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Generalized q-Difference Equations for q-Hypergeometric Polynomials with Double q-Binomial Coefficients

Jian Cao [1], Hari M. Srivastava [2,3,4,5,*], Hong-Li Zhou [1] and Sama Arjika [6,7]

1 School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China; 21caojian@hznu.edu.cn (J.C.); 2019111008035@stu.hznu.edu.cn (H.-L.Z.)
2 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
3 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
4 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
5 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
6 Department of Mathematics and Informatics, University of Agadez, Agadez P.O. Box 199, Niger; rjksama2008@gmail.com
7 International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, P.O. Box 072, Cotonou 50, Benin
* Correspondence: harimsri@math.uvic.ca

Abstract: In this paper, we apply a general family of basic (or q-) polynomials with double q-binomial coefficients as well as some homogeneous q-operators in order to construct several q-difference equations involving seven variables. We derive the Rogers type and the extended Rogers type formulas as well as the Srivastava-Agarwal-type bilinear generating functions for the general q-polynomials, which generalize the generating functions for the Cigler polynomials. We also derive a class of mixed generating functions by means of the aforementioned q-difference equations. The various results, which we have derived in this paper, are new and sufficiently general in character. Moreover, the generating functions presented here are potentially applicable not only in the study of the general q-polynomials, which they have generated, but indeed also in finding solutions of the associated q-difference equations. Finally, we remark that it will be a rather trivial and inconsequential exercise to produce the so-called (p,q)-variations of the q-results, which we have investigated here, because the additional forced-in parameter p is obviously redundant.

Keywords: homogeneous q-difference operator; double q-binomial coefficients; q-difference equations; q-hypergeometric polynomials; generating functions

MSC: Primary 05A30; 33D15; 33D45; Secondary 05A40; 11B65

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1. Introduction

In this paper, we adopt the notation and terminology for the basic (or q-) hypergeometric series as in [1,2]. Throughout this paper, we assume that q is a fixed nonzero real or complex number and $|q| < 1$. The q-shifted factorial and its compact factorial forms are defined for any real or complex parameter a, a_1, a_2, $\cdots$, a_r, respectively, as follows [1,2]:

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1}$$

and

$$(a_1,a_2,\cdots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m$$
$$\big(m \in \mathbb{N}_0 := \{0,1,2\cdots\} = \mathbb{N}\cup\{0\}\big).$$

We will also frequently use the following relation:

$$(aq^{-n};q)_n = \left(\frac{q}{a};q\right)_n (-a)^n \, q^{-n-\binom{n}{2}}. \tag{2}$$

The generalized q-binomial coefficients are defined as follows (see [1]):

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k} \, (-1)^k \, q^{\alpha k-\binom{k}{2}} \tag{3}$$

and

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} = \frac{(-q^{-\alpha};q)_k}{(-q;q)_k} \, q^{\alpha k-\binom{k}{2}} \qquad (\alpha \in \mathbb{C}), \tag{4}$$

so that

$$\binom{\alpha}{k} = \lim_{q\to 1-} \left\{ \begin{bmatrix} \alpha \\ k \end{bmatrix}_q \right\} \qquad (\alpha \in \mathbb{C})$$

for the familiar binomial coefficient.

The basic (or q-) hypergeometric function ${}_r\Phi_s$ in the variable z is defined by (see, for details, Slater ([3], Chap. 3) and Srivastava and Karlsson ([4], p. 347, Eq. (272)); see also [5]):

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ \\ b_1, b_2, \cdots, b_s; \end{matrix} \; q; z \right] = \sum_{n=0}^{\infty} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} \, \frac{z^n}{(q;q)_n}$$

when $r > s+1$. In particular, for $r = s+1$, we have:

$${}_{r+1}\Phi_r \left[\begin{matrix} a_1, a_2, \cdots, a_{r+1}; \\ \\ b_1, b_2, \cdots, b_r; \end{matrix} \; q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1}; q)_n}{(b_1, b_2, \cdots, b_r; q)_n} \, \frac{z^n}{(q;q)_n}.$$

We remark in passing that, in the recently-published survey-cum-expository review articles (see [6,7]), the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional forced-in parameter p being redundant or superfluous (see, for details, ([6], p. 340) and ([7], pp. 1511–1512)).

Chen et al. [8] introduced the homogeneous q-difference operator D_{xy} as follows:

$$D_{xy}\{f(x,y)\} := \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}, \tag{5}$$

which turns out to be suitable for dealing with the Cauchy polynomials. On the other hand, Wang and Cao [9] presented the following two extensions of Cigler's polynomials:

$$C_n^{(\alpha-n)}(x,y,b) = \sum_{k=0}^{n} (-1)^k \, q^{\binom{k}{2}} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q b^k \, \frac{(q;q)_n}{(q;q)_{n-k}} \, p_{n-k}(x,y) \tag{6}$$

and

$$\mathcal{D}_n^{(\alpha-n)}(x,y,b) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} \alpha \\ k \end{bmatrix}_q b^k \, \frac{(q;q)_n}{(q;q)_{n-k}} \left[(-1)^{n+k} q^{-\binom{n}{2}} \, p_{n-k}(y,x) \right], \tag{7}$$

where

$$p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = \left(\frac{y}{x};q\right)_n x^n$$

are the Cauchy polynomials.

Recently, Jia et al. [10] have introduced the following polynomials:

$$L_{\tilde{m},\tilde{n}}(\alpha, x, z, a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\tilde{m},\tilde{n})+\binom{k}{2}} \, (a;q)_k \, z^k \, x^{n-k} \tag{8}$$

with

$$\tau(\tilde{m},\tilde{n}) = \tilde{m}\binom{k}{2} - \tilde{n}\binom{k+1}{2}, \tag{9}$$

where $\tilde{m}$ and $\tilde{n}$ are real numbers. More recently, Cao et al. [11] introduced an extension of the above q-polynomials as follows:

$$\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}} \, (a;q)_k \; p_{n-k}(x,y) z^k \tag{10}$$

and gave the following result.

