mathematics

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Some New Concepts Related to Integral Operators and Inequalities on Coordinates in Fuzzy Fractional Calculus

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Abstract: In interval analysis, the fuzzy inclusion relation and the fuzzy order relation are two different concepts. Under the inclusion connection, convexity and non-convexity form a substantial link with various types of inequalities. Moreover, convex fuzzy-interval-valued functions are well known in convex theory because they allow us to infer more exact inequalities than convex functions. Most likely, integral operators play significant roles to define different types of inequalities. In this paper, we have successfully introduced the Riemann–Liouville fractional integrals on coordinates via fuzzy-interval-valued functions (FIVFs). Then, with the help of these integrals, some fuzzy fractional Hermite–Hadamard-type integral inequalities are also derived for the introduced coordinated convex FIVFs via a fuzzy order relation (FOR). This FOR is defined by φ-cuts or level-wise by using the Kulish–Miranker order relation. Moreover, some related fuzzy fractional Hermite–Hadamard-type integral inequalities are also obtained for the product of two coordinated convex fuzzy-interval-valued functions. The main results of this paper are the generalization of several known results.

Keywords: fuzzy Riemann–Liouville fractional integrals; Hermite–Hadamard-type inequality; fuzzy-interval-valued functions

Citation: Khan, M.B.; Santos-García, G.; Zaini, H.G.; Treanţă, S.; Soliman, M.S. Some New Concepts Related to Integral Operators and Inequalities on Coordinates in Fuzzy Fractional Calculus. *Mathematics* **2022**, *10*, 534. https://doi.org/10.3390/math10040534

Academic Editor: Christopher Goodrich

Received: 11 January 2022
Accepted: 6 February 2022
Published: 9 February 2022

1. Introduction

The classical version of Hermite–Hadamard inequality can be put in the following manner:

Let $\mathfrak{S}: K \to \mathbb{R}$ be a convex function on a convex set K and $\rho, \varsigma \in K$ with $\rho \leq \varsigma$. Then:

$$\mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\mathfrak{S}(\varpi)d\varpi \leq \frac{\mathfrak{S}(\rho)+\mathfrak{S}(\varsigma)}{2}. \tag{1}$$

In [1], Fejér looked at the key extensions of HH inequality, which is known as Hermite–Hadamard–Fejér inequality (HH–Fejér inequality):

Let $\mathfrak{S}: K \to \mathbb{R}$ be a convex function on a convex set K and $\rho, \varsigma \in K$ with $\rho \leq \varsigma$. Then:

$$\mathfrak{S}\left(\frac{\rho+\varsigma}{2}\right) \leq \frac{1}{\int_{\rho}^{\varsigma}\mathfrak{D}(\varpi)d\varpi}\int_{\rho}^{\varsigma}\mathfrak{S}(\varpi)\mathfrak{D}(\varpi)d\varpi \leq \frac{\mathfrak{S}(\rho)+\mathfrak{S}(\varsigma)}{2}\int_{\rho}^{\varsigma}\mathfrak{D}(\varpi)d\varpi. \tag{2}$$

If $\mathfrak{D}(\varpi) = 1$, then we obtain (1) from (2).

Many authors have focused their efforts to generalize inequalities (1) and (2); see [1–6]. They play an important role in convex analysis and may be a very strong tool for monitoring and quantifying mistakes. It is worth mentioning that Sarikaya et al. [7] used the Riemann–Liouville fractional integrals to develop new Hermite–Hadamard inequalities. Since then, many papers have been published that have extended various types of fractional integrals and have provided fresh and fascinating improvements of Hermite–Hadamard-type inequalities utilizing these integrals. For the Atangana–Baleanu fractional integral, Fernandez and Mohammed [8] constructed certain Hermite–Hadamard-type inequalities. In the context of fractional calculus, Mohammed and Abdeljawad [9] demonstrated novel Hermite–Hadamard-type inequalities with regard to functions with non-singular kernels. For more similar results, refs. [7–18] is a good place to look.

On the other hand, Moore [19] established the theory of interval analysis to increase the dependability of calculation outputs and autonomous operation error analysis. Interval analysis is a robust model for dealing with interval uncertainty, and it has been widely applied and broadened in fields such as control theory [20], dynamical game theory [21], and many more. Several well-known inequalities have recently been extended to interval-valued functions. Using the Hukuhara derivative, Chalco-Cano et al. [22] derived Ostrowski-type inequalities for interval-valued functions. The Minkowski and Beckenbach-type inequalities for interval-valued functions were discovered by Román-Flores et al. [23]. For fuzzy-interval-valued functions, Khan et al. [24–26] derived some new versions of Hermite–Hadamard-type inequalities and proved their validity with the help of non-trivial examples. Moreover, Khan et al. [27–29] discussed some novel types of Hermite–Hadamard-type inequalities in fuzzy-interval fractional calculus and proved that many classical versions are special cases of these inequalities.

For interval-valued functions, Liu et al. [30] demonstrated Hermite–Hadamard inequality using interval Riemann–Liouville-type fractional integrals. Zhao et al. [31,32] discovered Hermite–Hadamard inequalities for interval-valued coordinated functions very recently. Budak et al. [33] introduced several novel Hermite–Hadamard inequalities and defined Riemann–Liouville-type fractional integrals for interval-valued coordinated functions. Recently, Khan et al. [34] introduced the new class of convexity in fuzzy-interval calculus, which consists of coordinated convex fuzzy-interval-valued functions, and with the support of these classes, some Hermite–Hadamard-type inequalities are obtained via newly defined fuzzy-interval double integrals. For more information related to interval-valued and fuzzy-interval-valued functions, see [35–65].

Inspired by ongoing research work, we provide a novel class of Hermite–Hadamard-type inequalities for coordinated convex fuzzy-interval-valued functions through fuzzy-interval Riemann–Liouville-type fractional integrals. Motivated by the work of Khan et al. [27,28,34] and Budak et al. [33], we obtain Hermite–Hadamard-type inequalities for the products of two fuzzy-interval-valued coordinated functions.

2. Preliminaries

Let $\mathbb{R}_I$ and $\mathbb{F}_0$ be the collection of all closed and bounded intervals and fuzzy intervals of $\mathbb{R}$. We use $\mathbb{R}_I^+$ to represent the set of all positive intervals. The collection of all Riemann integrable real valued functions, Riemann integrable *IV-Fs* and fuzzy Riemann integrable *F-IV-Fs* over $[\rho, \varsigma]$, is denoted by $\mathcal{R}_{[\rho,\varsigma]}$, $\mathcal{IR}_{[\rho,\varsigma]}$, and $\mathcal{FR}_{([\rho,\varsigma])}$ respectively. For more conceptions on interval-valued functions and fuzzy-interval-valued functions, see [43–47]. Moreover, the inclusion " $\subseteq$ " means that:

$r \subseteq \eta$ if and only if $[r_*, r^*] \subseteq [\eta_*, \eta^*]$, if and only if $\eta_* \leq r_*$, $r^* \leq \eta^*$, for all $[r_*, r^*], [\eta_*, \eta^*] \in \mathbb{R}_I$.

Remark 1. [46] *The relation " $\leq_I$ " defined on $\mathbb{R}_I$ by $[r_*, r^*] \leq_I [\eta_*, \eta^*]$ if and only if $r_* \leq \eta_*, r^* \leq \eta^*$, for all $[r_*, r^*], [\eta_*, \eta^*] \in \mathbb{R}_I$, it is an order relation.*

Proposition 1. [47] *Let $\mathbb{F}_0$ be a set of fuzzy numbers. If $\mathcal{r}, \eta \in \mathbb{F}_0$, then relation " $\preccurlyeq$ " defined on $\mathbb{F}_0$ by:*

$$\mathcal{r} \preccurlyeq \eta \text{ if and only if } [\mathcal{r}]^{\varphi} \leq_I [\eta]^{\varphi}, \text{ for all } \varphi \in [0,1], \tag{3}$$

this relation is known as a partial order relation.

Definition 1. [16,48] *Let $\mathfrak{S}: [\rho, \varsigma] \to \mathbb{R}_I^+$ be an interval-valued function and $\mathfrak{S} \in \mathcal{IR}_{[\rho,\varsigma]}$. Then, interval Riemann–Liouville-type integrals of $\mathfrak{S}$ are defined as:*

$$\mathcal{J}_{\rho^+}^{\alpha}\, \mathfrak{S}(y) = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} \mathfrak{S}(\mathrm{t})\, d\mathrm{t} \quad (y > \rho), \tag{4}$$

$$\mathcal{J}_{\varsigma^-}^{\alpha}\, \mathfrak{S}(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\varsigma} (\mathrm{t} - y)^{\alpha-1} \mathfrak{S}(\mathrm{t})\, d\mathrm{t} \quad (y < \varsigma) \tag{5}$$

where $\alpha > 0$ and Γ is the gamma function.

Recently, Allahviranloo et al. [49] *defined the following fuzzy interval Riemann–Liouville fractional integral integrals:*

Definition 2. *Let $\alpha > 0$ and $L([\rho, \varsigma], \mathbb{F}_0)$ be the collection of all Lebesgue measurable fuzzy-IVFs on $[\rho, \varsigma]$. Then, the fuzzy interval left and right Riemann–Liouville fractional integral of $\widetilde{\mathfrak{S}} \in L([\rho, \varsigma], \mathbb{F}_0)$ with order $\alpha > 0$ are defined by:*

$$\mathcal{J}_{\rho^+}^{\alpha}\, \widetilde{\mathfrak{S}}(y) = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} \widetilde{\mathfrak{S}}(\mathrm{t})\, d\mathrm{t}, \quad (y > \rho), \tag{6}$$

and:

$$\mathcal{J}_{\varsigma^-}^{\alpha}\, \widetilde{\mathfrak{S}}(y) = \frac{1}{\Gamma(\alpha)} \int_{y}^{\boldsymbol{b}} (\mathrm{t} - y)^{\alpha-1} \widetilde{\mathfrak{S}}(\mathrm{t})\, d\mathrm{t}, \quad (y < \varsigma) \tag{7}$$

respectively, where $\Gamma(y) = \int_0^{\infty} \mathrm{t}^{y-1} e^{-\mathrm{t}}\, d\mathrm{t}$ is the Euler gamma function. The fuzzy interval left and right Riemann–Liouville fractional integral y based on left and right end point functions can be defined, that is:

$$\begin{aligned} \left[\mathcal{J}_{\rho^+}^{\alpha}\, \widetilde{\mathfrak{S}}(y)\right]^{\varphi} &= \tfrac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} \mathfrak{S}_{\varphi}(\mathrm{t})\, d\mathrm{t} \\ &= \tfrac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} [\mathfrak{S}_*(\mathrm{t}, \varphi), \mathfrak{S}^*(\mathrm{t}, \varphi)]\, d\mathrm{t}, (y > \rho), \end{aligned} \tag{8}$$

where:

$$\mathcal{J}_{\rho^+}^{\alpha}\, \mathfrak{S}_*(y, \varphi) = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} \mathfrak{S}_*(\mathrm{t}, \varphi)\, d\mathrm{t}, \quad (y > \rho), \tag{9}$$

and:

$$\mathcal{J}_{\rho^+}^{\alpha}\, \mathfrak{S}^*(y, \varphi) = \frac{1}{\Gamma(\alpha)} \int_{\rho}^{y} (y - \mathrm{t})^{\alpha-1} \mathfrak{S}^*(\mathrm{t}, \varphi)\, d\mathrm{t}, \qquad (y > \rho), \tag{10}$$

Similarly, we can define right Riemann–Liouville fractional integral $\widetilde{\mathfrak{S}}(y)$ based on left and right end point functions.

