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Fixed Point Results for Cirić and Almost Contractions in Convex *b*-Metric Spaces

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Abstract: We establish a fixed point theorem for Cirić contraction in the context of convex b-metric spaces. Furthermore, we ensure that there is a fixed point for the maps satisfying the condition (B) (a kind of almost contraction) in convex b-metric spaces and demonstrate its uniqueness as well. Supporting examples to substantiate the generality of the proved results are given.

Keywords: b-metric spaces; convex metric spaces; almost contraction

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1. Introduction and Preliminaries

The history of fixed point theory goes back a century, to the well-known work of Banach. Since the introduction of this simple but very powerful result of nonlinear analysis, the field of fixed-point theory has been expanded in several possible directions. Cirić [1] introduced the notion of quasi-contraction in 1974 and set out a generalization of the Banach contraction principle. In the sequel, many authors worked in this particular direction and announced some new contractions as an extension of the Banach contraction. The weak contraction defined by Berinde [2] is one of them, and it is vital to note that weak contraction and quasi-contraction are independent of one another. However, the class of weak contraction includes the large class of quasi-contraction. In 2008, Berinde [3] renamed it almost contraction. Furthermore, Babu et al. [4] worked on the open problem posed by Berinde [2] and consequently introduced the maps satisfying the condition (B).

In another direction, many authors extended this contraction principle by giving some ambient structure to the space. In this series, Bakhtin [5] introduced the concept of b-metric spaces, which was extensively defined by Czerwik [6] to enlarge the domain of the Banach contraction. As b-metric is not continuous in the topology generated by its basis, many researchers have been devoted to this space and established several fixed point theorems (for example, see [7–19]). Takahashi [20] established the concept of a convex structure in 1970 and coined the term "convex metric space" to describe a metric space with a convex structure. In the course of the last five decades, many scholars have investigated various properties of convex metric spaces and discussed whether a fixed point for non-expansive maps exists in these spaces (refer to [21–26]). Recently, Chen et al. [27] defined the notion of convex b-metric spaces and proved Banach and Kannan's type fixed point theorems in those spaces. Here, motivated by this idea, we establish several fixed point theorems for Cirić contraction as well as for the maps satisfying the condition (B) in a convex b-metric space and present some supporting examples for the proved results.

First, we recall the basic definitions and results, which are required in the sequel to prove our main results. Throughout, real number sets and natural number sets are indicated, respectively, by $\mathbb{R}$ and $\mathbb{N}$.

Definition 1 ([5,6])**.** *Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ be a given real number. Suppose $b_m : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a mapping satisfying the following axioms for all $\mu, \eta, \zeta \in \mathcal{X}$:*

(1) $b_m(\mu, \eta) = 0$ *if and only if* $\mu = \eta$;
(2) $b_m(\mu, \eta) = b_m(\eta, \mu)$;
(3) $b_m(\mu, \eta) \leq s[b_m(\mu, \zeta) + b_m(\zeta, \eta)]$.

Then, the mapping b_m is said to be a b-metric and the pair $(\mathcal{X}, b_m)$ is called a b-metric space.

The convergent and Cauchy sequence in the context of b-metric spaces is defined as follows:

Definition 2 ([14])**.** *Let $(\mathcal{X}, b_m)$ be a b-metric space. A sequence $\{\mu_n\}$ in $\mathcal{X}$ is said to be*

(1) *Convergent in $\mathcal{X}$, if there exists $\mu \in \mathcal{X}$ such that $b_m(\mu_n, \mu) \to 0$ as $n \to \infty$.*
(2) *Cauchy in $\mathcal{X}$, if for each $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that $b_m(\mu_n, \mu_m) < \epsilon$ for all $n, m > p$.*

The b-metric space $(\mathcal{X}, b_m)$ is called complete if every Cauchy sequence $\{\mu_n\} \subset \mathcal{X}$ is convergent in $\mathcal{X}$.

Definition 3 ([20])**.** *Let $(\mathcal{X}, b_m)$ be a b-metric space and $\Omega : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ be a continuous mapping. Then, Ω is called the convex structure on $\mathcal{X}$ if*

$$b_m(\mu, \Omega(\eta, \zeta, \sigma)) \leq \sigma b_m(\mu, \eta) + (1 - \sigma) b_m(\mu, \zeta). \tag{1}$$

The b-metric space $(\mathcal{X}, b_m)$ equipped with a convex structure Ω on $\mathcal{X}$ is called a convex b-metric space, and it is denoted by the triplet $(\mathcal{X}, b_m, \Omega)$. One can refer to [27] to see the examples of convex b-metric spaces. We offer one more example:

Example 1. *Let $\mathcal{X} = \mathbb{R}_0^{+^n}$ be the set of all ordered n-tuples of non-negative real numbers, and $b_m(\mu, \eta) = \sum_{i \in I}[(\mu_i - \eta_i)^2 + |\mu_i - \eta_i|]$ for all $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathcal{X}$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathcal{X}$, where $I = \{1, 2, \ldots, n\}$. Here, we observe that*

1. $b_m(\mu, \eta) = 0$ *iff* $\mu = \eta$;
2. $b_m(\mu, \eta) = b_m(\eta, \mu)$;
3. $b_m(\mu, \eta) \leq 2[b_m(\mu, \zeta) + b_m(\zeta, \eta)]$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ *as*

$$\begin{aligned}
b_m(\mu, \eta) &= \sum_{i \in I}[(\mu_i - \eta_i)^2 + |\mu_i - \eta_i|] \\
&= \sum_{i \in I}\Big[((\mu_i - \zeta_i) + (\zeta_i - \eta_i))^2 + |(\mu_i - \zeta_i) + (\zeta_i - \eta_i)|\Big] \\
&\leq \sum_{i \in I}\Big[2\Big((\mu_i - \zeta_i)^2 + (\zeta_i - \eta_i)^2\Big) + |(\mu_i - \zeta_i)| + |(\zeta_i - \eta_i)|\Big] \\
&\leq 2\sum_{i \in I}\Big[\Big((\mu_i - \zeta_i)^2 + (\zeta_i - \eta_i)^2\Big) + |(\mu_i - \zeta_i)| + |(\zeta_i - \eta_i)|\Big] \\
&= 2\left[\sum_{i \in I}[(\mu_i - \zeta_i)^2 + |\mu_i - \zeta_i|] + \sum_{i \in I}[(\zeta_i - \eta_i)^2 + |\zeta_i - \eta_i|]\right] \\
&= 2[b_m(\mu, \zeta) + b_m(\zeta, \eta)].
\end{aligned}$$

Let $\Omega : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ stand for the mapping given by

$$\Omega(\mu, \eta; \sigma) = \sigma\mu + (1 - \sigma)\eta,$$

for any $\mu, \eta \in \mathcal{X}$ and $\sigma \in [0, 1]$. Then, it follows,

$$\begin{aligned}
b_m(\zeta,\Omega(\mu,\eta;\sigma)) &= \sum_{i\in I}(\zeta_i-(\sigma\mu_i+(1-\sigma)\eta_i))^2+\sum_{i\in I}|\zeta_i-(\sigma\mu_i+(1-\sigma)\eta_i)| \\
&\le \sum_{i\in I}(\sigma|\zeta_i-\mu_i|+(1-\sigma)|\zeta_i-\eta_i|)^2+\sum_{i\in I}|\sigma|\zeta_i-\mu_i|+(1-\sigma)|\zeta_i-\eta_i|| \\
&\le \sum_{i\in I}\sigma^2(\zeta_i-\mu_i)^2+\sum_{i\in I}(1-\sigma)^2(\zeta_i-\eta_i)^2+2\sigma(1-\sigma)\sum_{i\in I}|\zeta_i-\mu_i|\cdot|\zeta_i-\eta_i| \\
&\quad +\sum_{i\in I}\sigma|\zeta_i-\mu_i|+\sum_{i\in I}(1-\sigma)|\zeta_i-\eta_i| \\
&\le \sum_{i\in I}\sigma^2(\zeta_i-\mu_i)^2+\sum_{i\in I}(1-\sigma)^2(\zeta_i-\eta_i)^2+\sigma(1-\sigma)\sum_{i\in I}((\zeta_i-\mu_i)^2+(\zeta_i-\eta_i)^2) \\
&\quad +\sum_{i\in I}\sigma|\zeta_i-\mu_i|+\sum_{i\in I}(1-\sigma)|\zeta_i-\eta_i| \\
&= \sigma\sum_{i\in I}[(\zeta_i-\mu_i)^2+|\zeta_i-\mu_i|]+(1-\sigma)\sum_{i\in I}[(\zeta_i-\eta_i)^2+|\zeta_i-\eta_i|] \\
&= \sigma b_m(\zeta,\mu)+(1-\sigma)b_m(\zeta,\eta).
\end{aligned}$$

As a result, we can designate the triplet $(\mathcal{X},b_m,\Omega)$ a convex b-metric space. The metric triangle inequality, however, is not satisfied by b_m, for example,

$$b_m(\overline{2},\overline{4})=6>b_m(\overline{2},\overline{3})+b_m(\overline{2},\overline{4})=4,$$

where $\overline{q}$ denotes the n-tuple $(q,1,1,\ldots,1)\in\mathcal{X}=\mathbb{R}_0^{+^n}$ and thus, $(\mathcal{X},b_m,\Omega)$ is not a metric space.

