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Stability Results of Mixed Type Quadratic-Additive Functional Equation in β-Banach Modules by Using Fixed-Point Technique

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Abstract: We aim to introduce the quadratic-additive functional equation (shortly, QA-functional equation) and find its general solution. Then, we study the stability of the kind of Hyers-Ulam result with a view of the aforementioned functional equation by utilizing the technique based on a fixed point in the framework of β-Banach modules. We here discuss our results for odd and even mappings as well as discuss the stability of mixed cases.

Keywords: quadratic-additive functional equation; fixed point approach; β-Banach module; Hyers-Ulam stability

MSC: 39B52; 39B82; 46H25

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1. Introduction

In 1940, Ulam [1] inquired about the stability of groups of homomorphisms: "What is an additive mapping in close range to an additive mapping of a group and a metric group?" In the next year, Hyers [2] responded affirmatively to the above query for more groups, assuming that Banach spaces are the groups. Rassias [3] extended Hyers' theorem by accounting for the unbounded Cauchy difference. Gavruta [4] has demonstrated the stability of Hyers-Ulam-Rassias with its enhanced control function. This stability finding is the stability of Hyers-Ulam-Rassias functional equations. Baker [5] utilized the Banach fixed point theorem to provide a Hyers-Ulam stability result.

Cădariu and Radu used the fixed point approach to prove the stability of the Cauchy functional equation in 2002. They planned to use the fixed-point alternative theorem [6] in β-normed spaces to achieve an accurate solution and error estimate. In 2003, this novel method was used in two consecutive publications [7,8], to get general stability in Hyers-Ulam in the functional equation of Jensen. The paper [9] also made the ECIT 2002 lecture possible. Many subsequent works employed the fixed point alternative to get generalized findings in many functional equations in various domains of Hyers-Ulam stability. The reader is given the following books and research articles that describe the progress made in the problem of Ulam over the last 70 years (see, for example [10–16]). The functional equations

$$\phi(a+b) = \phi(a) + \phi(b) \tag{1}$$

and

$$\phi(a+b) + \phi(a-b) = 2\phi(a) + 2\phi(b) \tag{2}$$

are known as additive functional equation and quadratic functional equation, respectively. Each additive and quadratic solution of a functional equation, in particular, must be an additive mapping and a quadratic mapping. Singh et al. [17] discussed the asymptotic stability of fractional order differential equations in the framework of Banach spaces.

In [18], Czerwik showed the stability of the quadratic functional Equation (2). Skof has been shown for the function $\phi : N_1 \to N_2$, where N_1 is normed space and N_2 is Banach space (see [19]), a stability issue in the Hyers-Ulam approach for Equation (2). Skof's theorem is still true if an Abelian group replaces the domain N_1, according to Cholewa [20].

Grabiec has generalized the above results in [21]. The quadratic functional equation is useful for distinguishing inner product spaces(for example, see [22–24]). The further generalization of Th.M. Rassias' theorem was provided by Găvruţa [4]. Several papers and monographs on different generalizations and applications of stability of the Hyers–Ulam–Rassias have also been published over the last three decades for several functional equations and mappings (see [25–35]).

In this work, we introduce a new kind of generalized quadratic-additive functional equation is

$$\begin{aligned}\sum_{1\le i<j\le n} \quad & \varphi\left(-v_i - v_j + \sum_{k=1;i\neq j\neq k}^{n} v_k\right) \\ = \quad & \left(\frac{n^2-9n+16}{2}\right)\sum_{1\le i<j\le n}\varphi(v_i+v_j) \\ & -\left(\frac{n^3-11n^2+26n-16}{2}\right)\sum_{i=1}^{n}\frac{\varphi(v_i)+\varphi(-v_i)}{2} \\ & -\left(\frac{n^3-11n^2+30n-20}{2}\right)\sum_{i=1}^{n}\frac{\varphi(v_i)-\varphi(-v_i)}{2} \end{aligned} \tag{3}$$

where $n \geq 4$, and obtain its general solutions. The main objective of this work is to examine the stability of a similar type of Hyers–Ulam theorem for the quadratic-additive functional equation in β-Banach modules on a Banach algebra by utilizing fixed point theory.

Throughout, in this work, we consider $\mathbb{K}$ refers either $\mathbb{R}$ or $\mathbb{C}$ and a real number β with $0 < \beta \leq 1$. We can directly utilize the definition of β-normed space in [36] to proceed our main results.

Theorem 1 ([6])**.** *If a complete generalized metric space is* (Y, d) *and* $F : \mathrm{Y} \to \mathrm{Y}$ *is a strictly contractive function with the Lipschitz constant* $0 < L < 1$*,*

$$i.e.,\ d(Fv_1, Fv_2) \leq Ld(v_1, v_2),\ \ \textit{for all } v_1, v_2 \in \mathrm{Y}.$$

Then for each given $v \in \mathrm{Y}$*, either*

$$d(F^m v, F^{m+1} v) = \infty,\ \ \textit{for all } m \geq 0,$$

or there is a positive integer m_0 *satisfies*

(1) $d(F^m v, F^{m+1} v) < \infty$, *for all* $m \geq m_0$;
(2) *the sequence* $\{F^m v\}$ *converges to a fixed point* w^* *of* F;
(3) w^* *is the only one fixed point of* F *in* $\mathrm{Y}^* = \{w \in \mathrm{Y}\ |d(F^{m_0} v, w) < \infty\}$;
(4) $d(w, w^*) \leq \frac{1}{1-L} d(w, Fw)$, *for all* $w \in \mathrm{Y}^*$.

2. Solution of the Quadratic-Additive functional equation

Here, we derive the general solution of (3). Let us assume that V and W are real vector spaces.

Theorem 2. *If an odd mapping $\varphi : V \to W$ satisfies the functional Equation (3) for all $v_1, v_2, \cdots, v_n \in V$, then the function φ is additive.*

Proof. Suppose that the mapping $\varphi : V \to W$ is odd. Since the oddness of φ, which satisfies the property $\varphi(-v) = -\varphi(v)$, for all $v \in V$. Using oddness property in Equation (3), we simply obtain

$$\sum_{1 \le i < j \le n} \varphi\left(-v_i - v_j + \sum_{k=1; i \ne j \ne k}^{n} v_k\right) = \left(\frac{n^2 - 9n + 16}{2}\right) \sum_{1 \le i < j \le n} \varphi(v_i + v_j) - \left(\frac{n^3 - 11n^2 + 30n - 20}{2}\right) \sum_{i=1}^{n} \varphi(v_i) \tag{4}$$

for all $v_1, v_2, \cdots, v_n \in V$. Setting $v_1 = v_2 = \cdots = v_n = 0$ in (4), we have $\varphi(0) = 0$. Replacing $v_1 = v_2 = v$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in Equation (4), we get

$$\varphi(2v) = 2\varphi(v) \tag{5}$$

for all $v \in V$. Interchanging $2v$ instead v in (5), we obtain

$$\varphi(2^2 v) = 2^2 \varphi(v) \tag{6}$$

for all $v \in V$. Again, switching v by $2v$ in (6), we have

$$\varphi(2^3 v) = 2^3 \varphi(v) \tag{7}$$

for all $v \in V$. Thus, for any non-negative integer $n \ge 1$, we can generalize the result that

$$\varphi(2^n v) = 2^n \varphi(v) \tag{8}$$

for all $v \in V$. Therefore, the function φ is odd, it has the solution of the Cauchy additive functional equation's solution. So that the function φ is additive. Moreover, interchanging $(v_1, v_2, \cdots, v_n)$ with $(v_1, v_2, 0, \cdots, 0)$ in (4), we can obtain the Equation (1). Hence the proof is now completed. □

