(X) + h\,\mathrm{Var}[\mathcal{L}_\nu(t)] + k\,\frac{t^\nu}{\Gamma(\nu+1)}}} \tag{15}$$

with $h = (c^2 - 2c\mu_A\lambda_A + 2c\mu_R\lambda_R + \lambda_A^2\mu_A^2 + \lambda_R^2\mu_R^2)$ and $k = \left(\mathbb{E}[A_i^2]\lambda_A + \mathbb{E}[R_i^2]\lambda_R\right)$.

Then, setting $A_\nu(s) = \mathrm{Var}(X) + k\frac{s^\nu}{\Gamma(\nu+1)}$, we can write

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] = \frac{A_\nu(s) + h\,\mathrm{Cov}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)]}{\sqrt{A_\nu(s) + h\,\mathrm{Var}[\mathcal{L}_\nu(s)]}\sqrt{A_\nu(t) + h\,\mathrm{Var}[\mathcal{L}_\nu(t)]}} \tag{16}$$

where a more compact expression can be

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] = H_1(s,t,\nu) + H_2(s,t,\nu)\mathrm{Corr}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)] \tag{17}$$

with

$$H_1(s,t,\nu) = \frac{A_\nu(s)}{\sqrt{A_\nu(s) + h\,\mathrm{Var}[\mathcal{L}_\nu(s)]}\sqrt{A_\nu(t) + h\,\mathrm{Var}[\mathcal{L}_\nu(t)]}}$$

and

$$H_2(s,t,\nu) = \frac{H_1(s,t,\nu)}{A_\nu(s)}\sqrt{\mathrm{Var}[\mathcal{L}_\nu(s)\mathrm{Var}[\mathcal{L}_\nu(t)]} = \frac{h}{\sqrt{\frac{A_\nu(s)+h\,\mathrm{Var}[\mathcal{L}_\nu(s)]}{\mathrm{Var}[\mathcal{L}_\nu(s)]}}\sqrt{\frac{A_\nu(t)+h\,\mathrm{Var}[\mathcal{L}_\nu(t)]}{\mathrm{Var}[\mathcal{L}_\nu(t)]}}}.$$

By taking into account (12) and (13), the asymptotic behaviors when $t \to \infty$ can be written for suitable constants $c_1(s,\nu)$ and $c_2(s,\nu)$:

$$\mathrm{Var}(\mathcal{L}_\nu(t)) \sim c_1(s,\nu)t^{2\nu} \quad \textit{and} \quad A_\nu(t) \sim c_2(s,\nu)t^\nu,$$

from which we can derive that

$$H_1(s,t,\nu) \sim C_1(s,\nu)t^{-\nu}$$

and

$$H_2(s,t,\nu) \sim C_2(s,\nu)$$

with $C_1(s,\nu)$ and $C_2(s,\nu)$ as suitable constants depending on s and ν. Finally, we obtain that the asymptotic behavior, when $t \to \infty$, for fixed s, of the correlation function is

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] \sim C_1(s,\nu)t^{-\nu} + C_2(s,\nu)\mathrm{Corr}[\mathcal{L}_\nu(t), \mathcal{L}_\nu(s)] \tag{18}$$

showing the long-range dependence behavior also for $S_\nu^f(t)$ involving that of $\mathcal{L}_\nu(t)$, i.e.,

$$\mathrm{Corr}[S_\nu^f(t), S_\nu^f(s)] \sim \left(C_1(s,\nu) + C_2(s,\nu)\frac{B(\nu+1,\nu)}{c_\nu(s)}\right)t^{-\nu}, \quad t \to \infty.$$

This completes the proof by identifying

$$K(s,\nu) = \left(C_1(s,\nu) + C_2(s,\nu)\frac{B(\nu+1,\nu)}{c_\nu(s)}\right).$$

□

5. The Ruin Probability

Now, in order to focus on the main quantity of interest in a financial context of application, i.e., the ruin probability, we first define the *conditioned ruin time*

$$\tau_\nu(x) = \inf\{t \geq 0 : S^f_{\nu,t} < 0 | X = x\}$$

for the risk process $\{S^f_{\nu,t}\}_{t\geq 0}$ defined by (1) and conditioned on the event $X = x$. In this paper, we specifically focus on the *integrated ruin time*

$$\tau^f_\nu = \int_0^\infty \tau_\nu(x) f(x) dx = \mathbb{E}[\tau_\nu(X)] \tag{19}$$

that is also the *mean* of the ruin time with respect to the random initial surplus. Note that if $f(x) = \delta_x$, i.e., the whole unitary probability mass concentrated in a fixed $x \in \mathbb{R}^+$ and $\nu = 1$, then $\tau^{\delta_x} = \tau(x)$ is the classical ruin probability. The relationship between $\tau(x)$ and $\tau_\nu(x)$ is derived by applying a subordination operator between the probability laws, i.e., $\forall\, t > 0$,

$$\mathbb{P}(\tau_\nu(x) < t) = \int_0^\infty \mathbb{P}(\tau(x) < z) f_\nu(z,t) dz$$

with $f_\nu(z,t)$ density of $\mathcal{L}_\nu(t)$. Clearly, we have $\mathbb{P}(\tau_\nu(x) < t) \neq \mathbb{P}(\tau(x) < t)$.

We denote by

$$\varphi_\nu(x) = \mathbb{E}[\mathbf{1}(\tau^f_\nu < \infty) | X = x] = \mathbb{E}[\mathbf{1}(\tau_\nu(x) < \infty)] \tag{20}$$

the *infinite-horizon ruin probability* conditioned by $X = x$.