Proposition 1 (see [11])**.** *Let $f(\alpha,x,y,a,z,\tilde{r},\tilde{s})$ be a seven-variable analytic function in a neighborhood of*

$$(\alpha,x,y,a,z,\tilde{r},\tilde{s}) = (0,0,0,0,0,0,0) \in \mathbb{C}^7.$$

Then $f(\alpha,x,y,a,z,\tilde{r},\tilde{s})$ can be expanded in terms of $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,a,z,\tilde{r},\tilde{s})$ if and only if f satisfies the following q-difference equation:

$$\begin{aligned}
&(x-q^{-1}y)\Big\{f(\alpha,x,y,a,z,\tilde{r},\tilde{s}) - f(\alpha,x,y,a,q^2z,\tilde{r},\tilde{s})\Big\} \\
&\quad = q^{\alpha-\tilde{r}}z\Big\{f(\alpha,x,q^{-1}y,a,zq^{\tilde{r}-\tilde{s}},\tilde{r},\tilde{s}) - f(\alpha,qx,y,a,zq^{\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\} \\
&\qquad + q^{-\tilde{r}-1}(1-aq^{\alpha})z\Big\{f(\alpha,x,yq^{-1},a,zq^{1+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s}) - f(\alpha,qx,y,a,zq^{1+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\} \\
&\qquad - azq^{-\tilde{r}-2}\Big\{f(\alpha,x,yq^{-1},a,zq^{2+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s}) - f(\alpha,qx,y,a,zq^{2+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\}.
\end{aligned} \tag{11}$$

Our present investigation is motivated essentially by the earlier works by Jia et al. [10] and by Cao et al. [11]. Our aim here is to introduce and study the following further extension of the above-mentioned q-polynomials:

$$\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}} \, \frac{(a,b;q)_k}{(c;q)_k} \; p_{n-k}(x,y) z^k, \tag{12}$$

where $\tau(\tilde{r},\tilde{s})$ is defined as in (9).

Zhou and Luo [12] obtained some new generating functions for the q-Hahn polynomials and their proofs are based upon the homogeneous q-difference operator. Saad and Abdlhusein [13] utilized the Cauchy operator in proving some identities involving the homogeneous Rogers-Szegö polynomials. However, we found it to be difficult to continue to calculate and generalize the above-mentioned authors' results for general q-polynomials with more parameters (see, for example, [10,12–15]).

It is natural to ask whether some general q-hypergeometric polynomials exist, which are solutions of certain generalized q-difference equations. The novelty of this paper is to search and find these generalized q-difference equations that are satisfied by some of the general q-hypergeometric polynomials, which we have investigated in this paper. The methods and techniques, which we have presented and used here, have produced potentially useful generalizations of the above-mentioned results (see, for details, [10,12–15]). Derivations of various known or new particular cases of our results are indicated in Remark 1.

Remark 1. *The general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ defined in (12) provide a generalized and unified form of the Hahn polynomials and the Al-Salam-Carlitz polynomials. Some of these special cases of the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ are being listed below.*

1. *Upon setting $y=0$ and $b=c=0$, the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ defined in (12) would reduce to (8) (see [10])*

$$\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,0,z,a,b,c) = L_{\tilde{r},\tilde{s}}(\alpha,x,z,a). \tag{13}$$

2. *If we put*
$$(\alpha, \tilde{r}, \tilde{s}, x, y, z, a) = (\infty, 0, 0, y, x, -z, -q, 0, 0),$$
the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, x, y, z, a, b, c)$ *reduce to the trivariate q-polynomials* $F_n(x, y, z; q)$ *(see [16]):*
$$\tilde{L}_n^{(0,0)}(\infty, y, x, -z, -q) = (-1)^n q^{\binom{n}{2}} F_n(x, y, z; q). \tag{14}$$
3. *Upon setting*
$$\alpha = n \in \mathbb{Z} \qquad \text{and} \qquad (\tilde{r}, \tilde{s}, a, b, c, x, y, z) = (0, -1, -yq, 0, 0, 1, 0, x),$$
the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, x, y, z, a, b, c)$ *reduce to the polynomials* $\rho_e(n, y, x, q)$ *(see [10]):*
$$\tilde{L}_n^{(0,-1)}(n, 1, 0, x, -qy) = \rho_e(n, y, x, q). \tag{15}$$
4. *If we set*
$$(\alpha, \tilde{r}, \tilde{s}, y, a, b, c) = (\infty, -1, 0, 0, -q, 0, 0),$$
the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, x, y, z, a, b, c)$ *reduce to the homogeneous Rogers-Szegö polynomials* $h_n(x, y|q)$ *(see [17]):*
$$\tilde{L}_n^{(-1,0)}(\infty, x, y, 1, -q) = h_n(x, y|q). \tag{16}$$
5. *By choosing*
$$(\alpha, \tilde{r}, \tilde{s}, a, b, c, x, y) = (\infty, -1, 0, -q, 0, 0, xq^{-n}, 0),$$
the q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, x, y, z, a, b, c)$ *reduce to the Rogers-Szegö polynomials* $g_n(z, x|q)$ *(see [17]):*
$$\tilde{L}_n^{(-1,0)}(\infty, xq^{-n}, 0, z, -q) = g_n(z, x|q). \tag{17}$$

The rest of this paper is organized as follows. In Section 2, we establish the main results for the q-difference equations involving seven variables for the general q-polynomials. In Section 3, we obtain the generating function of the general q-polynomials by the method of q-difference equations. In Section 4, we derive the Rogers-type formula for the general q-polynomials by using the q-difference equations. In Section 5, we present a mixed generating function for the general q-polynomials by means of the q-difference equations. We also consider the Srivastava-Agarwal-type bilinear generating functions for the general q-polynomials in Section 5 itself. In Section 6, we derive a transformation identity involving a Hecke-type series for the general q-polynomials. Finally, in Section 7, we present several remarks and observations that are based upon the results and findings in this paper.

2. Fundamental Theorem

In this section, we first state and prove the following fundamental theorem.