Theorem 3. *If an even mapping $\varphi : V \to W$ satisfies the functional Equation (3) for all $v_1, v_2, \cdots, v_n \in V$, then the function φ is quadratic.*

Proof. Suppose that the mapping $\varphi : V \to W$ is even. Since the evenness of φ, which satisfies the property $\varphi(-v) = \varphi(v)$, $v \in V$. Using evenness property in Equation (3), we simply obtain

$$\sum_{1 \le i < j \le n} \varphi\left(-v_i - v_j + \sum_{k=1; i \ne j \ne k}^{n} v_k\right) = \left(\frac{n^2 - 9n + 16}{2}\right) \sum_{1 \le i < j \le n} \varphi(v_i + v_j) - \left(\frac{n^3 - 11n^2 + 26n - 16}{2}\right) \sum_{i=1}^{n} v_i \tag{9}$$

for all $v_i \in V; i = 1, 2, \cdots, n$. Setting $v_1 = v_2 = \cdots = v_n = 0$ in (9), we get $\varphi(0) = 0$. Interchanging $(v_1, v_2, \cdots, v_n)$ with $(v, v, 0, \cdots, 0)$ in (9), we have

$$\varphi(2v) = 2^2 \varphi(v) \tag{10}$$

for all $v \in V$. Switching $2v$ instead of v in (10), we get

$$\varphi(2^2 v) = 2^4 \varphi(v) \tag{11}$$

for all $v \in V$. Interchanging v with $2v$ in (11), we have

$$\varphi(2^3 v) = 2^6 \varphi(v) \tag{12}$$

for all $v \in V$. Thus, for any integer $n \geq 1$, we can generalize the result that

$$\varphi(2^n v) = 2^{2n} \varphi(v) \tag{13}$$

for all $v \in V$. Therefore, if the function φ is even, it has the solution of the Euler quadratic functional equation's solution. Moreover, changing $(v_1, v_2, \cdots, v_n)$ with $(v_1, v_2, 0, \cdots, 0)$ in (9), we can get the functional Equation (2). Hence, the proof is now completed. $\square$

Theorem 4. *If a function $\varphi : V \to W$ satisfies $\varphi(0) = 0$ and the functional Equation (3) for all $v_1, v_2, \cdots, v_n \in V$ if and only if there exists a mapping $Q : V \times V \to W$ which is symmetric bi-additive and a mapping $A : V \to W$ is additive such that $\varphi(v) = Q(v, v) + A(v)$ for all v in V.*

Proof. It is trivial. $\square$

3. Main Results

Here, we investigated the stability (in the sense of Hyers-Ulam stability) of (3) in β-Banach modules by utilizing a fixed point approach for three different cases. Moreover, we can divide this section into three subsections. In Section 3.1, we get the stability outcomes for odd case; in Section 3.2, we get the stability outcomes for even case; in Section 3.3, we examined our main outcomes of the function Equation (3) for the mixed case.

Before proceed, let us consider B^* is a unital Banach algebra with $\|\cdot\|_{B^*}$, $B_1^* = \{s \in B^* | \ \|s\|_{B^*} = 1\}$, W is a β-normed left Banach B^*-module and V is a β-normed left B^*-module.

We utilize the below abbreviations for a mapping $\varphi : V \to W$:

$$\begin{aligned}
\Theta_s \varphi(v_1, v_2, \cdots, v_n) : \quad = \quad & \sum_{1 \leq i < j \leq n} \varphi\left(-sv_i - sv_j + \sum_{k=1; i \neq j \neq k}^{n} sv_k \right) \\
& - \left(\frac{n^2 - 9n + 16}{2} \right) \sum_{1 \leq i < j \leq n} \varphi(sv_i + sv_j) \\
& + \left(\frac{n^3 - 11n^2 + 26n - 16}{2} \right) s^2 \sum_{i=1}^{n} \frac{\varphi(v_i) + \varphi(-v_i)}{2} \\
& + \left(\frac{n^3 - 11n^2 + 30n - 20}{2} \right) s \sum_{i=1}^{n} \frac{\varphi(v_i) - \varphi(-v_i)}{2}
\end{aligned}$$

for all $v_1, v_2, \cdots, v_n \in V$ and $s \in B_1^*$.

3.1. Stability Results: When φ is Odd

Theorem 5. *Let a mapping $\psi : V^n \to [0, \infty)$ such that*

$$\lim_{m \to \infty} \frac{1}{|2|^{m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0, \quad \forall\, v_1, v_2, \cdots, v_n \in V. \tag{14}$$

Let $\varphi : V \to W$ be an odd mapping such that

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \psi(v_1, v_2, \cdots, v_n), \quad \forall\, v_1, v_2, \cdots, v_n \in V, \tag{15}$$

and $s \in B_1^*$. If there is $0 < L < 1$ (L is a Lipschitz constant) satisfies

$$v \to \phi(v) = \frac{\psi(v, v, 0, \cdots, 0)}{(2n - 6)}$$

and

$$\phi(2v) \leq |2|^{\beta} L\phi(v) \tag{16}$$

for all $v \in V$, then there exists a unique additive mapping $A_1 : V \to W$ satisfies

$$\|A_1(v) - \varphi(v)\|_{\beta} \leq \frac{\phi(v)}{|2|^{\beta} - |2|^{\beta} L}, \quad v \in V. \tag{17}$$

Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for every $v \in V$, then A_1 is B^*-linear, i.e., $A_1(sv) = sA_1(v)$ for all $v \in V$ and all $s \in B^*$.

Proof. Letting $s = 1$, and $v_1 = v_2 = v$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\begin{aligned} \|2(2n-6)\varphi(v) - (2n-6)\varphi(2v)\|_{\beta} &\leq \psi(v, v, 0, \cdots, 0) \\ \left\| \varphi(v) - \frac{\varphi(2v)}{2} \right\|_{\beta} &\leq L\phi(v), \quad v \in V. \end{aligned} \tag{18}$$

Consider the set

$$Y := \{a | a : V \to W, a(0) = 0\}$$

and define the generalized metric on Y as below:

$$d(a, b) = \inf\{\lambda \in [0, \infty) | \|a(v) - b(v)\|_{\beta} \leq \lambda\phi(v), \ \forall\, v \in V\}. \tag{19}$$

Easily, we can verify that (Y, d) is a complete generalized metric space (see [20]).

