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Article

Blurry Definability

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Abstract: I begin the study of a hierarchy of (hereditarily) $<\kappa$-blurrily ordinal definable sets. Here for a cardinal κ, a set is $<\kappa$-blurrily ordinal definable if it belongs to an OD set of cardinality less than κ, and it is hereditarily so if it and each member of its transitive closure is. I show that the class of hereditarily $<\kappa$-blurrily ordinal definable sets is an inner model of ZF. It satisfies the axiom of choice iff it is a κ-c.c. forcing extension of HOD, and HOD is definable inside it (even if it fails to satisfy the axiom of choice). Of particular interest are cardinals λ such that some set is hereditarily $<\lambda$-blurrily ordinal definable but not hereditarily $<\kappa$-blurrily ordinal definable for any cardinal $\kappa < \lambda$. Such cardinals I call leaps. The main results concern the structure of leaps. For example, I show that if λ is a limit of leaps, then the collection of all hereditarily $<\lambda$-blurrily ordinal definable sets is a model of ZF in which the axiom of choice fails. Using forcing, I produce models exhibiting various leap constellations, for example models in which there is a (regular/singular) limit leap whose cardinal successor is a leap. Many open questions remain.

Keywords: ordinal definability; HOD; forcing; axiom of choice

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1. Introduction

The concept of ordinal definability is a core notion in set theory, introduced by Gödel, and developed by Myhill and Scott. Classically, a set a is ordinal definable, or OD, iff there are a formula $\varphi(x,\vec{y})$ and ordinals $\vec{\alpha}$ such that a is the unique set such that $\varphi(a,\vec{\alpha})$ holds. Although it is a classical fact due to Tarski that in general, the class of all definable sets is itself not definable, allowing ordinal parameters results in a definable concept. A lot of research on questions of definability has been done already in the first half of the twentieth century, and there is a lot of recent research as well, which I will refer to later, but let me mention [1] now for a very current article relating to the abovementioned result due to Tarski.

The collection of ordinal definable sets is not necessarily transitive. For example, the collection of all subsets of ω is definable, but there may be subsets of ω which are not ordinal definable. Classically, the way around this defect is to define that a set a is *hereditarily ordinal definable* if it is OD and every member of its transitive closure is also OD. The collection of all hereditarily ordinal definable sets is denoted HOD, and is known to be an inner model in which the axiom of choice holds, even if V is merely a ZF model.

Fairly recently, Hamkins and Leahy [2] introduced the concept of *ordinal algebraicity*: a set a is ordinal algebraic if there is a finite OD set A such that $a \in A$. They showed that the hereditary version of this concept, the class of all hereditarily ordinal algebraic sets, is no different than HOD.

Taking the logical next step, in a recent manuscript, Tzouvaras [3] argues that one can capture what Russell meant when he used the adjective "typical" in a somewhat paradoxical statement, by defining a set to be *nontypical*, or NT, or $\mathsf{NT}_{\aleph_1}$, if it belongs to a *countable* OD set.

I follow a more liberal approach here, and I hope to convince the reader that this results in a very rich and interesting hierarchy of definability notions. I use the notation $\overline{\overline{a}}$ for the cardinality of the set a in the following.

Definition 1 (ZF)**.** *Let κ be a cardinal. A set a is* $<\kappa$-blurrily ordinal definable, *or* $<\kappa$-OD, *if there is an* OD *set A such that $a \in A$ and $\overline{\overline{A}} < \kappa$. As usual, I will also write* $<\kappa$-OD *for the class of all sets that are* $<\kappa$-OD.

This definition is to be understood in ZF, i.e., A is required to have a cardinality, and this cardinality is required to be less than κ. Nevertheless, I will mostly work in ZFC models, where every set has a cardinality, and the concepts are a little more natural. Clearly, $<\kappa$-OD $= \emptyset$ for $\kappa < 2$. The idea behind the concept is this: when a is $<\kappa$-OD, we may not be able to single out a as the *only* set with a certain property (with respect to some ordinal), as was the case with ordinal definability, but we may narrow it down so that a is one of fewer than κ many sets with that property, just as though our memory of the exact identity of a became a little blurry. The set A in Definition 1 can hence be thought of as a blurry definition of a .

Thus, a set is OD iff it is <2-OD, and it is ordinal algebraic iff it is $<\omega$-OD. The abovementioned "nontypical" sets are the ones that are $<\omega_1$-OD in the current terminology. The hereditary versions of the $<\kappa$-OD sets are defined in the usual way. I denote the transitive closure of a set x by $\mathsf{TC}(x)$.

Definition 2. *Let κ be a cardinal. A set a is* hereditarily $<\kappa$-blurrily ordinal definable, *denoted* $<\kappa$-HOD, *iff* $\mathsf{TC}(\{a\}) \subseteq$ $<\kappa$-OD. *As before, I also write* $<\kappa$-HOD *for the class of all* $<\kappa$-HOD *sets.*

Again, <0-HOD $=$ <1-HOD $= \emptyset$ (this is why I will always assume that $\kappa > 1$), <2-HOD $=$ HOD, $<\omega$-HOD is the collection of all hereditarily algebraic sets, and $<\omega_1$-HOD is the class of the hereditarily nontypical sets, or $\mathsf{HNT}_{\aleph_1}$, in the terminology of Tzouvaras. I refrained from employing the notation HNT_κ, which was used by Tzouvaras, but only to rule out the relevance of the case $\kappa > \omega_1$; he argues that in this case, the resulting concept would be unnatural, contradicting his interpretation of nontypicality. Thus, for $\kappa > \aleph_1$, we are not dealing with nontypicality, and I call it blurry definability instead.

One simple fact that I will frequently use is that a set a is $<\kappa$-HOD iff a is $<\kappa$-OD and $a \subseteq <\kappa$-HOD.

This article is a first exploration of the structure of this hierarchy of blurry definability and the associated hierarchy of classes $<\kappa$-HOD. Although in the existing literature, the focus was always on one particular κ (namely 2, ω or ω_1), here, I am more interested in questions such as *at which κ does a new set become hereditarily $<\kappa$-blurrily definable? What can the class of such κ's be like? What can be said in general about $<\kappa$-HOD and its relationship to* HOD, *say?* There are many questions such as this that are worthwhile.

The article is organized as follows.

In Section 2, I prove some general ZF/ZFC results about $<\kappa$-HOD: that it is an inner model (Proposition 2) and that it satisfies a "blurry" version of the axiom of choice (Theorem 2). Turning to the question how close HOD is to $<\kappa$-HOD, I show that HOD satisfies Hamkins' λ-approximation and -cover properties in $<\kappa$-HOD, for every cardinal $\lambda \geq \kappa$ (Theorem 3), that HOD and $<\kappa$-HOD have the same cardinals and cofinalities above κ (Proposition 6), and that $<\kappa$-HOD has no fresh sequences over HOD whose length has cofinality at least κ (Proposition 4). Two of the main results in this section are Theorems 5 and 7, which state:

- Let κ be an infinite cardinal. If $<\kappa$-HOD satisfies the axiom of choice then it is a set forcing extension of HOD by a κ-c.c. forcing notion.
- Let $\lambda \geq 2$ be a cardinal. Let $\kappa \geq \lambda$ be regular. Then HOD is definable in $<\lambda$-HOD using $\mathcal{P}(\kappa) \cap$ HOD as a parameter.

I also make a connection to Woodin's work on the HOD conjecture by showing that if δ is extendible, then the HOD hypothesis implies that $[\text{HOD}]^{<\delta} \subseteq <\delta$-HOD (Proposition 8). I further introduce the concept of a leap, which is a κ where a new set gets into $<\kappa$-HOD, and I prove some basic facts about the class of leaps: that it is closed, that the least leap, if existent, is a successor cardinal, and that successor leaps are successor cardinals (Lemma 2). A maybe surprising result is Theorem 10:

- If λ is a limit leap, then $<\lambda$-HOD does not satisfy the axiom of choice; more precisely, $\bar{\lambda}$-AC fails in $<\lambda$-HOD, where λ is the $\bar{\lambda}$-th leap.

Section 3 is concerned with the effects of forcing on blurry HOD and the leap structure. I prove preservation facts for belonging to blurry HOD, as well as results on not adding to blurry HOD. Specific forcing notions I investigate are Cohen forcing, a forcing due to Kanovei and Lyubetsky, forcing with certain homogeneous Souslin trees and Příkrý forcing. Strewn into this section are some relative consistency results, constructing models where the least leap is the successor of a regular limit cardinal (Corollary 7), where the cardinal successor of a regular limit leap is a leap (Theorem 14), where the least leap is the successor of a singular limit cardinal (Theorem 16), and where the cardinal successor of a singular limit leap is a leap (Theorem 17).

I close with Section 4, containing some open problems and directions for future research.

I would like to thank the two referees for thoroughly reading an earlier manuscript of this article, and for providing very useful feedback.

2. ZF(C) Results

In this section, I will prove some results on the structure of $<\kappa$-HOD—in some cases, I will work in ZF, and in others, I will also assume the axiom of choice.

2.1. Basic Observations

Let us begin by making some simple observations.

Proposition 1. *Let $2 \leq \kappa < \lambda$ be cardinals.*

(1) OD $\subseteq <\kappa$-OD $\subseteq <\lambda$-OD.
(2) $<\kappa$-HOD *is transitive, and* HOD $\subseteq <\kappa$-HOD $\subseteq <\lambda$-HOD.
(3) OD $\cap H_\kappa \subseteq <\kappa$-HOD.

(4) *Under* AC, $H_\kappa \subseteq {<}(2^{<\kappa})^+$-HOD.
(5) *So under* AC, V *is the increasing union* $\bigcup_{\kappa \in \mathrm{Card}} {<}\kappa$-HOD.

Proof. Points (1) and (2) are obvious.

For point (3), let $a \in H_\kappa$ be OD. To show that $a \in {<}\kappa$-HOD, we must check that every $x \in \mathsf{TC}(\{a\})$ is ${<}\kappa$-OD. However, $\mathsf{TC}(\{a\})$ is definable from a and hence OD, and the cardinality of $\mathsf{TC}(\{a\})$ is less than κ, as $a \in H_\kappa$. So $\mathsf{TC}(\{a\})$ witnesses that x is ${<}\kappa$-OD.

For point (4), note that the cardinality of H_κ is $2^{<\kappa}$. So H_κ is an OD set of size less than $(2^{<\kappa})^+$ that witnesses that a is ${<}(2^{<\kappa})^+$-OD, for every $a \in H_\kappa$. It follows that every $a \in H_\kappa$ is ${<}(2^{<\kappa})^+$-HOD, as claimed.

Point (5) is again obvious. □

Hamkins-Leahy [2] introduced the notion of ordinal algebraicity by defining that a set a is ordinal algebraic iff it belongs to a finite OD set, and denoted the class of all ordinal algebraic sets by OA. They further defined a to be hereditarily ordinal algebraic if $\mathsf{TC}(\{a\}) \subseteq \mathsf{OA}$, and denoted the class of all hereditarily ordinal algebraic sets by HOA. Thus, using the notation introduced here,

$$\mathsf{OA} = {<}\omega\text{-}\mathsf{OD} \text{ and } \mathsf{HOA} = {<}\omega\text{-}\mathsf{HOD}.$$

Theorem 1 (Hamkins-Leahy [2])**.** ${<}\omega$-HOD $=$ HOA $=$ HOD.

Proposition 2 (ZF)**.** *Let* $\kappa \geq 2$ *be a cardinal. Then* ${<}\kappa$-HOD *is an inner model, i.e., a transitive class containing the ordinals such that the* ZF *axioms hold when relativized to it.*

Proof. If $2 \leq \kappa \leq \omega$, then by Proposition 1 (2) and Theorem 1, HOD $\subseteq {<}\kappa$-HOD $\subseteq {<}\omega$-HOD $=$ HOD, so that ${<}\kappa$-HOD $=$ HOD certainly is an inner model (even satisfying the axiom of choice).

So let us assume that $\kappa > \omega$. It suffices to show that ${<}\kappa$-HOD satisfies the following condition: for every $u \subseteq {<}\kappa$-HOD, there is a transitive $v \in {<}\kappa$-HOD such that $u \subseteq v$ and $\mathsf{Def}(\langle v, \in\rangle) \subseteq {<}\kappa$-HOD (see [4], Thm. 13.9; the stated condition implies the assumption of the theorem.) Here is the short proof that this condition is satisfied:

So let $u \subseteq {<}\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap {<}\kappa$-HOD. Clearly, v is transitive, OD (so also ${<}\kappa$-OD), contained in ${<}\kappa$-HOD, so $v \in {<}\kappa$-HOD, and $u \subseteq v$. To show that $\mathsf{Def}(\langle v, \in\rangle) \subseteq {<}\kappa$-HOD, let $\varphi(x, \vec{y})$ be a formula, and let $\vec{a} = a_0, \ldots, a_{n-1} \in v$. We must show that $z = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{a})\} \in {<}\kappa$-HOD. Since $z \subseteq v \subseteq {<}\kappa$-HOD, it suffices to show that z is ${<}\kappa$-OD. For each $i < n$, let A_i be an OD set containing a_i such that $\overline{\overline{A_i}} < \kappa$. Since $a_i \in v$, we may assume that each $b \in A_i$ is in v, by adding this requirement to the definition of A_i if necessary, and this can be done for every $i < n$. Then z is in the set

$$B = \{w \mid \exists b_0 \in A_0 \ldots \exists b_{n-1} \in A_{n-1} \quad w = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{b})\}\},$$

B is OD, as $A_0, \ldots, A_{n-1}$ and v are, and obviously, $\overline{\overline{B}} \leq \overline{\overline{A_0}} \cdot \ldots \cdot \overline{\overline{A_{n-1}}} < \kappa$, as κ is an infinite cardinal. □

2.2. Blurry Choice

Even though ${<}\kappa$-HOD is an inner model, we do not know that it satisfies AC, see [5,6], where it is shown that ${<}\omega_1$-HOD (there referred to as HNT) may be strictly between HOD and V, while satisfying or not satisfying AC. However, it does satisfy a "blurry" form of AC, as we shall see presently.

However, first, I would like to say a few words about the definability of ${<}\kappa$-OD. I will work with some simple Gödelization of formulas as natural numbers. If φ is an actual formula, then I denote its Gödel number by $\ulcorner\varphi\urcorner$. When working inside a model of set theory, I will also sometimes write $\ulcorner\varphi\urcorner$ for a natural number in the sense of the model which it believes to be the code of a formula (even though there may be no actual formula

φ of which it would be the code, e.g., if $\ulcorner\varphi\urcorner$ is a nonstandard number), and it is useful to think of $\ulcorner\varphi\urcorner$ just as a formula in the sense of the model. There is a set-theoretic Σ_1 formula Sat which defines the satisfaction relation in any model M of set theory. Working inside a model M of set theory, if $\ulcorner\varphi\urcorner$ is a formula in the sense of M, $\mathfrak{A}$ is a model of the corresponding language, and s is an assignment of the free variables of $\ulcorner\varphi\urcorner$ with values in the universe of $\mathfrak{A}$, then $\mathsf{Sat}(\mathfrak{A}, \ulcorner\varphi\urcorner, s)$ expresses that $\mathfrak{A} \models \varphi[s]$. We can now define that $a \in {<}\kappa\text{-}\mathsf{OD}$ iff there is a formula $\ulcorner\varphi(x,y)\urcorner$ in the language of set theory with two free variables (so we are quantifying over a certain set of natural numbers) such that for some ordinals α and $\beta < \alpha$, the set $A = \{z \mid \mathsf{Sat}(\langle \mathrm{V}_\alpha, \in\rangle, \ulcorner\varphi\urcorner, \langle z, \beta\rangle)\}$ has cardinality less than κ, and $a \in A$. The reason that this works is that if this property holds, then the set A is OD, since A is defined using the parameters α, $\ulcorner\varphi\urcorner$ and β, which all are ordinals (even if $\ulcorner\varphi\urcorner$ may be a nonstandard natural number in the particular model we are working in—Sat is an actual formula.) Vice versa, if there is a set $A = \{z \mid \varphi(z,\beta)\}^M$ of M-cardinality less than κ, where φ is an actual formula, and $a \in A$, then by the Levy reflection theorem, there is an α such that in M, $A = \{z \mid \mathsf{Sat}(\langle \mathrm{V}_\alpha, \in\rangle, \ulcorner\varphi\urcorner, \langle z, \beta\rangle)\}$. I will often be a little sloppy in the notation, and write "$\mathfrak{A} \models \varphi[s]$" in place of the more precise, but maybe less suggestive "$\mathsf{Sat}(\mathfrak{A}, \ulcorner\varphi\urcorner, s)$." I also used the fact that in ZF, any finite sequence of ordinals can be coded by a single ordinal, using Gödel's pairing function, for example. I will tacitly keep using this throughout.

Theorem 2 (ZF)**.** *Let $\kappa > 1$ be a cardinal. Then, whenever $\mathrm{C} \in {<}\kappa\text{-}\mathsf{HOD}$ is a set consisting of nonempty sets, there is a function $f : \mathrm{C} \longrightarrow ([\bigcup \mathrm{C}]^{<\kappa})^{\mathrm{V}}$ such that $f \in {<}\kappa\text{-}\mathsf{HOD}$, and such that for every $c \in \mathrm{C}$, $\emptyset \neq f(c) \subseteq c$.*

Remark: So while in this situation, AC would guarantee a function that picks, for every $c \in C$, an element of c, which is equivalent to picking a *one element subset* of c, the theorem guarantees the existence of a function in ${<}\kappa$-HOD that picks, for every $c \in C$, a nonempty subset of c of size less than κ (as computed in V).