2. Main Results

Theorem 1. *Suppose $\Gamma:(\mathcal{X},b_m,\Omega)\to(\mathcal{X},b_m,\Omega)$ is a quasi-contraction, that is, Γ satisfies*

$$b_m(\Gamma\mu,\Gamma\eta)\le k\max\{b_m(\mu,\eta),b_m(\mu,\Gamma\mu),b_m(\eta,\Gamma\eta),b_m(\mu,\Gamma\eta),b_m(\eta,\Gamma\mu)\},\tag{2}$$

for all $\mu,\eta\in\mathcal{X}$ and some $k\in(0,1)$, where $(\mathcal{X},b_m,\Omega)$ is a complete convex b-metric space with $s>1$. Let $\mu_n=\Omega(\mu_{n-1},\Gamma\mu_{n-1};\sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0\in\mathcal{X}$ with the property $b_m(\mu_0,\Gamma\mu_0)<\infty$, where $0\le\sigma_{n-1}<1$ for each $n\in\mathbb{N}$. If $k<\min\left\{\frac{1}{s^2(s+1)},\frac{1}{s^4}\right\}$ and $0\le\sigma_{n-1}<\min\left\{\frac{1}{s^2}-(s+1)k,\frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}\right\}$ for each $n\in\mathbb{N}$, then Γ has a fixed point in $\mathcal{X}$ that is unique.

Proof. As Ω is a convex structure, then

$$b_m(\mu_n,\mu_{n+1})=b_m(\mu_n,\Omega(\mu_n,\Gamma\mu_n;\sigma_n))\le(1-\sigma_n)b_m(\mu_n,\Gamma\mu_n),\ \ n\in\mathbb{N},$$

and

$$\begin{aligned}
b_m(\mu_n,\Gamma\mu_n) &\le sb_m(\mu_n,\Gamma\mu_{n-1})+sb_m(\Gamma\mu_{n-1},\Gamma\mu_n) \\
&\le sb_m(\Omega(\mu_{n-1},\Gamma\mu_{n-1};\sigma_{n-1}),\Gamma\mu_{n-1})+sk\max\{b_m(\mu_{n-1},\mu_n), \\
&\quad b_m(\mu_{n-1},\Gamma\mu_{n-1}),b_m(\mu_n,\Gamma\mu_n),b_m(\mu_{n-1},\Gamma\mu_n),b_m(\mu_n,\Gamma\mu_{n-1})\} \\
&\le s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1})+sk\max\{(1-\sigma_{n-1})b_m(\mu_{n-1},\Gamma\mu_{n-1}), \\
&\quad b_m(\mu_{n-1},\Gamma\mu_{n-1}),b_m(\mu_n,\Gamma\mu_n),b_m(\mu_{n-1},\Gamma\mu_n),b_m(\mu_n,\Gamma\mu_{n-1})\} \\
&\le s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1})+sk\max\{b_m(\mu_{n-1},\Gamma\mu_{n-1}),b_m(\mu_n,\Gamma\mu_n), \\
&\quad sb_m(\mu_{n-1},\mu_n)+sb_m(\mu_n,\Gamma\mu_n),b_m(\Omega(\mu_{n-1},\Gamma\mu_{n-1};\sigma_{n-1}),\Gamma\mu_{n-1})\} \\
&\le s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1})+sk\max\{b_m(\mu_{n-1},\Gamma\mu_{n-1}),b_m(\mu_n,\Gamma\mu_n), \\
&\quad s(1-\sigma_{n-1})b_m(\mu_{n-1},\Gamma\mu_{n-1})+sb_m(\mu_n,\Gamma\mu_n),\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1})\} \\
&\le s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1})+sk\max\{b_m(\mu_{n-1},\Gamma\mu_{n-1}),
\end{aligned}$$

$$
\begin{aligned}
& s(1-\sigma_{n-1})b_m(\mu_{n-1},\Gamma\mu_{n-1}) + sb_m(\mu_n,\Gamma\mu_n)\} \\
\leq\ & s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1}) + sk\max\{b_m(\mu_{n-1},\Gamma\mu_{n-1}), \\
& sb_m(\mu_{n-1},\Gamma\mu_{n-1}) + sb_m(\mu_n,\Gamma\mu_n)\} \\
=\ & s\sigma_{n-1}b_m(\mu_{n-1},\Gamma\mu_{n-1}) + s^2kb_m(\mu_{n-1},\Gamma\mu_{n-1}) + s^2kb_m(\mu_n,\Gamma\mu_n) \\
=\ & [s\sigma_{n-1}+s^2k]b_m(\mu_{n-1},\Gamma\mu_{n-1}) + s^2kb_m(\mu_n,\Gamma\mu_n).
\end{aligned}
$$

This implies

$$(1-s^2k)b_m(\mu_n,\Gamma\mu_n) \leq [s\sigma_{n-1}+s^2k]b_m(\mu_{n-1},\Gamma\mu_{n-1})$$

$$
\begin{aligned}
b_m(\mu_n,\Gamma\mu_n) &\leq \frac{s\sigma_{n-1}+s^2k}{1-s^2k}b_m(\mu_{n-1},\Gamma\mu_{n-1}) \\
&< \frac{1}{s}b_m(\mu_{n-1},\Gamma\mu_{n-1}),
\end{aligned}
$$

with inequalities $k < \min\left\{\frac{1}{s^2(s+1)}, \frac{1}{s^4}\right\}$ and $0 \leq \sigma_{n-1} < \min\left\{\frac{1}{s^2} - (s+1)k, \frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}\right\}$ holding for all $n \in \mathbb{N}$. Thus,

$$
\begin{aligned}
b_m(\mu_n,\Gamma\mu_n) &< \frac{1}{s}b_m(\mu_{n-1},\Gamma\mu_{n-1}) \\
&< \frac{1}{s^2}b_m(\mu_{n-2},\Gamma\mu_{n-2}) \\
&\ \vdots \\
&< \frac{1}{s^n}b_m(\mu_0,\Gamma\mu_0). \qquad (3)
\end{aligned}
$$

Since $b_m(\mu_0,\Gamma\mu_0) < \infty$, then, by applying $n \to \infty$ in (3), we obtain

$$\lim_{n\to\infty} b_m(\mu_n,\Gamma\mu_n) = 0.$$

Thus,

$$b_m(\mu_n,\mu_{n+1}) \leq (1-\sigma_n)b_m(\mu_n,\Gamma\mu_n) \Longrightarrow \lim_{n\to\infty} b_m(\mu_n,\mu_{n+1}) = 0.$$

Moreover, we have to prove that the sequence $\{\mu_n\}$ is Cauchy. Conversely, assume that $\{\mu_n\}$ is a non-Cauchy sequence, we can obtain a positive ϵ and the subsequences $\{\mu_{m_\lambda}\}$ and $\{\mu_{n_\lambda}\}$ of $\{\mu_n\}$, such that m_λ is the smallest cardinal with $m_\lambda > n_\lambda > \lambda$,

$$b_m(\mu_{m_\lambda},\mu_{n_\lambda}) \geq \epsilon,$$

and

$$b_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) < \epsilon.$$

Then, it follows that

$$\epsilon \leq b_m(\mu_{m_\lambda},\mu_{n_\lambda}) \leq s\big[b_m(\mu_{m_\lambda},\mu_{n_\lambda+1}) + b_m(\mu_{n_\lambda+1},\mu_{n_\lambda})\big],$$

which gives

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to\infty} b_m(\mu_{m_\lambda},\mu_{n_\lambda+1}).$$

Moreover, we obtain

$$
\begin{aligned}
b_m(\mu_{m_\lambda},\mu_{n_\lambda+1}) &= b_m\big(\Omega(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1};\sigma_{m_\lambda-1}),\mu_{n_\lambda+1}\big) \\
&\leq \sigma_{m_\lambda-1}b_m(\mu_{m_\lambda-1},\mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1})b_m(\Gamma\mu_{m_\lambda-1},\mu_{n_\lambda+1}) \\
&\leq \sigma_{m_\lambda-1}b_m(\mu_{m_\lambda-1},\mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1})s[b_m(\Gamma\mu_{m_\lambda-1},\Gamma\mu_{n_\lambda+1}) \\
&\quad +b_m(\Gamma\mu_{n_\lambda+1},\mu_{n_\lambda+1})]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sigma_{m_\lambda-1}b_m(\mu_{m_\lambda-1},\mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1})sb_m(\Gamma\mu_{n_\lambda+1},\mu_{n_\lambda+1}) \\
&\quad +(1-\sigma_{m_\lambda-1})sk\max\{b_m(\mu_{m_\lambda-1},\mu_{n_\lambda+1}), b_m(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1}), \\
&\quad b_m(\mu_{n_\lambda+1},\Gamma\mu_{n_\lambda+1}), b_m(\mu_{m_\lambda-1},\Gamma\mu_{n_\lambda+1}), b_m(\mu_{n_\lambda+1},\Gamma\mu_{m_\lambda-1})\} \\
&\leq s\sigma_{m_\lambda-1}b_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) + s\sigma_{m_\lambda-1}b_m(\mu_{n_\lambda},\mu_{n_\lambda+1}) \\
&\quad +(1-\sigma_{m_\lambda-1})sb_m(\Gamma\mu_{n_\lambda+1},\mu_{n_\lambda+1}) \\
&\quad +(1-\sigma_{m_\lambda-1})sk\max\{sb_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) + sb_m(\mu_{n_\lambda},\mu_{n_\lambda+1}), \\
&\quad b_m(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1}), b_m(\mu_{n_\lambda+1},\Gamma\mu_{n_\lambda+1}), sb_m(\mu_{m_\lambda-1},\mu_{n_\lambda+1}) \\
&\quad +sb_m(\mu_{n_\lambda+1},\Gamma\mu_{n_\lambda+1}), sb_m(\mu_{n_\lambda+1},\mu_{m_\lambda-1}) + sb_m(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1})\} \\
&\leq s\sigma_{m_\lambda-1}b_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) + s\sigma_{m_\lambda-1}b_m(\mu_{n_\lambda},\mu_{n_\lambda+1}) \\
&\quad +(1-\sigma_{m_\lambda-1})sb_m(\Gamma\mu_{n_\lambda+1},\mu_{n_\lambda+1}) \\
&\quad +(1-\sigma_{m_\lambda-1})sk\max\{sb_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) + sb_m(\mu_{n_\lambda},\mu_{n_\lambda+1}), \\
&\quad b_m(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1}), b_m(\mu_{n_\lambda+1},\Gamma\mu_{n_\lambda+1}), s^2b_m(\mu_{m_\lambda-1},\mu_{n_\lambda}) \\
&\quad s^2b_m(\mu_{n_\lambda},\mu_{n_\lambda+1}) + sb_m(\mu_{n_\lambda+1},\Gamma\mu_{n_\lambda+1}), s^2b_m(\mu_{n_\lambda+1},\mu_{n_\lambda}) \\
&\quad +s^2b_m(\mu_{n_\lambda},\mu_{m_\lambda-1}) + sb_m(\mu_{m_\lambda-1},\Gamma\mu_{m_\lambda-1})\} \\
&< s\sigma_{m_\lambda-1}\epsilon + sk(1-\sigma_{m_\lambda-1})\max\{s\epsilon, s^2\epsilon\} \\
&= s\epsilon\Big(\sigma_{m_\lambda-1}(1-s^2k) + s^2k\Big) \\
&< s\epsilon\left(\frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}(1-s^2k) + s^2k\right) \\
&= s\epsilon\left(\frac{\frac{1}{s^2}-s^2k}{1-s^2k}(1-s^2k) + s^2k\right) = \frac{\epsilon}{s}.
\end{aligned}
$$