Next, we define a function $F : Y \to Y$ by

$$(Fa)(v) = \frac{1}{2} a(2v), \quad \forall\ a \in Y,\ v \in V. \tag{20}$$

Let $a, b \in Y$ and an arbitrary constant $\lambda \in [0, \infty)$ with $d(a, b) < \lambda$. Utilizing the definition of d, we obtain

$$\|a(v) - b(v)\|_{\beta} \leq \lambda\phi(v), \tag{21}$$

for all $v \in V$. By the given hypothesis and the last inequality, one has

$$\left\| \frac{1}{2} a(2v) - \frac{1}{2} b(2v) \right\|_{\beta} \leq \lambda L\phi(v) \tag{22}$$

for all $v \in V$. Hence,

$$d(Fa, Fb) \leq Ld(a, b).$$

From inequality (18), we get

$$d(F\varphi, \varphi) \leq \frac{1}{|2|^{\beta}}. \tag{23}$$

From Theorem 1, F has an unique fixed point $A_1 : V \to W$ in $Y^* = \{a \in Y |\, d(a, b) < \infty\}$ satisfies

$$A_1(v) := \lim_{m \to \infty} (F^m \varphi)(v) = \lim_{m \to \infty} \frac{1}{2^m} \varphi(2^m v) \tag{24}$$

and $A_1(2v) = 2A_1(v)\ \forall\, v \in V$. Also, using (23), we get

$$\begin{aligned} d(A_1,\varphi) &\le \frac{1}{1-L} d(F\varphi,\varphi) \\ &\le \frac{1}{1-L}\frac{1}{|2|^{\beta}} \\ &\le \frac{1}{|2|^{\beta}-|2|^{\beta}L}. \end{aligned} \tag{25}$$

Hence, inequality (17) valid for all $v \in V$.
Now, we want to prove that the function A_1 is additive. Using the inequalities (14), (15) and (24), we obtain

$$\begin{aligned} \|\Theta_1 A_1(v_1,v_2,\cdots,v_n)\|_{\beta} &= \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \|\Theta_1 \varphi(2^m v_1, 2^m v_2, \cdots, 2^m v_n)\|_{\beta} \\ &\le \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0, \end{aligned}$$

that is,

$$\begin{aligned} \sum_{1\le i<j\le n} \varphi\left(-v_i - v_j + \sum_{k=1;i\ne j\ne k}^{n} v_k\right) &= \left(\frac{n^2-9n+16}{2}\right) \sum_{1\le i<j\le n} \varphi(v_i+v_j) \\ &\quad - \left(\frac{n^3-11n^2+30n-20}{2}\right) \sum_{i=1}^{n} \varphi(v_i) \end{aligned}$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore, by Theorem 2, the function A_1 is odd.
Finally, we have to show that the function A_1 is unique. Let us consider that there exists an odd mapping $A_1^{'} : V \to W$ satisfies (17). Since

$$d(\varphi, A_1^{'}) \le \frac{1}{|2|^{\beta}(1-L)}$$

and $A_1^{'}$ is additive, we get $A_1^{'} \in Y^*$ and $(FA_1^{'})(v) = \frac{1}{2}A_1^{'}(2v) = A_1(v)$ for all $v \in V$, i.e., $A_1^{'}$ is a fixed point of F in Y^*. Clearly, $A_1^{'} = A_1$.
Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for every $v \in V$, then using the proof of [3], A_1 is $\mathbb{R}$-linear.
Switching $v_1 = v_2 = v$ and $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\begin{aligned} &\|(2n-6)\varphi(2sv) - (n^3-11n^2+34n-32)\varphi(sv) + (n^3-11n^2+30n-20)s\varphi(v)\|_{\beta} \\ &\qquad \le \psi(v,v,0,\cdots,0) \end{aligned} \tag{26}$$

for all $v \in V$ and all $s \in B_1^*$. Thus, using definition of A_1 and the inequalities (14) and (26), we get

$$\begin{aligned} &\|(2n-6)A_1(2sv) - (n^3-11n^2+34n-32)A_1(sv) + (n^3-11n^2+30n-20)sA_1(v)\|_{\beta} \\ &= \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \|(2n-6)\varphi(2^{m+1}sv) - (n^3-11n^2+34n-32)\varphi(2^m sv) \\ &\qquad + (n^3-11n^2+30n-20)s\varphi(2^m v)\|_{\beta} \\ &\le \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \psi(2^m v, 2^m v, 0, \cdots, 0) = 0 \end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. So,

$$(2n-6)A_1(2sv) - (n^3-11n^2+34n-32)A_1(sv) + (n^3-11n^2+30n-20)sA_1(v) = 0$$

for all $v \in V$ and all $s \in B_1^*$. Since A_1 is additive, we get $A_1(sv) = sA_1(v)$ for all $v \in V$ and all $s \in B_1^* \cup \{0\}$.
Since A_1 is $\mathbb{R}$-linear, let $s \in B^* \backslash \{0\}$. Then $A_1(sv) = sA_1(v)$ for all $v \in V$ and $s \in B^*$. Hence, A_1 is B^*-linear. □

Corollary 1. *If an odd function $\varphi : V \to W$ such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \left(\sum_{i=1}^{n} \|v_i\|_\beta^w \right), \quad v_1, v_2, \cdots, v_n \in V, \tag{27}$$

and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v)\|_\beta \leq \frac{\left(\alpha + 2\gamma \|v\|_\beta^w\right)}{(2n-6)\left(|2|^\beta - |2|^{\beta w}\right)}, \quad v \in V,$$

where $0 < w < 1$, $\alpha, \gamma \in [0, \infty)$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear.

Proof. By putting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left(\sum_{i=1}^{n} \|v\|_\beta^w \right)$$

and $L = |2|^{\beta(w-1)}$ in Theorem 5, we obtain our needed result. □

Corollary 2. *Let $w > 0$ such that $nw < 1$ and $\alpha, \gamma \in \mathbb{R}_+$, and let $\varphi : V \to W$ be an odd mapping such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw} \right], \quad v_1, v_2, \cdots, v_n \in V,$$

and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v)\|_\beta \leq \frac{\left(\alpha + 2\gamma \|v\|_\beta^{nw}\right)}{(2n-6)\left(|2|^\beta - |2|^{\beta nw}\right)} \tag{28}$$

for all $v \in V$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear.

Proof. By letting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw} \right]$$

and $L = |2|^{\beta(nw-1)}$ in Theorem 5 , we obtain our needed result. □

Theorem 6. *Let a mapping $\psi : V^n \to [0, \infty)$ such that*

$$\lim_{m \to \infty} |2|^{m\beta} \psi(2^{-m}v_1, 2^{-m}v_2, \cdots, 2^{-m}v_n) = 0 \tag{29}$$

for all $v_1, v_2, \cdots, v_n \in V$. Let $\varphi : V \to W$ be an odd mapping satisfies (15). If there is $0 < L < 1$ such that

$$v \to \phi(v) = \frac{\psi(v, v, 0, \cdots, 0)}{(2n-6)}$$

and

$$\phi(v) \leq |2|^{-\beta} L \phi(2v) \tag{30}$$

for all $v \in V$, then there exists a unique additive mapping $A_1 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v)\|_\beta \leq \frac{L}{|2|^\beta - |2|^\beta L}\phi(v), \quad v \in V. \tag{31}$$

Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear.

Proof. Letting $s = 1$ and $v_1 = v_2 = v$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\|2(2n-6)\varphi(v) - (2n-6)\varphi(2v)\|_\beta \leq \psi(v, v, 0, \cdots, 0) \tag{32}$$

for all $v \in V$. Interchanging v with $\frac{v}{2}$ in (32), we have

$$\left\|2\varphi\left(\frac{v}{2}\right) - \varphi(v)\right\|_\beta \leq L\phi(v) \tag{33}$$

for all $v \in V$. Assume the set

$$\mathrm{Y} := \{a | a : V \to W, a(0) = 0\}$$

and define the generalized metric on Y as below:

$$d(a, b) = \inf\{\lambda \in [0, \infty) | \|a(v) - b(v)\|_\beta \leq \lambda\phi(v), \ \forall\, v \in V\}. \tag{34}$$

Easily, we can verify that (Y, d) is a complete generalized metric space (see [20]).