Proof. For every $c \in {<}\kappa\text{-}\mathsf{HOD}$, let us say that a triple $\langle \alpha, \beta, \ulcorner\varphi\urcorner\rangle$ is a *code* for c if $\ulcorner\varphi\urcorner$ is the Gödel number of a formula in the language of set theory with the two free variables v_0 and v_1, and, letting $Z = \{x \mid \mathrm{V}_\alpha \models \varphi(x,\beta)\}$, we have that $\overline{\overline{Z}} < \kappa$ and $c \in Z$. Every element of ${<}\kappa$-HOD has such a code, and the codes are canonically well-ordered, for example using the lexicographical order.

Now, given $\mathrm{C} \in {<}\kappa\text{-}\mathsf{HOD}$ as in the theorem, define a function $f = f^{\mathrm{C}} : \mathrm{C} \longrightarrow [\bigcup \mathrm{C}]^{<\kappa}$ as follows. Given $c \in \mathrm{C}$, or any nonempty set $c \subseteq {<}\kappa\text{-}\mathsf{HOD}$, really, let $\tau_c = \langle \alpha_c, \beta_c, \ulcorner\varphi_c\urcorner\rangle$ be the least code of any element of c, and let

$$f(c) = \{x \in c \mid \mathrm{V}_{\alpha_c} \models \varphi_c(x, \beta_c)\}.$$

Thus, letting τ_c be a code for $d \in c$, we have that $d \in f(c) \subseteq c$ and $\overline{\overline{f(c)}} < \kappa$. Therefore, it remains to show that $f \in {<}\kappa\text{-}\mathsf{HOD}$. To this end, since $\mathrm{C} \in {<}\kappa\text{-}\mathsf{HOD}$, we can let A be an OD set such that $\mathrm{C} \in A$ and $\overline{\overline{A}} < \kappa$. By adding a requirement to the definition of A if necessary, we may assume that every $D \in A$ is a set of nonempty sets, and that $D \in {<}\kappa\text{-}\mathsf{HOD}$. We can then define a function f^D as above. Then, the set

$$\{f^D \mid D \in A\}$$

has cardinality less than κ, and it contains $f = f^{\mathrm{C}}$. So f is ${<}\kappa$-OD. To see that $f \in {<}\kappa\text{-}\mathsf{HOD}$, it remains to show that $f \subseteq {<}\kappa\text{-}\mathsf{HOD}$. Every member of f is a pair of the form $\langle c, f(c)\rangle$, where $c \in \mathrm{dom}(f)$. It suffices to show that $c, f(c) \in {<}\kappa\text{-}\mathsf{HOD}$, since ${<}\kappa$-HOD is closed under ordered pairs. Now $c \in \mathrm{dom}(f) = \mathrm{C} \in {<}\kappa\text{-}\mathsf{HOD}$, so $c \in {<}\kappa\text{-}\mathsf{HOD}$, since ${<}\kappa$-HOD is transitive. In addition, $f(c)$ is of the form $\{x \in c \mid \mathrm{V}_{\alpha_c} \models \varphi_c(x, \beta_c)\}$. So $f(c) \subseteq c \in {<}\kappa\text{-}\mathsf{HOD}$, and so, $f(c) \subseteq {<}\kappa\text{-}\mathsf{HOD}$, again since ${<}\kappa$-HOD is transitive. Therefore, all we must show is that $f(c) \in {<}\kappa\text{-}\mathsf{OD}$. However, this follows by an argument which is standard by now: since $c \in {<}\kappa\text{-HOD}$, there is an OD set B of cardinality less than κ, with $c \in B$. Since c is a nonempty set and $c \in {<}\kappa\text{-HOD}$, we may assume that B consists of nonempty sets which belong to ${<}\kappa$-HOD, by adding these requirements to the definition of B, if necessary. We can now let E be the set consisting of all sets of the form

$$\{x \in b \mid \mathrm{V}_{\alpha_b} \models \varphi_b(x, \beta_b)\}$$

where $b \in B$. The set E is OD, has size less than κ, since B has, and since $c \in B$, we have that $f(c) \in E$. This shows that $f(c) \in {<}\kappa\text{-HOD}$, as required. □

Remark 1. *The proof of the previous theorem shows that there is a* V*-definable prewellorder R of ${<}\kappa$-HOD such that for every $x \in {<}\kappa\text{-HOD}$, the set of y which are not comparable with x has cardinality less than κ. In other words, this prewellorder splits ${<}\kappa$-HOD into equivalence classes, each in ${<}\kappa$-HOD, of size less than κ, and these equivalence classes are strongly well-ordered, meaning that the collection of predecessors of any given set forms a set.*

A defect of this is that ${<}\kappa$-HOD may not know that these equivalence classes have cardinality less than κ. This can be fixed, however, if κ has a nice closure property.

Proposition 3 (AC)**.** *Let κ be a strong limit cardinal. Suppose that $a \in {<}\kappa\text{-HOD}$, and that $\overline{\overline{a}}^{\mathrm{V}} < \kappa$. Then $\overline{\overline{a}}^{<\kappa\text{-HOD}} = \overline{\overline{a}}^{\mathrm{V}}$.*

Proof. Let $\bar{\kappa} = \overline{\overline{a}} < \kappa$. It suffices to show that there is a bijection between a and $\bar{\kappa}$ in ${<}\kappa$-HOD.

There are $\bar{\kappa}^{\bar{\kappa}}$ many bijections between $\bar{\kappa}$ and a. Each of them is in ${<}\kappa$-HOD, as I will show presently. First, let A be an OD set of cardinality $\lambda < \kappa$, such that $a \in A$. We may assume that every element of A belongs to ${<}\kappa$-HOD and has cardinality $\bar{\kappa}$, since a has these properties, so that we otherwise may add these requirements to the definition of A. Consider the set

$$B = \{f \mid \text{there is a } b \in A \text{ such that } f : \bar{\kappa} \longrightarrow b \text{ is a bijection}\}.$$

For every fixed $b \in A$, there are $\bar{\kappa}^{\bar{\kappa}} < \kappa$ such bijections, and there are fewer than κ many elements of A, so in total, the cardinality of B is $\lambda \cdot \bar{\kappa}^{\bar{\kappa}} < \kappa$. In addition, since A is OD, so is B. It follows that $B \subseteq {<}\kappa\text{-HOD}$, because if $f \in B$, then $f \subseteq {<}\kappa\text{-HOD}$, as $f : \bar{\kappa} \longrightarrow b$, for some $b \in B$, so $f \subseteq \bar{\kappa} \times b \subseteq {<}\kappa\text{-HOD}$. Hence, there is a bijection between $\bar{\kappa}$ and a in ${<}\kappa$-HOD. □

Corollary 1. *If κ is a strong limit cardinal, then ${<}\kappa$-HOD satisfies the following form of the Axiom of Choice: for every collection C of nonempty sets, there is a function $f : \mathrm{C} \longrightarrow [\bigcup \mathrm{C}]^{<\kappa}$ such that for every $c \in \mathrm{C}$, $\emptyset \neq f(c) \subseteq c$.*

2.3. How Close Is HOD *to Blurry* HOD*?*

The main theme in this subsection is to find ways in which HOD and ${<}\kappa$-HOD are close to each other. The following definition captures two important forms of closeness, introduced by Hamkins [7] (Def. 2).

Definition 3. *Let $M \subseteq N$ be transitive classes, and let κ be a cardinal in N.*

M satisfies the κ-cover property *in N if for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^{N} < \kappa$, there is a set $c \in M$ such that $a \subseteq c$ and $\overline{\overline{c}}^{M} < \kappa$. M satisfies the* strong κ-cover property *if this is true for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^{\mathrm{V}} < \kappa$.*

Let $a \in N$ be a set with $a \subseteq M$. A set of the form $a \cap c$, where $c \in M$ and $\overline{\overline{c}}^{M} < \kappa$, is called a κ-approximation *to a in M. The set a is said to be* κ-approximated in M *if every κ-approximation* *to a in M belongs to M. M satisfies the κ-*approximation property *in N if whenever $a \in N$ with $a \subseteq M$ is κ-approximated in M, then $a \in M$.*

Note that in the notation of this definition, if $\kappa < \lambda$ are cardinals in N, and M satisfies the κ-approximation property in N, then M also satisfies the λ-approximation property in N, because if $x \in N$, $x \subseteq M$ is λ-approximated in M, then it is also κ-approximated in M.

Theorem 3. *Let $\kappa \leq \lambda$ be infinite cardinals. Then* HOD *satisfies the strong λ-cover property and the λ-approximation property in $<\kappa$-*HOD*.*

Proof. To verify the strong λ-cover property, let $a \in <\kappa\text{-HOD}$, $a \subseteq \text{HOD}$, with $\gamma = \overline{\overline{a}} < \lambda$. As a side remark, note that any subset of HOD is well-ordered by the canonical well-ordering of HOD, restricted to the set, and hence, it has a cardinality in V, even without assuming the axiom of choice.

Let A be OD with $a \in A$ and $\overline{\overline{A}} < \kappa$. Since $a \subseteq \text{HOD}$ and $\overline{\overline{a}} = \gamma$, we may assume that for all $b \in A$, $b \subseteq \text{HOD}$ and $\overline{\overline{b}} = \gamma$, since these requirements may be added to the definition of A if necessary. Set $c = \bigcup A$. Then $\overline{\overline{c}} \leq \gamma \cdot \overline{\overline{A}} < \lambda$, c is OD, and $c \subseteq \text{HOD}$. Thus, $c \in \text{HOD}$, and clearly, $a \subseteq c$. Since AC holds in HOD, c has a cardinality in HOD, and hence, $\overline{\overline{c}}^{\text{HOD}} < \lambda$, because if it were the case that $\overline{\overline{c}}^{\text{HOD}} \geq \lambda$, then λ would be collapsed as a cardinal. This verifies the strong λ-cover property. Note that consequently, since $\text{HOD} \subseteq <\kappa\text{-HOD}$, it is also true in $<\kappa$-HOD that the cardinality of a is less than λ.

Concerning the λ-approximation property, by the remark after Definition 3, we may assume that $\lambda = \kappa$, i.e., it suffices to prove the κ-approximation property. So let $a \in <\kappa\text{-HOD}$, $a \subseteq \text{HOD}$ be κ-approximated in HOD. Let A be OD, with $a \in A$ and $\overline{\overline{A}} < \kappa$. We may assume that every $b \in A$ is a subset of HOD that is κ-approximated in HOD. Let $T = \bigcup A$. Then T is OD and $T \subseteq \text{HOD}$, so T is in HOD.

For every $c \in ([T]^{<\kappa})^{\text{HOD}}$, the set

$$A \sqcap c = \{b \cap c \mid b \in A\}$$

is an OD subset of HOD, and hence an element of HOD. Moreover, the function $F : ([T]^{<\kappa})^{\text{HOD}} \longrightarrow \text{HOD}$ defined by

$$F(c) = A \sqcap c$$

belongs to HOD as well.

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \text{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\text{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\text{HOD}}$. Thus, $A \sqcap \Delta \in \text{HOD}$. Note that if $b, b' \in A$ are distinct, then $b \cap \Delta \neq b' \cap \Delta$.

Consequently, for $\Delta \subseteq c \in ([T]^{<\kappa})^{\text{HOD}}$ and $\bar{b} \in A \sqcap \Delta$, there is a unique $\bar{b}' \in A \sqcap c$ such that $\bar{b}' \cap \Delta = \bar{b}$. Therefore, we can define in HOD:

$$B(\bar{b}) = \bigcup\{\bar{b}' \mid \exists c \in ([T]^{<\kappa})^{\text{HOD}} \quad (\Delta \subseteq c \text{ and } \bar{b}' \in A \sqcap c \text{ and } \bar{b}' \cap \Delta = \bar{b})\}.$$

It follows that $B(\bar{b})$ is the unique $b \in A$ such that $b \cap \Delta = \bar{b}$. Since for $b \in A$, $b = B(b \cap \Delta)$, it follows that

$$A = \{B(\bar{b}) \mid \bar{b} \in A \sqcap \Delta\}$$

and hence, $A \in \text{HOD}$. In particular, $a \in \text{HOD}$. $\square$

As a side, note that the fact that HOD satisfies the ω-approximation property in $<\omega$-HOD implies directly that $\text{HOD} = <\omega\text{-HOD}$, reproving the Hamkins-Leahy result, Theorem 1. The approximation property also sheds a light on fresh sequences in blurry HOD over HOD. The terminology follows Hamkins [8].

Definition 4 (Hamkins)**.** *Let $M \subseteq N$ be inner models, and let θ be an ordinal. A sequence $a : \theta \longrightarrow M$ is a* fresh sequence in N over M *if $a \in N$, $a \notin M$, but for all $\xi < \theta$, $a{\restriction}\xi \in M$.*

Note that for there to be a fresh sequence of length θ, the ordinal θ must be a limit. The following proposition expresses another way in which $<\kappa$-HOD is close to HOD. It will turn out later that it is optimal, in a sense.

Proposition 4. *Let κ be an infinite cardinal. If θ is a limit ordinal with $\mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\theta) \geq \kappa$, then $<\kappa$-HOD has no length θ sequence that is fresh over HOD.*

Proof. This follows from the fact that HOD satisfies the κ-approximation property in $<\kappa$-HOD, by Theorem 3—here is the argument. Let $f : \theta \longrightarrow \mathsf{HOD}$, $f \in {<\kappa}\text{-}\mathsf{HOD}$, be such that for every $\alpha < \theta$, $f{\restriction}\alpha \in \mathsf{HOD}$. As a set, f is a subset of HOD, and the point is that f is κ-approximated in HOD. To see this, let $a \in \mathsf{HOD}$, $\overline{\overline{a}}^{\mathsf{HOD}} < \kappa$. Since $\mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\theta) \geq \kappa$, it follows that $\{\xi < \theta \mid \langle \xi, f(\xi)\rangle \in a\}$ is bounded in θ, say by α. Then, $f \cap a = (f{\restriction}\alpha) \cap a \in \mathsf{HOD}$. $\square$

The fact that HOD satisfies the κ-approximation and -cover properties in $<\kappa$-HOD suggests that the relationship between these models is in some sense similar to that between a ground model and its forcing extension by a forcing with a closure point below κ, see [7] (Lemma 13), although $<\kappa$-HOD may not satisfy the axiom of choice. In fact, using a criterion for when a ZFC model of set theory is a set forcing extension of an inner model proved in Bukovský [9], it can be seen that $<\kappa$-HOD actually is a forcing extension of HOD by a set-sized κ-c.c. forcing, if $<\kappa$-HOD satisfies ZFC.

Definition 5 ([9] (1.6))**.** *Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathsf{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}}^{M_1} < \kappa$.*

The remarkable main theorem of [9] is the following.

Theorem 4 (ZFC)**.** *Suppose M is a transitive inner model of* ZFC*, and κ is an infinite cardinal. Then the following conditions are equivalent:*

1. *V is a forcing extension of M by a κ-c.c. forcing notion.*
2. *$\mathsf{Apr}_{M,\mathrm{V}}(\kappa)$ holds.*

The point now is:

Proposition 5. *Let κ be a cardinal. Then $\mathsf{Apr}_{\mathsf{HOD},<\kappa\text{-}\mathsf{HOD}}(\kappa)$ holds.*

Proof. I will prove a more general condition, namely that if $f : d \longrightarrow \mathsf{HOD}$ is a function in $<\kappa$-HOD with $d \in \mathsf{HOD}$, then there is in HOD a function $g : d \longrightarrow \mathsf{HOD}$ such that for every $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathsf{HOD}} < \kappa$. To see this, let f be as described. Let F be OD with $f \in F$ and $\overline{\overline{F}}^{\mathrm{V}} < \kappa$, and such that for every $h \in F$, $h : d \longrightarrow \mathsf{HOD}$. Define a function g with domain d by

$$g(x) = \{h(x) \mid h \in F\}.$$

Then $g \in \mathsf{HOD}$, and for $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathsf{HOD}} < \kappa$. So g is as wished. $\square$

Consequently, we obtain the following result.

Theorem 5. *Let κ be an infinite cardinal. Then the following are equivalent:*

1. *$<\kappa$-HOD satisfies the axiom of choice.*
2. *$<\kappa$-HOD is a set forcing extension of HOD by a κ-c.c. forcing notion.*

Proof. The direction from 1 to 2 follows by Proposition 5 and Theorem 4. For the converse, since HOD satisfies the axiom of choice, so does $<\kappa$-HOD if the latter is a set forcing extension of the former. $\square$

Even if $<\kappa$-HOD fails to satisfy the axiom of choice, Bukovský's condition is useful.

Proposition 6. *Let κ be an infinite cardinal. Then HOD and $<\kappa$-HOD have the same cardinals and cofinalities above κ, in the following sense:*

(a) *If λ is a limit ordinal such that $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$, then $\mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$.*
(b) *For $\lambda \geq \kappa$, λ is regular in HOD iff λ is regular in $<\kappa$-HOD.*
(c) *$\mathrm{Card}^{\mathsf{HOD}} \setminus \kappa = \mathrm{Card}^{<\kappa\text{-}\mathsf{HOD}} \setminus \kappa$.*

In fact, this is true of any two models M (in place of HOD) and N (in place of $<\kappa$-HOD) such that $\mathsf{Apr}_{M,N}(\kappa)$ holds; for (c), we assume that the axiom of choice holds in M.

Proof. I will use only that $\mathsf{Apr}_{\mathsf{HOD},<\kappa\text{-}\mathsf{HOD}}(\kappa)$ holds, by Proposition 5.

To prove (a), assume the contrary. Then $\bar{\lambda} = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda) < \mathrm{cf}^{\mathsf{HOD}}(\lambda)$. Let $f : \bar{\lambda} \longrightarrow \lambda$ be cofinal, with $f \in <\kappa$-HOD. By $\mathsf{Apr}_{\mathsf{HOD},<\kappa\text{-}\mathsf{HOD}}(\kappa)$, let $g : \bar{\lambda} \longrightarrow \mathcal{P}(\lambda)$, $g \in$ HOD, so that for all $\xi < \bar{\lambda}$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}}^{\mathsf{HOD}} < \kappa$. Since $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$, it follows that $\sup g(\xi) < \lambda$. In HOD, define $h : \bar{\lambda} \longrightarrow \lambda$ by $h(\xi) = \sup g(\xi)$. Since $f(\xi) \leq g(\xi)$, $\mathrm{ran}(g)$ is cofinal in λ, contradicting that $\bar{\lambda} < \mathrm{cf}^{\mathsf{HOD}}(\lambda)$, and showing (a).