In [2], it was proven that such a probability is the same of that of the *integer* case for $\nu = 1$,i.e., $\varphi_\nu(x) \equiv \varphi(x) = \mathbb{E}[\mathbf{1}(\tau(x) < \infty)]$, and here we also prove it by using the definition of a sequence of *finite*-horizon ruin times.

Proposition 5. *We have*

$$\mathbb{P}(\tau_\nu(x) < \infty) = \mathbb{P}(\tau(x) < \infty).$$

Proof. For $T > 0$, let

$$\tau_\nu(x,T) = \begin{cases} \tau_\nu(x), \text{ if } \tau_\nu(x) \in (0,T) \\ T, \text{ otherwise} \end{cases} \tag{21}$$

be the *finite-horizon* ruin time with fractional order $\nu \in (0,1)$ and let

$$\tau(x,T) = \begin{cases} \tau(x), \text{ if } \tau(x) \in (0,T) \\ T, \text{ otherwise} \end{cases} \tag{22}$$

be the *finite-horizon* ruin time in the integer case. We assume that $\tau_\nu(x,T)$ is absolutely continuous random variable with density $f_{\tau_\nu(x,T)}(s), \quad s > 0$, the same for $\tau(x,T)$ with density $f_{\tau(x,T)}(s)$. Let $f_{\tau(x)}(s)$ and $f_{\tau_\nu(x)}(s)$ the density of $\tau(x)$ and $\tau_\nu(x)$, respectively. For $t \in (0,T)$, consider the event $A_{t,T} = \{\tau_\nu(x,T) < t\}$ whose probability is given by

$$\mathbb{P}(A_{t,T}) = \mathbb{P}(\tau_\nu(x,T) < t) = \int_0^\infty \mathbb{P}(\tau(x,T) < z) f_\nu(z,t) dz,$$

with $f_\nu(z,t)$ the density of $\mathcal{L}_\nu(t)$. We have that the random times $\tau_\nu(x,T)$ and $\tau(x,T)$ are such that

$$\tau_\nu(x,T) \leq \tau_\nu(x), \qquad \tau(x,T) \leq \tau(x), \quad \forall T > 0,$$

where the equality holds only in the case $\tau_\nu(x)$ and $\tau(x)$ are bounded themselves, respectively. Moreover, for $T \to \infty$, $\tau_\nu(x,T) \to \tau_\nu(x)$ and $\tau(x,T) \to \tau(x)$ weakly. Hence, we also have that, for $T \to \infty$, $A_{t,T} \to A_{t,\infty} = \{\tau_\nu(x) < t\}$. Furthermore, we can apply the Fubini theorem, and we can write

$$\begin{aligned}\mathbb{P}(\tau_\nu(x) < t) &= \lim_{T\to\infty} \mathbb{P}(\tau_\nu(x,T) < t) = \lim_{T\to\infty} \int_0^\infty \mathbb{P}(\tau(x,T) < z) f_\nu(z,t) dz \\ &= \lim_{T\to\infty} \int_0^\infty \int_0^z f_{\tau(x,T)}(s) ds f_\nu(z,t) dz = \lim_{T\to\infty} \int_0^\infty \int_s^\infty f_\nu(z,t) dz f_{\tau(x,T)}(s) ds \\ &= \int_0^\infty \int_s^\infty f_\nu(z,t) dz \lim_{T\to\infty} f_{\tau(x,T)}(s) ds = \int_0^\infty \mathbb{P}(\mathcal{L}_\nu(t) > s) f_{\tau(x)}(s) ds.\end{aligned}$$

Then, for $t \to \infty$, the last equation becomes:

$$\mathbb{P}(\tau_\nu(x) < \infty) = \lim_{t\to\infty} \int_0^\infty \mathbb{P}(\mathcal{L}_\nu(t) > s) f_{\tau(x)}(s) ds = \mathbb{P}(\tau(x) < \infty)$$

by using $\lim_{t\to\infty} \mathbb{P}(\mathcal{L}_\nu(t) > s) = 1, \quad \forall s > 0$.

Hence, we obtained the thesis. $\square$

6. The Integrated Infinite-Horizon Ruin Probability

As a consequence of the results of the previous section, it appears reasonable to study the distribution of the *infinite-horizon ruin* time $\tau_\nu(x)$ defined as

$$\varphi_\nu(x;t) = \mathbb{E}[\mathbf{1}(\tau_\nu(x) < t)] = \mathbb{P}(\tau_\nu(x) < t) = \lim_{T\to\infty} \mathbb{P}(\tau_\nu(x,T) < t) \tag{23}$$

and its integrated version as defined in what follows.

We define the *integrated (or expectation respect to the initial surplus X) infinite-horizon ruin distribution* as, for $t > 0$,

$$\psi_\nu^f(t) = \int_0^\infty \varphi_\nu(x;t) f(x) dx = \mathbb{E}[\varphi_\nu(X;t)] \tag{24}$$

with

$$\varphi_\nu(x;t) = \mathbb{E}[\mathbf{1}(\tau_\nu^f < t)|X = x] = \mathbb{E}[\mathbf{1}(\tau_\nu(x) < t)] \tag{25}$$

where $\mathbf{1}(\cdot)$ is the indicator function. Moreover, from (24) and (25), we have:

$$\psi_\nu^f(t) = \mathbb{E}[\mathbf{1}(\tau_\nu^f < t)] = \mathbb{P}(\tau_\nu^f < t). \tag{26}$$

Note that the *conditioned* ruin distribution $\varphi_1(x;t)$ in (25) coincides with the ruin probability $\psi(x)$ of [3] for $t \to \infty$, and for $f(x) = \delta_x$, i.e.,

$$\lim_{t\to\infty} \varphi_1(x;t) = \lim_{t\to\infty} \psi_\nu^{\delta_x}(t) = \psi(x)$$

of [3], with δ_x the delta distribution function centered in x, corresponding to the case of $\mathbb{P}(X = x) = 1$. We highlight that the proposed model in this paper is the fractional generalization of the previous ones in which the surplus risk process, conditioned by the fixed initial capital x, is here represented by the case of $S_{\nu,t}^{\delta_x}$.