Theorem 1. [27] *Let $\widetilde{\mathfrak{S}}: [c, d] \to \mathbb{F}_0$ be a convex fuzzy-IVF on $[c, d]$, whose φ-cuts establish the series* of IVFs $\mathfrak{S}_{\varphi}: [\mu, \nu] \subset \mathbb{R} \to {\mathbb{R}_C}^+$ *are given by $\mathfrak{S}_{\varphi}(y) = [\mathfrak{S}_*(y, \varphi), \mathfrak{S}^*(y, \varphi)]$ for all $y \in [\mu, \nu]$ and for all $\varphi \in [0, 1]$. If $\widetilde{\mathfrak{S}} \in L([c, d], \mathbb{F}_0)$, then:*

$$\widetilde{\mathfrak{S}}\left(\frac{\mu+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \widetilde{\mathfrak{S}}(\nu) \widetilde{+} \mathcal{I}_{\nu^-}^{\alpha}\ \widetilde{\mathfrak{S}}(\mu)\right] \preccurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\nu)}{2}. \tag{11}$$

Theorem 2. [27] *Let $\widetilde{\mathfrak{S}}, \tilde{\mathcal{J}} : [\mu, \nu] \to \mathbb{F}_0$ be two convex FIVFs. Then, from φ-cuts, we establish the series of IVFs $\mathfrak{S}_\varphi, \mathcal{J}_\varphi : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}_I^+$ are given by $\mathfrak{S}_\varphi(x) = [\mathfrak{S}_*(x,\varphi), \mathfrak{S}^*(x,\varphi)]$ and $\mathcal{J}_\varphi(x) = [\mathcal{J}_*(x,\varphi), \mathcal{J}^*(x,\varphi)]$ for all $x \in [\mu, \nu]$ and for all $\varphi \in [0,1]$. If $\widetilde{\mathfrak{S}} \,\widetilde{\times}\, \tilde{\mathcal{J}} \in L([\mu,\nu], \mathbb{F}_0)$ is fuzzy Riemann integrable, then:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \widetilde{\mathfrak{S}}(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\nu) + \mathcal{I}_{\nu^-}^{\alpha} \widetilde{\mathfrak{S}}(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\mu)\right] \\ &\qquad \preccurlyeq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\widetilde{\mathcal{M}}(\mu,\nu) \widetilde{+} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\widetilde{\mathcal{N}}(\mu,\nu),\end{aligned} \tag{12}$$

and:

$$\begin{aligned}&\widetilde{\mathfrak{S}}\left(\frac{\mu+\nu}{2}\right) \widetilde{\times}\, \tilde{\mathcal{J}}\left(\frac{\mu+\nu}{2}\right) \\ &\qquad \preccurlyeq \frac{\Gamma(\alpha+1)}{4(\nu-\mu)^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \widetilde{\mathfrak{S}}(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \widetilde{\mathfrak{S}}(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\mu)\right] \\ &+ \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\widetilde{\mathcal{M}}(\mu,\nu) + \frac{1}{2}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\widetilde{\mathcal{N}}(\mu,\nu),\end{aligned} \tag{13}$$

where $\widetilde{\mathcal{M}}(\mu,\nu) = \widetilde{\mathfrak{S}}(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\mu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\nu)$, $\widetilde{\mathcal{N}}(\mu,\nu) = \widetilde{\mathfrak{S}}(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\nu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}(\mu)$, $\mathcal{M}_\varphi(\mu,\nu) = \left[\mathcal{M}_*\big((\mu,\nu),\varphi\big), \mathcal{M}^*\big((\mu,\nu),\varphi\big)\right]$, *and* $\mathcal{N}_\varphi(\mu,\nu) = \left[\mathcal{N}_*\big((\mu,\nu),\varphi\big), \mathcal{N}^*\big((\mu,\nu),\varphi\big)\right]$.

2.1. Fuzzy-Interval-Valued Double Integrals and Convexity

For coordinated interval-valued function $\mathfrak{S}(x,y)$ and coordinated fuzzy-interval-valued function $\widetilde{\mathfrak{S}}(x,y)$, interval and fuzzy-interval Riemann–Liouville-type integrals are defined as:

Definition 3. [34] *A function $\widetilde{\mathfrak{S}}$: $\Delta = [\rho, \varsigma] \times [\mu, \nu] \to \mathbb{F}_0$ is called a fuzzy-interval double integrable (FD-integrable) on Δ if there exists $\tilde{B} \in \mathbb{F}_0$ such that, for each ϵ, there exists $\delta > 0$ such that:*

$$d\big(S(\widetilde{\mathfrak{S}}, P, \delta, \Delta), \tilde{B}\big) < \epsilon,$$

for every Riemann sum of $\widetilde{\mathfrak{S}}$ corresponding to $P \in \mathcal{P}(\delta, \Delta)$ and for arbitrary choice $\big(\eta_i, w_j\big) \in [x_{i-1}, x_i] \times \left[\varpi_{j-1}, \varpi_j\right]$ for $1 \le i \le k$ and $1 \le j \le n$. Then, we say that $\tilde{B}$ is the FR-integral of $\widetilde{\mathfrak{S}}$ on Δ and is denote by $\tilde{B} = (FD)\int_\rho^\varsigma \int_\mu^\nu \widetilde{\mathfrak{S}}(x,\varpi) d\varpi dx$ or $\tilde{B} = (FD)\iint_\Delta \widetilde{\mathfrak{S}} dA$.

Definition 4. [34] *A fuzzy-interval-valued map $\widetilde{\mathfrak{S}}$: $\Delta = [\rho, \varsigma] \times [\mu, \nu] \to \mathbb{F}_0$ is called FIVF on coordinates. Then, from φ-cuts, we establish the series of IVFs $\mathfrak{S}_\varphi : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ on coordinates are given by $\mathfrak{S}_\varphi(x,\varpi) = \left[\mathfrak{S}_*\big((x,\varpi),\varphi\big), \mathfrak{S}^*\big((x,\varpi),\varphi\big)\right]$ for all $(x,\varpi) \in \Delta$. Here, for each $\varphi \in [0,1]$, the end point real valued functions $\mathfrak{S}_*(.,\varphi), \mathfrak{S}^*(.,\varphi) : (x,\varpi) \to \mathbb{R}$ are called lower and upper functions of $\mathfrak{S}_\varphi$.*

Definition 5. [34] *Let $\widetilde{\mathfrak{S}} : \Delta \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a coordinate FIVF. Then $\widetilde{\mathfrak{S}}(x,y)$ is said to be continuous at $(x,y) \in \Delta = [\rho,\varsigma] \times [\mu,\nu]$, if for each $\varphi \in [0,1]$, both end point functions $\mathfrak{S}_*\big((x,y),\varphi\big)$ and $\mathfrak{S}^*\big((x,y),\varphi\big)$ are continuous at $(x,y) \in \Delta$.*

Definition 6. [34] *Let $\widetilde{\mathfrak{S}} : \Delta \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a coordinate fuzzy-IVF. Then $\widetilde{\mathfrak{S}}(x,y)$ is said to be continuous at $(x,y) \in \Delta = [\rho,\varsigma] \times [\mu,\nu]$, if for each $\varphi \in [0,1]$, both end point functions $\mathfrak{S}_*\big((x,y),\varphi\big)$ and $\mathfrak{S}^*\big((x,y),\varphi\big)$ are continuous at $(x,y) \in \Delta$.*

Definition 7. [34] *Let* $\widetilde{\mathfrak{S}}: \Delta \subset \mathbb{R}^2 \to \mathbb{F}_0$ *be a fuzzy-IVF on coordinates. Then fuzzy double integral of* $\widetilde{\mathfrak{S}}$ *over* $\Delta = [\rho, \varsigma] \times [\mu, \nu]$, *denoted by* $(FD)\int_\rho^\varsigma \int_\mu^\nu \widetilde{\mathfrak{S}}(x, y)dydx$, *it is defined level-wise by:*

$$\begin{aligned}\left[(FD)\int_\rho^\varsigma \int_\mu^\nu \widetilde{\mathfrak{S}}(x, y)dydx\right]^\varphi &= (ID)\int_\rho^\varsigma \int_\mu^\nu \mathfrak{S}_\varphi(x, y)dydx \\ &= \left\{\int_\rho^\varsigma \int_\mu^\nu \mathfrak{S}\big((x, y), \varphi\big)dydx : \mathfrak{S}\big((x, y), \varphi\big) \in \mathfrak{D}_\Delta\right\},\end{aligned} \tag{14}$$

for all $\varphi \in [0, 1]$, *where* $\mathfrak{D}_\Delta$ *is the collection of end point functions of IVFs on* Δ. $\widetilde{\mathfrak{S}}$ *is* FD*-integrable over* Δ *if* $(FD)\int_\rho^\varsigma \int_\mu^\nu \widetilde{\mathfrak{S}}(x, y)dydx \in \mathbb{F}_0$. *Note that, if both end point functions are Lebesgue-integrable, then* $\widetilde{\mathfrak{S}}$ *is fuzzy double Aumann-integrable function over* Δ.

Theorem 3. [34] *Let* $\widetilde{\mathfrak{S}}: \Delta \subset \mathbb{R}^2 \to \mathbb{F}_0$ *be a fuzzy-IVF on coordinates, whose* φ*-cuts establish the series of IVFs* $\mathfrak{S}_\varphi: \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ *are given by* $\mathfrak{S}_\varphi(x, y) = \left[\mathfrak{S}_*\big((x, y), \varphi\big), \mathfrak{S}^*\big((x, y), \varphi\big)\right]$ *for all* $(x, y) \in \Delta = [\rho, \varsigma] \times [\mu, \nu]$ *and for all* $\varphi \in [0, 1]$. *Then* $\widetilde{\mathfrak{S}}$ *is* FD*-integrable over* Δ *if and only if* $\mathfrak{S}_*(x, \varphi)$ *and* $\mathfrak{S}^*(x, \varphi)$ *are both* D*-integrable over* Δ. *Moreover, if* $\widetilde{\mathfrak{S}}$ *is* FD*-integrable over* Δ, *then:*

$$\begin{aligned}\left[(FD)\int_\rho^\varsigma \int_\mu^\nu \widetilde{\mathfrak{S}}(x, y)dydx\right]^\varphi &= \left[(D)\int_\rho^\varsigma \int_\mu^\nu \mathfrak{S}_*\big((x, y), \varphi\big)dydx\,, (D)\int_\rho^\varsigma \int_\mu^\nu \mathfrak{S}^*\big((x, y), \varphi\big)dydx\right] \\ &= (ID)\int_\rho^\varsigma \int_\mu^\nu \mathfrak{S}_\varphi(x, y)dydx\end{aligned} \tag{15}$$

for all $\varphi \in [0, 1]$.

The family of all FD*-integrable of fuzzy-IVFs over coordinates and* D*-integrable functions over coordinates are denoted by* $\mathcal{F}\mathfrak{D}_\Delta$ *and* $\mathfrak{D}_{(\Delta, \varphi)}$, *for all* $\varphi \in [0, 1]$.

Definition 8. [34] *The FIVF* $\widetilde{\mathfrak{S}}: \Delta \to \mathbb{F}_0$ *is said to be coordinated convex FIVF on* Δ *if:*

$$\begin{aligned}&\widetilde{\mathfrak{S}}(\varepsilon\rho + (1-\varepsilon)\varsigma, s\mu + (1-s)\nu) \\ &\quad \preccurlyeq \varepsilon s\widetilde{\mathfrak{S}}(\rho, \mu)\widetilde{+}\varepsilon(1-s)\widetilde{\mathfrak{S}}(\rho, \nu)\widetilde{+}(1-\varepsilon)s\widetilde{\mathfrak{S}}(\varsigma, \mu)\widetilde{+}(1-\varepsilon)(1-s)\widetilde{\mathfrak{S}}(\varsigma, \nu),\end{aligned} \tag{16}$$

for all $(\rho, \varsigma), (\mu, \nu) \in \Delta$, *and* $\varepsilon, s \in [0, 1]$, *where* $\widetilde{\mathfrak{S}}(x) \succcurlyeq \widetilde{0}$. *If inequality (16) is reversed, then* $\widetilde{\mathfrak{S}}$ *is called coordinate concave FIVF on* Δ.

Lemma 1. [34] *Let* $\widetilde{\mathfrak{S}}: \Delta \to \mathbb{F}_0$ *be a coordinated FIVF on* Δ. *Then,* $\widetilde{\mathfrak{S}}$ *is coordinated convex FIVF on* Δ *if and only if there exist two coordinated convex FIVFs* $\widetilde{\mathfrak{S}}_x: [\mu, \nu] \to \mathbb{F}_0$, $\widetilde{\mathfrak{S}}_x(w) = \widetilde{\mathfrak{S}}(x, w)$ *and* $\widetilde{\mathfrak{S}}_y: [\rho, \varsigma] \to \mathbb{F}_0$, $\widetilde{\mathfrak{S}}_y(z) = \widetilde{\mathfrak{S}}(z, y)$.