Claim (b) follows immediately from (a).

Turning to (c), it suffices to show that every cardinal $\lambda \geq \kappa$ in HOD is also a cardinal in $<\kappa$-HOD. Assume the contrary, and let λ be the least cardinal of HOD that is greater than or equal to κ and not a cardinal in $<\kappa$-HOD. Since κ itself is a cardinal in $<\kappa$-HOD, it follows that $\lambda > \kappa$, and obviously, λ must be a successor cardinal in HOD. However, then, λ is regular in HOD, by the axiom of choice, so $\lambda = \mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$, showing that λ is a cardinal in $<\kappa$-HOD after all. $\square$

I would like to give an application of the approximation and cover properties of HOD in $<\kappa$-HOD, namely that HOD is definable in $<\kappa$-HOD. In view of Theorem 5, this is clear if $<\kappa$-HOD is a model of the axiom of choice, modulo Laver's ground model definability result [10]. Therefore, the main case of interest is going to be that it is not. Note that this result is not at all obvious; HOD is highly non-absolute to inner models—see, for example, [11]. It can be viewed as expressing a certain closeness between $<\kappa$-HOD and V, because it says that $<\kappa$-HOD can tell what is hereditarily ordinal definable in V. I will use the following theorem, the proof of which is a variation of an argument due to Hamkins to prove Laver's abovementioned result, as exhibited in more detail in Reitz [12] (Lemma 7.2). The new point is that the ambient model is not required to satisfy any form of the axiom of choice (so that it will be possible to apply the result in $<\kappa$-HOD.)

Theorem 6. *Let W be a transitive model satisfying ZF. Let κ be a regular cardinal in W. Let $\mathcal{M}, \mathcal{M}' \in W$ be transitive models of ZF without replacement, with $\theta = \mathcal{M} \cap \mathrm{On} = \mathcal{M}' \cap \mathrm{On}$, and each satisfying: if $\bar{\kappa} \leq \kappa$ and $r \subseteq \bar{\kappa} \times \bar{\kappa}$ is such that $\langle \bar{\kappa}, r \rangle$ a well-order, then there are an ordinal α and a function $\pi : \bar{\kappa} \longrightarrow \alpha$ such that $\pi : \langle \bar{\kappa}, r \rangle \longrightarrow \langle \alpha, < \rangle$ is an isomorphism, and every set of ordinals has a monotone enumeration. Moreover, suppose that every $a \in \mathcal{M} \cup \mathcal{M}'$ is well-orderable in W.*

Let $W_\theta = (\mathrm{V}_\theta)^W$, and suppose that both $\mathcal{M}$ and $\mathcal{M}'$ satisfy the κ-cover and approximation properties in W_θ. Suppose, moreover, that $\mathcal{P}(\kappa) \cap \mathcal{M} = \mathcal{P}(\kappa) \cap \mathcal{M}'$, and that $(\kappa^+)^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}'} = (\kappa^+)^W$.

Then it follows that $\mathcal{P}(\theta) \cap \mathcal{M} = \mathcal{P}(\theta) \cap \mathcal{M}'$.

Proof. As a first step, I will show

$(*)$ *If $A \in W$ is a bounded subset of θ and $\overline{\overline{A}}^{W} < \kappa$, then there is a $B \in \mathcal{M} \cap \mathcal{M}'$ such that $\overline{\overline{B}} \leq \kappa$ in any of the models $\mathcal{M}$, $\mathcal{M}'$ and W, and $A \subseteq B$.*

To prove $(*)$, let $A \subseteq \alpha < \theta$, and fix well-orderings $<_{\mathcal{M}}$, $<_{\mathcal{M}'}$ in W of $\mathcal{P}(\alpha)^{\mathcal{M}}$, $\mathcal{P}(\alpha)^{\mathcal{M}'}$, respectively. By the power set axiom in $\mathcal{M}/\mathcal{M}'$, these are sets in $\mathcal{M}/\mathcal{M}'$, so by assumption, they are well-orderable in W. Note that $A \in W_\theta$.

Define in W a weakly increasing sequence $\langle A_\xi \mid \xi < \kappa \rangle$ of subsets of α of cardinality less than κ in W with the following properties: $A \subseteq A_0$, if ξ is even (including when ξ is a limit ordinal), then $A_\xi \in \mathcal{M}$, and if ξ is odd, then $A_\xi \in \mathcal{M}'$. The construction is straightforward at successor stages, using that $\mathcal{M}$ and $\mathcal{M}'$ satisfy the κ-cover property in W_θ. To make the sequence definable in W, we can always pick the $<_{\mathcal{M}}/<_{\mathcal{M}'}$-least covering set. At limit stage $\lambda < \kappa$, let $\bar{A}_\lambda = \bigcup_{\xi<\lambda} A_\xi$. Since we do not assume that W satisfies the axiom of choice, let me be careful in concluding that in W, $\bar{A}_\lambda$ has cardinality less than κ. Assume this were not the case. Then $\Omega = \mathrm{otp}(\bar{A}_\lambda) \geq \kappa$. Let $e : \bar{A}_\lambda \longrightarrow \Omega$ be the inverse of the monotone enumeration of $\bar{A}_\lambda$. For every $\xi < \lambda$, we inductively know that $\overline{\overline{A_\xi}}^{W} < \kappa$, so $\mathrm{otp}(A_\xi) < \kappa$. The sequence $\langle A_\xi \mid \xi < \lambda \rangle$ is in W, and so is the sequence $\langle \mathrm{otp}(A_\xi) \mid \xi < \lambda \rangle$. Since κ is regular in W, it follows that this sequence of ordinals is bounded in κ, say by $\bar{\kappa}$. For each $\xi < \lambda$, let $f_\xi : \bar{\kappa} \longrightarrow A_\xi \cup \{0\}$ be defined by letting $f_\xi(\zeta)$ be the ζ-th element of A_ξ according to its monotone enumeration, if $\zeta < \mathrm{otp}(A_\xi)$, and let $f_\xi(\zeta) = 0$ otherwise. In W, the function $h : \lambda \times \bar{\kappa} \longrightarrow \Omega$ can now be defined by

$$h(\xi, \zeta) = e(f_\xi(\zeta)).$$

This function is onto $\Omega \geq \kappa$, but $\lambda, \bar{\kappa} < \kappa$, contradicting that κ is a cardinal in W. So $\bar{A}_\lambda$ has cardinality less than κ in W, and one can define A_λ to be the $<_{\mathcal{M}}$-least set in $\mathcal{P}(\alpha) \cap \mathcal{M}$ covering $\bar{A}_\lambda$ and with size less than κ in $\mathcal{M}$.

This defines the sequence $\langle A_\xi \mid \xi < \kappa \rangle$. Let $B = \bigcup_{\xi<\kappa} A_\xi$. A similar argument to the above shows that $\overline{\overline{B}}^{W} \leq \kappa$. If not, then $\Omega = \mathrm{otp}(B) \geq (\kappa^+)^W$. However, for each $\xi < \kappa$, $\mathrm{otp}(A_\xi) < \kappa$. Again, letting $e : B \longrightarrow \Omega$ be the inverse of the monotone enumeration of B and letting, for each $\xi < \kappa$, $f_\xi : \kappa \longrightarrow A_\xi \cup \{0\}$ be defined by letting $f_\xi(\zeta)$ be the ζ-th element of A_ξ if $\zeta < \mathrm{otp}(A_\xi)$ and $f_\xi(\zeta) = 0$ otherwise, we can define $h : \kappa \times \kappa \longrightarrow \Omega$ by $h(\xi, \zeta) = e(f_\xi(\zeta))$, resulting in a function in W with domain $\kappa \times \kappa$ whose range contains $(\kappa^+)^W$, a contradiction.

To see that $B \in \mathcal{M}$, note that B is κ-approximated in $\mathcal{M}$, because if $C \in \mathcal{M}$ has $\overline{\overline{C}}^{\mathcal{M}} < \kappa$, then there is a $\xi < \kappa$ such that $C \cap B = C \cap A_\xi$, since κ is regular in W. So $C \cap B = C \cap A_\xi = C \cap A_{\xi+1} \in \mathcal{M}$, since at least one of A_ξ and $A_{\xi+1}$ is in $\mathcal{M}$. The same argument shows that $B \in \mathcal{M}'$, and claim $(*)$ is proven.

Following Hamkins' argument, the next step is to show that $\mathcal{M}$ and $\mathcal{M}'$ have the same sets of ordinals of cardinality less than κ. By symmetry, it suffices to show that given a set of ordinals $A \in \mathcal{M}$ of cardinality less than κ in $\mathcal{M}$, $A \in \mathcal{M}'$. Note first that A must be bounded in θ, since $\mathcal{M}$ satisfies ZF minus replacement. Let $B \in \mathcal{M} \cap \mathcal{M}'$ be such that $A \subseteq B$ and $\overline{\overline{B}} \leq \kappa$ in all three models. Let $\Omega = \mathrm{otp}(B) < \kappa^+$ (in the sense of any and all of the models involved). In $\mathcal{M}$, let $\bar{\kappa}$ be the cardinality of Ω, and let $h : \bar{\kappa} \longrightarrow \Omega$ be a bijection. Let $w = \{\langle \alpha, \beta \rangle \mid \alpha, \beta < \bar{\kappa} \wedge h(\alpha) < h(\beta)\}$. Since $\mathcal{P}(\kappa) \cap \mathcal{M} = \mathcal{P}(\kappa) \cap \mathcal{M}'$, it follows that $w \in \mathcal{M}'$. By assumption on $\mathcal{M}'$, there are a function h' and an ordinal Ω' in $\mathcal{M}'$ such that $h' : \langle \bar{\kappa}, w \rangle \longrightarrow \langle \Omega', < \rangle$. However, since $\langle \bar{\kappa}, w \rangle$ is actually a well-order, it must be that $h' = h$ and $\Omega' = \Omega$. The inverse of the monotone enumeration of B, $e : B \longrightarrow \Omega$ is also both in $\mathcal{M}$ and $\mathcal{M}'$. Let $\bar{A} = h^{-1} \circ e[A]$. Then $\bar{A} \in \mathcal{P}(\kappa) \cap \mathcal{M}$, so $\bar{A} \in \mathcal{M}'$, and so, $A = e^{-1} \circ h[\bar{A}] \in \mathcal{M}'$.

The proof can now be completed by showing that $\mathcal{M}$ and $\mathcal{M}'$ have the same sets of ordinals: let $A \in \mathcal{M}$ be a set of ordinals. It suffices to show that $A \in \mathcal{M}'$. As before, note that A is bounded in θ, so that $A \in W_\theta$. The point is now that A is κ-approximated in $\mathcal{M}'$.

Specifically, let $B \in \mathcal{M}'$ be a set of ordinals of cardinality less than κ in $\mathcal{M}'$. By the previous point, $B \in \mathcal{M}$. However, then, $A \cap B \in \mathcal{M}$, since both A and B are in $\mathcal{M}$. In addition, $A \cap B$ has cardinality less than κ in $\mathcal{M}$, so $A \cap B \in \mathcal{M}'$, again by the same point. It now follows by the κ-approximation property of $\mathcal{M}'$ in W_θ that $A \in \mathcal{M}'$. $\square$

It would be possible in the previous theorem to strengthen the assumptions on $\mathcal{M}$ and $\mathcal{M}'$ to ensure that every set in $\mathcal{M}/\mathcal{M}'$ is coded by a set of ordinals in $\mathcal{M}/\mathcal{M}'$, and to strengthen the conclusion to $\mathcal{M} = \mathcal{M}'$, as was done in the result due to Laver referenced above. However, the present form of the result seems to be a little more convenient here.

Theorem 7. *Let $\lambda \geq 2$ be a cardinal. Let $\kappa \geq \lambda$ be regular. Then* HOD *is definable in* $<\lambda$-HOD *using $\mathcal{P}(\kappa) \cap \mathsf{HOD}$ as a parameter.*

Remark: If $\lambda \leq \omega$, then HOD is trivially definable in $<\lambda$-HOD without parameters, since then, $<\lambda$-HOD $=$ HOD. For $\lambda > \omega$, note that by a result of Gitik [13], it is consistent with ZF (assuming the consistency of very large cardinals) that every uncountable cardinal is singular, so there may be no κ as stated. However, of course, ZFC implies that λ^+ is regular.

Proof. Let $\lambda \geq \omega$, and let $\kappa \geq \lambda$ be regular.

Let $z = \mathcal{P}(\kappa)^{\mathsf{HOD}}$. Working in $<\lambda$-HOD, and using the fact that κ is regular, we have that for every ordinal θ, if there is a transitive model $\mathcal{M}$ satisfying the conditions of Theorem 6 with $\theta = \mathrm{On} \cap \mathcal{M}$ and $\mathcal{P}(\kappa) \cap \mathcal{M} = z$, which satisfies the κ-approximation and cover properties in V_θ, such that $(\kappa^+)^{\mathcal{M}} = \kappa^+$, then for any other such model $\mathcal{N}$, we have that $\mathcal{P}(\theta) \cap \mathcal{M} = \mathcal{P}(\theta) \cap \mathcal{N}$. Let me denote this common value of $\mathcal{P}(\theta) \cap \mathcal{M}$ by S_θ. If such an $\mathcal{M}$ does not exist, then let S_θ be undefined. Let I be the class of θ such that S_θ is defined.

If $\theta > \kappa^+$ and θ is a limit ordinal in $<\lambda$-HOD then $\mathrm{V}_\theta^{\mathsf{HOD}}$ is a model as above, and so, $S_\theta = \mathcal{P}(\theta) \cap \mathrm{V}_\theta^{\mathsf{HOD}}$. This is because $(\kappa^+)^{\mathrm{V}_\theta^{\mathsf{HOD}}} = (\kappa^+)^{<\lambda\text{-}\mathsf{HOD}}$ by Proposition 6, and because HOD satisfies the κ-approximation and -cover properties in $<\lambda$-HOD by Theorem 3—this easily implies that $\mathrm{V}_\theta^{\mathsf{HOD}}$ satisfies the κ-approximation and -cover properties in $\mathrm{V}_\theta^{<\lambda\text{-}\mathsf{HOD}}$. The other conditions are also easily seen to be satisfied by $\mathrm{V}_\theta^{\mathsf{HOD}}$.

Still working in $<\lambda$-HOD, let $J = \{\theta \in I \mid \theta > \kappa^+ \text{ is a limit ordinal}\}$. We can define in $<\lambda$-HOD:

$$S = \bigcup_{\theta \in J} S_\theta.$$

Then $S = \mathcal{P}(\mathrm{On}) \cap \mathsf{HOD}$. It follows that the class $\hat{S}$ of sets coded by an element of S is equal to HOD. Here, let me say that a code is just a relation $r \subseteq \alpha \times \alpha$ (which can easily be coded as a set of ordinals), for some ordinal α, such that $\langle \alpha, r \rangle$ is well-founded and extensional. By Mostowski's theorem, there are a unique transitive set u and a unique bijective function $\pi : \alpha \longrightarrow u$ that constitutes an isomorphism between $\langle \alpha, r \rangle$ and $\langle u, \in \upharpoonright u \rangle$. The set coded by r is $\pi(0)$. Since HOD is a model of ZFC, every element a of it has a code $r \in \mathsf{HOD}$, which is in S, and consequently, $a \in \hat{S}$. Moreover, $S \subseteq \mathsf{HOD}$, and so, $\hat{S} \subseteq \mathsf{HOD}$. Thus, $\hat{S} = \mathsf{HOD}$ is definable in $<\lambda$-HOD using $\mathcal{P}(\kappa) \cap \mathsf{HOD}$ as a parameter, as claimed. $\square$

The original use of the approximation and cover properties in [7] was that the passage from the smaller to the larger model does not create new large cardinals of various types. I will just mention two instances of this phenomenon here. If $<\kappa$-HOD satisfies the axiom of choice, then all the results from [7] apply.

Proposition 7. *Let κ be an infinite cardinal, and let $\lambda \geq \kappa$ be inaccessible in* HOD.

(1) *If λ weakly compact in $<\kappa$-HOD, then it is weakly compact in* HOD.
(2) *If λ is measurable in $<\kappa$-HOD, then it is measurable in* HOD.

Proof. To see (1), let T be a λ-tree in HOD. Since λ is weakly compact in $<\kappa$-HOD, T has a cofinal branch b there. However, $<\kappa$-HOD has no fresh λ-sequence over HOD, by 4, so $b \in \mathsf{HOD}$.

For (2), the argument of [7] (Theorem 10) applies verbatim. □

This shows that $<\kappa$-HOD does not capture that much more of the large cardinal structure of V than HOD does. On the one hand, if κ is inaccessible, then $\mathrm{V}_\kappa \cap <\kappa\text{-HOD} = \mathrm{V}_\kappa$, so that large cardinal properties of V witnessed by V_κ are inherited by $<\kappa$-HOD. However, recall that it was shown in [14] (Theorem 4) that it is consistent that a supercompact cardinal λ is not weakly compact in HOD – so if $\kappa \leq \lambda$, then λ is not weakly compact in $<\kappa$-HOD either, by Proposition 7.

Finally, I would like to make a quick connection to Woodin's work on the HOD dichotomy. Without going into too many details, it can be formulated as follows.