Specifically, the study of (19) and (24) for several pdf $f(x)$ and ν is interesting to investigate which ν (discriminating for the choice of the time-scale) and which pdf $f(x)$ are suitable for minimizing the infinite-horizon ruin probability. Note that the ruin probability, obtained from (25) for $t \to \infty$, is a particular case of the expected discounted penalty function, which is also called the Gerber–Shiu function.

Here, by exploiting the results of Ragulina (see [3] and the references therein), we derive specific results for (24) and (25) related to the process (1).

6.1. The Model with Time-Space Multi-Layer Dividend Payment

We also extend our study to the case of multi-layer dividend payments, i.e., to the model:

$$S_{\nu,t}^{f,\mathbf{L}} = X - \sum_{j=1}^{k} c_j \int_0^{\mathcal{L}_\nu(t)} \mathbf{1}(l_{j-1} \le S_{\nu,w}^{f,\mathbf{L}} < l_j) dw + \sum_{i=1}^{N_{\nu,t}^A} A_i - \sum_{i=1}^{N_{\nu,t}^R} R_i, \qquad t > 0, k \ge 2, \quad (27)$$

in which the insurance company follows a k-layer dividend strategy payment taking into account also the premiums and claims regulated by the fractional compound Poisson processes $N_{\nu,t}^A$ and $N_{\nu,t}^R$, respectively. Here, $\mathbf{L} = (l_0, l_1, \ldots, l_{k-1})$, with $0 \le l_0 < l_1 < \cdots < l_{k-1} < \infty$, is the k-dimensional vector whose real-valued components represent the boundaries of the layers.

Now, we can apply our investigation strategy for the corresponding $\psi_\nu^f(t)$; however, an additional definition for the study of the ruin probability is required. For $z \in \mathbb{R}$, we define the space *shifted* $\psi_\nu^f(z;t)$, i.e.,

$$\psi_\nu^f(z;t) = \begin{cases} \psi_\nu^f(y;t) & = \int_0^\infty \varphi_\nu(x+y;t) f(x) dx, \quad \text{for} \quad z = y \ge 0,\, y \in \mathbb{R}^+, \\ \psi_\nu^f(-y;t) & = \int_y^\infty \varphi_\nu(x-y;t) f(x) dx, \quad \text{for} \quad z = -y < 0,\, y \in \mathbb{R}^+. \end{cases} \quad (28)$$

This is the space mean (respect to X) of the infinite-horizon ruin probability of $S_{\nu,t}^f$ process of model (1) when the random initial surplus X is shifted by z, i.e.,

$$\psi_\nu^f(z;t) = \mathbb{E}[\varphi_\nu(X+z;t)] \quad (29)$$

with

$$\varphi_\nu(x+z;t) = \mathbb{E}[\mathbf{1}(\tau_\nu^f(x+z) < t) | X = x]. \quad (30)$$

In addition, from (24), we specify that

$$\psi_\nu^f(0;t) \equiv \psi_\nu^f(t) = \int_0^\infty \varphi_\nu(x;t) f(x) dx \quad (31)$$

and, setting $l_0 = 0$, $l_k = \infty$, for the j-th layer, with $1 \le j \le k$, we define

$$\psi_\nu^{f,j}(z;t) = \begin{cases} \int_{l_{j-1}}^{l_j} \varphi_\nu(x+y;t) f(x) dx, & \text{for } z = y \ge 0, \\ \int_y^{l_j} \varphi_\nu(x-y;t) f(x) dx, & \text{for } z = -y < 0. \end{cases} \quad (32)$$

Finally, we have

$$\psi_\nu^f(z;t) = \sum_{j=1}^{k} \psi_\nu^{f,j}(z;t). \quad (33)$$

6.2. On the Stochastic Representation of the Fractional Poisson Process

In analogy to the classical representation of the Poisson process as the following random sum:

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}(T_1 + T_2 + \cdots + T_i < t)$$

with T_i i.i.d. r.v. exponentially distributed with parameter λ, we consider the time-change by $\mathcal{L}_\nu(t)$, and we can write that the following stochastic representation holds

$$N(\mathcal{L}_\nu(t)) = \sum_{i=1}^{\infty} \mathbf{1}(T_1 + T_2 + \cdots + T_i < \mathcal{L}_\nu(t)).$$

By taking into account that the density of $\mathcal{L}_\nu(t)$, i.e., $f_\nu(u,t) = \frac{\partial}{\partial u}\mathbb{P}(\mathcal{L}_\nu(t) \le u)$, has the state-Laplace transform such that ([23,27])

$$\int_0^\infty e^{-\lambda u} f_\nu(u,t)du = E_\nu(-\lambda t^\nu),$$

we have that the fractional Poisson process $N_{\nu,t}$ also admits the following stochastic representation (cf. [2,4,28–30]):

$$N_{\nu,t} = N(\mathcal{L}_\nu(t)) = \sum_{i=1}^{\infty} \mathbf{1}(\theta_1 + \theta_2 + \cdots + \theta_i < t)$$

with θ_i i.i.d. r.v. distributed as ν-Mittag–Leffler random variables. Specifically, this implies that the interarrival times between two successive jumps have the following distribution:

$$\mathbb{P}(\theta_i \le t) = 1 - E_\nu(-\lambda t^\nu), \qquad \forall i,$$

where $E_\nu(-\lambda t^\nu)$ is the Mittag–Leffler function (6) and λ is the rate of the Poisson process $N(t)$. We recall that the expectation of the Mittag–Leffler random variable θ_i is infinite, and, for this reason, it is not possible to define the rate of the fractional Poisson process by means of the reciprocal of the mean of the inter-arrival times as for the classical integer case. A way to proceed is to consider the expectation of the interarrival time θ_i $before$ a given time t, i.e.,

$$\mathbb{E}_t[\theta_i] := \int_0^t \mathbb{P}(\theta_i > s)ds = tE_{\nu,2}(-\lambda t^\nu) \tag{34}$$

where $E_{\nu,k}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\nu+k)}$ is the two-parameter Mittag–Leffler function. In this context, we consider the time-dependent rate of the fractional Poisson process $N_{\nu,t}$, and we define it as follows:

$$\Lambda_t = \frac{1}{\mathbb{E}_t[\theta_i]} = \frac{1}{tE_{\nu,2}(-\lambda t^\nu)}.$$

For the premiums and claims, we specifically denote the corresponding rates:

$$\Lambda_{A,t} = \frac{1}{\mathbb{E}_t[\theta_i^A]} = \frac{1}{tE_{\nu,2}(-\lambda_A t^\nu)}$$

with θ_i^A the generic interarrival time between two premiums, and

$$\Lambda_{R,t} = \frac{1}{\mathbb{E}_t[\theta_i^R]} = \frac{1}{tE_{\nu,2}(-\lambda_R t^\nu)}$$

with θ_i^R the generic interarrival time between two claims, respectively.

6.3. An Unifying Theorem

In addition to the previous definitions (24), (28) and (32), we denote by

$$\psi'_{\nu,f}(0;t) = (\psi_\nu^f)'(t) = \int_0^\infty \varphi'_\nu(x;t)f(x)dx \equiv \mathbb{E}[\varphi'_\nu(X;t)], \tag{35}$$

and for $1 \le j \le k-1$,

$$\psi'_{\nu,f,j}(0,t) = (\psi_\nu^{f,j})'(t) = \int_{l_{j-1}}^{l_j} \varphi'_\nu(x;t)f(x)dx. \tag{36}$$

Note that the prime symbol refers to x-derivative. Hence, the prime symbol in $\psi'_{\nu,f}(t)$ in the above equations is only a notation, while the prime in φ'_ν refers to the first $x-$derivative of φ_ν.

Theorem 1. *Let the integrated surplus process $(S^f_{\nu,t})_{t\geq 0}$ obey the model (1) and let the integrated surplus process $(S^{f,\mathbf{L}}_{\nu,t})_{t\geq 0}$ obey the model (27) with a multi-layer dividend strategy under the above assumptions. Moreover, let $F_R(r)$ and $F_A(a)$ be continuous distributions on $\mathbb{R}^+$ for the generic claim R and premium A, respectively. $\Lambda_{A,t}$ and $\Lambda_{R,t}$ are the rates of the fractional Poisson processes $N^A_{\nu,t}$ and $N^R_{\nu,t}$, respectively. Then, referring to $(S^f_{\nu,t})_{t\geq 0}$, for $t > 0$, the corresponding shifted $\psi^f_\nu(z,t)$ satisfies the following integro-differential representation:*

$$\begin{aligned} &c\psi'_{\nu,f}(0;t) + (\Lambda_{A,t} + \Lambda_{R,t})\psi_{\nu,f}(0;t) \\ &= \Lambda_{A,t}\int_0^\infty \psi_{\nu,f}(a;t)dF_A(a) + \Lambda_{R,t}\int_0^\infty \psi_{\nu,f}(-r;t)dF_R(r) + \Lambda_{R,t}(1 - F_{R,f}(t)), \end{aligned} \tag{37}$$

for any $f(x) \geq 0$, $\forall x \geq 0$, with $\int_0^\infty f(x)dx = 1$, where $F_{R,f}(t) = \int_0^\infty F_R(x;t)f(x)dx$ with $F_R(x;t) = \mathbb{P}(R < x, N^R_{\nu,t} = 1)$.

Then, referring to $(S^{f,\mathbf{L}}_{\nu,t})_{t\geq 0}$, the correspondent shifted $\psi_{\nu,f,j}(z;t)$, for $1 \leq j \leq k-1$, satisfies the following integro-differential representation:

$$\begin{aligned} &c_j\psi'_{\nu,f,j}(0;t) + (\Lambda_{A,t} + \Lambda_{R,t})\psi_{\nu,f,j}(0;t) \\ &= \Lambda_{A,t}\int_0^\infty \psi_{\nu,f,j}(a;t)dF_A(a) + \Lambda_{R,t}\int_0^{l_j} \psi_{\nu,f,j}(-r;t)dF_R(r) + \Lambda_{R,t}[\mathbb{P}(l_{j-1} \leq X \leq l_j) - F_{R,f,j}(t)], \end{aligned} \tag{38}$$

where $F_{R,f,j}(t) = \int_{l_{j-1}}^{l_j} F_R(x;t)f(x)dx$.

Finally, by identifying $S^f_{\nu,t} = S^{f,\mathbf{L}}_{\nu,t}$, for $j = k = 1$ and $l_0 = 0, l_1 = +\infty$, Equation (38) reduces to (37).

Before we give the proof of Theorem 1, we provide two Lemmas to be used in the proof.

In the next Lemma, we deal a result for the conditioned $\varphi_\nu(x;t)$ ruin distribution by inserting it in the presented setting for the specified process $S^f_{\nu,t}$ when $f(x) = \delta_x$, (cf. [3]). For $\nu = 1$, we provide a proof alternative to others already known (compare with [6,8,31]) here specialized for the ruin distribution function and for the fractional case.