Theorem 4. [34] *Let* $\widetilde{\mathfrak{S}}: \Delta \to \mathbb{F}_0$ *be a FIVF on* Δ. *Then, from* φ*-levels, we get the collection of IVFs* $\mathfrak{S}_\varphi: \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ *are given by:*

$$\mathfrak{S}_\varphi(x, \varpi) = \left[\mathfrak{S}_*\big((x, \varpi), \varphi\big), \mathfrak{S}^*\big((x, \varpi), \varphi\big)\right], \tag{17}$$

for all $(x, \varpi) \in \Delta$ *and for all* $\varphi \in [0, 1]$. *Then,* $\widetilde{\mathfrak{S}}$ *is coordinated convex FIVF on* Δ, *if and only if, for all* $\varphi \in [0, 1]$, $\mathfrak{S}_*\big((x, \varpi), \varphi\big)$ *and* $\mathfrak{S}^*\big((x, \varpi), \varphi\big)$ *are coordinated convex function*.

Example 1. *We consider the FIVFs* $\widetilde{\mathfrak{S}}: [0, 1] \times [0, 1] \to \mathbb{F}_0$ *defined by:*

$$\mathfrak{S}(x)(\sigma)=\begin{cases}\dfrac{\sigma}{2(6+e^{x})(6+e^{\varpi})}, & \sigma\in[0,2(6+e^{x})(6+e^{\varpi})]\\ \dfrac{4(6+e^{x})(6+e^{\varpi})-\sigma}{2(6+e^{x})(6+e^{\varpi})}, & \sigma\in(2(6+e^{x})(6+e^{\varpi}),4(6+e^{x})(6+e^{\varpi})]\\ 0\qquad, & otherwise,\end{cases}$$

Then, for each $\theta\in[0,1]$, we have $\mathfrak{S}_{\theta}(x)=[2\theta(6+e^{x})(6+e^{\varpi}),(4+2\theta)(6+e^{x})(6+e^{\varpi})\,]$. Since end point functions $\mathfrak{S}_{*}\big((x,\varpi),\theta\big)$, $\mathfrak{S}^{*}\big((x,\varpi),\theta\big)$ are coordinate concave functions for each $\theta\in[0,1]$. Hence, $\widetilde{\mathfrak{S}}(x,\varpi)$ is coordinate concave FIVF.

From Lemma 1 and Example 1, we can easily note the each convex FIVF is coordinated convex FIVF. However, the converse is not true.

Remark 2. *If one takes $\mathfrak{S}_{*}\big((x,\varpi),\varphi\big)=\mathfrak{S}^{*}\big((x,\varpi),\varphi\big)$ with $\varphi=1$, then $\mathfrak{S}$ is known as coordinated function if $\mathfrak{S}$ satisfies the coming inequality:*

$$\begin{aligned}&\mathfrak{S}(\varepsilon\rho+(1-\varepsilon)\varsigma,s\mu+(1-s)\nu)\\&\qquad\le\varepsilon s\mathfrak{S}(\rho,\mu)+\varepsilon(1-s)\mathfrak{S}(\rho,\nu)+(1-\varepsilon)s\mathfrak{S}(\varsigma,\mu)+(1-\varepsilon)(1-s)\mathfrak{S}(\varsigma,\nu)\end{aligned}$$

it is valid, which is defined by Dragomir [41].

Let one take $\mathfrak{S}_{*}\big((x,\varpi),\varphi\big)\neq\mathfrak{S}^{*}\big((x,\varpi),\varphi\big)$ with $\varphi=1$ and $\mathfrak{S}_{*}\big((x,\varpi),\varphi\big)$ is an affine function and $\mathfrak{S}^{*}\big((x,\varpi),\varphi\big)$ is a concave function. If coming inequality:

$$\begin{aligned}&\mathfrak{S}(\varepsilon\rho+(1-\varepsilon)\varsigma,s\mu+(1-s)\nu)\\&\qquad\supseteq\varepsilon s\mathfrak{S}(\rho,\mu)+\varepsilon(1-s)\mathfrak{S}(\rho,\nu)+(1-\varepsilon)s\mathfrak{S}(\varsigma,\mu)+(1-\varepsilon)(1-s)\mathfrak{S}(\varsigma,\nu),\end{aligned}$$

is valid, then $\mathfrak{S}$ is named as coordinated IVF which is defined by Zhao et al. [32]*, Definition 2 and Example 2.*

2.2. Fuzzy-Interval-Valued Fractional Integrals on Coordinated Functions

For coordinated interval-valued function $\mathfrak{S}(x,y)$ and coordinated fuzzy-interval-valued function $\widetilde{\mathfrak{S}}(x,y)$, double interval and double fuzzy-interval Riemann–Liouville-type integrals are defined as:

Definition 9. [33] *Let $\mathfrak{S}:\Delta\to\mathbb{R}_I^{+}$ and $\mathfrak{S}\in\mathcal{ID}_{\Delta}$. The double interval Riemann–Liouville-type integrals $\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}$, $\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}$, $\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}$, $\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}$ of $\mathfrak{S}$ order $\alpha,\beta>0$ are defined by:*

$$\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}\ \mathfrak{S}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{S}(\mathrm{t},\mathrm{s})\,ds dt\quad(x>\rho,y>\mu),\tag{18}$$

$$\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\ \mathfrak{S}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{S}(\mathrm{t},\mathrm{s})\,ds dt\quad(x>\rho,y<\nu)\tag{19}$$

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}\ \mathfrak{S}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\varsigma}\int_{\mu}^{y}(\mathrm{t}-x)^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{S}(\mathrm{t},\mathrm{s})\,ds dt\quad(x<\varsigma,y>\mu),\tag{20}$$

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\ \mathfrak{S}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\varsigma}\int_{y}^{\nu}(\mathrm{t}-x)^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{S}(\mathrm{t},\mathrm{s})\,ds dt\quad(x<\varsigma,y<\nu)\tag{21}$$

Now, with help of concepts related to fuzzy-interval-valued functions, which are given by Khan et al. [34]*, Allahviranloo et al.* [49]*, and Budak et al.* [33]*, we introduce the following fuzzy-interval-valued Riemann–Liouville double fractional integral of the function $\widetilde{\mathfrak{S}}(x,y)$ by:*

Definition 10. *Let $\widetilde{\mathfrak{S}}:\Delta\to\mathbb{F}_0$ and $\widetilde{\mathfrak{S}}\in\mathcal{FD}_{\Delta}$. The double fuzzy interval Riemann–Liouville-type integrals $\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}$, $\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}$, $\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}$, $\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}$ of $\mathfrak{S}$ order $\alpha,\beta>0$ are defined by:*

$$\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\,dsdt\quad(x>\rho,y>\mu), \tag{22}$$

$$\mathcal{J}_{\rho^{+},\nu^{-}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\,dsdt\quad(x>\rho,y<\nu) \tag{23}$$

$$\mathcal{J}_{\varsigma^{-},\mu^{+}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\varsigma}\int_{\mu}^{y}(\mathrm{t}-x)^{\alpha-1}(y-\mathrm{s})^{\beta-1}\widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\,dsdt\quad(x<\varsigma,y>\mu), \tag{24}$$

$$\mathcal{J}_{\varsigma^{-},\nu^{-}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\varsigma}\int_{y}^{\nu}(\mathrm{t}-x)^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\,dsdt\quad(x<\varsigma,y<\nu) \tag{25}$$

The fuzzy interval left and right Riemann–Liouville fractional integral of $\widetilde{\mathfrak{S}}$ based on left and right end point functions can be defined level-wise, that is, for all $\varphi\in[0,1]$, Equation (22), we have:

$$\begin{aligned}&\left[\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)\right]^{\varphi}\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{S}_{\varphi}\big((\mathrm{t},\mathrm{s}),\varphi\big)\,dsdt,\qquad(x>\rho,y>\mu),\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\big[\mathfrak{S}_{*}\big((\mathrm{t},\mathrm{s}),\varphi\big),\mathfrak{S}^{*}\big((\mathrm{t},\mathrm{s}),\varphi\big)\big]\,dsdt\quad(x>\rho,y>\mu)\end{aligned} \tag{26}$$

where:

$$\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\ \mathfrak{S}_{*}\big((x,y),\varphi\big)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{S}_{*}\big((\mathrm{t},\mathrm{s}),\varphi\big)\,dsdt,(x>\rho,y>\mu), \tag{27}$$

and:

$$\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\ \mathfrak{S}^{*}\big((x,y),\varphi\big)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{\mu}^{y}(x-\mathrm{t})^{\alpha-1}(y-\mathrm{s})^{\beta-1}\mathfrak{S}^{*}\big((\mathrm{t},\mathrm{s}),\varphi\big)\,dsdt,(x>\rho,y>\mu), \tag{28}$$

Now, for all $\varphi\in[0,1]$, Equation (23) we have:

$$\begin{aligned}&\left[\mathcal{J}_{\rho^{+},\nu^{-}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)\right]^{\varphi}\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{S}_{\varphi}\big((\mathrm{t},\mathrm{s}),\varphi\big)\,dsdt\quad(x>\rho,y<\nu)\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\big[\mathfrak{S}_{*}\big((\mathrm{t},\mathrm{s}),\varphi\big),\mathfrak{S}^{*}\big((\mathrm{t},\mathrm{s}),\varphi\big)\big]\,dsdt\quad(x>\rho,y<\nu)\end{aligned}$$

where:

$$\mathcal{J}_{\rho^{+},\nu^{-}}^{\alpha,\beta}\ \mathfrak{S}_{*}\big((x,y),\varphi\big)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{S}_{*}\big((x,y),\varphi\big)\,dsdt\quad(x>\rho,y<\nu) \tag{29}$$

and:

$$\mathcal{J}_{\rho^{+},\nu^{-}}^{\alpha,\beta}\ \mathfrak{S}^{*}\big((x,y),\varphi\big)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\rho}^{x}\int_{y}^{\nu}(x-\mathrm{t})^{\alpha-1}(\mathrm{s}-y)^{\beta-1}\mathfrak{S}^{*}\big((x,y),\varphi\big)\,dsdt\quad(x>\rho,y<\nu) \tag{30}$$

Similarly, for all $\varphi\in[0,1]$, Equation (24) we have:

$$\begin{aligned}&\mathcal{J}_{\varsigma^{-},\mu^{+}}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(x,y)\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\varsigma}\int_{\mu}^{y}(\mathrm{t}-x)^{\alpha-1}(y-\mathrm{s})^{\beta-1}\widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\,dsdt\quad(x<\varsigma,y>\mu),\end{aligned} \tag{31}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_\mu^y (\mathrm{t}-x)^{\alpha-1} (y-\mathrm{s})^{\beta-1} \left[\mathfrak{S}_*\big((\mathrm{t},\mathrm{s}),\varphi\big), \mathfrak{S}^*\big((\mathrm{t},\mathrm{s}),\varphi\big)\right] ds dt \quad (x<\varsigma, y>\mu)$$

where:

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\, \mathfrak{S}_*\big((x,y),\varphi\big) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_\mu^y (\mathrm{t}-x)^{\alpha-1} (y-\mathrm{s})^{\beta-1} \mathfrak{S}_*\big((x,y),\varphi\big)\, d\mathrm{s}d\mathrm{t} \quad (x<\varsigma, y>\mu),$$

and:

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\, \mathfrak{S}^*\big((x,y),\varphi\big) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_\mu^y (\mathrm{t}-x)^{\alpha-1} (y-\mathrm{s})^{\beta-1} \mathfrak{S}^*\big((x,y),\varphi\big)\, d\mathrm{s}d\mathrm{t} \quad (x<\varsigma, y>\mu),$$

Similarly, for all $\varphi \in [0,1]$, Equation (25) we have:

$$\begin{aligned} &\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\, \widetilde{\mathfrak{S}}(x,y) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_y^\nu (\mathrm{t}-x)^{\alpha-1} (\mathrm{s}-y)^{\beta-1} \widetilde{\mathfrak{S}}(\mathrm{t},\mathrm{s})\, ds dt \quad (x<\varsigma, y<\nu) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_y^\nu (\mathrm{t}-x)^{\alpha-1} (\mathrm{s}-y)^{\beta-1} \left[\mathfrak{S}_*\big((\mathrm{t},\mathrm{s}),\varphi\big), \mathfrak{S}^*\big((\mathrm{t},\mathrm{s}),\varphi\big)\right] ds dt \quad (x<\varsigma, y<\nu), \end{aligned} \tag{32}$$

where:

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\, \mathfrak{S}_*\big((x,y),\varphi\big) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_y^\nu (\mathrm{t}-x)^{\alpha-1} (\mathrm{s}-y)^{\beta-1} \mathfrak{S}_*\big((x,y),\varphi\big)\, d\mathrm{s}d\mathrm{t} \quad (x<\varsigma, y<\nu) \tag{33}$$

and:

$$\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\, \mathfrak{S}^*\big((x,y),\varphi\big) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\varsigma \int_y^\nu (\mathrm{t}-x)^{\alpha-1} (\mathrm{s}-y)^{\beta-1} \mathfrak{S}^*\big((x,y),\varphi\big)\, d\mathrm{s}d\mathrm{t} \quad (x<\varsigma, y<\nu) \tag{34}$$

Remark 3. *It can be easily noted that if $\mathfrak{S}_*\big((x,y),\varphi\big) \neq \mathfrak{S}^*\big((x,y),\varphi\big)$ with $\varphi = 1$, then from Definition 10, we achieve Definition 9.*

3. Fuzzy-Interval Fractional Hermite–Hadamard Inequalities

In this section, we shall continue with the following fractional HH inequality for convex fuzzy-IVFs, and we also give some fractional inequalities for the product of two coordinated convex fuzzy-IVF through FOR.