Thus, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to\infty} b_m(\mu_{m_\lambda},\mu_{n_\lambda+1}) < \frac{\epsilon}{s},$$

and hence our supposition, $\{\mu_n\}$ is a non-Cauchy sequence, is wrong; so, $\{\mu_n\}$ is Cauchy in $\mathcal{X}$. Then, due to the completeness of $\mathcal{X}$, there will be an element $\mu^* \in \mathcal{X}$ such that $\lim_{n\to\infty} b_m(\mu_n,\mu^*) = 0$.

Now, we will verify that $\Gamma\mu^* = \mu^*$. For this,

$$
\begin{aligned}
b_m(\mu^*,\Gamma\mu^*) &\leq sb_m(\mu^*,\mu_n) + sb_m(\mu_n,\Gamma\mu^*) \\
&\leq sb_m(\mu^*,\mu_n) + s^2b_m(\mu_n,\Gamma\mu_n) + s^2b_m(\Gamma\mu_n,\Gamma\mu^*) \\
&\leq sb_m(\mu^*,\mu_n) + s^2b_m(\mu_n,\Gamma\mu_n) + s^2k\max\{b_m(\mu_n,\mu^*), b_m(\mu_n,\Gamma\mu_n), \\
&\quad b_m(\mu^*,\Gamma\mu^*), b_m(\mu_n,\Gamma\mu^*), b_m(\mu^*,\Gamma\mu_n)\} \\
&\leq sb_m(\mu^*,\mu_n) + s^2b_m(\mu_n,\Gamma\mu_n) + s^2k\max\{b_m(\mu_n,\mu^*), b_m(\mu_n,\Gamma\mu_n), \\
&\quad b_m(\mu^*,\Gamma\mu^*), sb_m(\mu_n,\mu^*) + sb_m(\mu^*,\Gamma\mu^*), sb_m(\mu^*,\mu_n) + sb_m(\mu_n,\Gamma\mu_n)\}.
\end{aligned}
$$

Letting $n\to\infty$, we have

$$
\begin{aligned}
b_m(\mu^*,\Gamma\mu^*) &\leq s^2k\max\{b_m(\mu^*,\Gamma\mu^*), sb_m(\mu^*,\Gamma\mu^*)\} \\
&= s^3kb_m(\mu^*,\Gamma\mu^*) \\
&< b_m(\mu^*,\Gamma\mu^*).
\end{aligned}
$$

So, $b_m(\mu^*,\Gamma\mu^*) = 0 \implies \Gamma\mu^* = \mu^*$.

Hence, μ^* is a fixed point of Γ.

Now, it only remains to show that the fixed point μ^* is unique. For this, take a fixed point of Γ, say $q \in \mathcal{X}$ different from μ^*, then

$$0 < b_m(\mu^*,q) \quad = \quad b_m(\Gamma\mu^*,Tq)$$

$$\begin{aligned}
&\leq k\max\{b_m(\mu^*,q), b_m(\mu^*,\Gamma\mu^*), b_m(q,Tq), b_m(\mu^*,Tq), b_m(q,\Gamma\mu^*)\} \\
&\leq k\max\{b_m(\mu^*,q), sb_m(\mu^*,q)+sb_m(q,Tq), sb_m(q,\mu^*)+sb_m(\mu^*,\Gamma\mu^*)\} \\
&= skb_m(\mu^*,q) \\
&< \frac{1}{s^3}b_m(\mu^*,q) < b_m(\mu^*,q),
\end{aligned}$$

which is a contradiction. Therefore, $\mu^* = q$.

Hence, the proof. □

Lemma 1. *If $s \geq 2$, then $\min\left\{\frac{1}{s^2}-(s+1)k, \frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}\right\} = \frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}$, where $k > 0$.*

Proof. To begin with,

$$\frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k} = \frac{1}{s^2} - \frac{k\left(1-\frac{1}{s^2}\right)}{\frac{1}{s^2}-k}. \tag{4}$$

As, $s \geq 2 \Rightarrow \frac{1}{s}+\frac{2}{s^2} \leq 1 < 1+(s+1)k$, which gives $\frac{1-\frac{1}{s^2}}{\frac{1}{s^2}-k} > s+1$, then by (4) we obtain

$$\frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k} < \frac{1}{s^2}-(s+1)k.$$

Hence, the lemma. □

In view of the above lemma, we obtain the following result, which is an extension of Theorem 1 of [27].

Theorem 2. *Suppose $\Gamma : (\mathcal{X}, b_m, \Omega) \to (\mathcal{X}, b_m, \Omega)$ is a quasi-contraction, that is, Γ satisfies*

$$b_m(\Gamma\mu,\Gamma\eta) \leq k\max\{b_m(\mu,\eta), b_m(\mu,\Gamma\mu), b_m(\eta,\Gamma\eta), b_m(\mu,\Gamma\eta), b_m(\eta,\Gamma\mu)\}, \tag{5}$$

for all $\mu,\eta \in \mathcal{X}$ and some $k \in (0,1)$, where $(\mathcal{X}, b_m, \Omega)$ is a complete convex b-metric space with $s \geq 2$. Let $\mu_n = \Omega(\mu_{n-1}, \Gamma\mu_{n-1}; \sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0 \in \mathcal{X}$ with the property $b_m(\mu_0, \Gamma\mu_0) < \infty$, where $0 \leq \sigma_{n-1} < 1$ for each $n \in \mathbb{N}$. If $k < \frac{1}{s^4}$ and $0 \leq \sigma_{n-1} < \frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}$ for each $n \in \mathbb{N}$; then, Γ has a fixed point in $\mathcal{X}$ that is unique.

Proof. The proof follows the same lines as the proof of Theorem 1. □

Now, we present an example in support of the generality of the proved result over the existing one.

Example 2. *Let $\mathcal{X} = \mathbb{R}_0^+$ be the set of all non-negative real numbers, and $b_m(\mu,\eta) = (\mu-\eta)^2 + |\mu-\eta|$ for all $\mu,\eta \in \mathcal{X}$, $\Omega : \mathcal{X}\times\mathcal{X}\times[0,1] \to \mathcal{X}$ stands for the mapping defined as $\Omega(\mu,\eta,\sigma) = \sigma\mu + (1-\sigma)\eta$ for all $\mu,\eta \in \mathcal{X}$. Then, $(\mathcal{X}, b_m, \Omega)$ is a convex b-metric space with $s = 2$ (follows by taking n = 1 in Example 1). The map $\Gamma : \mathcal{X} \to \mathcal{X}$ is defined as follows*

$$\Gamma(\mu) = \begin{cases} \frac{\mu}{17}, & \mu \in [0,1) \\ \frac{\mu}{19}, & \mu \in [1,\infty). \end{cases}$$

Firstly, we prove that Γ satisfies the inequality (5). To prove this, we consider four cases.