Lemma 2. *Under the assumption of the net profit condition guaranteed by the submartingale property and the assumptions of the previous theorem, let the surplus process $(S^{\delta_x}_{\nu,t})_{t\geq 0}$ follow the model (1) for $f(x) = \delta_x$, with $F_R(r)$ and $F_A(a)$ continuous distribution functions on $\mathbb{R}^+$ for the claim and premium sizes, respectively. In this case, for $f(x) = \delta_x$, the ruin time distribution $\psi^f_\nu(t) = \psi^{\delta_x}_\nu(t)$ and $\psi^{\delta_x}_\nu(t) = \varphi_\nu(x;t)$ (a.s.) and satisfies the following integro-differential equation for $t > 0$:*

$$\begin{aligned} &c\varphi'_\nu(x;t) + (\Lambda_{A,t} + \Lambda_{R,t})\varphi_\nu(x;t) \\ &\quad = \Lambda_{A,t}\int_0^\infty \varphi_\nu(x+a;t)dF_A(a) + \Lambda_{R,t}\int_0^x \varphi_\nu(x-r;t)dF_R(r) + \Lambda_{R,t}(1 - F_R(x;t)), \\ &\qquad x \geq 0, \end{aligned} \tag{39}$$

with $\lim_{x\to\infty}\varphi_\nu(x;t) = 0$ and with the setting $\varphi_\nu(x;t) = 1$ for $x \leq 0$, $\forall t \geq 0$.

Proof. At first, we define the functions $\Phi_A(\cdot;t)$ and $\Phi_R(\cdot;t)$ as the expectation of the infinite-horizon ruin probabilities before t with an initial forward time-shift during which it is possible to observe the occurrence of an additional premium a and a reducing claim r with respect to the initial surplus, respectively. The forward time-shift, multiplied by c, affects the value of the initial capital x. Keeping in mind the dynamics of the process $S^{\delta_x}_{\nu,t}$ in (1), the functions Φ_A and Φ_R, for a forward time-shift $\vartheta > 0$, are defined as the following

$$\Phi_A(x - c\vartheta; t) = \int_0^\infty \varphi_\nu(x - c\vartheta + a; t)\mathrm{d}F_A(a), \tag{40}$$

$$\Phi_R(x - c\vartheta; t) = \left[\int_0^{x-c\vartheta} \varphi_\nu(x - c\vartheta - r; t)\mathrm{d}F_R(r) + \int_{x-c\vartheta}^\infty \mathrm{d}F_R(r)\right]. \tag{41}$$

They, for a zero time-shift, are such that

$$\Phi_A(x; t) = \int_0^\infty \varphi_\nu(x + a; t)\mathrm{d}F_A(a), \tag{42}$$

$$\Phi_R(x; t) = \left[\int_0^{x} \varphi_\nu(x - r; t)\mathrm{d}F_R(r) + \int_{x}^\infty \mathrm{d}F_R(r)\right]. \tag{43}$$

In particular, by considering the times of the first occurrence of a premium and of a claim, i.e., θ_1^A and θ_1^R, and their finite expectations given in (34), respectively, we have (a.s.)

$$\Phi_A(x - c\mathbb{E}_t[\theta_1^A]; t) = \int_0^\infty \varphi_\nu(x - c\mathbb{E}_t[\theta_1^A] + a; t)\mathrm{d}F_A(a) = \varphi_\nu(x; t) \tag{44}$$

$$\begin{aligned}\Phi_R(x - c\mathbb{E}_t[\theta_1^R]; t) &= \left[\int_0^{x-c\mathbb{E}_t[\theta_1^R]} \varphi_\nu(x - c\mathbb{E}_t[\theta_1^R] - r; t)\mathrm{d}F_R(r) + \int_{x-c\mathbb{E}_t[\theta_1^R]}^\infty \mathrm{d}F_R(r)\right] \\ &= \varphi_\nu(x; t).\end{aligned} \tag{45}$$

By applying the Lagrange theorem to the $C^1(\mathbb{R})$-functions $\Phi_A(x;t)$ and $\Phi_R(x;t)$ (cf. [1]), respectively, we can write that

$$\frac{\Phi_A(x;t) - \Phi_A(x - c\mathbb{E}_t[\theta_1^A]; t)}{c\mathbb{E}_t[\theta_1^A]} = \Phi'_A(x - c\theta_A; t), \quad \theta_A \in (0, \mathbb{E}_t[\theta_1^A]), \tag{46}$$

$$\frac{\Phi_R(x;t) - \Phi_R(x - c\mathbb{E}_t[\theta_1^R]; t)}{c\mathbb{E}_t[\theta_1^R]} = \Phi'_R(x - c\theta_R; t), \quad \theta_R \in (0, \mathbb{E}_t[\theta_1^R]). \tag{47}$$

By adding (46) and (47), we obtain, for $\theta \in (0, \mathbb{E}_t[\theta_1])$ with θ_1 the time of a first jump (due to the occurrence of a premium or a claim) and $\mathbb{E}_t[\theta_1] = \min\{\mathbb{E}_t[\theta_1^A], \mathbb{E}_t[\theta_1^R]\}$,

$$\begin{aligned}&\frac{\Phi_A(x;t)}{c\mathbb{E}_t[\theta_1^A]} + \frac{\Phi_R(x;t)}{c\mathbb{E}_t[\theta_1^R]} - \left(\frac{\Phi_A(x - c\mathbb{E}_t[\theta_1^A]; t)}{c\mathbb{E}_t[\theta_1^A]} + \frac{\Phi_R(x - c\mathbb{E}_t[\theta_1^R]; t)}{c\mathbb{E}_t[\theta_1^R]}\right) \\ &\qquad = \Phi'_A(x - c\theta; t) + \Phi'_R(x - c\theta; t).\end{aligned} \tag{48}$$