Theorem 5. *Let $\widetilde{\mathfrak{S}}\colon \Delta \to \mathbb{F}_0$ be a coordinate convex FIVF on Δ. Then, from φ-cuts, we establish that the series of IVFs $\mathfrak{S}_\varphi\colon \Delta \to \mathbb{R}_I^+$ are given by $\mathfrak{S}_\varphi(x,y) = \left[\mathfrak{S}_*\big((x,y),\varphi\big), \mathfrak{S}^*\big((x,y),\varphi\big)\right]$ for all $(x,y) \in \Delta$ and for all $\varphi \in [0,1]$. If $\widetilde{\mathfrak{S}} \in \mathcal{FD}_\Delta$, and then the following inequalities hold:*

$$\begin{aligned}
&\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\quad\preccurlyeq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}\left(\varsigma,\frac{\mu+\nu}{2}\right)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\widetilde{\mathfrak{S}}\left(\rho,\frac{\mu+\nu}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\nu\right)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]\\
&\quad\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\,\widetilde{\mathfrak{S}}(\rho,\mu)\right]\\
&\quad\preccurlyeq \frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\right]\\
&\quad\preccurlyeq \frac{\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\nu)}{4}.
\end{aligned}\tag{35}$$

If $\mathfrak{S}(x)$ coordinated concave FIVF then:

$$\begin{aligned}
&\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\quad\succcurlyeq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}\left(\varsigma,\frac{\mu+\nu}{2}\right)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\widetilde{\mathfrak{S}}\left(\rho,\frac{\mu+\nu}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\nu\right)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]\\
&\quad\succcurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\,\widetilde{\mathfrak{S}}(\rho,\mu)\right]\\
&\quad\succcurlyeq \frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\alpha}_{\rho^{+}}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}^{\beta}_{\mu^{+}}\,\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\right]\\
&\quad\succcurlyeq \frac{\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\nu)}{4}.
\end{aligned}\tag{36}$$

Proof. *Let $\widetilde{\mathfrak{S}}:[\rho,\varsigma]\to\mathbb{F}_0$ be a coordinated convex FIVF. Then, by hypothesis, we have*

$$4\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\preccurlyeq\widetilde{\mathfrak{S}}(\varepsilon\rho+(1-\varepsilon)\varsigma,\varepsilon\mu+(1-\varepsilon)\nu)\widetilde{+}\widetilde{\mathfrak{S}}\big((1-\varepsilon)\rho+\varepsilon\varsigma,(1-\varepsilon)\mu+\varepsilon\nu\big).$$

By using Theorem 4, for every $\varphi\in[0,1]$, we have:

$$\begin{aligned}
&4\mathfrak{S}_{*}\left(\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right),\varphi\right)\\
&\quad\le\mathfrak{S}_{*}\big((\varepsilon\rho+(1-\varepsilon)\varsigma,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big)+\mathfrak{S}_{*}\Big(\big((1-\varepsilon)\rho+\varepsilon\varsigma,(1-\varepsilon)\mu+\varepsilon\nu\big),\varphi\Big),\\
&4\mathfrak{S}^{*}\left(\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right),\varphi\right)\\
&\quad\le\mathfrak{S}^{*}\big((\varepsilon\rho+(1-\varepsilon)\varsigma,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big)+\mathfrak{S}^{*}\Big(\big((1-\varepsilon)\rho+\varepsilon\varsigma,(1-\varepsilon)\mu+\varepsilon\nu\big),\varphi\Big).
\end{aligned}$$

By using Lemma 1, we have:

$$\begin{aligned}
2\mathfrak{S}_{*}\left(\left(x,\frac{\mu+\nu}{2}\right),\varphi\right)&\le\mathfrak{S}_{*}\big((x,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big)+\mathfrak{S}_{*}\big((x,(1-\varepsilon)\mu+\varepsilon\nu),\varphi\big),\\
2\mathfrak{S}^{*}\left(\left(x,\frac{\mu+\nu}{2}\right),\varphi\right)&\le\mathfrak{S}^{*}\big((x,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big)+\mathfrak{S}^{*}\big((x,(1-\varepsilon)\mu+\varepsilon\nu),\varphi\big),
\end{aligned}\tag{37}$$

and:

$$2\mathfrak{S}_*\left(\left(\frac{\rho+\varsigma}{2},y\right),\varphi\right) \le \mathfrak{S}_*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big) + \mathfrak{S}_*\Big(\big((1-\varepsilon)\rho+t\varsigma,y\big),\varphi\Big),$$
$$2\mathfrak{S}^*\left(\left(\frac{\rho+\varsigma}{2},y\right),\varphi\right) \le \mathfrak{S}^*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big) + \mathfrak{S}^*\Big(\big((1-\varepsilon)\rho+t\varsigma,y\big),\varphi\Big). \tag{38}$$

From (37) and (38), we have:

$$2\left[\mathfrak{S}_*\left(\left(x,\frac{\mu+\nu}{2}\right),\varphi\right),\mathfrak{S}^*\left(\left(x,\frac{\mu+\nu}{2}\right),\varphi\right)\right]$$
$$\le_I \left[\mathfrak{S}_*\big((x,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big),\mathfrak{S}^*\big((x,\varepsilon\mu+(1-\varepsilon)\nu),\varphi\big)\right]$$
$$+\left[\mathfrak{S}_*\big((x,(1-\varepsilon)\mu+\varepsilon\nu),\varphi\big),\mathfrak{S}^*\big((x,(1-\varepsilon)\mu+\varepsilon\nu),\varphi\big)\right],$$

and:

$$2\left[\mathfrak{S}_*\left(\left(\frac{\rho+\varsigma}{2},y\right),\varphi\right),\mathfrak{S}^*\left(\left(\frac{\rho+\varsigma}{2},y\right),\varphi\right)\right]$$
$$\le_I \left[\mathfrak{S}_*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big),\mathfrak{S}^*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big)\right]$$
$$+\left[\mathfrak{S}_*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big),\mathfrak{S}^*\big((\varepsilon\rho+(1-\varepsilon)\varsigma,y),\varphi\big)\right],$$

It follows that:

$$\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right) \le_I \mathfrak{S}_\varphi(x,\varepsilon\mu+(1-\varepsilon)\nu) + \mathfrak{S}_\varphi(x,(1-\varepsilon)\mu+\varepsilon\nu), \tag{39}$$

and:

$$\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},y\right) \le_I \mathfrak{S}_\varphi(\varepsilon\rho+(1-\varepsilon)\varsigma,y) + \mathfrak{S}_\varphi(\varepsilon\rho+(1-\varepsilon)\varsigma,y) \tag{40}$$

Since $\mathfrak{S}_\varphi(x,.)$ *and* $\mathfrak{S}_\varphi(.,y)$ *are both coordinated convex-IVFs, then from inequality (11), for every* $\varphi\in[0,1]$*, inequalities (39) and (40) we have:*

$$\mathfrak{S}_{\varphi_x}\left(\frac{\mu+\nu}{2}\right) \le_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{J}^\beta_{\mu^+}\,\mathfrak{S}_{\varphi_x}(\nu) + \mathcal{J}^\beta_{\nu^-}\,\mathfrak{S}_{\varphi_x}(\mu)\right] \le_I \frac{\mathfrak{S}_{\varphi_x}(\mu)+\mathfrak{S}_{\varphi_x}(\nu)}{2}. \tag{41}$$

and:

$$\mathfrak{S}_{\varphi_y}\left(\frac{\rho+\varsigma}{2}\right) \le_I \frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^\alpha}\left[\mathcal{J}^\alpha_{\rho^+}\,\mathfrak{S}_{\varphi_y}(\varsigma) + \mathcal{J}^\alpha_{\varsigma^-}\,\mathfrak{S}_{\varphi_y}(\rho)\right] \le_I \frac{\mathfrak{S}_{\varphi_y}(\rho)+\mathfrak{S}_{\varphi_y}(\varsigma)}{2} \tag{42}$$

Since $\mathfrak{S}_{\varphi_x}(w)=\mathfrak{S}_\varphi(x,w)$*, then (41) can be written as:*

$$\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right) \le_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{J}^\alpha_{\mu^+}\,\mathfrak{S}_\varphi(x,\nu) + \mathcal{J}^\alpha_{\nu^-}\,\mathfrak{S}_\varphi(x,\mu)\right] \le_I \frac{\mathfrak{S}_\varphi(x,\mu)+\mathfrak{S}_\varphi(x,\nu)}{2}. \tag{43}$$

That is:

$$\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right)$$
$$\le_I \frac{\beta}{2(\nu-\mu)^\beta}\left[\int_\mu^\nu(\nu-s)^{\beta-1}\mathfrak{S}_\varphi(x,s)ds + \int_\mu^\nu(s-\mu)^{\beta-1}\mathfrak{S}_\varphi(x,s)ds\right] \le_I \frac{\mathfrak{S}_\varphi(x,\mu)+\mathfrak{S}_\varphi(x,\nu)}{2}.$$

Multiplying double inequality (43) by $\frac{\alpha(\varsigma-x)^{\alpha-1}}{2(\varsigma-\rho)^\alpha}$ *and integrating it with respect to* x *over* $[\rho,\varsigma]$*, we have:*

$$\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}\mathfrak{S}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)(\varsigma-x)^{\alpha-1}dx$$
$$\leq_I \int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(\varsigma-x)^{\alpha-1}(\nu-s)^{\beta-1}\mathfrak{S}_{\varphi}(x,s)\,dsdx+\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(\varsigma-x)^{\alpha-1}(s-\mu)^{\beta-1}\mathfrak{S}_{\varphi}(x,s)\,dsdx \tag{44}$$
$$\leq_I \frac{\alpha}{4(\varsigma-\rho)^{\alpha}}\left[\int_{\rho}^{\varsigma}(\varsigma-x)^{\alpha-1}\mathfrak{S}_{\varphi}(x,\mu)dx+\int_{\rho}^{\varsigma}(\varsigma-x)^{\alpha-1}\mathfrak{S}_{\varphi}(x,\nu)dx\right]$$

Again, multiplying double inequality (43) by $\frac{\alpha(x-\rho)^{\alpha-1}}{2(\varsigma-\rho)^{\alpha}}$ *and integrating it with respect to* x *over* $[\rho,\varsigma]$, *we have:*

$$\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}\mathfrak{S}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)(\varsigma-x)^{\alpha-1}dx$$
$$\leq_I \frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(x-\rho)^{\alpha-1}(\nu-s)^{\beta-1}\mathfrak{S}_{\varphi}(x,s)\,dsdx$$
$$+\frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(x-\rho)^{\alpha-1}(s-\mu)^{\beta-1}\mathfrak{S}_{\varphi}(x,s)\,dsdx \tag{45}$$
$$\leq_I \frac{\alpha}{4(\varsigma-\rho)^{\alpha}}\left[\int_{\rho}^{\varsigma}(x-\rho)^{\alpha-1}\mathfrak{S}_{\varphi}(x,\mu)dx+\int_{\rho}^{\varsigma}(x-\rho)^{\alpha-1}\mathfrak{S}_{\varphi}(x,\nu)dx\right]$$