Next, we can define a function $F : \mathrm{Y} \to \mathrm{Y}$ by

$$(Fa)(v) = 2a\left(\frac{v}{2}\right), \quad \forall\ a \in \mathrm{Y},\ v \in V. \tag{35}$$

Let $a, b \in \mathrm{Y}$ and an arbitrary constant $\lambda \in [0, \infty)$ with $d(a, b) < \lambda$.
Using the definition of d, we obtain

$$\|a(v) - b(v)\|_\beta \leq \lambda\phi(v), \tag{36}$$

for all $v \in V$. By the given hypothesis and the last inequality, one has

$$\left\|2a\left(\frac{v}{2}\right) - 2b\left(\frac{v}{2}\right)\right\|_\beta \leq \lambda L\phi(v) \tag{37}$$

for all $v \in V$. Hence,

$$d(Fa, Fb) \leq Ld(a, b).$$

From inequality (33), we get

$$d(F\varphi, \varphi) \leq \frac{L}{|2|^\beta}.$$

From Theorem 1, F has an unique fixed point $A_1 : V \to W$ in $\mathrm{Y}^* = \{a \in \mathrm{Y} |\ d(a, b) < \ \infty\}$ such that

$$A_1(v) := \lim_{m\to\infty} (F^m\varphi)(v) = \lim_{m\to\infty} 2^m \varphi\left(\frac{v}{2^m}\right) \tag{38}$$

and $A_1\left(\frac{v}{2}\right) = \frac{1}{2}A_1(v) \quad \forall\, v \in V$. Also,

$$d(A_1, \varphi) \leq \frac{1}{1-L} d(F\varphi, \varphi) \leq \frac{L}{|2|^{\beta} - |2|^{\beta} L}. \tag{39}$$

Hence, the inequality (31) valid for all $v \in V$.
Again, we want to show that the function A_1 is additive. Using the inequalities (29), (15) and (38), we obtain

$$\begin{aligned}\|\Theta_1 A_1(v_1, v_2, \cdots, v_n)\|_{\beta} &= \lim_{m\to\infty} |2|^{m\beta} \|\Theta_1 \varphi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right)\|_{\beta} \\ &\leq \lim_{m\to\infty} |2|^{m\beta} \psi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right) = 0,\end{aligned}$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore, by Theorem 2, the function A_1 is odd.
Finally, we have to show that the function A_1 is unique. Let us consider that there exists an odd mapping $A_1^{'} : V \to W$ satisfies (31). Since

$$d(\varphi, A_1^{'}) \leq \frac{L}{(1-L)|2|^{\beta}}$$

and $A_1^{'}$ is additive, we have $A_1^{'} \in Y^*$ and $(FA_1^{'})(v) = 2A_1^{'}\left(\frac{v}{2}\right) = A_1(v)$ for all $v \in V$, i.e., $A_1^{'}$ is a fixed point of F in Y^*. Clearly, $A_1^{'} = A_1$.
Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then using the proof of [3], A_1 is $\mathbb{R}$-linear.
Replacing $v_1 = v_2 = \frac{v}{2}$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\begin{aligned}\|(2n-6)\varphi(sv) - (n^3 - 11n^2 + 34n - 32)\varphi\left(\frac{sv}{2}\right) + (n^3 - 11n^2 + 30n - 20)s\varphi\left(\frac{v}{2}\right)\|_{\beta} \\ \leq \psi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)\end{aligned} \tag{40}$$

for all $v \in V$ and all $s \in B_1^*$. Thus, using definition of A_1, the inequalities (29) and (40), we get

$$\begin{aligned}&\|(2n-6)A_1(sv) - (n^3 - 11n^2 + 34n - 32)A_1\left(\frac{sv}{2}\right) \\ &+(n^3 - 11n^2 + 30n - 20)sA_1\left(\frac{v}{2}\right)\|_{\beta} \\ &\leq \lim_{m\to\infty} |2|^{m\beta} \psi\left(\frac{v}{2^{m+1}}, \frac{v}{2^{m+1}}, 0, \cdots, 0\right) = 0\end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. So,

$$\begin{aligned}(2n-6)A_1(sv) - (n^3 - 11n^2 + 34n - 32)A_1\left(\frac{sv}{2}\right) \\ +(n^3 - 11n^2 + 30n - 20)sA_1\left(\frac{v}{2}\right) = 0\end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. Since A_1 is additive, we get $A_1(sv) = sA_1(v)$ for all $v \in V$ and all $s \in B_1^* \cup \{0\}$.

Since A_1 is $\mathbb{R}$-linear, let $s \in B^* \backslash \{0\}$.

$$\begin{aligned} A_1(sv) &= A_1\left(\|s\|_{B^*} \cdot \frac{s}{\|s\|_{B^*}} v\right) \\ &= \|s\|_{B^*} \cdot A_1\left(\frac{s}{\|s\|_{B^*}} v\right) \\ &= \|s\|_{B^*} \cdot \frac{s}{\|s\|_{B^*}} A_1(v) \\ &= sA_1(v), \qquad v \in V, \quad s \in B^*. \end{aligned}$$

Hence, A_1 is B^*-linear. □

Corollary 3. *If $\varphi : V \to W$ is an odd mapping such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma\left(\sum_{i=1}^{n} \|v_i\|_\beta^w\right), \quad v_1, v_2, \cdots, v_n \in V, \tag{41}$$

and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v)\|_\beta \leq \frac{2\gamma\|v\|_\beta^w}{(2n-6)\left(|2|^{\beta w} - |2|^\beta\right)}$$

for all $v \in V$, where $w > 1$ and $\gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear.

Proof. By letting

$$\psi(v_1, v_2, \cdots, v_n) = \gamma\left(\sum_{i=1}^{n} \|v\|_\beta^w\right)$$

and $L = |2|^{\beta(1-w)}$ in Theorem 6, we obtain our needed outcome. □

Corollary 4. *If $\varphi : V \to W$ is an odd mapping such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma\left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw}\right]$$

for all $v_1, v_2, \cdots, v_n \in V$ and $b \in B_1^*$. Then there exists unique additive mapping $A_1 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v)\|_\beta \leq \frac{2\gamma\|v\|_\beta^{nw}}{(2n-6)\left(|2|^{\beta nw} - |2|^\beta\right)} \tag{42}$$

for all $v \in V$, where $w > 0$ and $\gamma \in \mathbb{R}_+$ with $nw > 1$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear.