Theorem 8 (HOD dichotomy theorem , Woodin [15] (Thm. 3.39))**.** *Suppose that δ is an extendible cardinal. Then exactly one of the following holds:*
(1) *Every regular cardinal $\geq \delta$ is ω-strongly measurable in* HOD*.*
(2) *No regular cardinal $\geq \delta$ is ω-strongly measurable in* HOD*. Furthermore,* HOD *is a weak extender model for the supercompactness of δ.*

Here, an inner model N of ZFC is called a weak extender model for the supercompactness of δ if for every $\gamma > \delta$, there is a δ-complete, normal, fine measure U on $\mathcal{P}_\delta(\gamma)$ such that $N \cap \mathcal{P}_\delta(\gamma) \in U$ and $U \cap N \in N$. To be able to follow, it is not necessary to know what it means that a cardinal is ω-strongly measurable in HOD; suffice it to say that it implies that the cardinal in question is measurable in HOD. The HOD hypothesis says that there is a proper class of regular cardinals which are not ω-strongly measurable in HOD. Thus, if the HOD hypothesis holds and there is an extendible cardinal δ, then HOD is a weak extender model for the supercompactness of δ. Here is the salient fact on weak extender models from Woodin's analysis.

Lemma 1 (Woodin [15] (Lemma 3.8))**.** *If N is a weak extender model for the supercompactness of δ, then N satisfies the δ-cover property in* V*.*

I can now make the connection to blurry HOD I was aiming at.

Proposition 8 (ZFC)**.** *Suppose δ is extendible. Then the* HOD *hypothesis implies that*

$$[\mathrm{HOD}]^{<\delta} \subseteq <\delta\text{-HOD}.$$

Proof. Let $a \subseteq$ HOD, with $\overline{\overline{a}} < \delta$. By our assumption, HOD is a weak extender model for the supercompactness of δ, so by Lemma 1, HOD satisfies the δ-cover property in V. This means that there is a $c \in$ HOD such that $a \subseteq c$ and $\overline{\overline{c}} < \delta$. Let $A = \mathcal{P}(c)$. Since δ is inaccessible, $\overline{\overline{A}} < \delta$. Since $c \in$ HOD, A is OD. In addition, clearly, $a \in A$, showing that $a \in <\delta$-HOD, as claimed. □

Recent work of Goldberg [16] reduces the assumption in Proposition 8 from an extendible cardinal to a strongly compact one.

2.4. Leaps

Clearly, in a model where V = HOD, we have that for every cardinal $\kappa > 1$, $\mathrm{V} = \mathrm{HOD} \subseteq <\kappa\text{-HOD} \subseteq \mathrm{V}$, so that HOD $= <\kappa$-HOD. On the other hand, if V $\neq$ HOD is a ZFC model, then since by Proposition 1 (5), $\mathrm{V} = \bigcup_{\kappa \in \mathrm{Card}} <\kappa\text{-HOD}$ is an increasing union, there must be stages γ at which $<\gamma$-HOD contains something new. These stages are obviously interesting, and I call them *leaps*.

Definition 6. *A cardinal $\lambda > 2$ is a* leap *if*

$$<\delta\text{-HOD} \subsetneq <\lambda\text{-HOD},$$

for every cardinal $\delta < \lambda$. I write $\langle \Lambda_\alpha \mid \alpha < \Theta\rangle$ for the monotone enumeration of the leaps (it is the empty sequence, i.e., Λ_0 is undefined, if there is no leap). A leap λ is a successor leap if $\lambda = \Lambda_{\xi+1}$, for some ξ, and it is a limit leap if it is of the form $\lambda = \Lambda_\xi$, where ξ is a limit ordinal.

Lemma 2. *Leaps have the following properties.*

(1) *The class of leaps is closed in the ordinals.*
(2) *Λ_0, if defined, is an uncountable successor cardinal.*
(3) *Successor leaps are successor cardinals.*

Proof. (1) is trivial.

(2): Λ_0 must be uncountable by Theorem 1. Now assume Λ_0 is a limit cardinal. Let a be $\in$-minimal in $<\Lambda_0$-HOD $\setminus$ HOD. So $a \subseteq$ HOD. Since $a \in <\Lambda_0$-OD, and Λ_0 is a limit cardinal, it follows that a is $<\lambda$-OD, for some $\lambda < \Lambda_0$. However, then $a \in <\lambda$-HOD $\setminus$ HOD (as $a \subseteq$ HOD $\subseteq <\lambda$-HOD). However, then λ is a leap less than Λ_0, a contradiction.

(3): Assume $\Lambda_{\xi+1}$ were a limit cardinal. Pick a $\in$-minimal in $<\Lambda_{\xi+1}$-HOD $\setminus <\Lambda_\xi$-HOD. Since $a \in <\Lambda_{\xi+1}$-HOD, it follows that $a \subseteq <\Lambda_{\xi+1}$-HOD, and hence that $a \subseteq <\Lambda_\xi$-HOD, by $\in$-minimality. In addition, since $a \in <\Lambda_{\xi+1}$-OD and $\Lambda_{\xi+1}$ is a limit cardinal, it follows that $a \in <\lambda$-OD, for some cardinal $\lambda < \Lambda_{\xi+1}$ which we may choose so that $\lambda \geq \Lambda_\xi$. Since $a \subseteq <\Lambda_\xi$-HOD, it follows that $a \subseteq <\lambda$-HOD, and so, since $a \in <\lambda$-OD, $a \in <\lambda$-HOD. However, since there is no leap in $(\Lambda_\xi, \lambda]$, $<\lambda$-HOD $=$ $<\Lambda_\xi$-HOD. So $a \in <\Lambda_\xi$-HOD, a contradiction. □

By Lemma 2, item (1), limit leaps occur automatically, in a sense. However, it is an interesting question whether a stronger property can hold.

Definition 7. *Say that a leap γ is a* big leap *if*

$$\left(\bigcup_{\delta<\gamma, \delta\in\mathrm{Card}} <\delta\text{-HOD}\right) \subsetneq <\gamma\text{-HOD}.$$

Successor leaps are always big leaps, but what can be said about limit leaps? Perhaps somewhat surprisingly, they are also big.

Theorem 9. *Every leap is big.*

Proof. Assume the contrary. Let λ be the least counterexample. Since successor leaps are clearly big leaps, λ must be a limit leap. Let $S = \{\kappa < \lambda \mid \kappa \text{ is a leap}\}$. So S is unbounded in λ, and by minimality of λ, every leap in S is a big leap. For $\kappa \in S$, let $\tau_\kappa = \langle \alpha_\kappa, \beta_\kappa, \ulcorner\varphi_\kappa\urcorner\rangle$, be the (lexicographically) least code for an element of $<\kappa$-HOD $\setminus \bigcup_{\bar\kappa\in\mathrm{Card}\cap\kappa} <\bar\kappa$-HOD, i.e., $A_\kappa = \{x \mid \mathsf{Sat}(\mathrm{V}_{\alpha_\kappa}, \ulcorner\varphi_\kappa\urcorner, \beta_\kappa)\}$ has cardinality less than κ, there is an $a \in A_\kappa$ such that $a \in <\kappa$-HOD $\setminus \bigcup_{\bar\kappa\in\mathrm{Card}\cap\kappa} <\bar\kappa$-HOD, and τ_κ is minimal with this property.

Note that the sequence $\langle \tau_\kappa \mid \kappa \in S\rangle$ is ordinal definable from the parameter λ. Now define

$$B_\kappa = A_\kappa \cap <\kappa\text{-HOD}$$

for $\kappa \in S$. The sequence $\vec{B} = \langle B_\kappa \mid \kappa \in S\rangle$ is then OD, and it is actually $<\lambda$-HOD, because every element of it is of the form $\langle \kappa, B_\kappa\rangle$, for some $\kappa \in S$. Now B_κ is OD, and $B_\kappa \subseteq <\kappa$-HOD $\subseteq <\lambda$-HOD. Thus, $B_\kappa \in <\lambda$-HOD, and so, $\langle \kappa, B_\kappa\rangle$ is $<\lambda$-HOD. Therefore, the whole sequence $\vec{B}$ is, as a set, contained in $<\lambda$-HOD, and it is OD, hence it is $<\lambda$-HOD.

However, clearly, $\vec{B} \notin <\kappa$-HOD, for $\kappa \in \mathrm{Card} \cap \lambda$, because we can pick $\kappa' \in S \setminus (\kappa+1)$, so that there is an $a \in B_{\kappa'}$ with $a \notin <\kappa$-HOD. Clearly then, $\vec{B}$ cannot belong to $<\kappa$-HOD, since $<\kappa$-HOD is transitive. □

Theorem 10. *If λ is a limit leap, then $<\lambda$-HOD does not satisfy the axiom of choice; more precisely, $\bar\lambda$-AC fails in $<\lambda$-HOD, where $\lambda = \Lambda_{\bar\lambda}$.*

Proof. Let us improve the sequence $\vec{B}$ of the proof of Theorem 9 slightly. First, let us work with
$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau'_\kappa = \langle \alpha'_\kappa, \beta'_\kappa, \ulcorner\varphi'_\kappa\urcorner\rangle$ be the least code for an $\in$-*minimal* element of $<\kappa\text{-HOD} \setminus <\kappa_-\text{-HOD}$ (in the same sense as in the previous proof), and let $A'_\kappa = \{x \mid \mathsf{Sat}(\mathrm{V}_{\alpha'_\kappa}, \ulcorner\varphi'_\kappa\urcorner, \beta'_\kappa)\}$. Let
$$B'_\kappa = \{x \in A'_\kappa \mid x \text{ is } \in\text{-minimal in } <\kappa\text{-HOD} \setminus <\kappa_-\text{-HOD}\}.$$

Therefore, for every $b' \in B'_\kappa$, $b' \in <\kappa\text{-HOD}$, $b' \notin <\kappa_-\text{-HOD}$, but $b' \subseteq <\kappa_-\text{-HOD}$; the latter by $\in$-minimality. As above, $\vec{B}' = \langle B'_\kappa \mid \kappa \in T\rangle$ is OD, and $\vec{B}'$ belongs to $<\lambda\text{-HOD}$. It is a sequence of nonempty sets, and I claim that it has no choice function in $<\lambda\text{-HOD}$: suppose it did. Let $\vec{b}' = \langle b'_\kappa \mid \kappa \in T\rangle \in <\lambda\text{-HOD}$ be such that for every $\kappa \in T$, $b'_\kappa \in B'_\kappa$. Since $\vec{b}' \in <\lambda\text{-HOD}$, it is $<\gamma\text{-OD}$, for some cardinal $\gamma < \lambda$. Let X witness this, i.e., let X be OD and of cardinality less than γ, so that $\vec{b}' \in X$. Let
$$Y = X \cap \prod_{\kappa \in T} B'_\kappa.$$

Clearly, Y is still OD, has cardinality less than γ, and has $\vec{b}' \in Y$.

Now pick $\kappa \in T$ such that $\gamma \leq \kappa_-$. I claim that $b'_\kappa \in <\kappa_-\text{-HOD}$, a contradiction. Specifically, let
$$Z = \{x(\kappa) \mid x \in Y\}.$$

This is an OD set of cardinality at most $\overline{\overline{Y}} < \gamma \leq \kappa_-$, and b'_κ belongs to it. So b'_κ is $<\kappa_-\text{-OD}$. In addition, since $b'_\kappa \in B'_\kappa$, we know that $b'_\kappa \subseteq <\kappa_-\text{-HOD}$. Thus, $b'_\kappa \in <\kappa_-\text{-HOD}$, as claimed. This contradiction completes the proof. □

In the notation of the previous proof, the question arises naturally, at which level δ a choice function for $\vec{B}'$ gets into $<\delta\text{-HOD}$. This may depend on cardinal arithmetic. Clearly, all such choice functions are in $<(2^\lambda)^+\text{-HOD}$. Can one be in $<\lambda^+\text{-HOD}$ already? We do not know.

Corollary 2. *In a* ZFC *model, the largest leap, if there is one, is a successor leap.*

Therefore, both the first and the last leap, if they exist, are successor cardinals.

3. Effects of Forcing on Blurry Definability

The main method for producing models of set theory with interesting leap structures is going to be forcing. In this section, I will investigate how $<\kappa\text{-HOD}^{\mathrm{V}}$ relates to $<\kappa\text{-HOD}^{\mathrm{V}[G]}$, where G is generic over V for certain notions of forcing. First, let me find ways to ensure that being in $<\kappa$-HOD is preserved, i.e., that $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$.

3.1. Preserving Blurry Definability

Proposition 9. *Suppose that $\mathbb{P}$ is a notion of forcing, G is generic for $\mathbb{P}$ over* V*, and* V *is a class in* $\mathrm{V}[G]$ *definable from ordinal parameters. Let κ be a cardinal in* $\mathrm{V}[G]$*. Then* $<\kappa\text{-OD}^{\mathrm{V}} \subseteq <\kappa\text{-OD}^{\mathrm{V}[G]}$*, and so,* $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$ *as well.*

Proof. Let $a \in <\kappa\text{-OD}$, and let A witness this, i.e., $A = \{x \mid \varphi(x, \alpha)\}$, for some formula φ and some ordinal α, where $a \in A$ and $\overline{\overline{A}} < \kappa$. Let $\psi(x, \beta)$ define V in $\mathrm{V}[G]$. Then $A = \{x \mid \psi(x, \beta) \wedge \varphi^{\{z \mid \psi(z,\beta)\}}(x, \alpha)\}^{\mathrm{V}[G]}$, and $\overline{\overline{A}}^{\mathrm{V}[G]} < \kappa$, so $a \in <\kappa\text{-OD}^{\mathrm{V}[G]}$. It follows immediately that $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$. □

The assumption that V is definable from ordinals in $\mathrm{V}[G]$ can be weakened as follows.

Proposition 10 (ZFC)**.** *Suppose that* $\mathbb{P}$ *is a notion of forcing,* G *is generic for* $\mathbb{P}$ *over* V*,* κ *is a cardinal in* $\mathrm{V}[G]$*, and* V *is definable in* $\mathrm{V}[G]$ *from a parameter in* $<\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$*. Then*

$$<\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}} \subseteq <\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$$

and so, $<\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}} \subseteq <\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]}$ *as well.*

Proof. Let $b \in <\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$ and $\psi(x,y)$ be a formula such that

$$\mathrm{V} = \{x \mid \psi(x,b)\}^{\mathrm{V}[G]}.$$

Since $b \in <\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$, we can also let B be $\mathsf{OD}^{\mathrm{V}[G]}$ so that $b \in B$ and B has cardinality less than κ in $\mathrm{V}[G]$.

Now, in V, let $a \in <\!\kappa\text{-}\mathsf{OD}$, and let A witness this, i.e., $A = \{x \mid \varphi(x,\alpha)\}$, for some formula φ and some ordinal α, where $a \in A$ and $\overline{\overline{A}}^{\mathrm{V}} < \kappa$. We must show that a is also $<\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$.

To this end, let $\overline{\overline{A}}^{\mathrm{V}[G]} = \bar{\kappa} < \kappa$, and let $\chi(x,y)$ be a formula and β an ordinal such that $B = \{x \mid \chi(x,\beta)^{\mathrm{V}[G]}\}$. We can now define

$$A' = \left\{z \mid \exists b' \exists w \quad (\chi(b',\beta) \wedge w = \{x \mid \varphi(x,\alpha)^{\{x \mid \psi(x,b')\}}\} \wedge \overline{\overline{w}} = \bar{\kappa} \wedge z \in w\right\}^{\mathrm{V}[G]}.$$

Clearly, $a \in A'$, as witnessed by $b' = b$ (and $w = \{x \mid \varphi(x,\alpha)^{\mathrm{V}}\}$), and $\overline{\overline{A'}} < \kappa$ in $\mathrm{V}[G]$, since in $V[G]$, A' is a union of fewer than κ sets, each of cardinality $\bar{\kappa} < \kappa$. Since A' is $\mathsf{OD}^{\mathrm{V}[G]}$, it follows that $a \in <\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$, as claimed. □

As a corollary, one obtains a preservation by small forcing result.

Corollary 3. *Let* κ *be a cardinal, and let* $\mathbb{P}$ *be a notion of forcing of cardinality* γ*, where* $2^{(2^\gamma)} < \kappa$*. If* G *is* $\mathbb{P}$*-generic over* V*, then*

$$<\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}} \subseteq <\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]}.$$

Proof. By a result of Woodin [17], V is (uniformly) definable in $\mathrm{V}[G]$ from the parameter $\mathcal{P}(\gamma)^{\mathrm{V}}$. This parameter is an element of $Z = \mathcal{P}(\mathcal{P}(\gamma))^{\mathrm{V}[G]}$. Since $\mathbb{P}$ is γ^+-c.c., it preserves cardinals above γ, so the cardinality of Z in $\mathrm{V}[G]$ is $2^{(2^\gamma)^{\mathrm{V}}} < \kappa$, so Z is $<\!\kappa\text{-}\mathsf{OD}^{\mathrm{V}[G]}$. The claim now follows from Proposition 10. □

3.2. Not Adding to Blurry HOD

Having seen how to preserve membership to $<\!\kappa$-HOD when passing to a forcing extension, let us now turn to the converse problem, ensuring that nothing is added to $<\!\kappa$-HOD. Homogeneity properties of the forcing notions in question will play an important role. The following is a folklore concept; I follow the terminology of [18].

Definition 8. *Let* $\mathbb{P}$ *be a forcing notion. For* $p \in \mathbb{P}$*, let the cone below* p *in* $\mathbb{P}$ *be the set*

$$\mathbb{P}_{\leq p} = \{q \in \mathbb{P} \mid q \leq p\}$$

equipped with the restriction of the ordering of $\mathbb{P}$*.*

$\mathbb{P}$ *is called* cone homogeneous *if for any two conditions* $p,q \in \mathbb{P}$*, there are* $p' \leq p$ *and* $q' \leq q$ *such that* $\mathbb{P}_{\leq p'}$ *and* $\mathbb{P}_{\leq q'}$ *are isomorphic.*

First, let me prove two general lemmas. I say that a forcing notion $\mathbb{P}$ is $<\!\kappa$-closed, where κ is a regular cardinal, if every decreasing sequence in $\mathbb{P}$ of length less than κ has a lower bound. The following lemma generalizes the well-known folklore fact that if G is generic for a cone homogenous forcing notion, then $\mathsf{HOD}^{V[G]} \subseteq V$—note that any forcing notion is $<\omega$-closed.