From (44), we have:

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\,\mathfrak{S}_{\varphi}\left(\varsigma,\frac{\mu+\nu}{2}\right)\right]$$
$$\leq_I \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^+,\mu^+}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\varsigma^-,\mu^+}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\right] \tag{46}$$
$$\leq_I \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)+\mathcal{J}_{\rho^+}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\right]$$

From (45), we have:

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}\left(\rho,\frac{\mu+\nu}{2}\right)\right]$$
$$\leq_I \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\varsigma^-,\mu^+}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)+\mathcal{J}_{\varsigma^-,\nu^-}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\rho,\mu)\right] \tag{47}$$
$$\leq_I \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)+\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\right]$$

Since from φ*-cuts, we obtain the collection of IVFs* $\mathfrak{S}_{\varphi}:\Delta\to\mathbb{R}_I^+$, *then we have:*

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\,\widetilde{\mathfrak{S}}\left(\varsigma,\frac{\mu+\nu}{2}\right)\right]$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^+,\mu^+}^{\alpha,\beta}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\tilde{+}\mathcal{J}_{\varsigma^-,\mu^+}^{\alpha,\beta}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\right] \tag{48}$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\tilde{+}\mathcal{J}_{\rho^+}^{\alpha}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\right]$$

and:

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\varsigma^-}^{\alpha}\ \widetilde{\mathfrak{S}}\left(\rho,\frac{\mu+\nu}{2}\right)\right]$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\varsigma^-,\mu^+}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}_{\varsigma^-,\nu^-}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\rho,\mu)\right] \tag{49}$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\varsigma^-}^{\alpha}\ \widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}_{\varsigma^-}^{\alpha}\ \widetilde{\mathfrak{S}}(\rho,\nu)\right]$$

Similarly, since $\widetilde{\mathfrak{S}}_y(z)=\widetilde{\mathfrak{S}}(z,y)$ *then, from the (42), (48), and (49), we have:*

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\ \widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\nu\right)\right]$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^+,\mu^+}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}_{\varsigma^-,\mu^+}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\rho,\nu)\right] \tag{50}$$
$$\preccurlyeq \frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\ \widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}_{\mu^+}^{\beta}\ \widetilde{\mathfrak{S}}(\varsigma,\nu)\right]$$

and:

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\alpha}}\left[\mathcal{J}_{\nu^-}^{\beta}\ \widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^+,\nu^-}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}_{\varsigma^-,\nu^-}^{\alpha,\beta}\ \widetilde{\mathfrak{S}}(\rho,\mu)\right] \tag{51}$$
$$\preccurlyeq \frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\nu^-}^{\beta}\ \widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}_{\nu^-}^{\beta}\ \widetilde{\mathfrak{S}}(\varsigma,\mu)\right]$$

After adding the inequalities (48), (49), (50), and (51), we will obtain as a result the second, third, and fourth inequalities of (35).

Now, from the left part of inequality (11), for all $\varphi\in[0,1]$ *we have:*

$$\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\leq_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\ \mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)+\mathcal{J}_{\nu^-}^{\beta}\ \mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)\right] \tag{52}$$

and:

$$\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\leq_I \frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\ \mathfrak{S}_{\varphi}\left(\varsigma,\frac{\mu+\nu}{2}\right)+\mathcal{J}_{\varsigma^-}^{\alpha}\ \mathfrak{S}_{\varphi}\left(\rho,\frac{\mu+\nu}{2}\right)\right] \tag{53}$$

Summing the inequalities (52) and (53), we obtain the following inequality:

$$\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\leq_I \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\ \mathfrak{S}_{\varphi}\left(\varsigma,\frac{\mu+\nu}{2}\right)+\mathcal{J}_{\varsigma^-}^{\alpha}\ \mathfrak{S}_{\varphi}\left(\rho,\frac{\mu+\nu}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\ \mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)+\mathcal{J}_{\nu^-}^{\beta}\ \mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]$$

Similarly, since for $\varphi\in[0,1]$, *we obtain the collection of IVFs* $\mathfrak{S}_{\varphi}:\Delta\to\mathbb{R}_I^+$, *then the above inequality can be as follows:*

$$\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)$$
$$\preccurlyeq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\ \widetilde{\mathfrak{S}}\left(\varsigma,\frac{\mu+\nu}{2}\right)\widetilde{+}\mathcal{J}_{\varsigma^-}^{\alpha}\ \widetilde{\mathfrak{S}}\left(\rho,\frac{\mu+\nu}{2}\right)\right]\widetilde{+}\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\ \widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\nu\right)\widetilde{+}\mathcal{J}_{\nu^-}^{\beta}\ \widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]. \tag{54}$$

This is the first inequality of (35).

Now, from the right part of inequality (11), for all $\varphi\in[0,1]$ *we have:*

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\mathfrak{S}_{\varphi}(\rho,\nu)+\mathcal{J}_{\nu^-}^{\beta}\ \mathfrak{S}_{\varphi}(\rho,\mu)\right]\leq_I \frac{\mathfrak{S}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\rho,\nu)}{2}. \tag{55}$$

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\mathfrak{S}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\nu^-}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\right]\leq_I\frac{\mathfrak{S}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)}{2}. \tag{56}$$

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\mathfrak{S}_{\varphi}(\varsigma,\mu)+\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\right]\leq_I\frac{\mathfrak{S}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)}{2} \tag{57}$$

$$\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\mathfrak{S}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\right]\leq_I\frac{\mathfrak{S}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)}{2} \tag{58}$$

Summing inequalities (55), (56), (57), and (58), and then taking multiplication of the resultant with $\frac{1}{4}$, we have:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\mathfrak{S}_{\varphi}(\varsigma,\mu)+\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)+\mathcal{J}_{\rho^+}^{\alpha}\mathfrak{S}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\varsigma^-}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\right]\\&\quad+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\mathfrak{S}_{\varphi}(\rho,\nu)+\mathcal{J}_{\nu^-}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\mu)+\mathcal{J}_{\mu^+}^{\beta}\mathfrak{S}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\nu^-}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\right]\\&\quad\leq_I\frac{\mathfrak{S}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)}{4}.\end{aligned}$$

Since from φ-cuts, we obtain the collection of IVFs $\mathfrak{S}_{\varphi}\colon\Delta\to\mathbb{R}_I^+$, then we have:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}_{\rho^+}^{\alpha}\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\mathcal{J}_{\varsigma^-}^{\alpha}\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}_{\rho^+}^{\alpha}\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}_{\varsigma^-}^{\alpha}\,\widetilde{\mathfrak{S}}(\rho,\nu)\right]\\&\quad\widetilde{+}\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\mu^+}^{\beta}\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\mathcal{J}_{\nu^-}^{\beta}\,\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\mathcal{J}_{\mu^+}^{\beta}\widetilde{\mathfrak{S}}(\varsigma,\nu)\widetilde{+}\mathcal{J}_{\nu^-}^{\beta}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\right]\\&\quad\preccurlyeq\frac{\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\nu)}{4}.\end{aligned} \tag{59}$$

This is last inequality of (35) and the result has been proven. □

Remark 4. *If one is to take $\alpha=1$ and $\beta=1$, then from (35), we achieve the coming inequality, see* [34]*:*

$$\begin{aligned}&\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\&\quad\preccurlyeq\frac{1}{2}\left[\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\widetilde{\mathfrak{S}}\left(x,\frac{\mu+\nu}{2}\right)dx\widetilde{+}\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},y\right)dy\right]\preccurlyeq\frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}\widetilde{\mathfrak{S}}(x,y)dydx\\&\quad\preccurlyeq\frac{1}{4(\varsigma-\rho)}\left[\int_{\rho}^{\varsigma}\widetilde{\mathfrak{S}}(x,\mu)\,dx\widetilde{+}\int_{\rho}^{\varsigma}\widetilde{\mathfrak{S}}(x,\nu)dx\right]+\frac{1}{4(\nu-\mu)}\left[\int_{\mu}^{\nu}\widetilde{\mathfrak{S}}(\rho,y)dy\,\widetilde{+}\int_{\mu}^{\nu}\widetilde{\mathfrak{S}}(\varsigma,y)dy\right]\\&\quad\preccurlyeq\frac{\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\nu)}{4}.\end{aligned} \tag{60}$$

Let one take $\mathfrak{S}_*((x,y),\varphi)$, which is an affine function, and $\mathfrak{S}^*((x,y),\varphi)$, which is concave function. If $\mathfrak{S}_*((x,y),\varphi)\neq\mathfrak{S}^*((x,y),\varphi)$ with $\varphi=1$, then Remark 2 is correct, and from (36), we acquire the following inequality, see [33]:

$$
\begin{aligned}
&\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\supseteq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}\left(\varsigma,\frac{\mu+\nu}{2}\right)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}\left(\rho,\frac{\mu+\nu}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\nu\right)+\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]\\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}\,\mathfrak{S}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}\,\mathfrak{S}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\,\mathfrak{S}(\rho,\mu)\right]\\
&\supseteq \frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}(\varsigma,\mu)\mathfrak{S}\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\mathfrak{S}(\rho,\mu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\mathfrak{S}(\rho,\nu)\right]\\
&+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}(\rho,\nu)\tilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)+\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}(\rho,\nu)+\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)\right]\\
&\supseteq \frac{\mathfrak{S}(\rho,\mu)+\mathfrak{S}(\varsigma,\mu)+\mathfrak{S}(\rho,\nu)+\mathfrak{S}(\varsigma,\nu)}{4}.
\end{aligned}
\tag{61}
$$

Let one take $\mathfrak{S}_{*}\big((x,y),\varphi\big)$, which is an affine function, and $\mathfrak{S}^{*}\big((x,y),\varphi\big)$, which is a concave function. If $\mathfrak{S}_{*}\big((x,y),\varphi\big)\neq\mathfrak{S}^{*}\big((x,y),\varphi\big)$ with $\varphi=1$, then Remark 2 is correct, and from (36), we acquire the following inequality, see [32]*:*

$$
\begin{aligned}
&\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\supseteq \frac{1}{2}\left[\frac{1}{\varsigma-\rho}\int_{\rho}^{\varsigma}\mathfrak{S}\left(x,\frac{\mu+\nu}{2}\right)dx+\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\mathfrak{S}\left(\frac{\rho+\varsigma}{2},y\right)dy\right]\supseteq\frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}\mathfrak{S}(x,y)dydx\\
&\supseteq \frac{1}{4(\varsigma-\rho)}\left[\int_{\rho}^{\varsigma}\mathfrak{S}(x,\mu)\,dx+\int_{\rho}^{\varsigma}\mathfrak{S}(x,\nu)dx\right]+\frac{1}{4(\nu-\mu)}\left[\int_{\mu}^{\nu}\mathfrak{S}(\rho,y)dy+\int_{\mu}^{\nu}\mathfrak{S}(\varsigma,y)dy\right]\\
&\supseteq \frac{\mathfrak{S}(\rho,\mu)+\mathfrak{S}(\varsigma,\mu)+\mathfrak{S}(\rho,\nu)+\mathfrak{S}(\varsigma,\nu)}{4}
\end{aligned}
\tag{62}
$$

If $\mathfrak{S}_{*}\big((x,y),\varphi\big)=\mathfrak{S}^{*}\big((x,y),\varphi\big)$ with $\varphi=1$, then from (35) we acquire the following inequality, see [40]*:*

$$
\begin{aligned}
&\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{4(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}\left(\varsigma,\frac{\mu+\nu}{2}\right)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}\left(\rho,\frac{\mu+\nu}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\nu\right)+\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\mu\right)\right]\\
&\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^{+},\mu^{+}}\,\mathfrak{S}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\mu^{+}}\,\mathfrak{S}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\,\mathfrak{S}(\rho,\mu)\right]\\
&\leq \frac{\Gamma(\alpha+1)}{8(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}(\varsigma,\mu)\mathfrak{S}\mathcal{J}^{\alpha}_{\rho^{+}}\,\mathfrak{S}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\mathfrak{S}(\rho,\mu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\,\mathfrak{S}(\rho,\nu)\right]\\
&+\frac{\Gamma(\beta+1)}{4(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}(\rho,\nu)\tilde{+}\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)+\mathcal{J}^{\beta}_{\mu^{+}}\,\mathfrak{S}(\rho,\nu)+\mathcal{J}^{\beta}_{\nu^{-}}\,\mathfrak{S}(\varsigma,\mu)\right]\\
&\leq \frac{\mathfrak{S}(\rho,\mu)+\mathfrak{S}(\varsigma,\mu)+\mathfrak{S}(\rho,\nu)+\mathfrak{S}(\varsigma,\nu)}{4}.
\end{aligned}
\tag{63}
$$