Theorem 1. *Let* $f(\alpha, x, y, a, b, c, z, \tilde{r}, \tilde{s})$ *be a nine-variable analytic function in a neighborhood of:*
$$(\alpha, x, y, a, b, c, z, \tilde{r}, \tilde{s}) = (0, 0, 0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^9.$$

Then $f(\alpha, x, y, a, b, c, z, \tilde{r}, \tilde{s})$ *can be expanded in terms of* $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, x, y, z, a, b, c)$ *if and only if the function* f *satisfies the following q-difference equation:*

$$
\begin{aligned}
&(x-q^{-1}y)\Big\{[f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})-f(\alpha,x,y,a,b,c,q^2z,\tilde{r},\tilde{s})]\\
&\quad -cq^{-1}[f(\alpha,x,y,a,b,c,qz,\tilde{r},\tilde{s})-f(\alpha,x,y,a,q^3z,\tilde{r},\tilde{s})]\Big\}\\
&=q^{\alpha-\tilde{r}}z\Big\{f(\alpha,x,q^{-1}y,a,b,c,zq^{\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})-f(\alpha,qx,y,a,b,c,zq^{\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\}\\
&\quad +q^{-\tilde{r}-1}(1-aq^{\alpha}-bq^{\alpha})z\Big\{f(\alpha,x,yq^{-1},a,b,c,zq^{1+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})-f(\alpha,qx,y,a,b,c,zq^{1+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\}\\
&\quad -(a+b-abq^{\alpha})zq^{-\tilde{r}-2}\Big\{f(\alpha,x,yq^{-1},a,b,c,zq^{2+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})-f(\alpha,qx,y,a,b,c,zq^{2+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\}\\
&\quad -abzq^{-\tilde{r}-3}\Big\{f(\alpha,x,yq^{-1},a,b,c,zq^{3+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})-f(\alpha,qx,y,a,b,c,zq^{3+\tilde{r}-\tilde{s}},\tilde{r},\tilde{s})\Big\}.
\end{aligned}
\tag{18}
$$

Remark 2. *For $b=c=0$ in Theorem 1, we can deduce Equation (11). Furthermore, if we set $y=0$ and $b=c=0$ in Theorem 1, we are led to the concluding remarks of Jia et al. [10].*

Lemma 1 (Hartogs's theorem)**.** *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $\mathbb{D}\in\mathbb{C}^n$, then it is holomorphic (analytic) in $\mathbb{D}$.*

In order to prove Theorem 1, we need the following fundamental property of functions of several complex variables (see, for example [18–20]; see also [21]).

Lemma 2 (see ([18], Proposition 1))**.** *If $f(x_1,x_2,\cdots,x_k)$ is analytic at the origin $(0,0,\cdots,0)\in\mathbb{C}^k$, then the function $f(x_1,x_2,\cdots,x_k)$ can be expanded in an absolutely convergent power series given by*

$$f(x_1,x_2,\cdots,x_k)=\sum_{n_1,n_2,\cdots,n_k=0}^{\infty}\Omega_{n_1,n_2,\cdots,n_k}\,x_1^{n_1}\,x_2^{n_2}\cdots x_k^{n_k}.$$

Proof of Theorem 1. In light of Hartogs theorem and the theory of functions of several complex variables, we assume that

$$f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=\sum_{k=0}^{\infty}A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\,z^k. \tag{19}$$

Firstly, by substituting from (19) into (18), we get:

$$
\begin{aligned}
&(x-q^{-1}y)\sum_{k=0}^{\infty}(1-q^{2k})(1-cq^{k-1})A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\,z^k\\
&=\sum_{k=0}^{\infty}\Big\{q^{\alpha+\tilde{r}(k-1)-\tilde{s}k}+q^{(\tilde{r}+1)(k-1)-\tilde{s}k}(1-bq^{\alpha}-aq^{\alpha})\\
&\quad -q^{(\tilde{r}+2)(k-1)-\tilde{s}k}(b+a-abq^{\alpha})+abq^{(\tilde{r}+3)(k-1)-\tilde{s}k}\Big\}\\
&\quad \cdot\Big\{A_k(\alpha,x,q^{-1}y,a,b,c,\tilde{r},\tilde{s})-A_k(\alpha,qx,y,a,b,c,\tilde{r},\tilde{s})\Big\}\,z^{k+1},
\end{aligned}
\tag{20}
$$

which readily yields

$$
\begin{aligned}
&(x-q^{-1}y)\sum_{k=0}^{\infty}(1-q^{2k})(1-cq^{k-1})A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\,z^k\\
&=\sum_{k=0}^{\infty}q^{\tilde{r}(k-1)-\tilde{s}k}\Big\{q^{\alpha}+q^{k-1}(1-bq^{\alpha}-aq^{\alpha})-q^{2k-2}(b+a-abq^{\alpha})+abq^{3k-3}\Big\}\\
&\quad \cdot\Big\{A_k(\alpha,x,q^{-1}y,a,b,c,\tilde{r},\tilde{s})-A_k(\alpha,qx,y,a,b,c,\tilde{r},\tilde{s})\Big\}\,z^{k+1}.
\end{aligned}
\tag{21}
$$

Upon equating the coefficients of $z^k \quad (k \in \mathbb{N})$ on both sides of the Equation (21), we see that

$$\begin{aligned}(x-q^{-1}y)(1-q^k)(1+q^k)(1-cq^{k-1})A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) \\ = q^{\tilde{r}(k-1)-\tilde{s}k}(q^\alpha+q^{k-1})(1-aq^{k-1})(1-bq^{k-1}) \\ \cdot \Big\{A_{k-1}(\alpha,x,q^{-1}y,a,b,c,\tilde{r},\tilde{s})-A_{k-1}(\alpha,qx,y,a,b,c,\tilde{r},\tilde{s})\Big\}\end{aligned} \tag{22}$$

or, equivalently, that

$$\begin{aligned}A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) &= q^{\tilde{r}(k-1)-\tilde{s}k}\frac{(q^\alpha+q^{k-1})(1-aq^{k-1})(1-bq^{k-1})}{(1-q^k)(1+q^k)(1-cq^{k-1})} \\ &\quad \cdot \frac{A_{k-1}(\alpha,x,q^{-1}y,a,b,c,\tilde{r},\tilde{s})-A_{k-1}(\alpha,qx,y,a,b,c,\tilde{r},\tilde{s})}{x-q^{-1}y} \\ &= q^{\alpha+\tilde{r}(k-1)-\tilde{s}k}\ \frac{(1+q^{-\alpha+k-1})(1-aq^{k-1})(1-bq^{k-1})}{(1-q^k)(1+q^k)(1-cq^{k-1})} \\ &\quad \cdot D_{xy}\{A_{k-1}(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\}.\end{aligned}$$