Lemma 3. *Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over* V*. Then*

$$<\kappa\text{-}\mathsf{HOD}^{V[G]} \subseteq V.$$

Proof. Assume the contrary, and let a be $\in$-minimal in $<\kappa\text{-}\mathsf{HOD}^{V[G]} \setminus V$. Then $a \subseteq V$, and there is an $\mathsf{OD}^{V[G]}$ set $A = \{x \mid \varphi(x,\rho)\}^{V[G]}$ such that $a \in A$ and $\overline{\overline{A}}^{V[G]} = \bar{\kappa} < \kappa$. Let $a = \dot{a}^G$, and let $p \in \mathbb{P}$ force all of these facts: that $\dot{a}$ is $\in$-minimal in $<\check{\kappa}\text{-}\mathsf{HOD}^{V[\dot{G}]} \setminus \check{V}$, that $\varphi(\dot{a},\check{\rho})$ holds, and that $\{x \mid \varphi(x,\check{\rho})\}$ has cardinality $\check{\bar{\kappa}}$.

In V, note that for every condition $q \le p$, there are $q_0, q_1 \le q$ and a set d such that $q_0 \Vdash \check{d} \in \dot{a}$ while $q_1 \Vdash \check{d} \notin \dot{a}$, because otherwise, q would decide, for every d, the statement "$\check{d} \in \dot{a}$". However, since $q \le p$, q also forces that $\dot{a} \subseteq V$. It would follow that $a = \{d \mid p \Vdash \check{d} \in \dot{a}\} \in V$, contradicting our assumption that p forces that $\dot{a} \notin \check{V}$.

This enables us to construct, in V, sequences $\langle q_s \mid s \in {}^{\le\bar{\kappa}}2\rangle$ and $\langle d_s \mid s \in {}^{<\bar{\kappa}}2\rangle$ with the following properties:

1. $q_\emptyset = p$,
2. $q_s \in \mathbb{P}$, and $s \subseteq t \implies q_t \le_{\mathbb{P}} q_s$,
3. $q_{s^\frown 0} \Vdash_{\mathbb{P}} \check{d}_s \in \dot{a}$ and $q_{s^\frown 1} \Vdash_{\mathbb{P}} \check{d}_s \notin \dot{a}$, for $s \in {}^{<\bar{\kappa}}2$.

To define q_s when $\mathrm{dom}(s)$ is a limit ordinal, we use that fact that this limit ordinal is at most $\bar{\kappa}$, which is less than κ, and that $\mathbb{P}$ is $<\kappa$-closed.

Let me temporarily fix a function $s : \bar{\kappa} \longrightarrow 2$ in V. Consider the set

$$D_s = \{p' \le p \mid \exists q' \le q_s \quad \mathbb{P}_{\le p'} \text{ is isomorphic to } \mathbb{P}_{\le q'}\}.$$

It is easily seen that D_s is dense below p, since $\mathbb{P}$ is cone homogeneous. It follows that there is a condition $p_s \in G \cap D_s$. Let $q_s' \le q_s$ and $\pi_s \in V$ witness this, i.e.,

$$\pi_s : \mathbb{P}_{\le p_s} \longrightarrow \mathbb{P}_{\le q_s'}$$

is an isomorphism. Then, $G_s = \pi_s[G]$ is generic for $\mathbb{P}_{\le q_s'}$. G_s generates a generic filter $\langle G_s\rangle$ in $\mathbb{P}$ that contains q_s, and since $\pi_s \in V$, it follows that

$$V[G] = V[\langle G_s\rangle] \models \varphi(\dot{a}^{\langle G_s\rangle}, \rho).$$

However, note that if $t \ne s$ is another member of ${}^{\bar{\kappa}}2$, then $\dot{a}^{\langle G_s\rangle} \ne \dot{a}^{\langle G_t\rangle}$, because if $\delta = \min\{i < \kappa \mid s(i) \ne t(i)\}$, then $q_s \Vdash_{\mathbb{P}} \check{d}_{s\restriction\delta} \in \dot{a} \iff q_t \Vdash_{\mathbb{P}} \check{d}_{s\restriction\delta} \notin \dot{a}$.

Since $\mathbb{P}$ is $<\kappa$-closed, it follows that $({}^{\bar{\kappa}}2)^V = ({}^{\bar{\kappa}}2)^{V[G]}$. Therefore, in $V[G]$, the set

$$\{\dot{a}^{\langle G_s\rangle} \mid s : \bar{\kappa} \to 2\}$$

is a $2^{\bar{\kappa}}$-sized subset of A. This contradicts that A has cardinality $\bar{\kappa}$ in $V[G]$. □

Lemma 4. *Let $\bar{\kappa}$ be a cardinal, and let $\mathbb{P}$ be a cone homogeneous forcing that preserves cardinals up to $\bar{\kappa}$, such that whenever G is $\mathbb{P}$-generic over* V*, then*

$$<\bar{\kappa}\text{-}\mathsf{HOD}^{V[G]} \subseteq V.$$

If $\mathbb{P}$ is $<\bar{\kappa}$-OD, then for any $\mathbb{P}$-generic filter G, we have that

$$<\bar{\kappa}\text{-}\mathsf{HOD}^{V[G]} \subseteq <\bar{\kappa}\text{-}\mathsf{HOD}^{V}.$$

Proof. Assume the contrary. Let a be $\in$-minimal in $<\bar{\kappa}\text{-}\mathsf{HOD}^{V[G]} \setminus <\bar{\kappa}\text{-}\mathsf{HOD}^{V}$. So $a \subseteq <\bar{\kappa}\text{-}\mathsf{HOD}^{V}$. We also know that $a \in V$. Since a is $<\bar{\kappa}\text{-}\mathsf{OD}^{V[G]}$, we can pick an $\mathsf{OD}^{V[G]}$ set

$A = \{\varphi(x,\rho)\}^{\mathrm{V}[G]}$ such that $a \in A$ and such that in $\mathrm{V}[G]$, $\overline{\overline{A}} = \delta < \bar{\kappa}$. By the cone homogeneity of $\mathbb{P}$ it follows that every condition in $\mathbb{P}$ forces that $\overline{\overline{A}} = \check{\delta}$—this is one of the folklore consequences of cone homogeneity of $\mathbb{P}$: if some condition $r \in \mathbb{P}$ forces a statement of the form $\psi(\check{a}_0, \ldots, \check{a}_n)$, then *every* condition in $\mathbb{P}$ forces this. This is easily seen, because otherwise we would have another condition, say s, forcing the negation of the statement. We could then find extensions $r' \leq r$ and $s' \leq s$ such that $\mathbb{P}_{\leq r'}$ is isomorphic to $\mathbb{P}_{\leq s'}$. If H is generic for $\mathbb{P}_{\leq r'}$, then H generates a filter H' which is generic for $\mathbb{P}$, and in $\mathrm{V}[H']$, $\varphi(a_0, \ldots, a_n)$ holds, since $r \in H'$. However, applying the isomorphism to H gives a $\mathbb{P}_{\leq s'}$-generic, say I, which also generates a $\mathbb{P}$-generic filter I', and in $\mathrm{V}[I']$, $\varphi(a_0, \ldots, a_n)$ fails, since $s \in I'$. However, clearly, $\mathrm{V}[H'] = \mathrm{V}[I']$, a contradiction. I will use this consequence of cone homogeneity in the future without further justification.

Furthermore, since $\mathbb{P}$ is $<\bar{\kappa}$-OD^{V}, there is a set $B = \{x \mid \psi(x,\tau)\}^{\mathrm{V}}$ such that $\mathbb{P} \in B$ and $\overline{\overline{B}} < \bar{\kappa}$. We may assume that all elements of B are cone homogeneous forcing notions that preserve cardinals up to $\bar{\kappa}$ and force that the cardinality of the set $\{x \mid \varphi(x,\check{\rho})\}$ is δ, since we can add these requirements to the definition of B if necessary.

Fixing a $\mathbb{Q} \in B$, consider the set

$$A_{\mathbb{Q}} = \{x \mid \ \Vdash_{\mathbb{Q}} \varphi(\check{x},\check{\rho})\}.$$

Since $\mathbb{Q}$ preserves cardinals up to $\bar{\kappa}$ and forces that the set $\{x \mid \varphi(x,\check{\rho})\}$ has cardinality δ, and since obviously, $\Vdash_{\mathbb{Q}} \check{A}_{\mathbb{Q}} \subseteq \{x \mid \varphi(x,\check{\rho})\}$, it follows that $A_{\mathbb{Q}}$ has cardinality at most δ in V. Moreover, we have that $a \in A_{\mathbb{P}}$, because $\Vdash_{\mathbb{P}} \varphi(\check{a},\check{\rho})$.

To conclude, the set

$$C = \bigcup_{\mathbb{Q} \in B} A_{\mathbb{Q}}$$

is OD^{V} and has cardinality at most $\overline{\overline{B}} \cdot \delta < \bar{\kappa}$, and it contains a, since $\mathbb{P} \in B$ and $a \in A_{\mathbb{P}}$. So a is $<\bar{\kappa}$-OD^{V}. Since $a \subseteq <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$, this shows that $a \in <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$, contradicting the choice of a. □

The following corollary again nicely generalizes a well-known folklore fact, that if $\mathbb{P}$ is a cone homogeneous forcing notion that is OD, and G is $\mathbb{P}$-generic over V, then $\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathsf{HOD}^{\mathrm{V}}$.

Corollary 4. *Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, $\bar{\kappa} \leq \kappa$ a cardinal such that $\mathbb{P}$ is $<\bar{\kappa}$-OD, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over* V*. Then*

$$<\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}}.$$

Proof. By Lemma 3, it follows that $<\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[\bar{G}]} \subseteq \mathrm{V}$, whenever $\bar{G}$ is $\mathbb{P}$-generic over V. Hence, Lemma 4 applies, immediately showing the claim. □

3.3. Cohen Forcing

By "Cohen forcing" I mean the standard forcing to add a subset to a regular cardinal κ by approximations of size less than κ, often denoted $\mathrm{Add}(\kappa, 1)$. In this case, we can do a little better.

Theorem 11 (ZFC)**.** *Let κ be an infinite regular cardinal such that $2^{<\kappa} < 2^{\kappa}$, and let G be generic for $\mathbb{P} = \mathrm{Add}(\kappa, 1)$. If $\bar{\kappa}$ is a cardinal less than or equal to 2^{κ} in $\mathrm{V}[G]$, then*

$$<\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}}.$$

Note: $\mathbb{P}$ is $(2^{<\kappa})^+$-c.c, so under the assumptions of the theorem, $\mathbb{P}$ preserves 2^{κ} as a cardinal, and we have that $(2^{\kappa})^{\mathrm{V}} = (2^{\kappa})^{\mathrm{V}[G]}$. The forcing collapses $2^{<\kappa}$ to κ and preserves all cardinals up to κ.

Proof. I take $\mathbb{P}$ to consist of all functions of the form $f : \alpha \longrightarrow \kappa$, where $\alpha < \kappa$, ordered by reverse inclusion. This forcing is $<\kappa$-closed, and hence does not add a new $<\kappa$-sequence of ground model elements.

(1) $<2^\kappa\text{-HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$.

Proof of (1)**.** Assume the contrary, and let $a \in <2^\kappa\text{-HOD}^{\mathrm{V}[G]} \setminus \mathrm{V}$ be $\in$-minimal. Then $a \subseteq \mathrm{V}$. Let $\dot{a} \in \mathrm{V}^{\mathbb{P}}$ be such that $a = \dot{a}^G$. Since $a \in <2^\kappa\text{-HOD}^{\mathrm{V}[G]}$, there is an $\mathrm{OD}^{\mathrm{V}[G]}$ set A such that $a \in A$ and $\overline{\overline{A}}^{\mathrm{V}[G]} < 2^\kappa$. Let $A = \{x \mid \varphi(x, \alpha)\}^{\mathrm{V}[G]}$, where α is some ordinal. Let $p \in G$ force that $\varphi(\dot{a}, \check{\alpha})$ holds and that $\dot{a}$ is contained in $\check{\mathrm{V}}$, but is not an element of $\check{\mathrm{V}}$.

Define

$$\mathcal{G} = \{H \in \mathrm{V}[G] \mid p \in H \subseteq \mathbb{P},\ H \text{ is a } \mathbb{P}\text{-generic filter over V, and } \mathrm{V}[H] = \mathrm{V}[G]\}.$$

It is easy to see that $\mathcal{G} \in \mathrm{V}[G]$.

(1.1) *If $H \in \mathcal{G}$, then $\dot{a}^H \in A$.*

Proof of (1.1)**.** Since $p \in H$, $\mathrm{V}[H] \models \varphi(\dot{a}^H, \alpha)$, so since $\mathrm{V}[H] = \mathrm{V}[G]$, $\mathrm{V}[G] \models \varphi(\dot{a}^H, \alpha)$, and this means that $\dot{a}^H \in A$. $\square$

The strategy is going to be to show that $\mathcal{G}$ is very rich: there are 2^κ filters in $\mathcal{G}$ that interpret $\dot{a}$ differently. This, together with (1.1), will produce a contradiction.

Let us now work in V for a while.

(1.2) *For every $q \leq p$, there are a set d and conditions $r_1, r_2 \leq q$ such that $r_1 \Vdash \check{d} \in \dot{a}$ and $r_2 \Vdash \check{d} \notin \dot{a}$*

Proof of (1.2)**.** A condition q constituting a counterexample would decide, for every d, the statement "$\check{d} \in \dot{a}$". It would also force that $\dot{a} \subseteq \check{V}$, since $q \leq p$. Hence, it would force that $\dot{a} \in \mathrm{V}$, while p forces that $\dot{a}$ is not in V, a contradiction. $\square$

Now, fix a condition $q \leq p$. Let $d(q)$ be such that there are $q_0, q_1 \leq q$ that decide "$(d(q))\check{} \in \dot{a}$" in different ways. Let A_q be a set of conditions such that

(a) for every $r \in A_q$, $r < q$,
(b) A_q is an antichain,
(c) A_q is maximal with (a) and (b),
(d) every $r \in A_q$ decides "$(d(q))\check{} \in \dot{a}$,"
(e) both of the sets

$$\begin{aligned} A_q^+ &= \{r \in A_q \mid r \Vdash_{\mathbb{P}} (d(q))\check{} \in \dot{a}\} \text{ and} \\ A_q^- &= \{r \in A_q \mid r \Vdash_{\mathbb{P}} (d(q))\check{} \notin \dot{a}\} \end{aligned}$$

have cardinality κ.

Let EVEN be the collection of the even ordinals below κ (including the limit ordinals), and let ODD consist of the odd ones. Let $e_q : \kappa \longrightarrow A_q$ be a bijection such that $e_q[\mathsf{EVEN}] = A_q^+$ and $e_q[\mathsf{ODD}] = A_q^-$.

Define a function $F : \mathbb{P}_{\leq p} \longrightarrow \mathbb{P}_{\leq p}$ by recursion, as follows. To begin, set $F(p) = p$. Given that $F(r) = s$ is defined, let, for $i < \kappa$,

$$F(r^\frown\langle i\rangle) = e_s(i).$$

Finally, if $\mathrm{dom}(r) = \lambda < \kappa$ is a limit ordinal and $F(r\restriction\alpha)$ has been defined for all $\mathrm{dom}(p) \leq \alpha < \lambda$, then set

$$F(r) = \bigcup_{\mathrm{dom}(p) \leq \alpha < \lambda} F(r\restriction\alpha).$$

(1.3) *$F : \mathbb{P}_{\leq p} \longrightarrow \mathbb{P}_{\leq p}$ is a dense embedding.*

Proof of (1.3)**.** Let $t \leq p$ be given. I will define a condition $s \leq p$ such that $F(s) \leq t$. This will be done by defining $s\restriction\alpha$, for $\mathrm{dom}(s) \geq \alpha \geq \mathrm{dom}(p)$. It will be maintained that either $F(s\restriction\alpha) \leq t$ (in which case we let $\alpha = \mathrm{dom}(s)$ and s is completely defined), or $t \leq F(s\restriction\alpha)$.

To begin, let $s\restriction\mathrm{dom}(p) = p$. We have $F(p) = p \geq t$.

Now suppose $\alpha > \mathrm{dom}(p)$ and for all $\beta < \alpha$, $s\restriction\beta$ has been defined.

If α is a limit ordinal, then $s\restriction\alpha$ is already determined as $\bigcup_{\beta<\alpha} s\restriction\beta$. Then, $F(s\restriction\alpha) = \bigcup_{\mathrm{dom}(p)\leq\beta<\alpha} F(s\restriction\beta)$. Inductively, we have that $F(s\restriction\beta) \geq t$, for every $\mathrm{dom}(p) \leq \beta < \alpha$, and so, $F(s\restriction\alpha) \geq t$ as well.

Now suppose $\alpha = \beta + 1$ is a successor ordinal. We have that $q = F(s\restriction\beta) \geq t$. If there is an $i < \kappa$ such that $F(s\restriction\beta^\frown i) \leq t$, then let $s(\beta) = i$, and we are done. If not, then no condition in A_q is below t, and since A_q is a maximal antichain below q, there is a (unique) $r \in A_q$ with $t \leq r$—there cannot be more than one such r, since A_q is an antichain, and there must be at least one such r, because A_q is a maximal antichain below q: some $r \in A_q$ must be compatible with t, and $r \not\leq t$, so it must be that $t \leq r$, since $\mathbb{P}$ is a tree. Letting r be the unique condition in A_q with $t \leq r$, let $e_q(i) = r$ and set $s(\beta) = i$. Then $F(s\restriction\alpha) = e_q(i) = r \geq t$.