Theorem 6. *Let $\tilde{\mathfrak{S}},\tilde{\mathcal{J}}:\Delta\to\mathbb{F}_0$ be a coordinate convex FIVFs on Δ. Then, from φ-cuts, we establish that the series of IVFs $\mathfrak{S}_{\varphi},\mathcal{J}_{\varphi}:\Delta\to\mathbb{R}_I^{+}$ are given by $\mathfrak{S}_{\varphi}(x,y)=\big[\mathfrak{S}_{*}\big((x,y),\varphi\big),\mathfrak{S}^{*}\big((x,y),\varphi\big)\big]$ and $\mathcal{J}_{\varphi}(x,y)=\big[\mathcal{J}_{*}\big((x,y),\varphi\big),\mathcal{J}^{*}\big((x,y),\varphi\big)\big]$ for all $(x,y)\in\Delta$ and for all $\varphi\in[0,1]$. If $\tilde{\mathfrak{S}}\tilde{\times}\tilde{\mathcal{J}}\in\mathcal{FO}_{\Delta}$, then the following inequalities hold:*

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu)\right]$$
$$\widetilde{+}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\right]$$
$$\preccurlyeq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\widetilde{K}(\rho,\varsigma,\mu,\nu)\widetilde{+}\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\widetilde{L}(\rho,\varsigma,\mu,\nu)$$
$$\widetilde{+}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\widetilde{+}\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu) \quad (64)$$

If $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ are both coordinate concave FIVFs on Δ, then the above inequality can be written as:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu)\right]$$
$$\widetilde{+}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\right]$$
$$\succcurlyeq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\widetilde{K}(\rho,\varsigma,\mu,\nu)\widetilde{+}\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\widetilde{L}(\rho,\varsigma,\mu,\nu)$$
$$\widetilde{+}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\widetilde{+}\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu) \quad (65)$$

where:

$$\widetilde{K}(\rho,\varsigma,\mu,\nu)=\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\,\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu)\widetilde{+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\,\widetilde{+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu),$$

$$\widetilde{L}(\rho,\varsigma,\mu,\nu)=\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu)\,\widetilde{+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\,\widetilde{+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu),$$

$$\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)=\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\,\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu)\widetilde{+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\,\widetilde{+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu),$$

$$\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)=\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\nu)\,\widetilde{+}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\nu)\widetilde{+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\varsigma,\mu)\widetilde{+}\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\widetilde{\mathcal{J}}(\rho,\mu)\,,$$

and for each $\varphi\in[0,1]$, $\widetilde{K}(\rho,\varsigma,\mu,\nu)$, $\widetilde{L}(\rho,\varsigma,\mu,\nu)$, $\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)$, and $\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)$ are defined as follows:

$$K_{\varphi}(\rho,\varsigma,\mu,\nu)=\left[K_*\big((\rho,\varsigma,\mu,\nu),\varphi\big),K^*\big((\rho,\varsigma,\mu,\nu),\varphi\big)\right],$$

$$L_{\varphi}(\rho,\varsigma,\mu,\nu)=\left[L_*\big((\rho,\varsigma,\mu,\nu),\varphi\big),L^*\big((\rho,\varsigma,\mu,\nu),\varphi\big)\right],$$

$$\mathcal{M}_{\varphi}(\rho,\varsigma,\mu,\nu)=\left[\mathcal{M}_*\big((\rho,\varsigma,\mu,\nu),\varphi\big),\mathcal{M}^*\big((\rho,\varsigma,\mu,\nu),\varphi\big)\right],$$

$$\mathcal{N}_{\varphi}(\rho,\varsigma,\mu,\nu)=\left[\mathcal{N}_*\big((\rho,\varsigma,\mu,\nu),\varphi\big),\mathcal{N}^*\big((\rho,\varsigma,\mu,\nu),\varphi\big)\right].$$

Proof. *Let $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ both be coordinated convex FIVFs on $[\rho,\varsigma]\times[\mu,\nu]$. Then:*

$$\widetilde{\mathfrak{S}}(\varepsilon\rho+(1-\varepsilon)\varsigma,s\mu+(1-s)\nu)$$
$$\preccurlyeq\varepsilon s\widetilde{\mathfrak{S}}(\rho,\mu)\widetilde{+}\varepsilon(1-s)\widetilde{\mathfrak{S}}(\rho,\nu)\widetilde{+}(1-\varepsilon)s\widetilde{\mathfrak{S}}(\varsigma,\mu)\widetilde{+}(1-\varepsilon)(1-s)\widetilde{\mathfrak{S}}(\varsigma,\nu),$$

and:

$$\widetilde{\mathcal{J}}(\varepsilon\rho+(1-\varepsilon)\varsigma,s\mu+(1-s)\nu)$$
$$\preccurlyeq\varepsilon s\widetilde{\mathcal{J}}(\rho,\mu)\widetilde{+}\varepsilon(1-s)\widetilde{\mathcal{J}}(\rho,\nu)\widetilde{+}(1-\varepsilon)s\widetilde{\mathcal{J}}(\varsigma,\mu)\widetilde{+}(1-\varepsilon)(1-s)\widetilde{\mathcal{J}}(\varsigma,\nu).$$

Since $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ are both coordinated convex FIVFs, then by Lemma 1, there exist:

$$\widetilde{\mathfrak{S}}_x : [\mu, \nu] \to \mathbb{F}_0,\ \widetilde{\mathfrak{S}}_x(y) = \widetilde{\mathfrak{S}}(x, y), \qquad \tilde{\mathcal{J}}_x : [\mu, \nu] \to \mathbb{F}_0,\ \tilde{\mathcal{J}}_x(y) = \tilde{\mathcal{J}}(x, y),$$

Since $\widetilde{\mathfrak{S}}_x$, and $\tilde{\mathcal{J}}_x$ are FIVFs, then by inequality (12), we have:

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{J}^\beta_{\mu^+}\,\widetilde{\mathfrak{S}}_x(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\nu) \widetilde{+} \mathcal{J}^\beta_{\nu^-}\widetilde{\mathfrak{S}}_x(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\mu)\right]$$
$$\preccurlyeq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\widetilde{\mathfrak{S}}_x(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\mu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}_x(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\nu)\right)$$
$$\widetilde{+}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\widetilde{\mathfrak{S}}_x(\mu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\nu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}_x(\nu) \,\widetilde{\times}\, \tilde{\mathcal{J}}_x(\mu)\right).$$

Now for all for all $\varphi \in [0, 1]$, we have:

$$\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{J}^\beta_{\mu^+}\,\mathfrak{S}_{\varphi_x}(\nu) \times \mathcal{J}_{\varphi_x}(\nu) + \mathcal{J}^\beta_{\nu^-}\mathfrak{S}_{\varphi_x}(\mu) \times \mathcal{J}_{\varphi_x}(\mu)\right]$$
$$\leq_I \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_{\varphi_x}(\mu) \times \mathcal{J}_{\varphi_x}(\mu) + \mathfrak{S}_{\varphi_x}(\nu) \times \mathcal{J}_{\varphi_x}(\nu)\right)$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_{\varphi_x}(\mu) \times \mathcal{J}_{\varphi_x}(\nu) + \mathfrak{S}_{\varphi_x}(\nu) \times \mathcal{J}_{\varphi_x}(\mu)\right).$$

That is:

$$\frac{\beta}{2(\nu-\mu)^\beta}\left[\int_\mu^\nu (\nu-y)^{\beta-1}\mathfrak{S}_\varphi(x,y) \times \mathcal{J}_\varphi(x,y)\,dy + \int_\mu^\nu (y-\mu)^{\beta-1}\mathfrak{S}_\varphi(x,y) \times \mathcal{J}_\varphi(x,y)\,dy\right]$$
$$\leq_I \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(x,\mu) \times \mathcal{J}_\varphi(x,\mu) + \mathfrak{S}_\varphi(x,\nu) \times \mathcal{J}_\varphi(x,\nu)\right) \tag{66}$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(x,\mu) \times \mathcal{J}_\varphi(x,\nu) + \mathfrak{S}_\varphi(x,\nu) \times \mathcal{J}_\varphi(x,\mu)\right).$$

Multiplying double inequality (66) by $\frac{\alpha(\varsigma-x)^{\alpha-1}}{2(\varsigma-\rho)^\alpha}$ and integrating it with respect to x over $[\rho, \varsigma]$, we get:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^\alpha(\nu-\mu)^\beta}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu) \times \mathcal{J}_\varphi(\varsigma,\nu) + \mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\mu) \times \mathcal{J}_\varphi(\varsigma,\mu)\right]$$
$$\leq_I \frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^\alpha}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^\alpha_{\rho^+}\,\mathfrak{S}_\varphi(\varsigma,\mu) \times \mathcal{J}_\varphi(\varsigma,\mu) + \mathcal{J}^\alpha_{\rho^+}\,\mathfrak{S}_\varphi(\varsigma,\nu) \times \mathcal{J}_\varphi(\varsigma,\nu)\right) \tag{67}$$
$$+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^\alpha_{\rho^+}\,\mathfrak{S}_\varphi(\varsigma,\mu) \times \mathcal{J}_\varphi(\varsigma,\nu) + \mathcal{J}^\alpha_{\rho^+}\,\mathfrak{S}_\varphi(\varsigma,\nu) \times \mathcal{J}_\varphi(\varsigma,\mu)\right).$$

Again, multiplying double inequality (66) by $\frac{\alpha(x-\rho)^{\alpha-1}}{2(\varsigma-\rho)^\alpha}$ and integrating it with respect to x over $[\rho, \varsigma]$, we gain:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^\alpha(\nu-\mu)^\beta}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu) \times \mathcal{J}_\varphi(\rho,\nu) + \mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}_\varphi(\rho,\mu) \times \mathcal{J}_\varphi(\rho,\mu)\right]$$
$$\leq_I \frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^\alpha}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^\alpha_{\varsigma^-}\,\mathfrak{S}_\varphi(\rho,\mu) \times \mathcal{J}_\varphi(\rho,\mu) + \mathcal{J}^\alpha_{\varsigma^-}\,\mathfrak{S}_\varphi(\rho,\nu) \times \mathcal{J}_\varphi(\rho,\nu)\right) \tag{68}$$
$$+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^\alpha_{\varsigma^-}\,\mathfrak{S}_\varphi(\rho,\mu) \times \mathcal{J}_\varphi(\rho,\nu) + \mathcal{J}^\alpha_{\varsigma^-}\,\mathfrak{S}_\varphi(\rho,\nu) \times \mathcal{J}_\varphi(\rho,\mu)\right).$$

Summing (67) and (68), we have:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu) \\ +\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\end{bmatrix}$$
$$\begin{aligned}&\le_I \frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\\ &+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\\ &+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\\ &+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\end{aligned} \tag{69}$$

Now, again with the help of integral inequality (12) for the first two integrals on the right-hand side of (69), we have the following relation:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\\ &\le_I \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\\ &+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right).\end{aligned} \tag{70}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\\ &\le_I \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\ &+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right).\end{aligned} \tag{71}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\\ &\le_I \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\ &+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right).\end{aligned} \tag{72}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\rho^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^-}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\\ &\le_I \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\\ &+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right).\end{aligned} \tag{73}$$

From (70)–(73) and inequality (69), we have

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu) \\ +\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\end{bmatrix}$$

$$\leq_I \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L_{\varphi}(\rho,\varsigma,\mu,\nu)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}_{\varphi}(\rho,\varsigma,\mu,\nu)$$

Since from φ-cuts, we obtain the collection of IVFs $\mathfrak{S}_{\varphi},\mathcal{J}_{\varphi}\colon \Delta\to\mathbb{R}_I^+$, then the above inequality can be written in the form of inequality (64). Hence, the result has been proven. □