By iterating this process, we find that

$$A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) = q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\ \frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}\cdot D_{xy}^k\{A_0(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\},$$

which, upon letting

$$f(\alpha,x,y,a,b,c,0,\tilde{r},\tilde{s}) = A_0(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) = \sum_{n=0}^{\infty}\mu_n p_n(x,y),$$

yields

$$\begin{aligned}A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) \quad &= q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\ \tfrac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k} \\ &\cdot \textstyle\sum_{n=0}^{\infty}\ \mu_n\ \frac{(q;q)_n}{(q;q)_{n-k}}\ p_{n-k}(x,y).\end{aligned} \tag{23}$$

We thus obtain

$$\begin{aligned}f(\alpha,x,y,z,a,b,c,\tilde{r},\tilde{s}) &= \sum_{k=0}^{\infty} q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k} \\ &\quad \cdot \sum_{n=0}^{\infty}\mu_n\ \frac{(q;q)_n}{(q;q)_{n-k}}\ p_{n-k}(x,y)z^k \\ &= \sum_{n=0}^{\infty}\mu_n\ \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}\alpha\\k\end{bmatrix}_{-q}\ q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}\frac{(a,b;q)_k}{(c;q)_k}p_{n-k}(x,y)\ z^k \\ &= \sum_{n=0}^{\infty}\ \mu_n\ \tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c).\end{aligned}$$

Secondly, if $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$ can be expanded in terms of $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$, we can verify that the function $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$ satisfies Equation (18). The proof of Theorem 1 is now complete. □

3. Generating Functions of the General q-Polynomials

In this section, we first give a generating function of the general q-polynomials by the method of q-difference equations as the application of our main results.

Theorem 2. *The following assertion holds true*:

$$\begin{aligned} &\sum_{n=0}^{\infty} \tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{t^n}{(q;q)_n} \\ &= \frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,(zt)^k \qquad (|xt|<1). \end{aligned} \tag{24}$$

As a special case of Theorem 2, if we take $\tilde{r}=\tilde{s}=0$, we are led to Corollary 1 below.

Corollary 1. *For* $\max\{|xt|,|ztq^\alpha|\}<1$, *it is asserted that*

$$\sum_{n=0}^{\infty}\tilde{L}_n^{(0,0)}(\alpha,x,y,z,a,b,c)\,\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix}-q^{-\alpha},a,b; & \\ & q;ztq^\alpha \\ -q,c; & \end{matrix}\right]. \tag{25}$$

Proof of Theorem 2. Denoting by $f(\alpha,x,y,a,,b,c,z,\tilde{r},\tilde{s})$ the right-hand side of the Equation (24), we can rewrite equivalently as follows:

$$\begin{aligned} f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s}) &= \sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}\,q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,z^k\cdot\frac{t^k\,(yt;q)_\infty}{(xt;q)_\infty} \\ &= \sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,z^k\,D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}. \end{aligned} \tag{26}$$

Now, letting:

$$f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=\sum_{k=0}^{\infty}A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\,z^k$$

and

$$A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})=q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}\,D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}, \tag{27}$$

we obtain:

$$A_0(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})=\frac{(yt;q)_\infty}{(xt;q)_\infty} \tag{28}$$

and

$$f(\alpha,x,y,a,b,c,0,\tilde{r},\tilde{s})=A_0(\alpha,x,y,a,\tilde{r},\tilde{s}).$$

Thus, upon substituting from (28) into (27), we find that

$$\begin{aligned} A_k(\alpha,x,y,a,b,c,\tilde{r},\tilde{s}) &= q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k} \\ &\cdot D_{xy}^k\{A_0(\alpha,x,y,a,b,c,\tilde{r},\tilde{s})\}. \end{aligned} \tag{29}$$

It is easily observed that $f(\alpha,x,y,a,,b,c,z,\tilde{r},\tilde{s})$ is a nine-variable analytic function in a neighborhood of

$$(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=(0,0,0,0,0,0,0,0,0)\in\mathbb{C}^9.$$

Hence, $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$ can be expanded in terms of $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ as follows:

$$f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=\sum_{n=0}^{\infty}\mu_n\cdot\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c). \tag{30}$$

Setting $z=0$ and using the following relation:

$$\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,0,a,b,c)=p_n(x,y)$$

in the resulting equation, we get:

$$f(\alpha,x,y,a,b,c,0,\tilde{r},\tilde{s}) = \frac{(yt;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \mu_n \cdot p_n(x,y). \tag{31}$$

Finally, upon comparing the coefficients of $p_n(x,y)$, we find that

$$\mu_n = \frac{t^n}{(q;q)_n}.$$

Substituting the above equation into Equation (30), we deduce that $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$ equals the left-hand side of Equation (24). This evidently completes the proof of Theorem 2. □

Remark 3. *Setting $y = 0$ and $b = c = 0$ in (24), we get the following concluding remark in the earlier work [10]:*

$$\sum_{n=0}^{\infty} L_{\tilde{r},\tilde{s}}(\alpha,x,z,a)\,\frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a;q)_k}{(q^2;q^2)_k}q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,(zt)^k \qquad \big(|xt|<1\big). \tag{32}$$

In Equation (24), we let $\alpha \to \infty$ and set $\tilde{r} = \tilde{s} = 0$, $a = -q$, and $b = c = 0$. Then, upon interchanging x and y, and replacing z by $-z$, we get the following corollary.

Corollary 2 ([16], Theorem 2.6)**.** *For $|yt| < 1$, it is asserted that*

$$\sum_{n=0}^{\infty} F_n(x,y,z;q)\frac{(-1)^n\, q^{\binom{n}{2}}\, t^n}{(q;q)_n} = \frac{(xt,zt;q)_\infty}{(yt;q)_\infty}. \tag{33}$$

4. Rogers Type and Extended Rogers Type Formulas for the General q-Polynomials

In this section, we apply the main results to state and prove the Rogers type and the extended Rogers-type formulas for the general q-polynomials by using the q-difference equations, so that we can derive the Rogers formula for the trivariate q-polynomials.