Now clearly, this process must stop at some point, or else we end up with a function $s : \kappa \longrightarrow \kappa$ such that $s \subseteq t$, but $\mathrm{dom}(t) < \kappa$. This means that a condition s is reached with $s \leq t$, as claimed. □

Since $F \in \mathrm{V}$ is a dense embedding from $\mathbb{P}_{\leq p}$ to $\mathbb{P}_{\leq p}$, it follows that if $I \subseteq \mathbb{P}$ is a generic filter, then $F[I]$ generates a generic filter $\langle F[I]\rangle$, and $\mathrm{V}[I] = \mathrm{V}[\langle F[I]\rangle]$.

Now, for every $x \subseteq [\mathrm{dom}(p), \kappa)$, $x \in \mathrm{V}$, let $\pi_x : \mathbb{P} \longrightarrow \mathbb{P}$ be the automorphism of $\mathbb{P}$ defined as follows. First, π_x does not change the domains of conditions, i.e., for $q \in \mathbb{P}$, $\mathrm{dom}(\pi_x(q)) = \mathrm{dom}(q)$. In addition, for $i \in \mathrm{dom}(q)$, define

$$(\pi_x(q))(i) = \begin{cases} q(i) & \text{if } i \notin x, \\ q(i)+1 & \text{if } i \in x \text{ and } q(i) \text{ is even,} \\ q(i)-1 & \text{if } i \in x \text{ and } q(i) \text{ is odd.} \end{cases}$$

Note that $\pi_x \in \mathrm{V}$, and that the restriction of π_x to $\mathbb{P}_{\leq p}$ is an automorphism as well. It follows that

$$\mathrm{V}[G] = \mathrm{V}[\pi_x[G]] = \mathrm{V}[\langle F[\pi_x[G]]\rangle].$$

Thus, writing H_x for $\langle F[\pi_x[G]]\rangle$, we have that $H_x \in \mathcal{G}$, and so, by (1.1), $\dot{a}^{H_x} \in A$. However, if $x_0 \neq x_1$ are any two distinct subsets of $[\mathrm{dom}(p), \kappa)$, both in V, then $\dot{a}^{H_{x_0}} \neq \dot{a}^{H_{x_0}}$. This is because if we let $g = \bigcup G$ and $\delta = \min(x_0 \triangle x_1)$, then $d_{F(g\restriction\delta)}$ is going to belong to exactly one of $\dot{a}^{H_{x_0}}, \dot{a}^{H_{x_1}}$. Hence, we have that $\{\dot{a}^{H_x} \mid x \in \mathcal{P}(\kappa)^{\mathrm{V}}\} \subseteq A$, which means that the cardinality of A is at least 2^κ in $\mathrm{V}[G]$, a contradiction. □

Now let $\bar{\kappa} \leq 2^\kappa$ be a cardinal in $\mathrm{V}[G]$.

(2) $<\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$.

Proof of (2)**.** This is because $<\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[G]} \subseteq <2^\kappa$-$\mathsf{HOD}^{\mathrm{V}[G]}$, as $\bar{\kappa} \leq 2^\kappa$, and since $<2^\kappa$-$\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$ by (1). □

(3) $<\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$.

Proof of (3)**.** Again, assume the contrary. Let a be $\in$-minimal in $<\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[G]} \setminus <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$. Then $a \subseteq <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$. I will show that a is $<\bar{\kappa}$-OD^{V}, which then implies that $a \in <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}}$, contradicting the choice of a.

Let $a \in A = \{x \mid \varphi(x, \alpha)\}^{\mathrm{V}[G]}$, where $\overline{\overline{A}}^{\mathrm{V}[G]} < \bar{\kappa}$. Since $a \in <\bar{\kappa}$-$\mathsf{HOD}^{\mathrm{V}[G]}$, we know by (2) that $a \in \mathrm{V}$. By the weak homogeneity of $\mathbb{P}$, $\Vdash_{\mathbb{P}} \varphi(\check{a}, \check{\alpha})$. Define in V:

$$B = \{x \mid \ \Vdash_{\mathbb{P}} \varphi(\check{x}, \check{\alpha})\}.$$

Since $\mathbb{P}$ is OD, this is a definition of B using only ordinal parameters. Clearly, $a \in B \subseteq A$. So $\overline{\overline{B}}^{\mathrm{V}[G]} \leq \overline{\overline{A}}^{\mathrm{V}[G]} < \bar{\kappa}$. However, then, $\overline{\overline{B}}^{\mathrm{V}} < \bar{\kappa}$ as well (otherwise, there would be in V a surjection from B onto $\bar{\kappa}$, and this surjection would also exist in $\mathrm{V}[G]$, but in $\mathrm{V}[G]$, $\overline{\overline{B}} < \bar{\kappa}$). So $a \in {<}\bar{\kappa}\text{-}\mathsf{OD}^{\mathrm{V}}$, which yields the desired contradiction. $\square$

This completes the proof of the theorem. $\square$

Corollary 5. *Assume* $\mathrm{V} = L$, *and let* κ *be an infinite regular cardinal. If* G *is generic for* $\mathbb{P} = \mathrm{Add}(\kappa, 1)$, *then*

$$L = \mathsf{HOD}^{L[G]} = {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]} \subsetneq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]} = L[G].$$

In particular, $\Lambda_0^{L[G]} = \kappa^{++}$.

Proof. Again, view $\mathbb{P}$ as consisting of all functions of the form $f : \alpha \longrightarrow \kappa$, where $\alpha < \kappa$. $\mathbb{P}$ is κ-closed and κ^{+}-c.c., by the GCH in L, and hence preserves cardinals. Since ${<}\kappa^{+}\text{-}\mathsf{HOD}^{L}$ trivially is equal to L, we have that ${<}\kappa^{+}\text{-}\mathsf{HOD}^{L} \subseteq {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]}$. By Theorem 11, ${<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]} \subseteq {<}\kappa^{+}\text{-}\mathsf{HOD}^{L}$. However, clearly, $L \subseteq {<}\kappa^{+}\text{-}\mathsf{HOD}^{L} \subseteq L$, and we have seen that $\mathsf{HOD}^{L[G]} \subseteq {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]}$. Putting this together gives $L = \mathsf{HOD}^{L[G]} = {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]}$.

Now let us consider $\bigcup G$, a function from κ to κ, as a subset of $\kappa \times \kappa$. Then $\bigcup G \in \mathcal{P}(\kappa \times \kappa)^{L[G]}$, and $\mathcal{P}(\kappa \times \kappa)^{L[G]}$ is an $\mathsf{OD}^{L[G]}$ set of cardinality κ^{+}. Thus, $\bigcup G$ is ${<}\kappa^{++}\text{-}\mathsf{OD}^{L[G]}$, and since $\bigcup G \subseteq \kappa \times \kappa \subseteq \mathsf{OD}^{L[G]}$, it follows that $\bigcup G \in {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]}$. Therefore, since G is absolutely definable from $\bigcup G$, $G \in {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]}$. However, since ${<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]}$ is an inner model of $L[G]$, this implies directly that ${<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]} = L[G]$.

The statement about $\Lambda_0^{L[G]}$ is immediate. $\square$

This may be a good place to make a connection to ordinal indiscernibility and the Leibniz–Mycielsky axiom. For a nice summary of the history and some recent results on the subject, see [19], and for a connection to Ehrenfeucht's lemma, see [20]. Two sets a and b are called ordinal indiscernible if for every formula $\varphi(x, y)$ in the language of set theory and every ordinal α, $\varphi(a, \alpha)$ holds iff $\varphi(b, \alpha)$ holds. Since ordinal parameters are used in this definition, the concept of ordinal indiscernibility is first order definable, just as ordinal definability. The Leibniz–Mycielsky axiom says that no distinct elements a and b are ordinal indiscernible, i.e., whenever $a \neq b$, then a and b can be distinguished by a property with an ordinal parameter.

There is an obvious connection between ordinal indiscernibility and blurry ordinal definability: suppose that $I = \{a_i \mid i < \kappa\}$ is a set of κ many ordinal indiscernible elements. Then none of them can be ${<}\kappa$-OD, because if some a_i belongs to $\{x \mid \varphi(x, \alpha)\}$, then every a_i does, and so the set $\{x \mid \varphi(x, \alpha)\}$, if it contains one of the ordinal indiscernibles, must have cardinality at least κ. However, note that the sets in ${<}\kappa^{++}\text{-}\mathsf{HOD} \setminus {<}\kappa^{+}\text{-}\mathsf{HOD}$ obtained in the previous corollary are subsets of $\kappa \times \kappa$, hence essentially sets of ordinals. As such, they are clearly not ordinal indiscernibles, because given $a \neq b$, both sets of ordinals, if α is in the symmetric difference of a and b, then the formula "$\alpha \in x$" distinguishes a and b.

The process of adding Cohen subsets can be iterated transfinitely. The following theorem is just an example. It is interesting to note that iterated Cohen forcing has been put to good use in the context of ordinal definability in [21] as well.

Theorem 12. *Assume* $\mathrm{V} = L$. *Let* λ *be a cardinal, and let* $\langle\langle \mathbb{P}_i \mid i \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_i \mid i < \lambda\rangle\rangle$ *be the reverse Easton iteration whose only nontrivial stages are when* $i = \kappa$ *is an infinite regular cardinal,* $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{Q}}_\kappa = \mathrm{Add}(\kappa, 1)$. *Let* G *be* $\mathbb{P} = \mathbb{P}_\lambda$*-generic over* L. *Then:*

(a) *For regular* $\kappa < \lambda$, $L[G{\restriction}(\kappa+1)] = {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]}$, *and* $G(\kappa) \in {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G]} \setminus {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[G]}$. *So* κ^{++} *is a leap in* $L[G]$.

(b) $L = {<}\omega_1\text{-}\mathsf{HOD}^{L[G]}$, *so* ω_1 *is not a leap in* $L[G]$.

(c) *For any limit cardinal $\kappa \leq \lambda$, $G{\restriction}\kappa \in {<}\kappa^{++}\text{-HOD}^{L[G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[G]}$. So κ^{++} is a leap in $L[G]$.*

Remark 2. *Thus, in $L[G]$, if $\omega \leq \kappa \leq \lambda$, and either κ is regular and $\kappa < \lambda$, or κ is a limit cardinal, then κ^{++} is a leap. All limit cardinals up to λ are also leaps, but ω_1 is not.*

Proof. First, towards proving part (a), it is easy to see that

(1) $L[G{\restriction}(\kappa+1)] = {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$.

This is because $G{\restriction}(\kappa+1)$ is coded by a subset X of κ, and arguing in $L[G{\restriction}(\kappa+1)]$, X belongs to the OD set $\mathcal{P}(\kappa)$, which has cardinality less than κ^{++}. Thus, X is ${<}\kappa^{++}\text{-OD}^{L[G{\restriction}(\kappa+1)]}$, so since X is a set of ordinals, $X \in {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$, and because X codes $G{\restriction}(\kappa+1)$, it follows that $G{\restriction}(\kappa+1) \in {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$. Since ${<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$ is an inner model of $L[G{\restriction}(\kappa+1)]$, it follows that $L[G{\restriction}(\kappa+1)] \subseteq {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]} \subseteq L[G{\restriction}(\kappa+1)]$.

The same argument shows that

(2) $L[G{\restriction}(\kappa+1)] \subseteq {<}\kappa^{++}\text{-HOD}^{L[G]}$.

Therefore, we must prove the reverse inclusion. Of course, nothing needs to be shown if $\kappa^{+} = \lambda$; in this case, $L[G] = L[G{\restriction}(\kappa+1)]$, so we trivially have that ${<}\kappa^{++}\text{-HOD}^{L[G]} \subseteq L[G] = L[G{\restriction}(\kappa+1)]$. Therefore, let us assume that $\kappa^{+} < \lambda$.

Now, κ^{+} is the next nontrivial stage of forcing. From the point of view of $L[G{\restriction}(\kappa+1)]$, $L[G{\restriction}(\kappa^{+}+1)]$ is obtained by forcing with $\mathrm{Add}(\kappa^{+},1)$. Applying Theorem 11 in $L[G{\restriction}(\kappa+1)]$ yields:

(3) ${<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa^{+}+1)]} \subseteq {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$.

Now if $\kappa^{++} = \lambda$, then there is nothing left to show, because $L[G] = L[G{\restriction}(\kappa^{+}+1)]$, so the previous displayed formula says that ${<}\kappa^{++}\text{-HOD}^{L[G]} \subseteq {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$. Therefore, let us assume that $\kappa^{++} < \lambda$.

The passage from $L[G{\restriction}(\kappa^{+}+1)]$ to $L[G]$ is then the tail of the iteration, let us call it $\mathbb{P}_{\mathtt{tail}}$. The next nontrivial stage in $\mathbb{P}_{\mathtt{tail}}$ is κ^{++}, and the forcing is ${<}\kappa^{++}$-closed. Moreover, each iterand is weakly homogeneous in the sense of [18] (called *almost homogeneous* in [22]), which implies that $\mathbb{P}_{\mathtt{tail}}$ is weakly homogeneous; see [18] (Lemma 4). Weak homogeneity implies cone homogeneity, by [18] (Fact 1), so Corollary 4 can be applied in $L[G{\restriction}(\kappa^{+}+1)]$, showing that

(4) ${<}\kappa^{++}\text{-HOD}^{L[G]} \subseteq {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa^{+}+1)]}$.

Together with (3), this shows that

(5) ${<}\kappa^{++}\text{-HOD}^{L[G]} \subseteq {<}\kappa^{++}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$.

It follows that ${<}\kappa^{++}\text{-HOD}^{L[G]} \subseteq L[G{\restriction}(\kappa+1)]$, by (1).

The second claim in (a) is that $G(\kappa) \in {<}\kappa^{++}\text{-HOD}^{L[G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[G]}$. It is obvious that $G(\kappa) \in {<}\kappa^{++}\text{-HOD}^{L[G]}$, because $G(\kappa)$ is essentially a subset of κ, and hence an element of the OD set $\mathcal{P}(\kappa)$. To see that $G(\kappa)$ does not belong to ${<}\kappa^{+}\text{-HOD}^{L[G]}$, suppose it did. Corollary 4 would then imply that $G(\kappa) \in {<}\kappa^{+}\text{-HOD}^{L[G{\restriction}(\kappa+1)]}$, because the forcing leading from $L[G{\restriction}(\kappa+1)]$ to $L[G]$ is ${<}\kappa^{+}$-closed and cone homogeneous. However, then Theorem 11 would yield that $G(\kappa) \in {<}\kappa^{+}\text{-HOD}^{L[G{\restriction}\kappa]}$, which is absurd.

Part (b), that $L = {<}\omega_1\text{-HOD}^{L[G]}$, follows similarly, because ${<}\omega_1\text{-HOD}^{L[G{\restriction}(\omega+1)]} \subseteq L$ by Theorem 11, and because ${<}\omega_1\text{-HOD}^{L[G]} \subseteq {<}\omega_1\text{-HOD}^{L[G{\restriction}(\omega+1)]}$ since the forcing from $L[G{\restriction}(\omega+1)]$ to $L[G]$ is ${<}\omega_1$-closed and weakly homogeneous (again using Corollary 4.)

Finally, let us turn to (c). First, it is obvious that $G{\restriction}\kappa \in {<}\kappa^{++}\text{-HOD}^{L[G]}$, since it is (coded by a set) in $\mathcal{P}(\kappa)^{L[G]}$, which is $\mathrm{OD}^{L[G]}$ and has cardinality κ^{+} there.

Next, let me show that

(6) $G{\restriction}\kappa \notin {<}\kappa^{+}\text{-HOD}^{L[G{\restriction}\kappa]}$.

Assume the contrary. Let $\dot{G}$ be the canonical $\mathbb{P}_\kappa$-name for the generic filter, and let $p \in \mathbb{P}_\kappa$ force that $\varphi(\dot{G},\check{\rho})$ holds, for some formula φ and some ordinal ρ such that p also forces that $A = \{x \mid \varphi(x, \check{p})\}$ has cardinality at most $\check{\kappa}$. Arguing in the ground model L, let, for every ordinal $i < \kappa$, Π_i be the set of bijections between i and i. For regular γ, every $\pi_\gamma \in \Pi_\gamma$ induces an automorphism π^*_γ of $\mathrm{Add}(\gamma, 1)$, as follows: for $f \in \mathrm{Add}(\gamma, 1)$, f is a function from some ordinal $\alpha = \mathrm{dom}(f) < \gamma$ to γ. Then $\pi^*_\gamma(f) = \pi_\gamma \circ f$. It is easily seen that π^*_γ is an automorphism of $\mathrm{Add}(\gamma, 1)$. Let R be the set of regular cardinals less than κ. Let $\Pi = \{\langle \pi_\gamma \mid \gamma \in R \mid \text{for every } \gamma \in R,\, \pi_\gamma \in \Pi_\gamma \rangle$. Every sequence $\vec{\pi} \in \Pi$ induces an automorphism $\vec{\pi}^*$ of $\mathbb{P}_\kappa$ as follows. Given a condition $r = \langle r_i \mid i < \kappa \rangle \in \mathbb{P}_\kappa$, define $\vec{\pi}^*(r) = \langle q_i \mid i < \kappa \rangle$ by recursion on i: if i is not a regular cardinal, then $q_i = r_i$. If i is a regular cardinal, then r_i is a $\mathbb{P}_i$-name such that $r{\restriction}i \Vdash_{\mathbb{P}_i} r_i \in \mathrm{Add}(i, 1)$. Inductively, let us assume that we have already defined the automorphism $(\vec{\pi}{\restriction}i)^*$ of $\mathbb{P}_i$. This automorphism extends to a transformation of $\mathbb{P}_i$ names, which I also denote by $(\vec{\pi}{\restriction}i)^*$. We then have that $(\vec{\pi}{\restriction}i)^*(r{\restriction}i) \Vdash_{\mathbb{P}_i} (\vec{\pi}{\restriction}i)^*(r_i) \in \mathrm{Add}(i, 1)$. There is then a canonical $\mathbb{P}_i$-name τ such that $(\vec{\pi}{\restriction}i)^*(r{\restriction}i) \Vdash_{\mathbb{P}_i} \tau = \check{\pi}^*_i((\vec{\pi}{\restriction}i)^*(r_i)) \in \mathrm{Add}(i, 1)$. I define q_i to be this τ.