Proof. By taking

$$\psi(v_1, v_2, \cdots, v_n) = \gamma\left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw}\right]$$

and $L = |2|^{\beta(1-nw)}$ in Theorem 6, we obtain our needed outcome. □

3.2. Stability Results: When φ Is Even

Theorem 7. *Let a mapping $\psi : V^n \to [0, \infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{|2|^{2m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0 \tag{43}$$

for all $v_1, v_2, \cdots, v_n \in V$. Let $\varphi : V \to W$ be an even mapping with $\varphi(0) = 0$ such that (15). If there is $0 < L < 1$ such that

$$v \to \phi(v) = \frac{\psi(v, v, 0, \cdots, 0)}{(2n-4)}$$

and

$$\phi(2v) \le |2|^{2\beta} L \phi(v) \tag{44}$$

for all $v \in V$, then there exists a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - Q_2(v)\|_\beta \le \frac{\phi(v)}{|2|^{2\beta} - |2|^{2\beta} L} \tag{45}$$

for all $v \in V$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic, i.e., $Q_2(sv) = s^2 Q_2(v)$ for all $v \in V$ and all $s \in B^*$.

Proof. Letting $s = 1$ and $v_1 = v_2 = v$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\begin{aligned} \left\|(2n-4)\varphi(2v) - 2^2(2n-4)\varphi(v)\right\|_\beta &\le \psi(v, v, 0, \cdots, 0) \\ \left\|\frac{\varphi(2v)}{2^2} - \varphi(v)\right\|_\beta &\le L\phi(v), \quad v \in V. \end{aligned} \tag{46}$$

Consider the set $\mathrm{Y} := \{a | a : V \to W, a(0) = 0\}$ and define the generalized metric on Y as below:

$$d(a, b) = \inf\{\lambda \in [0, \infty) \mid \|a(v) - b(v)\|_\beta \le \lambda \phi(v), \ \forall\, v \in V\}. \tag{47}$$

Clearly, (Y, d) is a complete generalized metric space (see [20]).
We can define a function $F : \mathrm{Y} \to \mathrm{Y}$ by

$$(Fa)(v) = \frac{1}{2^2} a(2v), \quad \forall\ a \in \mathrm{Y},\ v \in V. \tag{48}$$

Let $a, b \in \mathrm{Y}$ and an arbitrary constant $\lambda \in [0, \infty)$ with $d(a, b) < \lambda$.
Using the definition of d, we obtain

$$\|a(v) - b(v)\|_\beta \le \lambda \phi(v), \tag{49}$$

for all $v \in V$. By the given hypothesis and the last inequality, one has

$$\left\|\frac{1}{2^2} a(2v) - \frac{1}{2^2} b(2v)\right\|_\beta \le \lambda L \phi(v) \tag{50}$$

for all $v \in V$. Hence,

$$d(Fa, Fb) \le L d(a, b).$$

By using the inequality (46) that

$$d(F\varphi, \varphi) \le \frac{1}{|2|^{2\beta}}.$$

Thus, by Theorem 1, F has a unique fixed point $Q_2 : V \to W$ in $\mathrm{Y}^* = \{a \in \mathrm{Y} |\, d(a, b) < \infty\}$ satisfies

$$Q_2(v) := \lim_{m\to\infty} (F^m \varphi)(v) = \lim_{m\to\infty} \frac{1}{2^{2m}} \varphi(2^m v) \tag{51}$$

and $Q_2(2v) = 2^2 Q_2(v)$ for all $v \in V$. Also,

$$\begin{aligned} d(Q_2, \varphi) &\leq \frac{d(F\varphi, \varphi)}{1-L} \\ &\leq \frac{1}{|2|^{2\beta} - |2|^{2\beta} L}. \end{aligned} \tag{52}$$

Thus, inequality (45) holds for all $v \in V$.
Now, we show that Q_2 is quadratic. By (43), (15) and (51), we have

$$\begin{aligned} \|\Theta_1 Q_2(v_1, v_2, \cdots, v_n)\|_\beta &= \lim_{m\to\infty} \frac{1}{|2|^{2m\beta}} \|\Theta_1 \varphi(2^m v_1, 2^m v_2, \cdots, 2^m v_n)\|_\beta \\ &\leq \lim_{m\to\infty} \frac{1}{|2|^{2m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0, \end{aligned}$$

that is,

$$\begin{aligned} \sum_{1\leq i<j\leq n} \varphi\left(-v_i - v_j + \sum_{k=1; i\neq j\neq k}^{n} v_k\right) &= \left(\frac{n^2 - 9n + 16}{2}\right) \sum_{1\leq i<j\leq n} \varphi(v_i + v_j) \\ &\quad - \left(\frac{n^3 - 11n^2 + 26n - 16}{2}\right) \sum_{i=1}^{n} \frac{\varphi(v_i) + \varphi(-v_i)}{2} \end{aligned}$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore, by Theorem 3, the function Q_2 is even. Next, we want to prove that the function Q_2 is unique. Consider there exists an another quadratic mapping $Q_2^{'} : V \to W$ satisfies the inequality (45). Then,

$$d(\varphi, Q_2^{'}) \leq \frac{1}{|2|^{2\beta} - |2|^{2\beta} L}$$

and $Q_2^{'}$ is quadratic, which gives $Q_2^{'} \in Y^*$ and $(FQ_2^{'})(v) = \frac{1}{2^2} Q_2^{'}(2v) = Q_2(v)$ for all $v \in V$, i.e., $Q_2^{'}$ is a fixed point of F in Y^*. Hence, $Q_2^{'} = Q_2$.
Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for every $v \in V$, then using the proof of [3], Q_2 is $\mathbb{R}$-quadratic.
Replacing $v_1 = v_2 = v$ and the remaining $v_3 = v_4 = \cdots = v_n = 0$ in (15), we get

$$\begin{aligned} \|(2n-4)\varphi(2sv) - (n^3 - 11n^2 + 34n - 32)\varphi(sv) + (n^3 - 11n^2 + 26n - 16)s^2\varphi(v)\|_\beta \\ \leq \psi(v, v, 0, \cdots, 0) \end{aligned} \tag{53}$$

for every $v \in V$ and all $s \in B_1^*$. Using definition of Q_2, (43) and (53), we have

$$\begin{aligned} &\|(2n-4)Q_2(2sv) - (n^3 - 11n^2 + 34n - 32)Q_2(sv) + (n^3 - 11n^2 + 26n - 16)s^2 Q_2(v)\|_\beta \\ &= \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \|(2n-4)\varphi(2^{m+1}sv) - (n^3 - 11n^2 + 34n - 32)\varphi(2^m sv) \\ &\qquad + (n^3 - 11n^2 + 26n - 16)s^2 \varphi(2^m v)\|_\beta \\ &\leq \lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \psi(2^m v, 2^m v, 0, \cdots, 0) = 0 \end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. So,

$$(2n-4)Q_2(2sv) - (n^3 - 11n^2 + 34n - 32)Q_2(sv) + (n^3 - 11n^2 + 26n - 16)s^2 Q_2(v) = 0$$

for all $v \in V$ and all $s \in B_1^*$. Since Q_2 is quadratic, we get $Q_2(sv) = s^2 Q_2(v)$ for all $v \in V$ and all $s \in B_1^* \cup \{0\}$. Since Q_2 is $\mathbb{R}$-quadratic, let $s \in B^* \backslash \{0\}$, then $Q_2(sv) = s^2 Q_2(v)$ for all $v \in V$ and all $s \in B^*$. Hence, Q_2 is B^*-quadratic. □

Corollary 5. *Let $\varphi : V \to W$ be an even function with $\varphi(0) = 0$ such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \left(\sum_{i=1}^{n} \|v_i\|_\beta^w \right) \tag{54}$$

for every $v_1, v_2, \cdots, v_n \in V$ and $s \in B_1^*$, then there is only one quadratic function $Q_2 : V \to W$ fulfils

$$\|\varphi(v) - Q_2(v)\|_\beta \leq \frac{\left(\alpha + 2\gamma \|v\|_\beta^w\right)}{(2n-4)\left(|2|^{2\beta} - |2|^{\beta w}\right)}, \quad v \in V.$$

where $0 < w < 2$, $\alpha, \gamma \in [0, \infty)$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic.