We first recall that Chen and Liu [22] studied the q-exponential operator as follows (see [17]):

$$T(bD_a) = \sum_{n=0}^{\infty}\frac{(bD_a)^n}{(q;q)_n}, \tag{34}$$

where the usual q-differential operator, or the q-derivative, is defined by

$$D_a f(a) = \frac{f(a)-f(qa)}{a}. \tag{35}$$

The following q-Leibniz rule for the q-derivative operator D_a is a variation of the q-binomial theorem (see [23]):

$$D_a^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot D_a^k\{f(a)\}D_a^{n-k}\Big\{g\big(aq^k\big)\Big\}, \tag{36}$$

where D_a^0 is understood as the identity operator.

The following important property of the q-derivative operator D_a is easily derivable.

Lemma 3. *For $|a\omega| < 1$, the following result holds true:*

$$D_a^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \omega^n \frac{(s/\omega;q)_n}{(as;q)_n} \frac{(as;q)_\infty}{(a\omega;q)_\infty}. \tag{37}$$

Lemma 4. *For $k \in \mathbb{N}_0$ and $|x\omega| < 1$, it is asserted that*

$$\begin{aligned} &T(tD_\omega)\left\{\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\omega^k\right\} \\ &= \frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\omega^k \sum_{j=0}^{k} \frac{(-1)^j\, q^{kj-\binom{j}{2}}(q^{-k},x\omega;q)_j\,(t/\omega)^j}{(y\omega,q;q)_j} \\ &\cdot {}_2\Phi_1\left[\begin{matrix} y/x,0; \\ y\omega q^j; \end{matrix}\; q;xt\right]. \end{aligned} \tag{38}$$

We now turn to the generalized Rogers-Szegö polynomials which are defined by (see [24,25]):

$$r_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k\, y^{n-k}, \tag{39}$$

where (see [25]):

$$r_n(x,y) = T(xD_y)\{y^n\}. \tag{40}$$

We are now in a position to state and prove the following Rogers-type formula for the general q-polynomials by using the q-difference equations.

Theorem 3. *For $\max\{|x\omega|,|xt|\} < 1$, the following Rogers-type formula holds true:*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tilde{L}_{n+m}^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\, \frac{t^n}{(q;q)_n}\, \frac{\omega^m}{(q;q)_m} \\ &= \frac{(y\omega;q)_\infty}{(x\omega;q)_\infty} \sum_{k=0}^{\infty}\sum_{j=0}^{k} q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\, \frac{(-q^{-\alpha},a,b;q)_k(\omega z)^k}{(-q,c;q)_k(q;q)_{k-j}} \\ &\cdot \frac{(x\omega;q)_j(t/\omega)^j}{(y\omega,q;q)_j}\, {}_2\Phi_1\left[\begin{matrix} y/x,0; \\ y\omega q^j; \end{matrix}\; q;xt\right]. \end{aligned} \tag{41}$$

Remark 4. *As a special case of Theorem 3, we let $\alpha \to \infty$ and set $\tilde{r} = \tilde{s} = 0$, $a = -q$, and $b = c = 0$ (41). Then, upon interchanging x and y, and replacing z by $-z$, we get the following corollary.*

Corollary 3 (see [16], Theorem 3.1)**.** *It is asserted that*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{n+m}(x,y,z;q)(-1)^{n+m}\, q^{\binom{n+m}{2}}\, \frac{t^n}{(q;q)_n}\, \frac{\omega^m}{(q;q)_m} \\ &= \frac{(x\omega,z\omega;q)_\infty}{(y\omega;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\, q^{\binom{j}{2}}(y\omega;q)_j(zt)^j}{(x\omega,z\omega,q;q)_j} \\ &\cdot {}_2\Phi_1\left[\begin{matrix} x/y,0; \\ x\omega q^j; \end{matrix}\; q;yt\right] \qquad (|\omega y| < 1). \end{aligned} \tag{42}$$

Proof of Theorem 3. Denoting the right-hand side of the Equation (24) by $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$, it can be written equivalently as follows:

$$\begin{aligned}
&f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})\\
&=\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,\frac{(-q^{-\alpha},a,b;q)_k(\omega z)^k}{(-q,c;q)_k(q;q)_{k-j}}\,\frac{(x\omega;q)_j\,(t/\omega)^j}{(y\omega,q;q)_j}\\
&\quad\cdot{}_2\Phi_1\left[\begin{matrix} y/x,0; \\ y\omega q^j; \end{matrix}\;q;xt\right]\\
&=\sum_{k=0}^{\infty} q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\frac{(-q^{-\alpha},a,b;q)_k z^k}{(q^2,c;q^2)_k}\sum_{j=0}^{k}\frac{(y\omega q^j;q)_\infty(q;q)_k\,\omega^{k-j}\,t^j}{(x\omega q^j;q)_\infty(q;q)_j(q;q)_{k-j}}\\
&\quad\cdot{}_2\Phi_1\left[\begin{matrix} y/x,0; \\ y\omega q^j; \end{matrix}\;q;xt\right]\\
&=\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}z^k T(tD_\omega)\left\{\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\,\omega^k\right\}\\
&=T(tD_\omega)\left\{\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}(\omega z)^k\right\}\quad(\text{by using (24)})\\
&=T(tD_\omega)\left\{\sum_{m=0}^{\infty}\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{\omega^m}{(q;q)_m}\right\}\\
&=\sum_{m=0}^{\infty}\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{1}{(q;q)_m}T(tD_\omega)\{\omega^m\}\quad(\text{by (40)})\\
&=\sum_{m=0}^{\infty}\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{r_m(t,\omega)}{(q;q)_m}\\
&=\sum_{m=0}^{\infty}\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{1}{(q;q)_m}\sum_{n=0}^{m}\begin{bmatrix} m\\ n\end{bmatrix}_q t^n\,\omega^{m-n}\\
&=\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)\,\frac{t^n}{(q;q)_n}\,\frac{\omega^{m-n}}{(q;q)_{m-n}}.
\end{aligned}$$

It is easily seen that $f(\alpha,x,y,a,,b,c,z,\tilde{r},\tilde{s})$ is a nine-variable analytic function in a neighborhood of:

$$(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=(0,0,0,0,0,0,0,0,0)\in\mathbb{C}^9.$$