By using (44) and (45) and recalling that $\mathbb{E}_t[\theta_1^A] = 1/\Lambda_{A,t}$ and $\mathbb{E}_t[\theta_1^R] = 1/\Lambda_{R,t}$, we also obtain

$$\Lambda_{A,t}\Phi_A(x;t) + \Lambda_{R,t}\Phi_R(x;t) - (\Lambda_{A,t} + \Lambda_{R,t})\varphi_\nu(x;t) = c\big(\Phi'_A(x - c\theta; t) + \Phi'_R(x - c\theta; t)\big). \tag{49}$$

Note that, by using the condition $\varphi_\nu(x;t) = 1$ for $x \le 0$, we directly can write:

$$\Phi_R(x; t) = \int_0^\infty \varphi_\nu(x - r; t)\mathrm{d}F_R(r). \tag{50}$$

Then,

$$\Phi'_A(x - c\theta; t) + \Phi'_R(x - c\theta; t) = \int_0^\infty \varphi'_\nu(x - c\theta + a; t)\mathrm{d}F_A(a) + \int_0^\infty \varphi'_\nu(x - c\theta - r; t)\mathrm{d}F_R(r), \tag{51}$$

and setting $j = a$ in the first integral at the RHS of (51) and $j = -r$ in the second integral at the RHS of (51), one has

$$\begin{aligned}
&\Phi'_A(x - c\theta; t) + \Phi'_R(x - c\theta; t) \\
&= \int_0^\infty \varphi'_\nu(x - c\theta + j; t)\mathrm{d}F_A(j) + \int_{-\infty}^0 \varphi'_\nu(x - c\theta + j; t)\mathrm{d}F_R(-j) \\
&= \int_{-\infty}^\infty \varphi'_\nu(x - c\theta + j; t)\mathrm{d}F_J(j)
\end{aligned} \tag{52}$$

with $F_J(j) = F_A(j)\mathbf{1}(j \geq 0) + F_R(-j)\mathbf{1}(j < 0)$ is the cumulative distribution function of a generic (positive or negative) jump J. A similar argument can be used to prove the same result for θ in any period of time between two successive jumps; hence, the result holds for any $\theta > 0$. Finally, we can identify

$$\int_{-\infty}^\infty \varphi'_\nu(x - c\theta + j; t)\mathrm{d}F_J(j) = \varphi'_\nu(x; t) \qquad (a.s.),$$

which can be validated with the following identity

$$\begin{aligned}
\int_{-\infty}^\infty \varphi_\nu(x - c\theta + j; t)\mathrm{d}F_J(j) &= \mathbb{E}_J[\mathbb{E}[\mathbf{1}(\tau_\nu(x - c\theta + j) < t)|S_{\nu,\theta} = x - c\theta + j]] \\
&= \mathbb{E}[\mathbf{1}(\tau_\nu(x - c\theta) < t)|S_{\nu,\theta} = x - c\theta] \\
&= \mathbb{E}[\mathbf{1}(\tau_\nu(x) < t)|S_0 = x] = \varphi_\nu(x; t) \qquad (a.s.).
\end{aligned} \tag{53}$$

Finally, for the RHS of (49), we obtain $\forall\theta > 0$

$$c(\Phi'_A(x - c\theta; t) + \Phi'_R(x - c\theta; t)) = c\varphi'_\nu(x; t). \tag{54}$$

Furthermore, by considering the LHS of (49), from (42) and (43), we also have

$$\begin{aligned}
\Lambda_{A,t}\Phi_A(x; t) + \Lambda_{R,t}\Phi_R(x; t) = \Lambda_{A,t} &\int_0^\infty \varphi_\nu(x + a; t)\mathrm{d}F_A(a) \\
&+\Lambda_{R,t}\left[\int_0^x \varphi_\nu(x - r; t)\mathrm{d}F_R(r) + (1 - F_R(x; t))\right].
\end{aligned} \tag{55}$$

By using (54) and (55) in (49) , the thesis (39) follows. □

Lemma 3. *Under the assumptions of the previous lemma and theorem, let the surplus process $(S^{\delta_x,\mathbf{L}}_{\nu,t})_{t\geq 0}$ follow the model (27) for $f(x) = \delta_x$, with $F_R(r)$ and $F_A(a)$ continuous distribution functions on $\mathbb{R}^+$ for the claim and premium sizes, respectively. In this case, for $f(x) = \delta_x$, and $t > 0$ the ruin probabilities $\psi^{\delta_x}_\nu(t) = \varphi_\nu(x; t)$ (a.s.) are differentiable and satisfy the following integro-differential equation on the intervals $[l_{j-1}, l_j]$ for $j = 1, \ldots, k-1$:*

$$\begin{aligned}
&c_j\varphi'_\nu(x; t) + (\Lambda_{A,t} + \Lambda_{R,t})\varphi_\nu(x; t) \\
&\quad = \Lambda_{A,t}\int_0^\infty \varphi_\nu(x + a; t)dF_A(a) + \Lambda_{R,t}\int_0^x \varphi_\nu(x - r; t)dF_R(r) + \Lambda_{R,t}(1 - F_R(x; t)) \\
&\qquad\qquad x \in [l_{j-1}, l_j]
\end{aligned} \tag{56}$$

with $l_0 = 0$, $l_k = +\infty$, $\lim_{x\to l_j^-}\varphi_\nu(x; t) = \lim_{x\to l_j^+}\varphi_\nu(x; t)$ for $1 \leq j \leq k-1$ and $\lim_{x\to\infty}\varphi_\nu(x; t) = 0$ and by setting $\varphi_\nu(x; t) = 1$ for $x \leq 0$, $\forall t \geq 0$.