Proof. By letting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left(\sum_{i=1}^{n} \|v\|_\beta^w \right)$$

and $L = |2|^{\beta(w-2)}$ in Theorem 7, we obtain our needed result. □

Corollary 6. *Let $w > 0$ such that $nw < 2$ and $\alpha, \gamma \in \mathbb{R}_+$, and let an even mapping $\varphi : V \to W$ and $\varphi(0) = 0$ such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw} \right]$$

for all $v_1, v_2, \cdots, v_n \in V$ and $s \in B_1^*$, then there exists a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - Q_2(v)\|_\beta \leq \frac{\left(\alpha + 2\gamma \|v\|_\beta^{nw}\right)}{(2n-4)\left(|2|^{2\beta} - |2|^{\beta nw}\right)} \tag{55}$$

for all $v \in V$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all fixed $v \in V$, then Q_2 is B^*-quadratic.

Proof. By letting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left(\sum_{i=1}^{n} \|v\|_\beta^w \right)$$

and $L = |2|^{\beta(nw-2)}$ in Theorem 7, we obtain our needed result. □

Theorem 8. *Let $\psi : V^n \to [0, \infty)$ be a function such that*

$$\lim_{m \to \infty} |2|^{2m\beta} \psi(2^{-m} v_1, 2^{-m} v_2, \cdots, 2^{-m} v_n) = 0 \tag{56}$$

for all $v_1, v_2, \cdots, v_n \in V$. Let $\varphi : V \to W$ be an even function with $\varphi(0) = 0$ such that (15). If there is $0 < L < 1$ satisfies

$$v \to \phi(v) = \frac{\psi(v, v, 0, \cdots, 0)}{(2n-4)}$$

and

$$\phi(v) \leq |2|^{-2\beta} L \phi(2v) \tag{57}$$

for all $v \in V$, then there exists a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - Q_2(v)\|_\beta \leq \frac{L}{|2|^{2\beta} - |2|^{2\beta} L} \phi(v), \quad v \in V. \tag{58}$$

Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic.

Proof. Letting $s = 1$ and $v_1 = v_2 = v$ and the remaining $v_3 = v_3 = \cdots = v_n = 0$ in (15), we get

$$\left\|(2n-4)\varphi(2v) - 2^2(2n-4)\varphi(v)\right\|_\beta \leq \psi(v, v, 0, \cdots, 0) \tag{59}$$

for all $v \in V$. Switching v by $\frac{v}{2}$ in (59), we have

$$\left\|2^2\varphi\left(\frac{v}{2}\right) - \varphi(v)\right\|_\beta \leq L\phi(v) \tag{60}$$

for all $v \in V$. Consider the set $Y := \{a | a : V \to W, a(0) = 0\}$ and define the generalized metric on Y as below:

$$d(a,b) = \inf\{\lambda \in [0, \infty) \mid \|a(v) - b(v)\|_\beta \leq \lambda\phi(v), \ \forall\, v \in V\}. \tag{61}$$

Clearly, (Y, d) is a complete generalized metric space (see [20]). Now, we define a function $F : Y \to Y$ by

$$(Fa)(v) = 2^2 a\left(\frac{v}{2}\right) \tag{62}$$

for all $v \in V$ and all $a \in Y$. Let $a, b \in Y$ and an arbitrary constant $\lambda \in [0, \infty)$ with $d(a,b) < \lambda$.
Using the definition of d, we get

$$\|a(v) - b(v)\|_\beta \leq \lambda\phi(v), \tag{63}$$

for all $v \in V$. By the given hypothesis and the last inequality, one has

$$\left\|2^2 a\left(\frac{v}{2}\right) - 2^2 b\left(\frac{v}{2}\right)\right\|_\beta \leq \lambda L\phi(v) \tag{64}$$

for all $v \in V$. Hence,

$$d(Fa, Fb) \leq Ld(a,b).$$

By utilizing inequality (60) that

$$d(F\varphi, \varphi) \leq \frac{L}{|2|^{2\beta}}.$$

Thus, by Theorem 1, F has a only one fixed point $Q_2 : V \to W$ in $Y^* = \{a \in Y \mid d(a,b) < \infty\}$ satisfies

$$Q_2(v) := \lim_{m\to\infty} (F^m\varphi)(v) = \lim_{m\to\infty} 2^{2m}\varphi\left(\frac{v}{2^m}\right) \tag{65}$$

and $Q_2\left(\frac{v}{2}\right) = \frac{1}{2^2} Q_2(v), \ \forall\, v \in V$. Also,

$$\begin{aligned} d(Q_2, \varphi) &\leq \frac{1}{1-L} d(F\varphi, \varphi) \\ &\leq \frac{L}{|2|^{2\beta} - |2|^{2\beta} L}. \end{aligned} \tag{66}$$

Thus, the inequality (58) holds for all $v \in V$.

Now, we show that Q_2 is quadratic. By (56), (15) and (65), we have

$$\begin{aligned} \|\Theta_1 Q_2(v_1, v_2, \cdots, v_n)\|_\beta &= \lim_{m\to\infty} |2|^{2m\beta} \left\|\Theta_1 \varphi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right)\right\|_\beta \\ &\leq \lim_{m\to\infty} |2|^{2m\beta} \psi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right) = 0, \end{aligned}$$

Therefore, by Theorem 3, the function Q_2 is even. Next, we want to prove that the function Q_2 is unique. Consider there is a quadratic function $Q_2^{'} : V \to W$ which fulfils the inequality (58). Then,

$$d(\varphi, Q_2^{'}) \leq \frac{L}{|2|^{2\beta} - |2|^{2\beta} L}$$

and $Q_2^{'}$ is quadratic, which gives $Q_2^{'} \in Y^*$ and $(FQ_2^{'})(v) = 2^2 Q_2^{'}(\frac{v}{2}) = Q_2(v)$ for every $v \in V$, i.e., $Q_2^{'}$ is a fixed point of F in Y^*. Hence, $Q_2^{'} = Q_2$.
Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then using the proof of [3], Q_2 is $\mathbb{R}$-quadratic.
Interchanging $(v_1, v_2, \cdots, v_n)$ with $\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$ in (15), we get

$$\begin{aligned} &\left\|(2n-4)\varphi(sv) - (n^3 - 11n^2 + 34n - 32)\varphi\left(\frac{sv}{2}\right)\right. \\ &\quad \left. + (n^3 - 11n^2 + 26n - 16)s^2 \varphi\left(\frac{v}{2}\right)\right\|_\beta \leq \psi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right) \end{aligned} \tag{67}$$

for all $v \in V$ and all $s \in B_1^*$. Using definition of Q_2, (56) and (67), we have