Hence, $f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})$ can be expanded in terms of $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ by Theorem 1 as follows:

$$f(\alpha,x,y,a,b,c,z,\tilde{r},\tilde{s})=\sum_{m,n=0}^{\infty}\mu_{m,n}\cdot\tilde{L}_m^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c).\qquad(43)$$

Letting $z = 0$ in Equation (43), we obtain:

$$\begin{aligned} f(\alpha, x, y, a, b, c, 0, \tilde{r}, \tilde{s}) &= \frac{(y\omega; q)_\infty}{(x\omega; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} x/y, 0; \\ \\ y\omega; \end{matrix} \, q; yt \right] \\ &= \sum_{n=0}^{\infty} \frac{p_n(x,y)t^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{p_m(x,yq^n)\omega^m}{(q;q)_m} \\ &= \sum_{m,n=0}^{\infty} \mu_{m,n} \cdot p_m(x,y). \end{aligned} \tag{44}$$

Comparing the coefficients of $p_m(x, y)$, we deduce that

$$\mu_{m,n} = \frac{t^n \omega^{m-n}}{(q;q)_n (q;q)_{m-n}}.$$

Substituting the above equation into Equation (43), we find that $f(\alpha, x, y, a, b, c, z, \tilde{r}, \tilde{s})$ is equal to the left-hand side of Equation (41). This completes the proof of Theorem 3. □

5. Mixed Generating Functions for the General q-Polynomials

The Hahn polynomials [26,27] (or the Al-Salam-Carlitz polynomials [28,29]) are defined as follows:

$$\phi_n^{(\sigma)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ n \end{bmatrix}_q (\sigma; q)_k \, x^k. \tag{45}$$

In the year 1989, Srivastava and Agarwal [30] utilized the method of transformation theory in order to establish the following result. More recently, Cao [29] used the decomposition technique of exponential operators to give an alternative proof. For more information about the Srivastava-Agarwal-type generating functions and other related results, the reader is referred to the works [13,26–31].

Lemma 5 (see [30], Eq. (3.20))**.** *It is asserted that*

$$\sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q)(\lambda;q)_n \frac{t^n}{(q;q)_n} = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} {}_2\Phi_1 \left[\begin{matrix} \lambda, \sigma; \\ \\ \lambda t; \end{matrix} \, q; xt \right] \tag{46}$$

$$\big(\max\{|t|, |xt|\} < 1\big).$$

In Theorem 4 below, we apply the main results to state and prove a mixed generating function for the general q-polynomials by making use of the q-difference equations.

Theorem 4. *For $|ut| < 1$, the following result holds true*:

$$\begin{aligned} &\sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q) \tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, u, v, z, a, b, c) \frac{t^n}{(q;q)_n} \\ &= \frac{(vt;q)_\infty}{(ut;q)_\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{m} \frac{(\sigma;q)_m \, x^m}{(q;q)_m} \frac{(q^{-m}, ut;q)_j q^j}{(vt,q;q)_j} \\ &\quad \cdot \frac{(-q^{-\alpha};q)_k (a,b;q)_k \, (tzq^j)^k}{(q^2,c;q^2)_k} q^{k\alpha + \tilde{r}\binom{k}{2} - \tilde{s}\binom{k+1}{2}}. \end{aligned} \tag{47}$$

In our proof of Theorem 4, the following q-Chu-Vandermonde formula will be needed.

Lemma 6 (q-Chu-Vandermonde sum [1], Eq. (II.6))**.** *The following q-summation holds true*:

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, a; \\ c; \end{matrix}\; q; q\right] = \frac{(c/a;q)_n}{(c;q)_n}\, a^n. \tag{48} $$

Remark 5. *If we let $\alpha \to \infty$, set $a = -q$ and $b = c = 0$, and $\tilde{r} = \tilde{s} = 0$, interchange u and v, and replace z by $-z$, in Theorem 4, we are led to the following corollary.*

Corollary 4 (Mixed Generating Function for the Trivariate q-Polynomials $F_n(x, y, z; q)$)**.** *The following mixed generating function holds true*:

$$ \sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q) F_n(u, v, z; q)\, \frac{(-1)^n\, q^{\binom{n}{2}}\, t^n}{(q;q)_n} = \frac{(\sigma x, ut, zt; q)_\infty}{(vt, x; q)_\infty}\, {}_4\Phi_3\left[\begin{matrix} \sigma, vt, 0, 0; \\ ut, zt, q/x; \end{matrix}\; q; q\right]. \tag{49} $$

Proof of Theorem 4. Equation (47) can be written equivalently as follows:

$$ \begin{aligned} \sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q) \tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha, u, v, z, a, b, c)\, \frac{t^n}{(q;q)_n} &= \sum_{m=0}^{\infty} \frac{(\sigma;q)_m\, x^m}{(q;q)_m} \sum_{j=0}^{m} \frac{(q^{-m};q)_j\, q^j}{(q;q)_j} \sum_{k=0}^{\infty} q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}} \\ &\quad \cdot \frac{(-q^{-\alpha};q)_k (a, b; q)_k\, z^k}{(q^2, c; q^2)_k}\, D_{uv}^k \left\{ \frac{(vtq^j;q)_\infty}{(utq^j;q)_\infty} \right\}. \end{aligned} \tag{50} $$

Now, if we use $g(\alpha, u, v, a, b, c, z, \tilde{r}, \tilde{s})$ to denote the right-hand side of (50), it is easy to see that $g(\alpha, u, v, a, , b, c, z, \tilde{r}, \tilde{s})$ satisfies (18). Thus, upon letting

$$ g(\alpha, u, v, a, b, c, z, \tilde{r}, \tilde{s}) = \sum_{k=0}^{\infty} B_k(\alpha, x, y, a, b, c, \tilde{r}, \tilde{s}) z^k $$

and

$$ \begin{aligned} B_k(\alpha, u, v, a, b, c, \tilde{r}, \tilde{s}) &= q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\, \frac{(-q^{-\alpha};q)_k (a, b; q)_k}{(q^2; c, q^2)_k} \\ &\quad \cdot D_{uv}^k \left\{ \sum_{m=0}^{\infty} \frac{(\sigma;q)_m\, x^m}{(q;q)_m} \sum_{j=0}^{m} \frac{(q^{-m};q)_j q^j}{(q;q)_j}\, \frac{(vtq^j;q)_\infty}{(utq^j;q)_\infty} \right\}, \end{aligned} \tag{51} $$