Remark 5. *If one is to take $\alpha=1$ and $\beta=1$, then from (64), we achieve the coming inequality, see* [34]*:*

$$\frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}\widetilde{\mathfrak{S}}(x,y)\,\widetilde{\times}\,\tilde{\mathcal{J}}(x,y)\,dy\,dx$$
$$\preccurlyeq \frac{1}{9}\widetilde{K}(\rho,\varsigma,\mu,\nu)\widetilde{+}\frac{1}{18}\left[\tilde{L}(\rho,\varsigma,\mu,\nu)+\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\right]\widetilde{+}\frac{1}{36}\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu) \quad (74)$$

Let one take $\mathfrak{S}_*((x,y),\varphi)$ as an affine function and $\mathfrak{S}^*((x,y),\varphi)$ as a concave function. If $\mathfrak{S}_*((x,y),\varphi)\neq\mathfrak{S}^*((x,y),\varphi)$ with $\varphi=1$ then, by Remark 2 and (65), we acquire the following inequality, see [33]*:*

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}(\varsigma,\nu)\times\mathcal{J}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}(\varsigma,\mu)\times\mathcal{J}(\varsigma,\mu)\right]$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}(\rho,\nu)\times\mathcal{J}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}(\rho,\mu)\times\mathcal{J}(\rho,\mu)\right]$$
$$\supseteq \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\rho,\varsigma,\mu,\nu)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\rho,\varsigma,\mu,\nu) \quad (75)$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\rho,\varsigma,\mu,\nu)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\rho,\varsigma,\mu,\nu)$$

Let one take $\mathfrak{S}_*((x,y),\varphi)$ as an affine function and $\mathfrak{S}^*((x,y),\varphi)$ as a concave function. If $\mathfrak{S}_*((x,y),\varphi)\neq\mathfrak{S}^*((x,y),\varphi)$ with $\varphi=1$, then by Remark 2 and (65), we acquire the following inequality, see [32]*:*

$$\frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}\mathfrak{S}(x,y)\times\mathcal{J}(x,y)\,dy\,dx$$
$$\supseteq \frac{1}{9}K(\rho,\varsigma,\mu,\nu)+\frac{1}{18}\left[L(\rho,\varsigma,\mu,\nu)+\mathcal{M}(\rho,\varsigma,\mu,\nu)\right]+\frac{1}{36}\mathcal{N}(\rho,\varsigma,\mu,\nu) \quad (76)$$

If $\mathfrak{S}_*((x,y),\varphi)=\mathfrak{S}^*((x,y),\varphi)$ and $\mathcal{J}_*((x,y),\varphi)=\mathcal{J}^*((x,y),\varphi)$ with $\varphi=1$, then from (64), we acquire the following inequality, see [50]*:*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}(\varsigma,\nu)\times\mathcal{J}(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}(\varsigma,\mu)\times\mathcal{J}(\varsigma,\mu)\right]\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}(\rho,\nu)\times\mathcal{J}(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}(\rho,\mu)\times\mathcal{J}(\rho,\mu)\right]\\
&\le\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\rho,\varsigma,\mu,\nu)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\rho,\varsigma,\mu,\nu)\\
&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\rho,\varsigma,\mu,\nu)+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\rho,\varsigma,\mu,\nu)
\end{aligned}\tag{77}$$

Theorem 7. *Let $\widetilde{\mathfrak{S}},\tilde{\mathcal{J}}:\Delta\to\mathbb{F}_0$ be a coordinate convex FIVFs on Δ. Then, from φ-cuts, we establish that the series of IVFs $\mathfrak{S}_{\varphi},\mathcal{J}_{\varphi}:\Delta\to\mathbb{R}_I^+$ are given by $\mathfrak{S}_{\varphi}(x,y)=\left[\mathfrak{S}_*\big((x,y),\varphi\big),\mathfrak{S}^*\big((x,y),\varphi\big)\right]$ and $\mathcal{J}_{\varphi}(x,y)=\left[\mathcal{J}_*\big((x,y),\varphi\big),\mathcal{J}^*\big((x,y),\varphi\big)\right]$ for all $(x,y)\in\Delta$ and for all $\varphi\in[0,1]$. If $\widetilde{\mathfrak{S}}\,\tilde{\times}\,\tilde{\mathcal{J}}\in\mathcal{FD}_{\Delta}$, then the following inequalities hold:*

$$\begin{aligned}
&4\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\tilde{\times}\,\tilde{\mathcal{J}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\preccurlyeq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\nu)\tilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\mu)\right]\\
&\tilde{+}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\rho,\nu)\tilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\widetilde{\mathfrak{S}}(\rho,\mu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\rho,\mu)\right]\\
&\tilde{+}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\tilde{K}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)
\end{aligned}\tag{78}$$

If $\widetilde{\mathfrak{S}}$ and $\tilde{\mathcal{J}}$ are both coordinate concave FIVFs on Δ, then the above inequality can be written as:

$$\begin{aligned}
&4\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\tilde{\times}\,\tilde{\mathcal{J}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\succcurlyeq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\nu)\tilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\mu)\right]\\
&\tilde{+}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\rho,\nu)\tilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\widetilde{\mathfrak{S}}(\rho,\mu)\,\tilde{\times}\,\tilde{\mathcal{J}}(\rho,\mu)\right]\\
&\tilde{+}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\tilde{K}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\\
&\tilde{+}\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)
\end{aligned}\tag{79}$$

where $\tilde{K}(\rho,\varsigma,\mu,\nu)$, $\tilde{L}(\rho,\varsigma,\mu,\nu)$, $\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)$, and $\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)$ are given in Theorem 6.

Proof. *Since* $\widetilde{\mathfrak{S}}, \tilde{\mathcal{J}} : \Delta \to \mathbb{F}_0$ *are two convex FIVFs, from inequality (13) and for each* $\varphi \in [0,1]$, *we have:*

$$\begin{aligned}
&2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\le_I \frac{\alpha}{2(\varsigma-\rho)^\alpha}\begin{bmatrix}\int_\rho^\varsigma(\varsigma-x)^{\alpha-1}\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(x,\frac{\mu+\nu}{2}\right)dx\\+\int_\rho^\varsigma(x-\rho)^{\alpha-1}\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(x,\frac{\mu+\nu}{2}\right)dx\end{bmatrix}\\
&+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)+\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\right)\\
&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)+\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\right)
\end{aligned} \tag{80}$$

and:

$$\begin{aligned}
&2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\le_I \frac{\beta}{2(\nu-\mu)^\beta}\begin{bmatrix}\int_\mu^\nu(\nu-y)^{\beta-1}\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)dy\\+\int_\mu^\nu(y-\mu)^{\beta-1}\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)dy\end{bmatrix}\\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)+\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\right)\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)+\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\right)
\end{aligned} \tag{81}$$

Adding (73) and (74), and then taking the multiplication of the resultant one by 2, we obtain:

$$\begin{aligned}
&8\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\le_I \frac{\alpha}{2(\varsigma-\rho)^\alpha}\begin{bmatrix}\int_\rho^\varsigma 2(\varsigma-x)^{\alpha-1}\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(x,\frac{\mu+\nu}{2}\right)dx\\+\int_\rho^\varsigma 2(x-\rho)^{\alpha-1}\mathfrak{S}_\varphi\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(x,\frac{\mu+\nu}{2}\right)dx\end{bmatrix}\\
&+\frac{\beta}{2(\nu-\mu)^\beta}\begin{bmatrix}\int_\mu^\nu 2(\nu-y)^{\beta-1}\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)dy\\+\int_\mu^\nu 2(y-\mu)^{\beta-1}\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},y\right)dy\end{bmatrix}\\
&+\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(2\mathfrak{S}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)+2\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\right)\\
&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(2\mathfrak{S}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)+2\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\right)\\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)+2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\right)\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)+2\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\mu\right)\right)
\end{aligned} \tag{82}$$

Again, with the help of integral inequality (13) and Lemma 1 for each integral on the right-hand side of (82), we have:

$$\begin{aligned}
&\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}2(\varsigma-x)^{\alpha-1}\mathfrak{S}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)dx\\
&\leq_{I}\frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(\varsigma-x)^{\alpha-1}(\nu-y)^{\beta-1}\mathfrak{S}_{\varphi}(x,y)\,dydx\right]\\
&\qquad+\frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(\varsigma-x)^{\alpha-1}(y-\mu)^{\beta-1}\mathfrak{S}_{\varphi}(x,y)\,dydx\right]\\
&+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}(\varsigma-x)^{\alpha-1}\Big(\mathfrak{S}_{\varphi}(x,\mu)\times\mathcal{J}_{\varphi}(x,\mu)+\mathfrak{S}_{\varphi}(x,\nu)\times\mathcal{J}_{\varphi}(x,\nu)\Big)dx\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}(\varsigma-x)^{\alpha-1}\Big(\mathfrak{S}_{\varphi}(x,\mu)\times\mathcal{J}_{\varphi}(x,\nu)+\mathfrak{S}_{\varphi}(x,\nu)\times\mathcal{J}_{\varphi}(x,\mu)\Big)dx\\
&=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\rho^{+},\nu^{-}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right]\\
&+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Big(\mathcal{J}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\Big)\\
&+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Big(\mathcal{J}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\Big).
\end{aligned}\tag{83}$$

$$\begin{aligned}
&\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}2(x-\rho)^{\alpha-1}\mathfrak{S}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_{\varphi}\left(x,\frac{\mu+\nu}{2}\right)dx\\
&\leq_{I}\frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(x-\rho)^{\alpha-1}(\nu-y)^{\beta-1}\mathfrak{S}_{\varphi}(x,y)\,dydx\right]\\
&+\frac{\alpha\beta}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\int_{\rho}^{\varsigma}\int_{\mu}^{\nu}(x-\rho)^{\alpha-1}(y-\mu)^{\beta-1}\mathfrak{S}_{\varphi}(x,y)\,dydx\right]\\
&+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}(x-\rho)^{\alpha-1}\Big(\mathfrak{S}_{\varphi}(x,\mu)\times\mathcal{J}_{\varphi}(x,\mu)+\mathfrak{S}_{\varphi}(x,\nu)\times\mathcal{J}_{\varphi}(x,\nu)\Big)dx\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\alpha}{2(\varsigma-\rho)^{\alpha}}\int_{\rho}^{\varsigma}(x-\rho)^{\alpha-1}\Big(\mathfrak{S}_{\varphi}(x,\mu)\times\mathcal{J}_{\varphi}(x,\nu)+\mathfrak{S}_{\varphi}(x,\nu)\times\mathcal{J}_{\varphi}(x,\mu)\Big)dx\\
&=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\varsigma^{-},\mu^{+}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{J}_{\varsigma^{-},\nu^{-}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right]\\
&+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Big(\mathcal{J}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathcal{J}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\Big)\\
&+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Big(\mathcal{J}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{J}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\Big).
\end{aligned}\tag{84}$$

$$\begin{aligned}
&\frac{\beta}{2(\nu-\mu)^{\beta}}\left[\int_{\mu}^{\nu}2(\nu-y)^{\beta-1}\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},y\right)dy\right]\\
&\leq_{I}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}_{\rho^{+},\mu^{+}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\varsigma^{-},\mu^{+}}^{\alpha,\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right]\\
&+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Big(\mathcal{J}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{J}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\Big)\\
&+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Big(\mathcal{J}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\Big).
\end{aligned}\tag{85}$$

$$\frac{\beta}{2(\nu-\mu)^{\beta}}\left[\int_{\mu}^{\nu}2(y-\mu)^{\beta-1}\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},y\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},y\right)dy\right]$$
$$\leq_{I}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\left[\mathcal{J}^{\alpha,\beta}_{\rho^{+},\nu^{-}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^{-},\nu^{-}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right]$$
$$+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathcal{J}^{\beta}_{\nu^{-}}\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathcal{J}^{\beta}_{\nu^{-}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\tag{86}$$
$$+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathcal{J}^{\beta}_{\nu^{-}}\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\beta}_{\nu^{-}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right).$$

and:

$$2\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)$$
$$\leq_{I}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\tag{87}$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)$$

$$2\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)$$
$$\leq_{I}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\tag{88}$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)$$

$$2\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)$$
$$\leq_{I}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\tag{89}$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)$$

$$2\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\nu\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},\mu\right)$$
$$\leq_{I}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho^{+}}\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{J}^{\alpha}_{\varsigma^{-}}\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\tag{90}$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)$$

$$2\mathfrak{S}_{\varphi}\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_{\varphi}\left(\rho,\frac{\mu+\nu}{2}\right)\tag{91}$$

$$\leq_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\mu)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\rho,\mu)+\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\nu)\right)$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\mu)\right),$$

$$\begin{gathered}2\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\\ \leq_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right]\\ +\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)+\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)\right)\\ +\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right)\end{gathered} \tag{92}$$

$$\begin{gathered}2\mathfrak{S}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\\ \leq_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right]\\ +\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)+\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)\right)\\ +\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right)\end{gathered} \tag{93}$$

and:

$$\begin{gathered}2\mathfrak{S}_\varphi\left(\varsigma,\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\rho,\frac{\mu+\nu}{2}\right)\\ \leq_I \frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left[\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\rho,\mu)\right]\\ +\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\rho,\mu)+\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\rho,\nu)\right)\\ +\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\rho,\mu)\right)\end{gathered} \tag{94}$$