1. *If $\mu, \eta \in [0,1)$, then*

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= \left(\frac{\mu}{17} - \frac{\eta}{17}\right)^2 + \left|\frac{\mu}{17} - \frac{\eta}{17}\right| \\ &\leq \frac{1}{17}\left[\frac{1}{17}(\mu-\eta)^2 + |\mu-\eta|\right] \\ &\leq \frac{1}{17}\left[(\mu-\eta)^2 + |\mu-\eta|\right] \\ &\leq \frac{1}{17}b_m(\mu,\eta). \end{aligned}$$

2. *If $\mu \in [0,1)$ and $\eta \in [1,\infty)$, then*

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= (T\mu - T\eta)^2 + |T\mu - T\eta| \\ &= \left(\frac{\mu}{17} - \frac{\eta}{19}\right)^2 + \left|\frac{\mu}{17} - \frac{\eta}{19}\right| \\ &= \frac{1}{17}\left[\frac{1}{17}\left(\mu - \frac{17}{19}\eta\right)^2 + \left|\mu - \frac{17}{19}\eta\right|\right]. \end{aligned}$$

If $\mu > \frac{17}{19}\eta$, then

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= \frac{1}{17}\left[\frac{1}{17}\left(\mu - \frac{17}{19}\eta\right)^2 + \left(\mu - \frac{17}{19}\eta\right)\right] \\ &< \frac{1}{17}\left[\left(\mu - \frac{17}{19}\eta\right)^2 + \left(\mu - \frac{17}{19}\eta\right)\right] \\ &\leq \frac{1}{17}\left[\left(\mu - \frac{\eta}{19}\right)^2 + \left(\mu - \frac{\eta}{19}\right)\right] \\ &= \frac{1}{17}b_m(\mu, \Gamma\eta). \end{aligned}$$

If $\mu < \frac{17}{19}\eta$, then

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= \frac{1}{17}\left[\frac{1}{17}\left(\frac{17}{19}\eta - \mu\right)^2 + \left(\frac{17}{19}\eta - \mu\right)\right] \\ &< \frac{1}{17}\left[\frac{1}{17}(\eta-\mu)^2 + (\eta-\mu)\right] \\ &< \frac{1}{17}\left[(\eta-\mu)^2 + (\eta-\mu)\right] \\ &= \frac{1}{17}b_m(\eta,\mu) = \frac{1}{17}b_m(\mu,\eta). \end{aligned}$$

3. *If $\mu \in [1,\infty)$ and $\eta \in [0,1)$, then as in the above case, we obtain*

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq \frac{1}{17}b_m(\eta, \Gamma\mu), \ \ if\ \eta > \frac{17}{19}\mu \\ and\ \ b_m(\Gamma\mu, \Gamma\eta) &\leq \frac{1}{17}b_m(\mu,\eta), \ \ if\ \eta < \frac{17}{19}\mu. \end{aligned}$$

4. *If $\mu, \eta \in [1,\infty)$*

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= \left(\frac{\mu}{19} - \frac{\eta}{19}\right)^2 + \left|\frac{\mu}{19} - \frac{\eta}{19}\right| \\ &\leq \frac{1}{19}\left[\frac{1}{19}(\mu-\eta)^2 + |\mu-\eta|\right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{19}\Big[(\mu-\eta)^2+|\mu-\eta|\Big] \\ &\leq \frac{1}{19}b_m(\mu,\eta)<\frac{1}{17}b_m(\mu,\eta), \end{aligned}$$

which shows that

$$b_m(\Gamma\mu,\Gamma\eta)\leq\frac{1}{17}\max\{b_m(\mu,\eta),b_m(\mu,\Gamma\mu),b_m(\eta,\Gamma\eta),b_m(\mu,\Gamma\eta),b_m(\eta,\Gamma\mu)\},$$

for all $\mu,\eta\in H$. Thus, Γ is satisfying the inequality (5) for $k=\frac{1}{17}<\frac{1}{s^4}$.

Now, choose an initial point $\mu_0\in\mathcal{X}$ and generate $\mu_n=\Omega(\mu_n,\Gamma\mu_{n-1};\sigma_{n-1})$ with $\sigma_{n-1}=\frac{1}{53}<\frac{\frac{1}{s^4}-k}{\frac{1}{s^2}-k}$. There are two possibilities for μ_0

1. *If $\mu_0<1$, then*

$$\begin{aligned} T\mu_0 &= \frac{\mu_0}{17} \\ \mu_1 &= \frac{1}{53}\mu_0+\frac{52}{53}T\mu_0=\frac{69}{901}\mu_0 \\ \mu_2 &= \frac{1}{53}\mu_1+\frac{52}{53}T\mu_1=\left(\frac{69}{901}\right)^2\mu_0 \\ &\vdots \\ \mu_n &= \frac{1}{53}\mu_{n-1}+\frac{52}{53}T\mu_{n-1}=\left(\frac{69}{901}\right)^n\mu_0. \end{aligned}$$

Clearly, $\mu_n\to 0$ as $n\to\infty$.

2. *If $\mu_0\geq 1$, then*

$$\begin{aligned} T\mu_0 &= \frac{\mu_0}{19} \\ \mu_1 &= \frac{1}{53}\mu_0+\frac{52}{53}T\mu_0=\frac{71}{1007}\mu_0. \end{aligned}$$

If $\mu_1\in[0,1)$, then as in the above case, $\mu_n\to 0$ as $n\to\infty$. If $\mu_1\in[1,\infty)$, then $\frac{\mu_2}{\mu_1}=\frac{1}{53}+\frac{52}{53}\cdot\frac{T\mu_1}{\mu_1}=\frac{71}{1007}$. Proceeding in a similar fashion, we can assume that $\mu_{n-1}\in[1,\infty)$, then we obtain,

$$\frac{\mu_n}{\mu_{n-1}}=\frac{1}{53}+\frac{52}{53}\cdot\frac{T\mu_{n-1}}{\mu_{n-1}}=\frac{71}{1007},$$

and

$$\frac{\mu_n}{\mu_0}=\frac{\mu_1}{\mu_0}\cdot\frac{\mu_2}{\mu_1}\cdots\frac{\mu_n}{\mu_{n-1}}=\left(\frac{71}{1007}\right)^n.$$

Hence $\lim_{n\to\infty}\mu_n=0$. Thus, Γ has a unique fixed point since all of the assumptions of Theorem 2 are fulfilled. It is worth mentioning that 0 would be the only fixed point of Γ in $\mathcal{X}$. The map Γ, on the other hand, fails to follow the contraction condition used in Theorem 1 of [27] at the point $\mu=\frac{999}{1000},\eta=\frac{1001}{1000}$, as we observe that

$$b_m(\mu,\eta)=\left(\frac{999}{1000}-\frac{1001}{1000}\right)^2+\left|\frac{999}{1000}-\frac{1001}{1000}\right|=\frac{501}{250000}=0.002004,$$

and

$$\begin{aligned} b_m(\Gamma\mu,\Gamma\eta) &= \left(\frac{1}{17}\cdot\frac{999}{1000}-\frac{1}{19}\cdot\frac{1001}{1000}\right)^2+\left|\frac{1}{17}\cdot\frac{999}{1000}-\frac{1}{19}\cdot\frac{1001}{1000}\right| \\ &= \frac{39889331}{6520562500}=0.006117>b_m(\mu,\eta). \end{aligned}$$

Therefore, Theorem 1 of [27] is not applicable to guarantee the existence of fixed point of map Γ.

The following result is a Chatterjea type fixed point theorem in the context of a convex b-metric space, which is a direct consequence of Theorem 2.

Corollary 1. *Suppose $\Gamma : (\mathcal{X}, b_m, \Omega) \to (\mathcal{X}, b_m, \Omega)$ is a Chatterjea-contraction, that is, Γ satisfies*

$$b_m(\Gamma\mu,\Gamma\eta) \le k[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)], \tag{6}$$

for all $\mu,\eta \in \mathcal{X}$ and some $k \in \left(0,\frac{1}{2}\right)$, where $(\mathcal{X}, b_m, \Omega)$ is a complete convex b-metric space with $s \ge 2$. Let $\mu_n = \Omega(\mu_{n-1},\Gamma\mu_{n-1};\sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0 \in \mathcal{X}$ with the property $b_m(\mu_0,\Gamma\mu_0) < \infty$, where $0 \le \sigma_{n-1} < 1$ for each $n \in \mathbb{N}$. If $k < \frac{1}{2s^4}$ and $0 \le \sigma_{n-1} < \frac{\frac{1}{s^4}-2k}{\frac{1}{s^2}-2k}$ for each $n \in \mathbb{N}$, then Γ has a fixed point in $\mathcal{X}$ that is unique.

Next, we present an example for the applicability of the above corollary.

Example 3. *Take the triplet $(\mathcal{X}, b_m, \Omega)$ as given in Example 2. Suppose $\Gamma : \mathcal{X} \to \mathcal{X}$ is defined as*

$$T(\mu) = \begin{cases} \frac{\mu}{33}, & \mu \in [0,1) \\ \frac{1}{35\mu}, & \mu \in [1,\infty). \end{cases}$$

Now, we prove that Γ satisfies the inequality (6) for $k = \frac{1}{33}$. For this, we discuss the following possible cases

1. *If $\mu,\eta \in [0,1)$, then the inequality (6) holds.*
2. *If $\mu \in [0,1)$ and $\eta \in [1,\infty)$, then*

$$\begin{aligned} b_m(\Gamma\mu,\Gamma\eta)-\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] &= \left[(\Gamma\mu-\Gamma\eta)^2+|\Gamma\mu-\Gamma\eta|\right] \\ &\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\ &= \left[\left(\frac{\mu}{33}-\frac{1}{35\eta}\right)^2+\left|\frac{\mu}{33}-\frac{1}{35\eta}\right|\right] \\ &\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\ &= \frac{1}{33}\left[\frac{1}{33}\left(\mu-\frac{33}{35\eta}\right)^2+\left|\mu-\frac{33}{35\eta}\right|\right] \\ &\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)]. \end{aligned}$$

If $\mu > \frac{33}{35\eta}$, then

$$\begin{aligned}
b_m(\Gamma\mu,\Gamma\eta)-\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] &= \frac{1}{33}\left[\frac{1}{33}\left(\mu-\frac{33}{35\eta}\right)^2+\left(\mu-\frac{33}{35\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&< \frac{1}{33}\left[\left(\mu-\frac{33}{35\eta}\right)^2+\left(\mu-\frac{33}{35\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&< \frac{1}{33}\left[\left(\mu-\frac{1}{35\eta}\right)^2+\left(\mu-\frac{1}{35\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&= \frac{1}{33}b_m(\mu,\Gamma\eta)-\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&= -\frac{1}{33}b_m(\eta,\Gamma\mu)\leq 0.
\end{aligned}$$

If $\mu<\frac{33}{35\eta}$, then

$$\begin{aligned}
b_m(\Gamma\mu,\Gamma\eta)-\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] &= \frac{1}{33}\left[\frac{1}{33}\left(\frac{33}{35\eta}-\mu\right)^2+\left(\frac{33}{35\eta}-\mu\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&= \frac{1}{33}\left[\frac{1}{33}\left(\frac{1}{\eta}-\mu\right)^2+\left(\frac{1}{\eta}-\mu\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&< \frac{1}{33}\left[\frac{1}{33}(\eta-\mu)^2+(\eta-\mu)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&< \frac{1}{33}\left[\left(\eta-\frac{\mu}{33}\right)^2+\left(\eta-\frac{\mu}{33}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&= \frac{1}{33}b_m(\eta,\Gamma\mu)-\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)] \\
&= -\frac{1}{33}b_m(\mu,\Gamma\eta)\leq 0.
\end{aligned}$$

which implies that $b_m(\Gamma\mu,\Gamma\eta)\leq\frac{1}{33}[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)]$ holds $\forall\ \mu\in[0,1),\eta\in[1,\infty)$.