we obtain

$$\begin{aligned}
&B_0(\alpha,u,v,a,b,c,\tilde{r},\tilde{s}) \\
&= \frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{m=0}^{\infty}\frac{(\sigma;q)_m\,x^m}{(q;q)_m}\sum_{j=0}^{m}\frac{(q^{-m},ut;q)_j q^j}{(vt,q;q)_j} \\
&= \frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{m=0}^{\infty}\frac{(\sigma;q)_m\,x^m}{(q;q)_m}{}_2\Phi_1\left[\begin{matrix} q^{-m},ut; \\ vt; \end{matrix}\; q;q\right] \quad \text{(by (48))} \\
&= \frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{m=0}^{\infty}\frac{(\sigma;q)_m\,x^m}{(q;q)_m}\frac{(v/u;q)_m(ut)^m}{(vt;q)_m} \\
&= \frac{(vt;q)_\infty}{(ut;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} v/u,\sigma; \\ vt; \end{matrix}\; q;uxt\right] \\
&= \sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)\,\frac{p_n(u,v)\,t^n}{(q;q)_n}
\end{aligned} \tag{52}$$

and

$$g(\alpha,u,v,a,b,c,0,\tilde{r},\tilde{s}) = B_0(\alpha,u,v,a,b,c,\tilde{r},\tilde{s}).$$

Upon substituting from Equation (52) into Equation (51), we get:

$$\begin{aligned}
B_k(\alpha,u,v,a,b,c,\tilde{r},\tilde{s}) = q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k} \\
D_{uv}^k\{B_0(\alpha,u,v,a,b,c,\tilde{r},\tilde{s})\}.
\end{aligned} \tag{53}$$

In light of the above identities, $g(\alpha,u,v,a,,b,c,z,\tilde{r},\tilde{s})$ satisfies Equation (18), so we have:

$$g(\alpha,u,v,a,b,c,z,\tilde{r},\tilde{s}) = \sum_{n=0}^{\infty}\mu_n\cdot\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,u,v,z,a,b,c). \tag{54}$$

Furthermore, we deduce that

$$\begin{aligned}
&g(\alpha,u,v,a,b,c,z,\tilde{r},\tilde{s}) \\
&= \sum_{k=0}^{n}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}\,q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,z^k\,D_{uv}^k\left\{\sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)\,\frac{p_n(u,v)\,t^n}{(q;q)_n}\right\} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-q^{-\alpha},a,b;q)_k}{(q^2,c;q^2)_k}\,q^{k\alpha+\tilde{r}\binom{k}{2}-\tilde{s}\binom{k+1}{2}}\,z^k\,\phi_n^{(\sigma)}(x|q)\,\frac{p_{n-k}(u,v)\,t^n}{(q;q)_{n-k}} \\
&= \sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)\,\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\begin{bmatrix} \alpha \\ k\end{bmatrix}_{-q} q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}\,\frac{(a,b;q)_k}{(c;q)_k}p_{n-k}(u,v)\,z^k \\
&= \sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)\,\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,u,v,z,a,b,c)\,\frac{t^n}{(q;q)_n}.
\end{aligned} \tag{55}$$

By comparing the coefficients of $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,u,v,z,a,b,c)$ on both sides of Equations (54) and (55), we obtain:

$$\mu_n = \phi_n^{(\sigma)}(x|q)\frac{t^n}{(q;q)_n}. \tag{56}$$

The proof of Theorem 4 is thus completed. □

6. A Transformation Identity Involving Hecke-Type Series for the General q-Polynomials

Jia and Zheng [32] proved a general expansion formula involving the Askey-Wilson polynomials by applying the Bailey transform and the Bressoud inversion.

Proposition 2 (see [32], Proposition 2.3)**.** *The following series identity holds true for suitably-bounded sequences* $\{\beta_n\}_{n\in\mathbb{N}_n}$ *and* $\{\delta_n\}_{n\in\mathbb{N}_n}$:

$$\sum_{n=0}^{\infty}\beta_n\,\delta_n=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,a/b;q)_n(b/a)^n}{(1-a)(bq,q;q)_n}\sum_{k=0}^{n}\frac{(1-bq^{2k})(aq^n,q^{-n};q)_k\,q^k}{(1-b)(bq^{n+1},bq^{1-n}/a;q)_k}\,\beta_k \cdot\sum_{r=0}^{\infty}\frac{(b/a;q)_r(b;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\,\delta_{r+n}. \tag{57}$$

In this section, we give an application of the above series identity (57).

Theorem 5. *For* $\max\{|aq|,|aq/\alpha\beta|\}<1$, *the following transformation identity holds true*:

$$\sum_{n=0}^{\infty}\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\n\end{bmatrix}_{-q}\frac{(\tilde{a},\tilde{b},\alpha,\beta;q)_n}{(\tilde{c};q)_n}\,q^{\tau(\tilde{r},\tilde{s})+\binom{n}{2}}\left(\frac{aq}{\alpha\beta}\right)^n P_{N-n}(x,y)\,z^n$$
$$=\frac{(aq/\alpha,aq/\beta;q)_\infty}{(aq,aq/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(\alpha,\beta,a;q)_n}{(1-a)(aq/\alpha,aq/\beta,q;q)_n}\left(\frac{aq}{\alpha\beta}\right)^n \cdot\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\k\end{bmatrix}_{-q}\frac{(aq^n,q^{-n},\tilde{a},\tilde{b};q)_k}{(\tilde{c};q)_k}\,q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}\,P_{N-k}(x,y)\,z^k. \tag{58}$$

In our proof of Theorem 5, the following q-Gauss sum will be needed.