Proof. Assuming $f(x) = \delta_x$ means that this is the case of a deterministic initial capital; hence, $1 = \mathbb{P}(X = x)$. The proof could be given following the proof of Theorem 1 in [3] (not in a simple way), but here it is sufficient to adapt the proof given in Lemma 2. Indeed, (56) is the same of (39) holding in each of interval $[l_{j-1}, l_j]$, for $j = 1, \ldots, k-1$: the role of these intervals affects only the alternative possible range of values for the initial capital x.

Finally, note that we refer to the one-sided derivatives of $\varphi_\nu(x;t)$ that is not differentiable at l_j ,$1 \leq j \leq k-1$, indeed its one-sided derivatives differ only at those points. □

Proof of Theorem 1. The proof is obtained by exploiting essentially the proof of Lemma 2. Equation (37) is obtained from the integro-differential Equation (39) valid for the conditioned $\varphi_\nu^{\delta_x}(x;t)$ function. The boundary conditions of Equation (39) imply the boundedness of $\psi_\nu^f(t)$ involved in (37). Moreover, the representation (37) is obtained multiplying both sides of (39) for $f(x)$ and integrating over $\mathbb{R}^+$ respect to x. Then, the use of Fubini theorem in the two double integrals on the right-hand-side and definitions (28) and (35) allow obtaining (37).

Similarly, the same procedure can also be applied to (56) specifically for the case of the process $(S_{\nu,t}^{f,\mathbf{L}})_{t\geq 0}$, by considering definitions (32) and (36) and by integrating (56) in each interval $[l_{j-1}, l_j]$ with respect to density $f(x)$. Hence, equation (38) follows.
It is easy to realize that equation (37) can be obtained from equation (38) for the only ($k=1$) unlimited layer defined by $l_0 = 0, l_1 = +\infty$, by also taking into account that definitions (32) and (36) reduce to definitions (28) and (35). □

Corollary 1. *Under the assumptions of Theorem 1, Equations (37) and (38) hold also for the infinite-horizon ruin probability τ_ν^f ($=\tau^f$ in law) for a given initial surplus density f.*

Proof. First, we recall that, from Proposition 5, $\mathbb{P}(\tau_\nu(x) < \infty) = \mathbb{P}(\tau(x) < \infty)$. Furthermore, by setting $\nu = 1$ and by taking into account that $E_{1,2}(-\lambda t) = \frac{1-e^{-\lambda t}}{\lambda t}$, from (34), we have that $\lim_{t\to\infty} \Lambda_{A,t} = \Lambda_A = \lambda_A$ and $\lim_{t\to\infty} \Lambda_{R,t} = \Lambda_R = \lambda_R, \forall t > 0$. Finally, by applying the limit for $t \to \infty$ to all functions in (37), we obtain

$$\begin{aligned} &c\psi'_{\nu,f} + (\Lambda_A + \Lambda_R)\psi_{\nu,f} \\ &= \Lambda_A \int_0^\infty \psi_{\nu,f}(a) dF_A(a) + \Lambda_R \int_0^\infty \psi_{\nu,f}(-r) dF_R(r) + \Lambda_R(1 - F_{R,f}) \end{aligned} \tag{57}$$

with $F_{R,f} = \int_0^\infty F_R(x) f(x) dx$.

Similarly, we also have the corresponding result to Equation (38), i.e.,

$$\begin{aligned} &c_j\psi'_{\nu,f,j} + (\Lambda_A + \Lambda_R)\psi_{\nu,f,j} \\ &= \Lambda_A \int_0^\infty \psi_{\nu,f,j}(a) dF_A(a) + \Lambda_R \int_0^{l_j} \psi_{\nu,f,j}(-r) dF_R(r) + \Lambda_R[\mathbb{P}(l_{j-1} \leq X \leq l_j) - F_{R,f,j}] \end{aligned} \tag{58}$$

with $F_{R,f,j} = \int_{l_{j-1}}^{l_j} F_R(x) f(x) dx$. □

Equations (39) and (56) can be analytically solved in the non fractional case when distribution functions of the claim and premium sizes are specified. For (37) and (38), a detailed investigation is required when the distributions of the claim and premium sizes and the density $f(x)$ are assigned.

7. Conclusions

In this contribution, we limit ourselves to the mathematical setting of proposed models; however, the results and the discussion for specified distribution functions and about theoretical and numerical comparisons will be the object of a future work. Indeed, we feel stimulated to work in this direction for the purpose of investigating, also quantitatively, (i) how the ruin probability changes when a random initial capital is considered in place of the assigned one; (ii) how the transient behavior of the surplus process changes for different values of the fractional order ν; (iii) the possible advantages derived from the multi-layer dividend payment strategy in these kinds of dynamics; and (iv) how to adapt these models to real data by means of specific techniques for estimating the involved parameters and by making use of extensive simulations.