3. *If $\mu\in[1,\infty)$ and $\eta\in[0,1)$, then as in the above case, we obtain that the inequality (6) holds.*

4. *If $\mu, \eta \in [1, \infty)$, then*

$$
\begin{aligned}
b_m(\Gamma\mu, \Gamma\eta) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \quad &= \quad \left[\left(\frac{1}{35\mu} - \frac{1}{35\eta}\right)^2 + \left|\frac{1}{35\mu} - \frac{1}{35\eta}\right|\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&\leq \quad \frac{1}{35}\left[\frac{1}{35}\left(\frac{1}{\mu} - \frac{1}{\eta}\right)^2 + \left|\frac{1}{\mu} - \frac{1}{\eta}\right|\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)].
\end{aligned}
$$

If $\eta > \mu$, then

$$
\begin{aligned}
b_m(\Gamma\mu, \Gamma\eta) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \quad &\leq \quad \frac{1}{35}\left[\frac{1}{35}\left(\mu - \frac{1}{\eta}\right)^2 + \left(\mu - \frac{1}{\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{35}\left[\frac{1}{35}\left(\mu - \frac{1}{35\eta}\right)^2 + \left(\mu - \frac{1}{35\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{35}\left[\left(\mu - \frac{1}{35\eta}\right)^2 + \left(\mu - \frac{1}{35\eta}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&\leq \quad \frac{1}{35}b_m(\mu, \Gamma\eta) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{33}b_m(\mu, \Gamma\eta) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&= \quad -\frac{1}{33}b_m(\eta, \Gamma\mu) \leq 0.
\end{aligned}
$$

If $\mu > \eta$, then

$$
\begin{aligned}
b_m(\Gamma\mu, \Gamma\eta) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \quad &\leq \quad \frac{1}{35}\left[\frac{1}{35}\left(\eta - \frac{1}{\mu}\right)^2 + \left(\eta - \frac{1}{\mu}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{35}\left[\frac{1}{35}\left(\eta - \frac{1}{35\mu}\right)^2 + \left(\eta - \frac{1}{35\mu}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{35}\left[\left(\eta - \frac{1}{35\mu}\right)^2 + \left(\eta - \frac{1}{35\mu}\right)\right] \\
&\quad -\frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&\leq \quad \frac{1}{35}b_m(\eta, \Gamma\mu) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\
&< \quad \frac{1}{33}b_m(\eta, \Gamma\mu) - \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)]
\end{aligned}
$$

$$= \quad -\frac{1}{33} b_m(\mu, \Gamma\eta) \leq 0,$$

which shows that

$$b_m(\Gamma\mu, \Gamma\eta) \leq \frac{1}{33}[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)],$$

for all $\mu, v \in \mathcal{X}$. Let us choose an initial point $\mu_0 \in \mathcal{X}$ and generate $\mu_n = \Omega(\mu_n, \Gamma\mu_{n-1}; \sigma_{n-1})$ with $\sigma_{n-1} = \frac{1}{101} < \frac{\frac{1}{s^4}-2k}{\frac{1}{s^2}-2k}$. Now, to ensure the uniqueness of a fixed point, we will consider the following choices of μ_0.

1. *If $\mu_0 < 1$, then*

$$\begin{aligned} \Gamma\mu_0 &= \frac{\mu_0}{33} \\ \mu_1 &= \frac{1}{101}\mu_0 + \frac{100}{101}\Gamma\mu_0 = \frac{133}{3333}\mu_0 \\ \mu_2 &= \frac{1}{101}\mu_1 + \frac{100}{101}\Gamma\mu_1 = \left(\frac{133}{3333}\right)^2 \mu_0 \\ &\vdots \\ \mu_n &= \frac{1}{101}\mu_{n-1} + \frac{100}{101}\Gamma\mu_{n-1} = \left(\frac{133}{3333}\right)^n \mu_0. \end{aligned}$$

Letting $n \to \infty$, we have $\mu_n \to 0$.

2. *If $\mu_0 \geq 1$, then*

$$\begin{aligned} \Gamma\mu_0 &= \frac{1}{35\mu_0} \\ \mu_1 &= \frac{1}{101}\mu_0 + \frac{100}{101}\Gamma\mu_0 \\ &= \frac{1}{101}\mu_0 + \frac{100}{101} \cdot \frac{1}{35\mu_0} \\ \frac{\mu_1}{\mu_0} &= \frac{1}{101} + \frac{100}{3535} \cdot \frac{1}{\mu_0^2} \leq \frac{135}{3535}. \end{aligned}$$

If $\mu_1 \in [0, 1)$, then as in the above case, $\mu_n \to 0$ as $n \to \infty$. If $\mu_1 \in [1, \infty)$, then $\frac{\mu_2}{\mu_1} = \frac{1}{101} + \frac{100}{3535} \cdot \frac{1}{\mu_1^2} \leq \frac{135}{3535}$. Proceeding in a similar fashion, we can assume that $\mu_{n-1} \in [1, \infty)$, then we obtain,

$$\frac{\mu_n}{\mu_{n-1}} = \frac{1}{101} + \frac{100}{3535} \cdot \frac{1}{\mu_{n-1}^2} \leq \frac{135}{3535},$$

and

$$\frac{\mu_n}{\mu_{n-1}} = \frac{\mu_1}{\mu_0} \cdot \frac{\mu_2}{\mu_1} \cdots \frac{\mu_n}{\mu_{n-1}} \leq \left(\frac{135}{3535}\right)^n.$$

Here, also $\mu_n \to 0$ as $n \to \infty$.

Thus, the sequence $\mu_n \to 0$ and hence by Corollary 1, the map Γ has a unique fixed point in $\mathcal{X}$. Notice that 0 is a fixed point of Γ in $\mathcal{X}$, and this fixed point is unique as well. To examine the uniqueness, suppose that there is a fixed point of Γ different from 0, say $q \in [1, \infty)$. Then,

$$q = Tq = \frac{1}{35q} \implies q = \frac{1}{\sqrt{35}} < 1,$$

which is an anomaly. Therefore, the only fixed point of Γ in $\mathcal{X}$ is 0.

The following theorem ensures the existence and uniqueness of a fixed point for the map Γ satisfying the condition (B) in a convex b-metric space.

Theorem 3. *Suppose $\Gamma : (\mathcal{X}, b_m, \Omega) \to (\mathcal{X}, b_m, \Omega)$ is a map satisfying the condition (B), that is, Γ satisfies*

$$b_m(\Gamma\mu, \Gamma\eta) \leq kb_m(\mu, \eta) + L\min\{b_m(\mu, \Gamma\mu), b_m(\eta, \Gamma\eta), b_m(\mu, \Gamma\eta), b_m(\eta, \Gamma\mu)\}, \tag{7}$$

for all $\mu, \eta \in \mathcal{X}$ and some $k \in (0,1)$ with $L \geq 0$, where $(\mathcal{X}, b_m, \Omega)$ is a complete convex b-metric space with $s > 1$. Let $\mu_n = \Omega(\mu_{n-1}, \Gamma\mu_{n-1}; \sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0 \in \mathcal{X}$ with the property $b_m(\mu_0, \Gamma\mu_0) < \infty$, where $0 \leq \sigma_{n-1} < 1$ for each $n \in \mathbb{N}$. If $k < \frac{1}{s^3}$ and $0 \leq \sigma_{n-1} < \frac{\frac{1}{s^3} - k}{1-k+L}$ for each $n \in \mathbb{N}$, then Γ has a fixed point in $\mathcal{X}$ that is unique.