Lemma 7 (q-Gauss sum [1], Eq. (II.8))**.** *The following q-summation formula holds true*:

$${}_2\Phi_1\left[\begin{matrix}a,b;\\c;\end{matrix}\;q;\frac{c}{ab}\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}\qquad\left(\left|\frac{c}{ab}\right|<1\right). \tag{59}$$

Proof of Theorem 5. Upon setting $b=0$,

$$\beta_n=\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\n\end{bmatrix}_{-q}\frac{(\tilde{a},\tilde{b};q)_n}{(\tilde{c};q)_n}\,q^{\tau(\tilde{r},\tilde{s})+\binom{n}{2}}\,P_{N-n}(x,y)\,z^n$$

and

$$\delta_n=(\alpha,\beta;q)_n\left(\frac{aq}{\alpha\beta}\right)^n$$

in (57), we obtain

$$\sum_{n=0}^{\infty}\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\n\end{bmatrix}_{-q}\frac{(\tilde{a},\tilde{b},\alpha,\beta;q)_n}{(\tilde{c};q)_n}\,q^{\tau(\tilde{r},\tilde{s})+\binom{n}{2}}\left(\frac{aq}{\alpha\beta}\right)^n P_{N-n}(x,y)\,z^n$$
$$=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(\alpha,\beta,a;q)_n(aq/\alpha\beta)^n}{(1-a)(q;q)_n(aq;q)_{2n}}\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\k\end{bmatrix}_{-q}\frac{(aq^n,q^{-n},\tilde{a},\tilde{b};q)_k}{(\tilde{c};q)_k}$$
$$\cdot\,q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}P_{N-k}(x,y)\,z^k\sum_{r=0}^{\infty}\frac{(\alpha q^n,\beta q^n;q)_r}{(aq^{1+2n},q;q)_r}\left(\frac{aq}{\alpha\beta}\right)^r$$
$$=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(\alpha,\beta,a;q)_n}{(1-a)(q;q)_n(aq;q)_{2n}}\left(\frac{aq}{\alpha\beta}\right)^n\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\k\end{bmatrix}_{-q}\frac{(aq^n,q^{-n},\tilde{a},\tilde{b};q)_k}{(\tilde{c};q)_k}$$
$$\cdot\,q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}P_{N-k}(x,y)\,z^k\,{}_2\Phi_1\left[\begin{matrix}\alpha q^n,\beta q^n;\\ aq^{1+2n};\end{matrix}\;q;\frac{aq}{\alpha\beta}\right]. \tag{60}$$

Thus, by applying the q-Gauss sum (48) in the right-hand side of the above equation, we find that

$$\sum_{n=0}^{\infty}\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\n\end{bmatrix}_{-q}\frac{(\tilde{a},\tilde{b},\alpha,\beta;q)_n}{(\tilde{c};q)_n}\,q^{\tau(\tilde{r},\tilde{s})+\binom{n}{2}}\left(\frac{aq}{\alpha\beta}\right)^n P_{N-n}(x,y)\,z^n$$
$$=\frac{(aq/\alpha,aq/\beta;q)_\infty}{(aq,aq/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(\alpha,\beta,a;q)_n}{(1-a)(aq/\alpha,aq/\beta,q;q)_n}\left(\frac{aq}{\alpha\beta}\right)^n$$
$$\cdot\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\tilde{\alpha}\\k\end{bmatrix}_{-q}\frac{(aq^n,q^{-n},\tilde{a},\tilde{b};q)_k}{(\tilde{c};q)_k}\,q^{\tau(\tilde{r},\tilde{s})+\binom{k}{2}}P_{N-k}(x,y)\,z^k,$$

which completes the proof of the result asserted by Theorem 5. □

Remark 6. *In Theorem 5, we set $z=q$ and let $N,\tilde{\alpha},\alpha,\beta\to\infty$. Then, upon putting $\tilde{r}=0$, $\tilde{s}=1$, $y=0$, $x=1$, and $\tilde{b}=0$ in Theorem 5, we can deduce the following result*:

$$\sum_{n=0}^{\infty}\frac{(\tilde{a};q)_n}{(\tilde{c},-q,q;q)_n}\,a^n$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}(1-aq^{2n})(a;q)_n\,a^n}{(1-a)(q;q)_n}\sum_{k=0}^{n}\frac{(aq^n,q^{-n},\tilde{a};q)_k}{(\tilde{c},-q,q;q)_k}\,q^k$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}(1-aq^{2n})(a;q)_n\,a^n}{(1-a)(q;q)_n}\,{}_3\Phi_2\left[\begin{matrix}q^{-n},aq^n,\tilde{a};\\ \tilde{c},-q;\end{matrix}\;q;q\right]. \tag{61}$$

7. Further Remarks and Observations

In our present investigation, we have made use of a general family of basic (or q-) polynomials, together with double q-binomial coefficients, as well as some homogeneous q-operators with a view to constructing several q-difference equations involving seven variables. We have derived the Rogers and the extended Rogers-type formulas as well as the Srivastava-Agarwal type bilinear generating functions for the q-polynomials considered in this paper, which generalize the generating functions for the Cigler polynomials. We have also derived a class of mixed generating functions by means of the above-mentioned q-difference equations.

In addition to the remarks and observations concerning the novelty and generality of the q-hypergeometric polynomials and their associated q-difference equations, which we have investigated in the preceding sections, by appropriately using the list of special cases presented in Remark 1, the various results which we have derived in this paper for the general q-polynomials $\tilde{L}_n^{(\tilde{r},\tilde{s})}(\alpha,x,y,z,a,b,c)$ defined in (12) would apply to derive the corresponding results for each of the q-polynomials listed in Remark 1. Indeed, as it is widely recognized, studies involving q-generating functions can lead naturally to interesting and useful properties of the q-polynomial sequences which they generate. Moreover, as pointed out in the monograph by Srivastava and Karlsson ([4], pp. 350–351), the widely- and extensively-investigated families of q-series and q-polynomials have been demonstrated to be useful in a wide variety of fields such as, for example, number theory and partition theory, Lie theory, quantum mechanics and particle physics, non-linear electric circuit theory, combinatorial analysis, and so on. Our results for a significantly wide class of q-polynomials are potentially useful in some of these fields. With a view to motivating the interested readers toward the theory and widespread applications of various families of q-series, q-polynomials, as well as q-difference and q-derivative operators, we have chosen here to include references (see, for example, [33–45]) to various related developments in recent years.

We remark in conclusion that, in the recently-published survey-cum-expository review articles by Srivastava (see [6,7]), the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional forced-in parameter p being redundant or superfluous (see, for details, ([6], p. 340) and ([7], pp. 1511–1512)). This remarkable demonstration by Srivastava (see [6,7]) will surely apply to any attempt to produce the rather straightforward (p,q)-variations of the results that we have presented herein.

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