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References

1. Boikov, A.V. The Cramér-Lundberg model with stochastic premium process. *Theory Probab. Appl.* **2003**, *47*, 489–493. [CrossRef]
2. Kumar, A.; Leonenko, N.; Pichler, A. Fractional risk process in insurance. *Math Finan Econ.* **2020**, *14*, 43–65. [CrossRef]
3. Ragulina, O. The risk model with stochastic premiums and a multi-layer dividend strategy. *Mod. Stoch. Theory Appl.* **2019**, *6*, 285–309. [CrossRef]
4. Meerschaert, M.M.; Nane, E.; Vellaisamy, P. The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.* **2011**, *16*, 1600–1620. [CrossRef]
5. Kerss, A.; Leonenko, N.; Sikorskii, A. Fractional Skellam processes with applications to finance. *Fract. Calc. Appl. Anal.* **2014**, *17*, 532–551. [CrossRef]
6. Mishura, Y.S.; Ragulina, O.Y.; Stroev, O.M. Analytic property of infinite-horizon survival probability in a risk model with additional funds. *Theory Probab. Math. Stat.* **2015**, *91*, 131–143. [CrossRef]
7. Mishura, Y.; Ragulina, O.; Stroev, O. Practical approaches to the estimation of the ruin probability in a risk model with additional funds. *Mod. Stoch. Theory Appl.* **2014**, *1*, 167–180. [CrossRef]
8. Rolski, T.; Schmidli, H.; Schmidt, V.; Teugels, J. *Stochastic Processes for Insurance and Finance*; John Wiley and Sons: Chichester, UK, 1999.
9. Peng, X.; Su, W.; Zhang, Z. On a perturbed compound Poisson risk model under a periodic threshold-type dividend strategy. *J. Ind. Manag. Optim.* **2020**, *16*, 1967–1986. [CrossRef]
10. Sottinen, T.; Alos, E.; Azmoodeh, E.; Di Nunno, G. Long-Memory Models in Mathematical Finance. *Front. Appl. Math. Stat.* **2021**, *7*. [CrossRef]
11. Abundo, M.; Pirozzi, E. Fractionally Integrated Gauss-Markov processes and applications. *Commun. Nonlinear Sci. Numer. Simulat.* **2021**, *101*, 105862. [CrossRef]
12. Pirozzi, E. Colored noise and a stochastic fractional model for correlated inputs and adaptation in neuronal firing. *Biol. Cybern.* **2018**, *112*, 25–39. [CrossRef] [PubMed]
13. Ascione, G.; Leonenko, N.; Pirozzi, E. Time-Non-Local Pearson Diffusions. *J. Stat. Phys.* **2021**, *183*, 48. [CrossRef]
14. Leonenko, N.; Pirozzi, E. First passage times for some classes of fractional time-changed diffusions. *Stoch. Anal. Appl.* **2021**, 1–29 . [CrossRef]
15. Ascione, G.; Leonenko, N.; Pirozzi, E. Non-local Solvable Birth–Death Processes. *J. Theor. Probab.* **2021**. 1–40. [CrossRef]
16. Ascione, G.; Leonenko, N.; Pirozzi, E. Fractional immigration-death processes. *J. Math. Anal. Appl.* **2021**, *495*, 124768. [CrossRef]
17. Ascione, G.; Leonenko, N.; Pirozzi, E. Fractional Erlang queues. *Stoch. Process. Their Appl.* **2020**, *130*, 3249–3276. [CrossRef]
18. Ascione, G.; Leonenko, N.; Pirozzi, E. Fractional Queues with Catastrophes and Their Transient Behaviour. *Mathematics* **2018**, *6*, 159. [CrossRef]
19. Bertoin, J. *Subordinators: Examples and applications, in Lectures on Probability Theory and Statistics*; Springer: Berlin/Heidelberg, Germany, 1999; pp. 1–91.
20. Aletti, G.; Leonenko, N.N.; Merzbach, E. Fractional Poisson processes and martingales. *J. Stat. Phys.* **2018**, *170*, 700–730. [CrossRef]
21. Cahoy, D.O.; Uchaikin, V.V.; Woyczynski, W.A. Parameter estimation for fractional Poisson processes. *J. Stat. Plan. Inference* **2010**, *140*, 3106–3120. [CrossRef]
22. Laskin N. Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.* **2003**, *8*, 201–213. [CrossRef]
23. Meerschaert, M.M.; Straka, P. Inverse stable subordinators. *Math. Model. Nat. Phenom.* **2013**, *8*, 1–16. [CrossRef] [PubMed]
24. Garrappa, R.; Kaslik, E.; Popolizio, M. Evaluation of Fractional Integrals and Derivatives of Elementary Functions: Overview and Tutorial. *Mathematics* **2019**, *7*, 407. [CrossRef]
25. Mainardi, F. Why the Mittag–Leffler Function Can Be Considered the Queen Function of the Fractional Calculus? *Entropy* **2020**, *22*, 1359. [CrossRef] [PubMed]
26. Leonenko, N.N.; Meerschaert, M.M.; Schilling, R.L.; Sikorskii, A. Correlation structure of time-changed Lévy processes. *Commun. Appl. Ind. Math.* **2014**, *6*, e-483. [CrossRef]
27. Bingham, N.H. Limit theorems for occupation times of Markov processes. *Z. Wahrscheinlichkeitstheorie Verwandte Geb.* **1971**, *17*, 1–22 [CrossRef]
28. Huillet T.E. On Mittag–Leffler distributions and related stochastic processes. *J. Comput. Appl. Math.* **2016**, *296*, 181–211. [CrossRef]
29. Mainardi, F.; Gorenflo, R.; Vivoli, A. Renewal processes of Mittag–Leffler and Wright type. *Fract. Calc. Appl. Anal.* **2005**, *8*, 7–38.
30. Mainardi, F.; Gorenflo, R.; Scalas, E. A fractional generalization of the Poisson processes. *Vietnam J. Math.* **2004**, *32*, 53–64.
31. Mishura, Y.; Ragulina, O. *Ruin Probabilities: Smoothness, Bounds, Supermartingale Approach*; ISTE Press-Elsevier: London, UK, 2016.