Proof. As Ω is a convex structure, then

$$b_m(\mu_n, \mu_{n+1}) = b_m(\mu_n, \Omega(\mu_n, \Gamma\mu_n; \sigma_n)) \leq (1 - \sigma_n)b_m(\mu_n, \Gamma\mu_n),\ n \in \mathbb{N},$$

and

$$\begin{aligned}
b_m(\mu_n, \Gamma\mu_n) &\leq sb_m(\mu_n, \Gamma\mu_{n-1}) + sb_m(\Gamma\mu_{n-1}, \Gamma\mu_n) \\
&\leq sb_m(\Omega(\mu_{n-1}, \Gamma\mu_{n-1}; \sigma_{n-1}), \Gamma\mu_{n-1}) + skb_m(\mu_{n-1}, \mu_n) \\
&\quad + sL\min\{\, b_m(\mu_{n-1}, \Gamma\mu_{n-1}), b_m(\mu_n, \Gamma\mu_n), b_m(\mu_{n-1}, \Gamma\mu_n), b_m(\mu_n, \Gamma\mu_{n-1})\} \\
&\leq s\sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) + sk(1 - \sigma_{n-1})b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&\quad + sL\min\{\, b_m(\mu_{n-1}, \Gamma\mu_{n-1}), sb_m(\mu_n, \mu_{n-1}) + sb_m(\mu_{n-1}, \Gamma\mu_n), \\
&\quad b_m(\mu_{n-1}, \Gamma\mu_n), b_m(\Omega(\mu_{n-1}, \Gamma\mu_{n-1}; \sigma_{n-1}), \Gamma\mu_{n-1})\} \\
&\leq s\sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) + sk(1 - \sigma_{n-1})b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&\quad + sL\min\{b_m(\mu_{n-1}, \Gamma\mu_{n-1}), b_m(\mu_{n-1}, \Gamma\mu_n), \sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1})\} \\
&\leq s\sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) + sk(1 - \sigma_{n-1})b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&\quad + sL\min\{sb_m(\mu_{n-1}, \Gamma\mu_{n-1}) + sb_m(\Gamma\mu_{n-1}, \Gamma\mu_n), \sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1})\} \\
&= s\sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) + sk(1 - \sigma_{n-1})b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&\quad + sL\sigma_{n-1}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&= [\sigma_{n-1}(s - sk + sL) + sk]b_m(\mu_{n-1}, \Gamma\mu_{n-1}) \\
&< \frac{1}{s^2}b_m(\mu_{n-1}, \Gamma\mu_{n-1}),
\end{aligned}$$

with inequalities $k < \frac{1}{s^3}$ and $0 \leq \sigma_{n-1} < \frac{\frac{1}{s^3} - k}{1-k+L}$ holding for all $n \in \mathbb{N}$. Thus,

$$b_m(\mu_n, \Gamma\mu_n) < \frac{1}{s^2}b_m(\mu_{n-1}, \Gamma\mu_{n-1}) < b_m(\mu_{n-1}, \Gamma\mu_{n-1}). \tag{8}$$

This implies that $\{b_m(\mu_n, \Gamma\mu_n)\}$ is a sequence of non-negative real numbers that is non-increasing. Consequently, there exists a non-negative real number δ such that

$$\lim_{n\to\infty} b_m(\mu_n, \Gamma\mu_n) = \delta.$$

Now, it is to be shown that $\delta = 0$. On the contrary if $\delta > 0$, then by applying $n \to \infty$ in (8), we obtain

$$\delta \leq \frac{1}{s^2}\delta,$$

which is a contradiction as $s > 1$ and hence δ is zero, that is,

$$\lim_{n\to\infty} b_m(\mu_n, \Gamma\mu_n) = 0.$$

Thus,

$$b_m(\mu_n, \mu_{n+1}) \leq (1 - \sigma_n) b_m(\mu_n, \Gamma\mu_n) \Longrightarrow \lim_{n\to\infty} b_m(\mu_n, \mu_{n+1}) = 0.$$

Moreover, we have to prove that the sequence $\{\mu_n\}$ is Cauchy. Conversely, assume that $\{\mu_n\}$ is a non-Cauchy sequence, we can obtain a positive ϵ and and the subsequences $\{\mu_{m_\lambda}\}$ and $\{\mu_{n_\lambda}\}$ of $\{\mu_n\}$, such that m_λ is the smallest cardinal with $m_\lambda > n_\lambda > \lambda$,

$$b_m(\mu_{m_\lambda}, \mu_{n_\lambda}) \geq \epsilon,$$

and

$$b_m(\mu_{m_\lambda - 1}, \mu_{n_\lambda}) < \epsilon.$$

Then, we conclude that

$$\epsilon \leq b_m(\mu_{m_\lambda}, \mu_{n_\lambda}) \leq s\big[b_m(\mu_{m_\lambda}, \mu_{n_\lambda+1}) + b_m(\mu_{n_\lambda+1}, \mu_{n_\lambda})\big],$$
$$\Rightarrow \frac{\epsilon}{s} \leq \limsup_{\lambda\to\infty} b_m(\mu_{m_\lambda}, \mu_{n_\lambda+1}).$$

Now, we consider that

$$\begin{aligned}
b_m(\mu_{m_\lambda}, \mu_{n_\lambda+1}) &= b_m\big(\Omega(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1}; \sigma_{m_\lambda-1}), \mu_{n_\lambda+1}\big) \\
&\leq \sigma_{m_\lambda-1} b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1}) b_m(\Gamma\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) \\
&\leq \sigma_{m_\lambda-1} b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1}) s[b_m(\Gamma\mu_{m_\lambda-1}, \Gamma\mu_{n_\lambda+1}) \\
&\quad + b_m(\Gamma\mu_{n_\lambda+1}, \mu_{n_\lambda+1})] \\
&\leq \sigma_{m_\lambda-1} b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) + (1-\sigma_{m_\lambda-1}) s b_m(\Gamma\mu_{n_\lambda+1}, \mu_{n_\lambda+1}) \\
&\quad + (1-\sigma_{m_\lambda-1}) s[k b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) + L\min\{b_m(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1}), \\
&\quad b_m(\mu_{n_\lambda+1}, \Gamma\mu_{n_\lambda+1}), b_m(\mu_{m_\lambda-1}, \Gamma\mu_{n_\lambda+1}), b_m(\mu_{n_\lambda+1}, \Gamma\mu_{m_\lambda-1})\}] \\
&\leq s\sigma_{m_\lambda-1} b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda}) + s\sigma_{m_\lambda-1} b_m(\mu_{n_\lambda}, \mu_{n_\lambda+1}) \\
&\quad + (1-\sigma_{m_\lambda-1}) s b_m(\Gamma\mu_{n_\lambda+1}, \mu_{n_\lambda+1}) \\
&\quad + (1-\sigma_{m_\lambda-1}) s[k\{s b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda}) + s b_m(\mu_{n_\lambda}, \mu_{n_\lambda+1})\} \\
&\quad + \min\{b_m(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1}), b_m(\mu_{n_\lambda+1}, \Gamma\mu_{n_\lambda+1}), s b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda+1}) \\
&\quad + s b_m(\mu_{n_\lambda+1}, \Gamma\mu_{n_\lambda+1}), s b_m(\mu_{n_\lambda+1}, \mu_{m_\lambda-1}) + s b_m(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1})\}] \\
&\leq s\sigma_{m_\lambda-1} b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda}) + s\sigma_{m_\lambda-1} b_m(\mu_{n_\lambda}, \mu_{n_\lambda+1}) \\
&\quad + (1-\sigma_{m_\lambda-1}) s b_m(\Gamma\mu_{n_\lambda+1}, \mu_{n_\lambda+1}) \\
&\quad + (1-\sigma_{m_\lambda-1}) s[k\{s b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda}) + s b_m(\mu_{n_\lambda}, \mu_{n_\lambda+1})\} \\
&\quad + \min\{b_m(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1}), b_m(\mu_{n_\lambda+1}, \Gamma\mu_{n_\lambda+1}), s^2 b_m(\mu_{m_\lambda-1}, \mu_{n_\lambda}) \\
&\quad + s^2 b_m(\mu_{n_\lambda}, \mu_{n_\lambda+1}) + s b_m(\mu_{n_\lambda+1}, \Gamma\mu_{n_\lambda+1}), s^2 b_m(\mu_{n_\lambda+1}, \mu_{n_\lambda}) \\
&\quad + s^2 b_m(\mu_{n_\lambda}, \mu_{m_\lambda-1}) + s b_m(\mu_{m_\lambda-1}, \Gamma\mu_{m_\lambda-1})\}] \\
&< s\sigma_{m_\lambda-1}\epsilon + s(1-\sigma_{m_\lambda-1})[ks\epsilon + L\min\{0, 0, s^2\epsilon, s^2\epsilon\}] \\
&= s\epsilon\big(\sigma_{m_\lambda-1}(1-sk) + sk\big) \\
&< s\epsilon\left(\frac{\frac{1}{s^3} - k}{1-k+L}(1-sk) + sk\right) \\
&= s\epsilon\left(\frac{\frac{1}{s^2} - sk}{s-sk+sL}(1-sk) + sk\right) \\
&\leq s\epsilon\left(\frac{\frac{1}{s^2} - sk}{1-sk}(1-sk) + sk\right) = \frac{\epsilon}{s}.
\end{aligned}$$

Thus, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{\lambda\to\infty} b_m(\mu_{m_\lambda}, \mu_{n_\lambda+1}) < \frac{\epsilon}{s},$$

and hence, our supposition, μ_n is a non-Cauchy sequence, is wrong and so $\{\mu_n\}$ is Cauchy in $\mathcal{X}$.
Due to the completeness of $\mathcal{X}$, there will be an element $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} b_m(\mu_n, \mu^*) = 0$$