$$\begin{aligned} &\left\|(2n-4)Q_2(sv) - (n^3 - 11n^2 + 34n - 32)Q_2\left(\frac{sv}{2}\right)\right. \\ &\left. + (n^3 - 11n^2 + 26n - 16)s^2 Q_2\left(\frac{v}{2}\right)\right\|_\beta \\ &\quad \leq \lim_{m\to\infty} |2|^{2m\beta} \psi\left(\frac{v}{2^{m+1}}, \frac{v}{2^{m+1}}, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. So,

$$\begin{aligned} (2n-4)Q_2(sv) - (n^3 - 11n^2 + 34n - 32)Q_2\left(\frac{sv}{2}\right) \\ + (n^3 - 11n^2 + 26n - 16)s^2 Q_2\left(\frac{v}{2}\right) = 0 \end{aligned}$$

for all $v \in V$ and all $s \in B_1^*$. Since Q_2 is quadratic, we get $Q_2(sv) = s^2 Q_2(v)$ for all $v \in V$ and all $s \in B_1^* \cup \{0\}$. Since Q_2 is $\mathbb{R}$-quadratic, let $s \in B^* \backslash \{0\}$,

$$\begin{aligned} Q_2(sv) &= Q_2\left(\|s\|_{B^*} \cdot \frac{s}{\|s\|_{B^*}} v\right) \\ &= \|s\|_{B^*}^2 \cdot Q_2\left(\frac{s}{\|s\|_{B^*}} v\right) \\ &= \|s\|_{B^*}^2 \cdot \left(\frac{s}{\|s\|_{B^*}}\right)^2 Q_2(v) \\ &= s^2 Q_2(v), \quad v \in V, \end{aligned}$$

and all $s \in B^*$. Hence, Q_2 is B^*-quadratic. $\square$

Corollary 7. *Let $\varphi : V \to W$ be an even function with $\varphi(0) = 0$ such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma \left(\sum_{i=1}^{n} \|v_i\|_\beta^w \right), \quad v_1, v_2, \cdots, v_n \in V, \tag{68}$$

and $s \in B_1^*$, then there exists a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - Q_2(v)\|_\beta \leq \frac{2\gamma \|v\|_\beta^w}{(2n-4)\left(|2|^{\beta w} - |2|^{2\beta}\right)}, \quad v \in V.$$

where $w >$ and $\gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic.

Proof. By letting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left(\sum_{i=1}^{n} \|v\|_\beta^w \right)$$

and $L = |2|^{\beta(2-w)}$ in Theorem 8, we achieve our needed result. □

Corollary 8. *Let $\varphi : V \to W$ be an even function with $\varphi(0) = 0$ such that*

$$\|\Theta_s \varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma \left[\prod_{i=1}^{n} \|v_i\|_\beta^w + \sum_{i=1}^{n} \|v_i\|_\beta^{nw} \right]$$

for all $v_1, v_2, \cdots, v_n \in V$ and $b \in B_1^*$, then there exists a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - Q_2(v)\|_\beta \leq \frac{2\gamma \|v\|_\beta^{nw}}{(2n-4)\left(|2|^{\beta nw} - |2|^{2\beta}\right)}, \quad v \in V, \tag{69}$$

where $w > 0$ such that $nw > 2$ and $\gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic.

Proof. By putting

$$\psi(v_1, v_2, \cdots, v_n) = \alpha + \gamma \left(\sum_{i=1}^{n} \|v\|_\beta^w \right)$$

and $L = |2|^{\beta(2-nw)}$ in Theorem 8, we obtain our needed outcome. □

3.3. Stability Results for the Mixed Case

Theorem 9. *Let a mapping $\psi : V^n \to [0, \infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{|2|^{m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0, \lim_{m\to\infty} \frac{1}{|2|^{2m\beta}} \psi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0 \tag{70}$$

for all $v_1, v_2, \cdots, v_n \in V$. If a mapping $\varphi : V \to W$ and $\varphi(0) = 0$ such that (15). If there exists a constant $0 < L < 1$ satisfies

$$\begin{aligned} \psi(2v, 2v, 0, \cdots, 0) &\leq |2|^\beta L \psi(v, v, 0, \cdots, 0) \quad \text{and} \\ \psi(2v, 2v, 0, \cdots, 0) &\leq |2|^{2\beta} L \psi(v, v, 0, \cdots, 0) \end{aligned} \tag{71}$$

for all $v \in V$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta$$
$$\leq \frac{(\psi(v, v, 0, \cdots, 0) + \psi(-v, -v, 0, \cdots, 0))}{|2|^{2\beta} - |2|^{2\beta}L}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right]$$

for all $v \in V$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear and Q_2 is B^*-quadratic.

Proof. If we divide the function φ into two parts such as even and odd by letting

$$\varphi_e(v) = \frac{\varphi(v) + \varphi(-v)}{2} \quad \text{and} \quad \varphi_o(v) = \frac{\varphi(v) - \varphi(-v)}{2} \tag{72}$$

for $v \in V$, then $\varphi(v) = \varphi_e(v) + \varphi_o(v)$. Let

$$\chi(v_1, v_2, \cdots, v_n) = \frac{[\psi(v_1, v_2, \cdots, v_n) + \psi(-v_1, -v_2, \cdots, -v_n))}{2^\beta},$$

then by (70), (71) and (72), we have

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{|2|^{m\beta}}\chi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) &= 0;\\
\lim_{m\to\infty} \frac{1}{|2|^{2m\beta}}\chi(2^m v_1, 2^m v_2, \cdots, 2^m v_n) &= 0,\\
\chi(2v, 2v, 0, \cdots, 0) &\leq |2|^\beta L\chi(v, v, 0, \cdots, 0),\\
\text{and } \chi(2v, 2v, 0, \cdots, 0) &\leq |2|^{2\beta} L\chi(v, v, 0, \cdots, 0),\\
\|\Theta_s\varphi_o(v_1, v_2, \cdots, v_n)\|_\beta &\leq \chi(v_1, v_2, \cdots, v_n),\\
\|\Theta_s\varphi_e(v_1, v_2, \cdots, v_n)\|_\beta &\leq \chi(v_1, v_2, \cdots, v_n).
\end{aligned}$$

Hence, by Theorem 5 and 7, there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi_o(v) - A_1(v)\|_\beta \leq \frac{1}{(2n-6)|2|^\beta(1-L)}\chi(v, v, 0, \cdots, 0),$$

and

$$\|\varphi_e(v) - Q_2(v)\|_\beta \leq \frac{1}{(2n-4)|2|^{2\beta}(1-L)}\chi(v, v, 0, \cdots, 0)$$

for all $v \in V$. Therefore,

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \leq \|\varphi_o(v) - A_1(v)\|_\beta + \|\varphi_e(v) - Q_2(v)\|_\beta$$
$$\leq \left[\frac{1}{(2n-6)|2|^\beta(1-L)} + \frac{1}{(2n-4)|2|^{2\beta}(1-L)}\right]\chi(v, v, 0, \cdots, 0)$$
$$\leq \frac{1}{|2|^{2\beta} - |2|^{2\beta}L}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right](\psi(v, v, 0, \cdots, 0) + \psi(-v, -v, 0, \cdots, 0))$$

for all $v \in V$. □

Corollary 9. *Let $\varphi : V \to W$ be a function with $\varphi(0) = 0$ such that*

$$\|\Theta_s\varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \sum_{i=1}^{n} \|v_i\|_\beta^w, \quad v_1, v_2, \cdots, v_n \in V, \tag{73}$$

and every $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \leq \frac{2\left(\alpha + 2\gamma\|v\|_\beta^w\right)}{\left(|2|^{2\beta} - |2|^{\beta(w+1)}\right)}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right]$$

for all $v \in V$, where $0 < w < 1$ and $\alpha, \gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then A_1 is B^*-linear and Q_2 is B^*-quadratic.