From inequalities (83) to (94) and inequality (82), we have:

$$\begin{aligned}&8\mathfrak{S}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}_\varphi\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\ &\leq_I \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2(\varsigma-\rho)^\alpha(\nu-\mu)^\beta}\begin{bmatrix}\mathcal{I}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathcal{I}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)\\ +\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\rho,\mu)\end{bmatrix}\\ &+\left(\frac{2\alpha}{(\alpha+1)(\alpha+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left(\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)\right)\\ +\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left(\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\rho,\mu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right)\end{bmatrix}\\ &+2\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left(\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathcal{I}^{\beta}_{\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)\right)\\ +\frac{\Gamma(\beta+1)}{2(\nu-\mu)^\beta}\left(\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)+\mathcal{I}^{\beta}_{\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu)\right)\end{bmatrix}\end{aligned} \tag{95}$$

$$+2\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left[\begin{array}{l}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\end{array}\right]$$

$$+2\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left[\begin{array}{l}\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\\+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\end{array}\right]$$

$$+\frac{2\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}K_{\varphi}(\rho,\varsigma,\mu,\nu)++\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{2\beta}{(\beta+1)(\beta+2)}L_{\varphi}(\rho,\varsigma,\mu,\nu)$$

$$+\frac{2\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_{\varphi}(\rho,\varsigma,\mu,\nu)$$

$$+2\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_{\varphi}(\rho,\varsigma,\mu,\nu)$$

Again, with the help of integral inequality (12) and Lemma 1, for each integral on the right-hand side of (95), we have:

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\mathcal{I}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{I}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\&\qquad+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\mathcal{I}_{\nu^{-}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathcal{I}_{\nu^{-}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\\&\qquad\leq_{I}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}_{\varphi}(\rho,\varsigma,\mu,\nu)\end{aligned}\tag{96}$$

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\mathcal{I}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)+\mathcal{I}_{\mu^{+}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\&\qquad+\frac{\Gamma(\beta+1)}{2(\nu-\mu)^{\beta}}\left(\mathcal{I}_{\nu^{-}}^{\beta}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{I}_{\nu^{-}}^{\beta}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)\right)\\&\qquad\leq_{I}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}_{\varphi}(\rho,\varsigma,\mu,\nu)\end{aligned}\tag{97}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\mu)\times\mathcal{J}_{\varphi}(\varsigma,\mu)+\mathcal{I}_{\rho^{+}}^{\alpha}\,\mathfrak{S}_{\varphi}(\varsigma,\nu)\times\mathcal{J}_{\varphi}(\varsigma,\nu)\right)\\&\qquad+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\mu)+\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\nu)\right)\\&\qquad\leq_{I}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)K_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\alpha}{(\alpha+1)(\alpha+2)}L_{\varphi}(\rho,\varsigma,\mu,\nu)\end{aligned}\tag{98}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\\&\qquad+\frac{\Gamma(\alpha+1)}{2(\varsigma-\rho)^{\alpha}}\left(\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\mu)\times\mathcal{J}_{\varphi}(\rho,\nu)+\mathcal{I}_{\varsigma^{-}}^{\alpha}\,\mathfrak{S}_{\varphi}(\rho,\nu)\times\mathcal{J}_{\varphi}(\rho,\mu)\right)\\&\qquad\leq_{I}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{M}_{\varphi}(\rho,\varsigma,\mu,\nu)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}_{\varphi}(\rho,\varsigma,\mu,\nu)\end{aligned}\tag{99}$$

From (92) to (99) we have:

$$4\mathfrak{S}_{\varphi}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}_{\varphi}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\tag{100}$$

$$\leq_I \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}_\varphi(\varsigma,\nu)\times\mathcal{J}_\varphi(\varsigma,\nu)+\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}_\varphi(\varsigma,\mu)\times\mathcal{J}_\varphi(\varsigma,\mu) \\ +\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}_\varphi(\rho,\nu)\times\mathcal{J}_\varphi(\rho,\nu)+\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}_\varphi(\rho,\mu)\times\mathcal{J}_\varphi(\rho,\mu)\end{bmatrix}$$

$$+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K_\varphi(\rho,\varsigma,\mu,\nu)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L_\varphi(\rho,\varsigma,\mu,\nu)$$

$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}_\varphi(\rho,\varsigma,\mu,\nu)$$

$$+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}_\varphi(\rho,\varsigma,\mu,\nu)$$

That is:

$$4\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\widetilde{\times}\,\tilde{\mathcal{J}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)$$

$$\preccurlyeq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix} \mathcal{J}^{\alpha,\beta}_{\rho^+,\mu^+}\,\widetilde{\mathfrak{S}}(\varsigma,\nu)\,\widetilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\rho^+,\nu^-}\,\widetilde{\mathfrak{S}}(\varsigma,\mu)\,\widetilde{\times}\,\tilde{\mathcal{J}}(\varsigma,\mu) \\ \widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\widetilde{\mathfrak{S}}(\rho,\nu)\,\widetilde{\times}\,\tilde{\mathcal{J}}(\rho,\nu)\widetilde{+}\mathcal{J}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\widetilde{\mathfrak{S}}(\rho,\mu)\,\widetilde{\times}\,\tilde{\mathcal{J}}(\rho,\mu)\end{bmatrix}$$

$$\widetilde{+}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]\widetilde{K}(\rho,\varsigma,\mu,\nu)$$

$$\widetilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\rho,\varsigma,\mu,\nu)$$

$$\widetilde{+}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)$$

$$\widetilde{+}\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu)$$

Hence, the result has been proven. □

Remark 6. *If one is to take $\alpha=1$ and $\beta=1$, then from (78), we achieve the coming inequality, see* [34]:

$$4\,\widetilde{\mathfrak{S}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\widetilde{\times}\,\tilde{\mathcal{J}}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)$$
$$\preccurlyeq \frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_\rho^\varsigma\int_\mu^\nu \widetilde{\mathfrak{S}}(x,y)\,\widetilde{\times}\,\tilde{\mathcal{J}}(x,y)dydx\,\widetilde{+}\,\frac{5}{36}\widetilde{K}(\rho,\varsigma,\mu,\nu) \tag{101}$$
$$\widetilde{+}\,\frac{7}{36}\left[\tilde{L}(\rho,\varsigma,\mu,\nu)\widetilde{+}\widetilde{\mathcal{M}}(\rho,\varsigma,\mu,\nu)\right]\widetilde{+}\,\frac{2}{9}\widetilde{\mathcal{N}}(\rho,\varsigma,\mu,\nu).$$

Let one take $\mathfrak{S}_*((x,y),\varphi)$ as an affine function and $\mathfrak{S}^*((x,y),\varphi)$ as a convex function. If $\mathfrak{S}_*((x,y),\varphi)\neq\mathfrak{S}^*((x,y),\varphi)$ with $\varphi=1$, then from Remark 2 and (79), we acquire the following inequality, see [32]:

$$4\,\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)$$
$$\supseteq \frac{1}{(\varsigma-\rho)(\nu-\mu)}\int_\rho^\varsigma\int_\mu^\nu \mathfrak{S}(x,y)\times\mathcal{J}(x,y)dydx+\frac{5}{36}K(\rho,\varsigma,\mu,\nu) \tag{102}$$
$$+\frac{7}{36}\left[L(\rho,\varsigma,\mu,\nu)+\mathcal{M}(\rho,\varsigma,\mu,\nu)\right]+\frac{2}{9}\mathcal{N}(\rho,\varsigma,\mu,\nu).$$

Let one take $\mathfrak{S}_*\big((x,y),\varphi\big)$ *as an affine function and* $\mathfrak{S}^*\big((x,y),\varphi\big)$ *as a convex function. If* $\mathfrak{S}_*\big((x,y),\varphi\big) \neq \mathfrak{S}^*\big((x,y),\varphi\big)$ *with* $\varphi = 1$, *then from Remark 2 and (79), we acquire the following inequality, see* [33]*:*

$$\begin{aligned}
&4\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix}\mathcal{I}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}(\varsigma,\nu)\times\mathcal{J}(\varsigma,\nu)+\mathcal{I}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}(\varsigma,\mu)\times\mathcal{J}(\varsigma,\mu)\\ +\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}(\rho,\nu)\times\mathcal{J}(\rho,\nu)+\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}(\rho,\mu)\times\mathcal{J}(\rho,\mu)\end{bmatrix}\\
&+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\rho,\varsigma,\mu,\nu)
\end{aligned} \tag{103}$$

If $\mathfrak{S}_*\big((x,y),\varphi\big) = \mathfrak{S}^*\big((x,y),\varphi\big)$ *and* $\mathcal{J}_*\big((x,y),\varphi\big) = \mathcal{J}^*\big((x,y),\varphi\big)$ *with* $\varphi = 1$, *then from (78), we acquire the following inequality, see* [50]*:*

$$\begin{aligned}
&4\mathfrak{S}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\times\mathcal{J}\left(\frac{\rho+\varsigma}{2},\frac{\mu+\nu}{2}\right)\\
&\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\varsigma-\rho)^{\alpha}(\nu-\mu)^{\beta}}\begin{bmatrix}\mathcal{I}^{\alpha,\beta}_{\rho^+,\mu^+}\,\mathfrak{S}(\varsigma,\nu)\times\mathcal{J}(\varsigma,\nu)+\mathcal{I}^{\alpha,\beta}_{\rho^+,\nu^-}\,\mathfrak{S}(\varsigma,\mu)\times\mathcal{J}(\varsigma,\mu)\\ +\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\mu^+}\,\mathfrak{S}(\rho,\nu)\times\mathcal{J}(\rho,\nu)+\mathcal{I}^{\alpha,\beta}_{\varsigma^-,\nu^-}\,\mathfrak{S}(\rho,\mu)\times\mathcal{J}(\rho,\mu)\end{bmatrix}\\
&+\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}(\rho,\varsigma,\mu,\nu)\\
&+\left[\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}(\rho,\varsigma,\mu,\nu)
\end{aligned} \tag{104}$$

4. Conclusions and Future Plans

The major goal for this study was to introduce new fuzzy fractional operators and to show some Hermite–Hadamard-type integral inequalities with the help of Lemma 1 for introduced coordinated convex fuzzy-interval-valued functions via newly defined fuzzy fractional integrals. We also demonstrated that the newly established inequalities could be transformed into Hermite–Hadamard-type inequalities for classical convex functions, for fuzzy-interval-valued and interval-valued functions [24,27,30], and for coordinated convex fuzzy-interval-valued and interval-valued functions [32–34,41,42,50] via Riemann integrals and fuzzy Riemann integrals, respectively, without having to prove each one separately. This is a novel and intriguing topic, and future research will be able to find equivalent inequalities for various types of convexity and coordinated m-convexity by using different fractional integral operators.

Author Contributions: Conceptualization, M.B.K.; methodology, M.B.K.; validation, S.T., M.S.S. and H.G.Z.; formal analysis, G.S.-G.; investigation, M.S.S.; resources, S.T.; data curation, H.G.Z.; writing—original draft preparation, M.B.K., G.S.-G. and H.G.Z.; writing—review and editing, M.B.K. and S.T.; visualization, H.G.Z.; supervision, M.B.K. and M.S.S.; project administration,

M.B.K.; funding acquisition, G.S.-G., M.S.S. and H.G.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The research of Santos-García was funded by the Spanish MINECO project TRACES TIN2015–67522–C3–3–R. This work was also funded by Taif University Researchers Supporting Project (number TURSP-2020/345), Taif University, Taif, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research.

Conflicts of Interest: The authors declare no conflict of interest.

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