It is now easy to see that if $\vec{\pi}, \vec{\sigma}$ are distinct elements of Π, then $\Vdash_{\mathbb{P}_\kappa} \vec{\pi}^*(\dot{G}) \neq \vec{\sigma}^*(\dot{G})$. The point is that there are κ^+ many elements of Π which fix p. To see this, note that for $\sigma \in R$, $\mathbb{P}_\sigma$ is σ-c.c. (if σ is a successor cardinal, then $\mathbb{P}_\sigma$ is forcing equivalent to $\mathbb{P}_{\bar{\sigma}+1}$, where $\sigma = \bar{\sigma}^+$). It follows that there is an ordinal $\beta_\sigma < \sigma$ such that

$$(p{\restriction}\sigma) \Vdash_{\mathbb{P}_\sigma} \mathrm{ran}(p(\sigma)) \subseteq \check{\beta}_\sigma.$$

Let

$$\Pi^p = \{\vec{\pi} \in \Pi \mid \forall \gamma \in R \quad \pi_\gamma{\restriction}\beta_\gamma = \mathrm{id}{\restriction}\beta_\gamma\}.$$

It follows that if $\vec{\pi} \in \Pi^p$, then $\vec{\pi}^*(p) = p$. Denoting equinumerosity by the symbol $\sim$, we have that

$$\kappa \sim \bigcup_{\alpha \in R} \Pi_\alpha \sim \sum_{\alpha \in R} 2^\alpha < \prod_{\alpha \in R} 2^\alpha \sim \overline{\overline{\Pi^p}}.$$

Therefore, there are κ^+ many automorphisms of $\mathbb{P}_\kappa$ fixing p and giving distinct images of $G{\restriction}\kappa$. However, if $\vec{\pi} \in \Pi^p$ is such an isomorphism, then

$$L[G{\restriction}\kappa] = L[\vec{\pi}^*[G{\restriction}\kappa]] \models \varphi(\vec{\pi}^*[G{\restriction}\kappa], \alpha),$$

that is, $\vec{\pi}^*[G{\restriction}\kappa] \in A$. However, $\mathbb{P}_\kappa$ preserves cardinals, so in $L[G{\restriction}\kappa]$, this set has cardinality κ^+, contradicting that it has cardinality at most κ.

Familiar arguments yield (c) now, using (6). Of course, there is nothing to prove if $\kappa = \lambda$, so let us assume that $\kappa < \lambda$. We are left to show that $G{\restriction}\kappa$ does not belong to $<\kappa^+$-$\mathsf{HOD}^{L[G]}$. Suppose it did. First, note that by Corollary 4

(7) $\quad <\kappa^+$-$\mathsf{HOD}^{L[G]} \subseteq <\kappa^+$-$\mathsf{HOD}^{L[G{\restriction}\kappa^+]}$.

since the forcing from $L[G{\restriction}\kappa^+]$ to $L[G]$ is $<\kappa^+$-closed and cone homogeneous. This would yield that $G{\restriction}\kappa \in <\kappa^+$-$\mathsf{HOD}^{L[G{\restriction}\kappa^+]}$. If κ is singular, then $L[G{\restriction}\kappa^+] = L[G{\restriction}\kappa]$, so we would have $G{\restriction}\kappa \in L[G{\restriction}\kappa]$, contradicting (6). In addition, if κ is regular, then the forcing leading from $L[G{\restriction}\kappa]$ to $L[G{\restriction}(\kappa+1)] = L[G{\restriction}\kappa^+]$ is $\mathrm{Add}(\kappa, 1)$, so by Theorem 11, $<\kappa^+$-$\mathsf{HOD}^{L[G{\restriction}\kappa^+]} \subseteq <\kappa^+$-$\mathsf{HOD}^{L[G{\restriction}\kappa]}$, leading to the same contradiction. $\square$

Therefore, this theorem allows us to control $<\kappa$-HOD for double successors of regular cardinals, and to introduce leaps at double successors of any cardinal.

3.4. The Forcing by Kanovei and Lyubetsky

Note that we are not able to introduce a leap at ω_1 using Cohen forcing. This can be achieved by the Kanovei-Lyubetsky forcing [23]. In this subsection, I only report on their results and give an application.

Their forcing is used in L, where it is a finite support product of an OD forcing $\mathbb{T}$ whose conditions are a certain collection of perfect trees, ordered by inclusion. This finite support product is denoted by $\mathbb{T}^{<\omega}$. It is a ccc forcing in L, and forcing with $\mathbb{T}^{<\omega}$ over L adds a sequence $\vec{x} = \langle x_i \mid i < \omega \rangle$ of reals such that in $L[\vec{x}]$, $\{x_i \mid i < \omega\}$ is the set of $\mathbb{T}$-generic reals over L.

Proposition 11. *Let $\vec{x}$ be a $\mathbb{T}^{<\omega}$-generic sequence over L. Then*

1. $\{x_i \mid i < \omega\}$ *is* $\mathsf{OD}^{L[\vec{x}]}$,
2. $\{x_i \mid i < \omega\} \subseteq {<}\omega_1\text{-}\mathsf{HOD}^{L[\vec{x}]}$,
3. $\{x_i \mid i < \omega\} \in {<}\omega_1\text{-}\mathsf{HOD}^{L[\vec{x}]}$,
4. $\vec{x} \notin {<}\omega_1\text{-}\mathsf{HOD}^{L[\vec{x}]}$,
5. *in* $L[\vec{x}]$, $L = \mathsf{HOD} = {<}\omega\text{-}\mathsf{HOD} \subsetneq {<}\omega_1\text{-}\mathsf{HOD} \subsetneq {<}\omega_2\text{-}\mathsf{HOD} = \mathrm{V}$. *In particular,* $\Lambda_0^{L[\vec{x}]} = \omega_1$.

Proof. Point 1 is because $\{\vec{x}\}$ is the set of all reals which are $\mathbb{T}$-generic over L—since $\mathbb{T}$ is OD^L, this is a definition of $\{\vec{x}\}$ in $L[\vec{x}]$ using ordinal parameters.

Point 2 follows from 1; for any $i < \omega$, by the first point, x_i is ${<}\omega_1\text{-}\mathsf{OD}^{L[\vec{x}]}$, and of course, x_i, being a real, is a subset of $\omega \times \omega$, and hence of $\mathsf{HOD}^{L[\vec{x}]}$.

Point 3 follows from 1 and 2.

Point 4 is because if $\vec{x} \in A$, and A is $\mathsf{OD}^{L[\vec{x}]}$, then A must be uncountable in $L[\vec{x}]$. Specifically, let $A = \{z \mid \varphi(z, \alpha)\}$, and let Γ be the canonical name for the generic sequence. Let $p \in \mathbb{T}^{<\omega}$ force that $\Gamma \in A$, i.e., that $\varphi(\Gamma, \check{\alpha})$ holds. Then for any automorphism $\pi \in L$ which perturbs the coordinates of conditions in $\mathbb{T}^{<\omega}$ that fixes the coordinates in the support of p, we have that $p \in \pi(\vec{x})$ and $L[\vec{x}] = L[\pi(\vec{x})] \models \varphi(\Gamma, \check{\alpha})$, so that $\pi(\vec{x}) \in A$. There are $2^\omega = \omega_1$ such automorphisms, and any two distinct automorphisms give different generics.

Point 5 follows because $\mathbb{T}^{<\omega}$ is weakly homogeneous and OD^L, so that $L \subseteq \mathsf{HOD}^{L[\vec{x}]} \subseteq L$. The other relationships follow immediately from the previous points. Note that $\vec{x} \in {<}\omega_2\text{-}\mathsf{HOD}^{L[\vec{x}]}$ because $\vec{x} \in ({}^\omega\mathbb{R})^{L[\vec{x}]}$, and this set has cardinality $\omega_1 < \omega_2$ in $L[\vec{x}]$. $\square$

This proposition shows that Proposition 4 is optimal in the case $\kappa = \omega_1$: in the model constructed, there are subsets of ω in ${<}\omega_1\text{-}\mathsf{HOD}$ that are fresh over HOD. Proposition 4 shows that there can be no fresh subset of ω_1 (or any κ with $\mathrm{cf}(\kappa) \geq \omega_1$) over HOD in ${<}\omega_1\text{-}\mathsf{HOD}$.

Note that if $\vec{x} = \langle x_i \mid i < \omega \rangle$ is $\mathbb{T}^{<\omega}$-generic, then $L[\vec{x}] = L(\mathbb{R})^{L[\vec{x}]}$. If we proceed by performing the iteration described in the previous subsection, then we end up with a model in which ω_1 is a leap, as well as all cardinals of the form κ^{++}, for $\kappa \geq \omega$.

Corollary 6. *Assume* $\mathrm{V} = L$. *Let $\vec{x}$ be a $\mathbb{T}^{<\omega}$-generic sequence over L. Working in $L[\vec{x}]$, let λ be a cardinal, and let $\langle \langle \mathbb{P}_i \mid i \leq \lambda \rangle, \langle \dot{\mathbb{Q}}_i \mid i < \lambda \rangle \rangle$ be the reverse Easton iteration whose only nontrivial stages are when $i = \kappa$ is an* uncountable *regular cardinal, $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{Q}}_\kappa = \mathrm{Add}(\kappa, 1)$. Let G be $\mathbb{P} = \mathbb{P}_\lambda$-generic over $L[\vec{x}]$. Then:*

(a) *For uncountable regular $\kappa < \lambda$, $L[\vec{x}][G \restriction (\kappa+1)] = {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$, and $G(\kappa) \in {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[\vec{x}][G]} \setminus {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$. So κ^{++} is a leap in $L[\vec{x}][G]$.*
(b) $L[\vec{x}] = {<}\omega_2\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$.
(c) *For any limit cardinal $\kappa \leq \lambda$, $G \restriction \kappa \in {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[\vec{x}][G]} \setminus {<}\kappa^{+}\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$. So κ^{++} is a leap in $L[\vec{x}][G]$.*
(d) $\{x_i \mid i < \omega\} \in {<}\omega_1\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$, *but* $\vec{x} \notin {<}\omega_1\text{-}\mathsf{HOD}^{L[\vec{x}][G]}$.
(e) ${<}\omega\text{-}\mathsf{HOD}^{L[\vec{x}][G]} = L$.

Remark 3. *Thus, in $L[\vec{x}][G]$, if $\omega_1 \leq \kappa \leq \lambda$, and either κ is regular and $\kappa < \lambda$, or κ is a limit cardinal, then κ^{++} is a leap. All limit cardinals up to λ are also leaps, and so are ω_1 and ω_2.*

For example, if $\lambda = \aleph_\omega$, then in $L[\vec{x}][G]$, all uncountable cardinals up to and including $\aleph_\omega$ are leaps, as is $\aleph_{\omega+2}$.

Proof (sketch). To prove (a) and (c), the proofs of the corresponding statements in Theorem 12 relativize, because $\vec{x}$ is essentially a real number and hence it is $<\omega_2$-$\mathsf{HOD}^{L[\vec{x}][G\upharpoonright\kappa]}$, for any cardinal κ, so Proposition 10 applies.

Turning to (b), the inclusion from left to right follows from the reason just given. For the converse direction, the forcing leading from $L[\vec{x}][G\upharpoonright(\omega_1+1)]$ to $L[\vec{x}][G]$ is $<\omega_2$-closed and weakly homogeneous, and the forcing leading from $L[\vec{x}]$ to $L[\vec{x}][G\upharpoonright(\omega_1+1)]$ is $\mathrm{Add}(\omega_1, 1)$, so a combination of Corollary 4 and Theorem 11 shows that $<\omega_2$-$\mathsf{HOD}^{L[\vec{x}][G]} \subseteq$ $<\omega_2$-$\mathsf{HOD}^{L[\vec{x}]} \subseteq L[\vec{x}]$.

For the first part of (d), we know from Proposition 11.3 that $\{x_i \mid i < \omega\}$ is $<\omega_1$-$\mathsf{HOD}^{L[\vec{x}]}$. The reason is that $\{x_i \mid i < \omega\}$ is the set of all $\mathbb{T}$-generic reals over L. Since $\mathbb{P}$ is countably closed, and hence does not add reals, the same definition of $\{x_i \mid i < \omega\}$ works in $L[\vec{x}][G]$. For the second part, by now familiar arguments show that if we assume the contrary, that $\vec{x} \in$ $<\omega_1$-$\mathsf{HOD}^{L[\vec{x}][G]}$, then $\vec{x} \in$ $<\omega_1$-$\mathsf{HOD}^{L[\vec{x}]}$ by Corollary 4 since $\mathbb{P}$ is countably closed and weakly homogeneous in $L[\vec{x}]$. However, this contradicts Proposition 11.4.

Part (e) follows because the combined forcing $\mathbb{T}^{<\omega} * \dot{\mathbb{P}}$ is cone homogeneous; use Lemma 3 with $\kappa = \omega$. $\square$

3.5. Forcing with Homogeneous Souslin Trees

The forcings we have investigated so far were not able to introduce a leap at successors of uncountable limit cardinals. In this subsection, we will achieve this. The forcing notions in question will be Souslin trees with certain homogeneity properties. The following definition uses terminology from [24].

Definition 9. *Let κ be a regular cardinal. A streamlined (or sequential) κ-tree is a set T of functions p such that the domain of p is an ordinal less than κ and the range of p is contained in κ, closed under restrictions to ordinals, ordered by inclusion, such that for every $\alpha < \kappa$, the α-th level of T, $T(\alpha) = \{p \in T \mid \mathrm{dom}(p) = \alpha\}$ has cardinality less than κ and is nonempty. If $p, q \in T$, then $p \perp q$ (p, q are incompatible) iff neither $p \subseteq q$ nor $q \subseteq p$. An antichain in T is a set $A \subseteq T$ of pairwise incompatible elements. T is a κ-Souslin tree if it has no antichain of cardinality κ. It is* coherent *if whenever $p, q \in T$, then the set $d(p, q) = \{i \in \mathrm{dom}(p) \cap \mathrm{dom}(q) \mid p(i) \neq q(i)\}$ is finite. It is* uniformly homogeneous *if whenever $p, q \in T$ and $\mathrm{dom}(p) \leq \mathrm{dom}(q)$, then the function $p * q = p \cup (q\upharpoonright(\mathrm{dom}(q) \setminus \mathrm{dom}(p))) \in T$. It is* uniformly coherent *if it is coherent and uniformly homogeneous.*

Remark 4. *Let T be a streamlined uniformly homogeneous κ-tree. Let $s, t \in T$ with $\mathrm{dom}(s) = \mathrm{dom}(t)$. Let $\pi_{s,t}$ be the following function defined on $T_{\geq s} = \{u \in T \mid s \subseteq u\}$:*

$$\pi_{s,t}(u) = t * u.$$

Then $\pi_{s,t} : T_{\geq s} \longrightarrow T_{\geq t}$ is an isomorphism. It follows that T is cone homogeneous, as a forcing notion.

Proof. Clearly, $\pi_{s,t} : T_{\geq s} \longrightarrow T_{\geq t}$, and it is order preserving. However, also, $\pi_{t,s} : T_{\geq t} \longrightarrow T_{\geq s}$ is order preserving, and it is the inverse function of $\pi_{s,t}$. It follows that $\pi_{s,t}$ is an isomorphism. $\square$

Remark 5. *If T is a κ-Souslin tree, where κ is an uncountable regular cardinal, and $\mathbb{P}$ is a forcing notion of cardinality less than κ, then whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over* V*, T remains a κ-Souslin tree in* V$[G]$*.*

Proof. If not, then let $\dot{f}$ be a $\mathbb{P}$-name and $p \in G$ a condition forcing that $\dot{f} : \check{\kappa} \longrightarrow \check{T}$ is an injective enumeration of an antichain in T. For every $\xi < \kappa$, there are a condition $q_\xi \in G$ and a $\check{t}_\xi$ such that $q_\xi \leq p$ and $q_\xi \Vdash_{\mathbb{P}} \dot{f}(\check{\xi}) = \check{t}_\xi$. Since κ is still a regular cardinal in V$[G]$, and since the cardinality of $\mathbb{P}$ is less than κ, there must be a $q \in \mathbb{P}$ such that for κ many $\xi < \kappa$, $q_\xi = q$. However, then, the set

$$A = \{t \mid \exists \xi < \kappa \quad q \Vdash_{\mathbb{P}} \dot{f}(\check{\xi}) = \check{t}\}$$

is an antichain in T of cardinality κ that exists in V, contradicting that T is Souslin there. $\square$

When forcing with a tree, the nodes that are higher up in the tree are stronger. In other words, I follow the Israel convention when forcing with trees.

Theorem 13. *Let κ be a regular uncountable cardinal, and let T be a streamlined, uniformly coherent κ-Souslin tree. Let $G \subseteq T$ be T-generic over* V. *Then:*

(1) $<\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$.

(2) *If T is $<\kappa^+\text{-}\mathsf{HOD}^{V[G]}$, then $G \in <\kappa^+\text{-}\mathsf{HOD}^{\mathrm{V}[G]}$.*

(3) *If $\bar{\kappa} \leq \kappa$ is a cardinal and T is $<\bar{\kappa}\text{-}\mathsf{OD}$, then $<\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-}\mathsf{HOD}^{\mathrm{V}}$.*

Proof. It is well-known that T is $<\kappa$-distributive, i.e., that forcing with T does not add a new sequence of ground model elements of length less than κ. The generic filter G itself can be identified with $b = \bigcup G$, a new such sequence of length κ, a function $b : \kappa \longrightarrow \kappa$ such that for all $\alpha < \kappa$, $b{\upharpoonright}\alpha \in T$, i.e., a cofinal branch of T. Since T is Souslin, it is also κ-c.c. It follows that forcing with T preserves all cardinals and cofinalities.