Corollary 10. *Let $\varphi : V \to W$ be a function with $\varphi(0) = 0$ such that*

$$\|\Theta_s\varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma \sum_{i=1}^{n} \|v_i\|_\beta^w \tag{74}$$

for all $v_1, v_2, \cdots, v_n \in V$ and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \leq \frac{4\gamma\|v\|_\beta^w}{\left(|2|^{2\beta} - |2|^{\beta w}\right)}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right]$$

for all $v \in V$, where $w > 2$ and $\gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic and A_1 is B^*-linear.

Theorem 10. *Let a mapping $\psi : V^n \to [0, \infty)$ such that*

$$\lim_{m\to\infty} |2|^{m\beta}\psi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right) = 0, \lim_{m\to\infty} |2|^{2m\beta}\psi\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right) = 0 \tag{75}$$

for all $v_1, v_2, \cdots, v_n \in V$. If a mapping $\varphi : V \to W$ with $\varphi(0) = 0$ such that (15). If there is a constant $0 < L < 1$ such that

$$\begin{aligned} \psi(v, v, 0, \cdots, 0) &\leq |2|^{-\beta} L\psi(2v, 2v, 0, \cdots, 0) \quad \text{and} \\ \psi(v, v, 0, \cdots, 0) &\leq |2|^{-2\beta} L\psi(2v, 2v, 0, \cdots, 0) \end{aligned} \tag{76}$$

for all $v \in V$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\begin{aligned} &\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \\ &\leq \frac{(\psi(v, v, 0, \cdots, 0) + \psi(-v, -v, 0, \cdots, 0))L}{|2|^{2\beta}(1-L)}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right] \end{aligned}$$

for all $v \in V$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic and A_1 is B^*-linear.

Corollary 11. *If $\varphi : V \to W$ is a function with $\varphi(0) = 0$ such that*

$$\|\Theta_s\varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \gamma \sum_{i=1}^{n} \|v_i\|_\beta^w \tag{77}$$

for every $v_1, v_2, \cdots, v_n \in V$ and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \leq \frac{4\gamma\|v\|_\beta^w}{\left(|2|^{\beta w} - |2|^{2\beta}\right)}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right]$$

for every $v \in V$, where $w > 2$ and $\gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic and A_1 is B^*-linear.

Corollary 12. *If $\varphi : V \to W$ is a function with $\varphi(0) = 0$ such that*

$$\|\Theta_s\varphi(v_1, v_2, \cdots, v_n)\|_\beta \leq \alpha + \gamma \sum_{i=1}^{n} \|v_i\|_\beta^w, \quad v_1, v_2, \cdots, v_n \in V, \tag{78}$$

and $s \in B_1^*$, then there exists a unique additive mapping $A_1 : V \to W$ and a unique quadratic mapping $Q_2 : V \to W$ satisfies

$$\|\varphi(v) - A_1(v) - Q_2(v)\|_\beta \leq \frac{2\left(\alpha + 2\gamma\|v\|_\beta^w\right)}{(|2|^{\beta(w+1)} - |2|^{2\beta})}\left[\frac{|2|^\beta}{(2n-6)} + \frac{1}{(2n-4)}\right]$$

for all $v \in V$, where $0 < w < 1$ and $\alpha, \gamma \in \mathbb{R}_+$. Moreover, if $\varphi(kv)$ is continuous in $k \in \mathbb{R}$ for all $v \in V$, then Q_2 is B^*-quadratic and A_1 is B^*-linear.

Remark 1. *If an even mapping $\varphi : \mathbb{R} \to V$ satisfies the functional Equation (3), then the below assertions holds:*

(1) $\varphi(m^{c/2}v) = m^c\varphi(v), \quad v \in \mathbb{R}, m \in \mathbb{Q}$ and $c \in \mathbb{Z}$.
(2) $\varphi(v) = v^2\varphi(1), \quad v \in \mathbb{R}$ if the function φ is continuous.

Example 1. *Let an even mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by: $\varphi(v) = \sum_{p=0}^{\infty} \frac{\psi(2^p v)}{2^{2p}}$ where*

$$\psi(v) = \begin{cases} \lambda v^2, & -1 < v < 1 \\ \lambda, & else, \end{cases} \tag{79}$$

then the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\Theta\varphi(v_1, v_2, \cdots, v_n)| \leq \left(\frac{n^4 - 8n^3 + 5n^2 + 34n - 32}{4}\right)\left(\frac{4}{3}\right)\lambda\left(\sum_{j=1}^{n} |v_j|^2\right) \tag{80}$$

for all $v_1, v_2, \cdots, v_n \in \mathbb{R}$, but doesn't exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\varphi(v) - Q_2(v)| \leq \delta|v|^2, \quad v \in \mathbb{R}, \tag{81}$$

where λ and δ is a constant.

Remark 2. *If an odd mapping $\varphi : \mathbb{R} \to V$ satisfies the functional Equation (3), then the below assertions holds:*

(1) $\varphi(m^c v) = m^c\varphi(v), \quad v \in \mathbb{R}, m \in \mathbb{Q}$ and $c \in \mathbb{Z}$.
(2) $\varphi(v) = v\varphi(1), \quad v \in \mathbb{R}$ if the function φ is continuous.

Example 2. *Let an odd mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by: $\varphi(v) = \sum_{p=0}^{\infty} \frac{\psi(2^p v)}{2^p}$ where*

$$\psi(v) = \begin{cases} \lambda v, & -1 < v < 1 \\ \lambda, & else, \end{cases} \tag{82}$$

then the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|\Theta\varphi(v_1, v_2, \cdots, v_n)| \leq 2\left(\frac{n^4 - 8n^3 + 5n^2 - 42n - 40}{4}\right)\lambda\left(\sum_{j=1}^{n} |v_j|\right) \tag{83}$$

for all $v_1, v_2, \cdots, v_n \in \mathbb{R}$, *but doesn't exist a additive mapping* $A_1 : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|\varphi(v) - A_1(v)| \leq \delta|v|, \quad v \in \mathbb{R}, \tag{84}$$

where λ *and* δ *is a constant.*

4. Conclusions

As of our knowledge, our findings in this study are novel in the field of stability theory. This is our antecedent endeavor to deal with a new type of mixed QA-functional equation. It is shown that the Equation (3) is equivalent to each other to conclude that their solution is both additive and quadratic mapping. The stability results of different forms of additive and quadratic functional equations are obtained by many mathematicians in various spaces. But, in this work, we have introduced mixed QA-functional Equation (3) and obtained its general solution in Section 2. The main aim of this work is to examine the Hyers-Ulam stability of (3), which has been achieved in Section 3.3 with the help of Section 3.1, where the function φ is odd; and Section 3.2, where the function φ is even, in β-Banach modules by using fixed point approach. By the Corollaries, we have discussed Hyers-Ulam stability for the factors of *sum of norms* and *sum of the product of norms*.

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