Now, we will verify that $\Gamma\mu^* = \mu^*$. For this,

$$\begin{aligned} b_m(\mu^*, \Gamma\mu^*) &\leq sb_m(\mu^*, \mu_n) + sb_m(\mu_n, \Gamma\mu^*) \\ &\leq sb_m(\mu^*, \mu_n) + s^2 b_m(\mu_n, \Gamma\mu_n) + s^2 b_m(\Gamma\mu_n, \Gamma\mu^*) \\ &\leq sb_m(\mu^*, \mu_n) + s^2 b_m(\mu_n, \Gamma\mu_n) + s^2[kb_m(\mu_n, \mu^*) + L\min\{b_m(\mu_n, \Gamma\mu_n), \\ &\quad b_m(\mu^*, \Gamma\mu^*), b_m(\mu_n, \Gamma\mu^*), b_m(\mu^*, \Gamma\mu_n)\}] \\ &\leq sb_m(\mu^*, \mu_n) + s^2 b_m(\mu_n, \Gamma\mu_n) + s^2[kb_m(\mu_n, \mu^*) + L\min\{b_m(\mu_n, \Gamma\mu_n), \\ &\quad b_m(\mu^*, \Gamma\mu^*), sb_m(\mu_n, \mu^*) + sb_m(\mu^*, \Gamma\mu^*), sb_m(\mu^*, \mu_n) + sb_m(\mu_n, \Gamma\mu_n)\}]. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$b_m(\mu^*, \Gamma\mu^*) \leq s^2 \min\{0, b_m(\mu^*, \Gamma\mu^*), sb_m(\mu^*, \Gamma\mu^*)\} = 0.$$

So, $b_m(\mu^*, \Gamma\mu^*) = 0 \implies \Gamma\mu^* = \mu^*$.
Hence, μ^* is a fixed point of Γ.
Now, we shall show that μ^* is unique. For this, take a fixed point of Γ, say $q(\neq \mu^*) \in \mathcal{X}$, then

$$\begin{aligned} 0 < b_m(\mu^*, q) &= b_m(\Gamma\mu^*, Tq) \\ &\leq kb_m(\mu^*, q) + L\min\{b_m(\mu^*, \Gamma\mu^*), b_m(q, Tq), b_m(\mu^*, Tq), b_m(q, \Gamma\mu^*)\} \\ &\leq kb_m(\mu^*, q) + L\min\{0, 0, sb_m(\mu^*, q) + sb_m(q, Tq), sb_m(q, \mu^*) + sb_m(\mu^*, \Gamma\mu^*)\} \\ &= kb_m(\mu^*, q) \\ &< \frac{1}{s^3} b_m(\mu^*, q) < b_m(\mu^*, q), \end{aligned}$$

which is a contradiction. Therefore, $\mu^* = q$.
Hence, the proof. □

If we take $L = 0$ in Theorem 3, then we obtain the following result.

Corollary 2. *Suppose $\Gamma : (\mathcal{X}, b_m, \Omega) \to (\mathcal{X}, b_m, \Omega)$ is a contraction mapping, that is, Γ satisfies*

$$b_m(\Gamma\mu, \Gamma\eta) \leq kb_m(\mu, \eta), \tag{9}$$

for all $\mu, \eta \in \mathcal{X}$ and some $k \in (0,1)$, where $(\mathcal{X}, b_m, \Omega)$ is a complete convex b-metric space with $s > 1$. Let $\mu_n = \Omega(\mu_{n-1}, \Gamma\mu_{n-1}; \sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0 \in \mathcal{X}$ with the property $b_m(\mu_0, \Gamma\mu_0) < \infty$, where $0 \leq \sigma_{n-1} < 1$ for each $n \in \mathbb{N}$. If $k < \frac{1}{s^3}$ and $0 \leq \sigma_{n-1} < \frac{\frac{1}{s^3} - k}{1-k}$ for each $n \in \mathbb{N}$, then Γ has a fixed point in $\mathcal{X}$ that is unique.

The following example illustrates the generality of Corollary 2 over Theorem 1 of [27].

Example 4. *Let $(\mathcal{X}, b_m, \Omega)$ be a triplet as defined in Example 2 and $\Gamma(\mu) = \frac{\mu}{9}$ for each $\mu \in \mathcal{X}$. Now, we prove that Γ satisfies the inequality (9) for $k = 1/9$. For this, take*

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= [(\Gamma\mu - \Gamma\eta)^2 + |\Gamma\mu - \Gamma\eta|] \\ &= \left[\left(\frac{\mu}{9} - \frac{\eta}{9}\right)^2 + \left|\frac{\mu}{9} - \frac{\eta}{9}\right|\right] \\ &= \frac{1}{9}\left[\frac{1}{9}(\mu - \eta)^2 + |\mu - \eta|\right] \\ &\leq \frac{1}{9}\left[(\mu - \eta)^2 + |\mu - \eta|\right] \\ &= \frac{1}{9} b_m(\mu, \eta), \end{aligned}$$

for all $\mu, \eta \in \mathcal{X}$. We choose an initial point $\mu_0 \in \mathcal{X}$ and generate $\mu_n = \Omega(\mu_n, \Gamma\mu_{n-1}; \sigma_{n-1})$ with $\sigma_{n-1} = \frac{1}{65} < \frac{\frac{1}{s^3} - k}{1-k}$ as follows

$$\begin{aligned} \Gamma\mu_0 &= \frac{\mu_0}{9} \\ \mu_1 &= \frac{1}{65}\mu_0 + \frac{64}{65}\Gamma\mu_0 = \frac{73}{585}\mu_0 \\ \mu_2 &= \frac{1}{65}\mu_1 + \frac{64}{65}\Gamma\mu_1 = \left(\frac{73}{585}\right)^2 \mu_0 \\ &\vdots \\ \mu_n &= \frac{1}{65}\mu_{n-1} + \frac{64}{65}\Gamma\mu_{n-1} = \left(\frac{73}{585}\right)^n \mu_0. \end{aligned}$$

Thus, $\lim_{n\to\infty} \mu_n = 0$. As the map Γ satisfies all the hypotheses of Corollary 2, hence, Γ has a unique fixed point 0 in $\mathcal{X}$. To investigate the uniqueness, let us take a fixed point say $q \in \mathcal{X}$ of Γ, which is different from zero, then

$$\begin{aligned} 0 < b_m(\mu^*, q) &= b_m(\Gamma\mu^*, Tq) \\ &= [(\Gamma\mu^* - Tq)^2 + |\Gamma\mu^* - Tq|] \\ &= \left[\left(\frac{\mu^*}{9} - \frac{q}{9}\right)^2 + \left|\frac{\mu^*}{9} - \frac{q}{9}\right|\right] \\ &= \frac{1}{9}\left[\frac{1}{9}(\mu^* - q)^2 + |\mu^* - q|\right] \\ &\leq \frac{1}{9}\left[(\mu^* - q)^2 + |\mu^* - q|\right] \\ &= \frac{1}{9} b_m(\mu^*, q), \end{aligned}$$

which is a contradiction. Thus, $\mu^* = q$.
Therefore, 0 is the only fixed point of Γ in $\mathcal{X}$. However, the contraction condition (9) is not satisfied by the mapping Γ for any $k < \frac{1}{s^4} = \frac{1}{16}$. Indeed, if we take $\mu = \frac{1}{5}$ and $\eta = \frac{1}{10}$, then

$$b_m(\mu, \eta) = \left(\frac{1}{5} - \frac{1}{10}\right)^2 + \left(\frac{1}{5} - \frac{1}{10}\right) = \frac{11}{100},$$

and

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &= \left(\frac{1}{45} - \frac{1}{90}\right)^2 + \left(\frac{1}{45} - \frac{1}{90}\right) = \frac{91}{8100} \\ &> \frac{81}{8100} = \frac{1}{100} = \frac{1}{11} \cdot \frac{11}{100} = \frac{1}{11} b_m(\mu, \eta). \end{aligned}$$

Thus, Theorem 1 of [27] is not applicable to ensure the existence and uniqueness of the fixed point of map Γ.

Proposition 1. *Let $(\mathcal{X}, b_m)$ be a b-metric space. Then, any map $\Gamma : \mathcal{X} \to \mathcal{X}$ satisfying the Chatterjea contraction satisfies the condition(B) if $k < \frac{1}{s(s+1)}$.*

Proof. Using the Chatterjea contractive condition and the property of b-metric, we observe that

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq k[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\ &\leq k[s b_m(\mu, \eta) + s b_m(\eta, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\ &\leq k\Big[s b_m(\mu, \eta) + s^2 b_m(\eta, \Gamma\mu) + s^2 b_m(\Gamma\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)\Big]. \end{aligned}$$

It follows that

$$b_m(\Gamma\mu, \Gamma\eta) \leq \frac{ks}{1 - ks^2} b_m(\mu, \eta) + \frac{k(s^2 + 1)}{1 - ks^2} b_m(\eta, \Gamma\mu). \tag{10}$$

In the similar manner, we also obtain

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq k[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\ &\leq k[s b_m(\mu, \Gamma\eta) + s b_m(\eta, \mu) + b_m(\mu, \Gamma\mu)] \\ &\leq k\Big[s b_m(\mu, \Gamma\eta) + s^2 b_m(\eta, \mu) + s^2 b_m(\mu, \Gamma\eta) + b_m(\Gamma\eta, \Gamma\mu)\Big], \end{aligned}$$

which yields

$$b_m(\Gamma\mu, \Gamma\eta) \leq \frac{ks}{1 - ks^2} b_m(\mu, \eta) + \frac{k(s^2 + 1)}{1 - ks^2} b_m(\mu, \Gamma\eta). \tag{11}$$

Again, using the Chatterjea contraction and the property of b-metric, we have the inequality

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq k[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\ &\leq k[s b_m(\mu, \eta) + s b_m(\eta, \Gamma\eta) + s b_m(\eta, \Gamma\eta) + b_m(\Gamma\eta, \Gamma\mu)]. \end{aligned}$$

That implies

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq \frac{ks}{1 - ks} b_m(\mu, \eta) + \frac{2ks}{1 - ks} b_m(\eta, \Gamma\eta) \\ &\leq \frac{ks}{1 - ks^2} b_m(\mu, \eta) + \frac{k(s^2 + 1)}{1 - ks^2} b_m(\eta, \Gamma\eta). \end{aligned} \tag{12}$$

With the similar argument, we obtain

$$\begin{aligned} b_m(\Gamma\mu, \Gamma\eta) &\leq k[b_m(\mu, \Gamma\eta) + b_m(\eta, \Gamma\mu)] \\ &\leq k[s b_m(\mu, \Gamma\mu) + s b_m(\Gamma\mu, \Gamma\eta) + s b_m(\eta, \Gamma\mu) + b_m(\mu, \Gamma\mu)], \end{aligned}$$

which gives

$$\begin{aligned} b_m(\Gamma\mu,\Gamma\eta) &\leq \frac{ks}{1-ks}b_m(\mu,\eta)+\frac{2ks}{1-ks}b_m(\mu,\Gamma\mu) \\ &\leq \frac{ks}{1-ks^2}b_m(\mu,\eta)+\frac{k(s^2+1)}{1-ks^2}b_m(\mu,\Gamma\mu). \end{aligned} \tag{13}$$

Now, by using Equations (10)–(13), we have

$$\begin{aligned} b_m(\Gamma\mu,\Gamma\eta) &\leq \frac{ks}{1-ks^2}b_m(\mu,\eta)+\frac{k(s^2+1)}{1-ks^2}\min\{b_m(\mu,\Gamma\mu),b_m(\eta,\Gamma\eta),b_m(\mu,\Gamma\eta),b_m(\eta,\Gamma\mu)\} \\ &\leq pb_m(\mu,\eta)+L\min\{b_m(\mu,\Gamma\mu),b_m(\eta,\Gamma\eta),b_m(\mu,\Gamma\eta),b_m(\eta,\Gamma\mu)\}, \end{aligned}$$

where $p=\frac{ks}{1-ks^2}<1$ $\left(as\ k<\frac{1}{s(s+1)}\right)$ and $L=\frac{k(s^2+1)}{1-ks^2}\geq 0$. Thus, the map Γ satisfies condition (B). □

Proposition 2. *If $s\geq 1$ and $k\in[0,1/2)$ such that $\sigma<\frac{\frac{1}{s^4}-\frac{k}{s^2}-k}{\frac{1+k}{s}-k}$, then $\sigma<\frac{\frac{1}{s^3}-p}{1-p+L}$, where $p=\frac{ks}{1-ks^2}$ and $L=\frac{k(s^2+1)}{1-ks^2}$.*

Proof. It is observed that

$$\begin{aligned} \sigma &< \frac{\frac{1}{s^4}-\frac{k}{s^2}-k}{\frac{1+k}{s}-k} \\ &= \frac{\frac{1-ks^2-ks^4}{s^4(1-ks^2)}}{\frac{1+k-sk}{s(1-ks^2)}} \\ &= \frac{\frac{(1-ks^2)-ks^4}{s^3(1-ks^2}}{\frac{(1-ks^2)-ks+ks^2+k}{1-ks^2}}, \end{aligned}$$

which yields that

$$\sigma<\frac{\frac{1}{s^3}-\frac{ks}{1-ks^2}}{1-\frac{ks}{1-ks^2}+\frac{k(s^2+1)}{1-ks^2}}\Rightarrow\sigma<\frac{\frac{1}{s^3}-p}{1-p+L}.$$

□

The following result is another description of Chatterjea fixed point theorem in a convex b-metric space.

Corollary 3. *Suppose $\Gamma:(\mathcal{X},b_m,\Omega)\to(\mathcal{X},b_m,\Omega)$ is a Chatterjea contraction mapping, that is, Γ satisfies*

$$b_m(\Gamma\mu,\Gamma\eta)\leq k[b_m(\mu,\Gamma\eta)+b_m(\eta,\Gamma\mu)], \tag{14}$$

for all $\mu,\eta\in\mathcal{X}$ and some $k\in\left(0,\frac{1}{2}\right)$, where $(\mathcal{X},b_m,\Omega)$ is a complete convex b-metric space with $s>1$. Let $\mu_n=\Omega(\mu_{n-1},\Gamma\mu_{n-1};\sigma_{n-1})$ be a sequence defined by choosing an initial point $\mu_0\in\mathcal{X}$ with the property $b_m(\mu_0,\Gamma\mu_0)<\infty$, where $0\leq\sigma_{n-1}<1$ for each $n\in\mathbb{N}$. If $k<\frac{1}{s^2(s^2+1)}$ and $0\leq\sigma_{n-1}<\frac{\frac{1}{s^4}-\frac{k}{s^2}-k}{\frac{1+k}{s}-k}$ for each $n\in\mathbb{N}$, then Γ has a fixed point in $\mathcal{X}$ that is unique.

Proof. As Γ satisfies Equation (14), then on account of Proposition 1, it will be satisfying condition (B); that is, Γ satisfies

$$b_m(\Gamma\mu,\Gamma\eta)\leq pb_m(\mu,\eta)+L\min\{b_m(\mu,\Gamma\mu),b_m(\eta,\Gamma\eta),b_m(\mu,\Gamma\eta),b_m(v,\Gamma\mu)\},$$

where $p = \frac{ks}{1-ks^2}$ and $L = \frac{k(s^2+1)}{1-ks^2}$.

Since $\sigma_{n-1} < \frac{\frac{1}{s^4}-\frac{k}{s^2}-k}{\frac{1+k}{s}-k}$ for each $n \in \mathbb{N}$, then Proposition 2 implies $\sigma_{n-1} < \frac{\frac{1}{s^3}-p}{1-p+L}$. Thus, Theorem 3 yields that Γ has a unique fixed point in $\mathcal{X}$. □

Example 5. *In Example 3, if we replace the map $\Gamma : \mathcal{X} \to \mathcal{X}$ with*

$$T(\mu) = \begin{cases} \frac{\mu}{21}, & \mu \in [0,1) \\ \frac{1}{23\mu}, & \mu \in [1,\infty). \end{cases}$$

Then, by following the steps as in example 3, it can be verified that the map Γ satisfies the inequality (14) for $k = \frac{1}{21} < \frac{1}{s^2(s^2+1)}$. Now, we take $\mu_0 \in \mathcal{X}$ and generate $\mu_n = \Omega(\mu_n, \Gamma\mu_{n-1}; \sigma_{n-1})$ with $\sigma_{n-1} = \frac{1}{161} < \frac{\frac{1}{s^4}-\frac{k}{s^2}-k}{\frac{1+k}{s}-k}$ as

1. *If $\mu_0 < 1$, then*

$$\begin{aligned} \Gamma\mu_0 &= \frac{\mu_0}{21} \\ \mu_1 &= \frac{1}{161}\mu_0 + \frac{160}{161}\Gamma\mu_0 = \frac{181}{3881}\mu_0 \\ \mu_2 &= \frac{1}{161}\mu_1 + \frac{160}{161}\Gamma\mu_1 = \left(\frac{181}{3881}\right)^2\mu_0 \\ &\vdots \\ \mu_n &= \frac{1}{161}\mu_{n-1} + \frac{160}{161}\Gamma\mu_{n-1} = \left(\frac{181}{3881}\right)^n\mu_0. \end{aligned}$$

That implies $\mu_n \to 0$ as $n \to \infty$.

2. *If $\mu_0 \geq 1$, then*

$$\begin{aligned} \Gamma\mu_0 &= \frac{1}{23\mu_0} \\ \mu_1 &= \frac{1}{161}\mu_0 + \frac{160}{161}\Gamma\mu_0 \\ &= \frac{1}{161}\mu_0 + \frac{160}{161}\cdot\frac{1}{23\mu_0} \\ \frac{\mu_1}{\mu_0} &= \frac{1}{161} + \frac{160}{3703}\cdot\frac{1}{\mu_0^2} \leq \frac{183}{3703}. \end{aligned}$$

If $\mu_1 \in [0,1)$, then as in above case, $\mu_n \to 0$ as $n \to \infty$. If $\mu_1 \in [1,\infty)$, then $\frac{\mu_2}{\mu_1} = \frac{1}{161} + \frac{160}{3703}\cdot\frac{1}{\mu_1^2} \leq \frac{183}{3703}$. Proceeding in a similar fashion, we can assume that $\mu_{n-1} \in [1,\infty)$, then we obtain,

$$\frac{\mu_n}{\mu_{n-1}} = \frac{1}{161} + \frac{160}{3703}\cdot\frac{1}{\mu_{n-1}^2} \leq \frac{183}{3703},$$

and

$$\frac{\mu_n}{\mu_0} = \frac{\mu_1}{\mu_0}\cdot\frac{\mu_2}{\mu_1}\cdots\frac{\mu_n}{\mu_{n-1}} \leq \left(\frac{183}{3703}\right)^n.$$

So, the sequence μ_n tends to zero as $n \to \infty$. Thus, all the hypotheses of Corollary 3 are satisfied, and hence, the map Γ has only one fixed point in $\mathcal{X}$, which is $\mu = 0$.

Remark 1. *It is clear from Examples 3 and 5 that Corollaries 1 and 3 are independent to each other.*

3. Conclusions

Recently, Chen et al. [27] defined the notion of convex b-metric spaces and proved Banach and Kannan's type fixed point theorems in convex b-metric spaces. Here, motivated by this idea, we established several fixed point theorems for Cirić contraction as well as for the maps satisfying the condition (B) in the context of a convex b-metric space and presented some supporting examples for the proved results.

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