To show (1), that $<\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$, let us assume the contrary, and let a be $\in$-minimal in $<\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \setminus \mathrm{V}$. So $a \subseteq \mathrm{V}$, and there is an $\mathsf{OD}^{\mathrm{V}[G]}$ set $A = \{x \mid \varphi(x, \rho)\}^{\mathrm{V}[G]}$ such that $a \in A$ and $\gamma = \overline{\overline{A}}^{\mathrm{V}[G]} < \kappa$. Let $a = \dot{a}^G$, and let $p \in G$ force all the above: that $\dot{a}$ is contained in V but not an element of V, that it belongs to A, and that A has cardinality γ. I will carry out the following construction in $\mathrm{V}[G]$. For every $q \geq p$ with $q \in G$, there is a $d_q \in \mathrm{V}$ and a $q^- \geq q$ such that

$$d_q \in a \iff (q^- \Vdash_T (\check{d}_q \notin \dot{a}))^{\mathrm{V}}.$$

This is because otherwise, q would decide membership to $\dot{a}$ for every element of V, and since q also forces that $\dot{a} \subseteq \mathrm{V}$, q would force that $\dot{a} \in \mathrm{V}$. Using this, one can now define a sequence $\langle\langle \alpha_i, q_i^-, d_i\rangle \mid i < \kappa\rangle$ such that $\langle \alpha_i \mid i < \kappa\rangle$ is a strictly increasing sequence of ordinals less than κ, $q_i^- \in T$, $q_i^- \geq b{\upharpoonright}\alpha_i$, and $b{\upharpoonright}\alpha_{i+1}$, q_i^- decide "$\check{d}_i \in \dot{a}$" in different ways. In the limit step, one uses the fact that κ remains regular in $\mathrm{V}[G]$.

For $i < \kappa$, let $b_i = q_i^- * b = \bigcup_{\mathrm{dom}(q_i^-) < \alpha < \kappa} q_i^- * b{\upharpoonright}\alpha$. By uniform coherence of T, b_i is a cofinal branch of T, so, since T is Souslin, $G_i = \{b_i{\upharpoonright}\alpha \mid \alpha < \kappa\}$ is T-generic over V. Moreover, if $i < j < \kappa$, then $\dot{a}^{G_i} \neq \dot{a}^{G_j}$. This is because $d_i \in \dot{a}^{G_i} \not\Longleftrightarrow d_i \in a \iff d_i \in \dot{a}^{G_j}$, as $q_j \geq b{\upharpoonright}\alpha_{i+1}$.

However, obviously, $\mathrm{V}[G_i] = \mathrm{V}[G]$, since b_i and b differ only on an initial segment. Hence, we have

$$\mathrm{V}[G] = V[G_i] \models \varphi(\dot{a}^{G_i}, \rho)$$

so that $\{\dot{a}^{G_i} \mid i < \kappa\} \subseteq A$. So A has cardinality κ in $\mathrm{V}[G]$, a contradiction.

Towards claim (2), the crucial point is that

$$[T]^{\mathrm{V}[G]} = \{b' \in \mathrm{V}[G] \mid b' \text{ is a cofinal branch through } T\}$$

has cardinality κ in $\mathrm{V}[G]$. First, note that if $b' \in [T]^{\mathrm{V}[G]}$, then $d(b, b') = \{i < \kappa \mid b(i) \neq b'(i)\}$ is finite. For suppose it were not. Let θ be the supremum of the first ω elements of $d(b, b')$. Clearly, θ cannot be less than κ, because if it were, then we would have that $b{\upharpoonright}\theta, b'{\upharpoonright}\theta \in T$, so that by coherency of T, $d(b{\upharpoonright}\theta, b'{\upharpoonright}\theta) < \omega$, a contradiction. Therefore, then it would have to be that $\theta = \kappa$, but that would mean that the cofinality of κ is collapsed to ω in $\mathrm{V}[G]$, which contradicts that T preserves cofinalities. Thus, every $b' \in [T]^{\mathrm{V}[G]}$ can be obtained from b by modifying it in finitely many places. Clearly, there are only up to κ many ways to do this. Therefore, the cardinality of $[T]^{\mathrm{V}[G]}$ is at most κ in $\mathrm{V}[G]$. However, it is also at least κ, because for every $q \in T$, $q * b \in [T]^{V[G]}$. A by now familiar argument shows that if T is $<\kappa^+$-$\mathrm{OD}^{V[G]}$, then b (and hence G) is $<\kappa^+$-$\mathrm{HOD}^{V[G]}$. Specifically, if A is $\mathrm{OD}^{V[G]}$, has cardinality less than κ^+ there, and contains T, then the set

$$\bigcup\{[S] \mid S \in A \text{ and } S \text{ is a } \kappa\text{-tree with } \overline{\overline{[S]}} = \kappa\}$$

is $\mathrm{OD}^{V[G]}$, has cardinality κ, and contains b. Every element of b is a pair of ordinals less than κ, so that clearly, $b \subseteq <\kappa^+$-$\mathrm{HOD}^{V[G]}$. Hence, $b \in <\kappa^+$-$\mathrm{HOD}^{V[G]}$. In addition, $G = \{b{\restriction}\alpha \mid \alpha < \kappa\}$ is definable in and hence a member of the inner model $<\kappa^+$-$\mathrm{HOD}^{V[G]}$.

Turning to item (3), let $\bar{\kappa}$ be as stated, and assume that T is $<\bar{\kappa}$-OD. By item (1), it follows that whenever G is T-generic over V, then $<\bar{\kappa}$-$\mathrm{HOD}^{V[G]} \subseteq V$. Hence, the assumptions of Lemma 4 are satisfied, and that lemma gives the result. □

It follows from Brodsky and Rinot [24] that in L, for every regular cardinal κ that is not weakly compact, there is a streamlined, uniformly coherent κ-Souslin tree. Since the relevant reference points are somewhat scattered within [24], here are the details. [24] (Thm. 6.1.(21)) says that if $V = L$ and κ is an uncountable regular cardinal that is not weakly compact, then the "parametric proxy principle" $P^\bullet(\kappa, 2, \sqsubseteq, \kappa, \mathcal{S}, 2)$ holds for $\mathcal{S} = \{E^\kappa_{\geq\chi} \mid \chi \in \mathrm{Reg}(\kappa) \text{ and } \kappa \text{ is } ({<}\chi)\text{-closed}\}$. Here, κ is $({<}\chi)$-closed iff for every $\lambda < \kappa$, $\lambda^{<\chi} < \kappa$, and the principle, with a subscript called ξ, is defined in [24] (Def. 5.9) . According to [24] (Convention 4.3), if the subscript is omitted, then it is understood to be κ. It is obvious from the definition that the principle becomes weaker if one shrinks $\mathcal{S}$ or expands any element of $\mathcal{S}$; see also [24] (Remark 4.11). Since κ trivially is $({<}\omega)$-closed, it follows that in L, $P^\bullet(\kappa, 2, \sqsubseteq, \kappa, \{\kappa\}, 2)$ holds, assuming that κ is an uncountable, regular cardinal that is not weakly compact in L. Now [24] (Thm. 6.35) concludes the existence of a uniformly coherent κ-Souslin tree (which has some additional properties that are not needed here) from this assumption. Hence, we obtain the following corollary.

Corollary 7. *Assume $V = L$, and let λ be an uncountable regular cardinal that is not weakly compact. Then there is a forcing extension $L[G]$ of L such that $\Lambda_0^{L[G]} = \lambda^+$.*

Proof. We can let T be the $<_L$-least streamlined, uniformly coherent λ-Souslin tree and let G be T-generic over L. By Theorem 13, $L[G]$ is as wished. □

In particular:

Corollary 8. *If ZFC is consistent with the existence of an inaccessible cardinal, then it is consistent that Λ_0 is the successor of an inaccessible cardinal.*

Note that Corollary 7 shows that Proposition 4 is optimal in the sense that in the model constructed, $<\lambda^+$-HOD has a subset of λ that is fresh over HOD. The proposition says that it cannot have such a subset of a cardinal of cofinality greater than or equal to λ^+.

It is also possible to combine forcing with a uniformly coherent Souslin tree with an iteration of Cohen forcings as in the proof of the following theorem.

Theorem 14. *If ZFC is consistent with the existence of an inaccessible cardinal, then ZFC is consistent with the existence of a regular (in fact inaccessible) limit leap whose successor cardinal is also a leap.*

Proof. Under the assumption of the theorem, ZFC is consistent with $V = L$ together with the existence of an inaccessible cardinal. Therefore, let us assume $V = L$, and let λ be inaccessible, but not weakly compact. Let T be the $<_L$-least streamlined uniformly coherent Souslin tree on λ, and let b be a T-generic branch over L. We know from Theorem 13 that in $L[b]$, b is one of altogether λ many cofinal branches of T.

In $L[b]$, let $\mathbb{P}$ be an iteration of Cohen forcings $\mathrm{Add}(\rho, 1)$, for regular $\rho < \lambda$, as described in Lemma 12.

In the terminology of [7] (Def. 12), the forcing $\mathbb{P}$ has a closure point at every regular cardinal $\delta < \lambda$, because $\mathbb{P}$ factors as $\mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$, where $\overline{\overline{\mathbb{P}_{\delta+1}}} = \delta$ and $\Vdash_{\mathbb{P}_{\delta+1}}$ "$\dot{\mathbb{Q}}$ is strategically $\leq\delta$-closed. As a result, by [7] (Lemma 13), letting G be $\mathbb{P}$-generic over $L[b]$, it follows that $L[b]$ satisfies the δ^+-approximation and -cover properties in $L[b][G]$. In particular, by the argument of the proof of Proposition 4, $L[b][G]$ has no fresh sequence of length λ over $L[b]$. This means that $\mathbb{P}$ cannot add a cofinal branch to T over $L[b]$, so that in $L[b][G]$, b is still one of λ many branches through T. However, by the special way T was chosen, it is $\mathsf{OD}^{L[b][G]}$. It follows that:

(A) $\quad b \in {<}\lambda^+\text{-}\mathsf{HOD}^{L[b][G]}$.

Since T adds no bounded subsets to λ, the iteration $\mathbb{P}$ of length λ is the same as defined in L or in $L[b]$ (the point is that bounded support is used in forming $\mathbb{P}_\lambda$), and $L[b][G] = L[G][b]$.

Imitating the argument of the proof of Theorem 12, let us check that

(B) $\quad$ *for every regular cardinal $\kappa < \lambda$,*

$$L[G\restriction(\kappa+1)] = {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G]}.$$

First, we have

(1) $\quad L[G\restriction(\kappa+1)] = {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa+1)]}$.

The inclusion from left to right holds because $G\restriction(\kappa+1)$ is (coded by) a subset X of κ, and arguing in $L[b][G\restriction(\kappa+1)]$, X belongs to the OD set $\mathcal{P}(\kappa)$, which has cardinality less than κ^{++}.

For the converse, we can change the order of forcing and write

$${<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa+1)]} = {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G\restriction(\kappa+1)][b]}.$$

Note that by Remark 5, T is still a λ-Souslin in $L[G\restriction(\kappa+1)]$, and it obviously also is uniformly coherent there. Moreover, T is ordinal definable in $L[G\restriction(\kappa+1)]$. Hence, Theorem 13 applies, showing that

$${<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G\restriction(\kappa+1)][b]} \subseteq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[G\restriction(\kappa+1)]} = L[G\restriction(\kappa+1)].$$

This shows the inclusion from right to left.

The same argument shows that

(2) $\quad L[G\restriction(\kappa+1)] \subseteq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G]}$.

Therefore, we must prove the reverse inclusion. We have $\kappa^+ < \lambda$, since λ is inaccessible.

The next nontrivial stage of forcing is κ^+. From the point of view of $L[b][G\restriction(\kappa+1)]$, $L[b][G\restriction(\kappa^+ + 1)]$ is obtained by forcing with $\mathrm{Add}(\kappa^+, 1)$. Applying Theorem 11 in $L[b][G\restriction(\kappa+1)]$ yields:

(3) $\quad {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa^+ + 1)]} \subseteq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa+1)]}$.

Furthermore, we have $\kappa^{++} < \lambda$, so the passage from $L[b][G\restriction(\kappa^+ + 1)]$ to $L[G]$, which we again call $\mathbb{P}_{\mathtt{tail}}$, has as its next nontrivial stage κ^{++}, and the forcing used is ${<}\kappa^{++}$-closed, as well as cone homogeneous, as we argued before. Hence, Corollary 4 can be applied in $L[G\restriction(\kappa^+ + 1)]$, showing that

(4) $\quad {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G]} \subseteq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa^+ + 1)]}$.

Together with (3), this shows that

(5) $\quad {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G]} \subseteq {<}\kappa^{++}\text{-}\mathsf{HOD}^{L[b][G\restriction(\kappa+1)]}$.

However, again,

$$<\kappa^{++}\text{-HOD}^{L[b][G\restriction(\kappa+1)]} = <\kappa^{++}\text{-HOD}^{L[G\restriction(\kappa+1)][b]} \subseteq L[G\restriction(\kappa+1)].$$

It follows that $<\kappa^{++}$-HOD$^{L[b][G]} \subseteq L[G\restriction(\kappa+1)]$, by (5). This proves (B).

Clearly, (B) shows that λ is a limit leap. Let us now say something about $<\lambda$-HOD$^{L[b][G]}$:

(C) $<\lambda$-HOD$^{L[b][G]} \subseteq L[G]$.

To see this, let us think of $L[b][G]$ as $L[G][b]$ instead, obtained by forcing with T over $L[G]$. At first sight, it may seem as though the claim follows immediately from Theorem 13 (1), but it is not clear that T is a Souslin tree in $L[G]$ - $\mathbb{P}_\lambda$ is not small forcing regarding λ! However, fortunately, the proof of the theorem still goes through—all it uses is that forcing with T preserves λ as a regular cardinal, that the maps $\pi_{s,t}$ of Remark 4 are isomorphisms, and that T is uniform. There is a small subtlety in the proof: we argued that (using the notation from the proof) $q_i^- * b$ is T-generic because it is a cofinal branch of T, and every cofinal branch in a Souslin tree is generic. Since T may not be Souslin in $L[G]$, one cannot argue that way now, but it is sufficient to know that, letting $s = b\restriction\mathrm{dom}(q_i^-)$, and letting $b_i = \pi_{s,q_i^-}[\{b\restriction\xi \mid \xi < \lambda\}]$, we have that π_{s,q_i^-} is an isomorphism between $T_{\geq s}$ and $T_{\geq q_i^-}$, so that b_i is generic for T. With these modifications, the proof goes through to prove the current claim.

Therefore, by (B) and (C), $b \in <\lambda^+$-HOD$^{L[b][G]} \setminus <\lambda$-HOD$^{L[b][G]}$, showing that λ^+ is a leap in $L[b][G]$, and λ is a limit leap, as wished. □

3.6. Příkrý Forcing

In the previous subsections, we saw how to produce models where $\Lambda_0 = \kappa^+$, for a regular limit cardinal κ, or where such a κ is a limit leap whose successor cardinal is also a leap. We will now see how this can be arranged for a singular limit cardinal. Recall Lemma 2.(2), which showed that Λ_0 must be a successor cardinal.

To this end, I will use Příkrý forcing [25]: given a measurable cardinal κ and a normal ultrafilter U on κ, the forcing notion consists of pairs $\langle s,T\rangle$, where s is a finite subset of κ, $T \in U$, and $s \subseteq \min(T)$. The ordering is defined by declaring that $\langle s_1,T_1\rangle \leq \langle s_0,T_0\rangle$ iff $T_1 \subseteq T_0$, $s_0 \subseteq s_1$ and $s_1 \setminus s_0 \subseteq T_0$. In this situation, $\langle s_1,T_1\rangle$ is called a *direct extension* of $\langle s_0,T_0\rangle$, denoted $\langle s_1,T_1\rangle \leq^* \langle s_0,T_0\rangle$, if $s_1 = s_0$. It is well-known that forcing with this poset adds a set C (called a Příkrý sequence), cofinal in κ, of order type ω, and interdefinable with the generic filter (using U as a parameter).

Theorem 15. *Let κ be a measurable cardinal, let U be a normal ultrafilter on κ, let $\mathbb{P}$ be the Příkrý forcing for U, and let G be $\mathbb{P}$-generic over* V. *Then*

(a) $<\kappa$-HOD$^{\mathrm{V}[G]} \subseteq \mathrm{V}$.

(b) *If U is $<\kappa^+$-OD$^{V[G]}$, then, letting C be the Příkrý sequence corresponding to G, $C \in <\kappa^+$-HOD$^{\mathrm{V}[G]}$.*

(c) *If $\bar{\kappa} \leq \kappa$ is a cardinal and U is $<\bar{\kappa}$-OD, then $<\bar{\kappa}$-HOD$^{\mathrm{V}[G]} \subseteq <\bar{\kappa}$-HOD$^{\mathrm{V}}$.*

Proof. Let C be the Příkrý sequence corresponding to G, i.e.,

$$C = \bigcup\{s \mid \exists T \quad \langle s,T\rangle \in G\}.$$

It is well-known that $\mathrm{V}[G] = \mathrm{V}[C]$. I will soon need the fact, due to Mathias [26], that a subset D of κ with order type ω is the Příkrý sequence corresponding to some $\mathbb{P}$-generic H iff for every $B \in U$, $D \setminus B$ is finite. I will also need a maximality fact, a proof of which can be found in [27] (but in a much broader context), saying that if D is the Příkrý sequence corresponding to some H which is $\mathbb{P}$-generic over V, and this H is in $\mathrm{V}[C]$ (equivalently, $D \in \mathrm{V}[C]$), then $D \setminus C$ is finite.

Let $\mathcal{G} = \mathcal{G}^{\mathrm{V}[C]}$ be the collection of Příkrý sequences D (again viewed as subsets of κ of order type ω) such that $\mathrm{V}[C] = \mathrm{V}[D]$.

(1) $\mathcal{G} = \{D \subseteq \kappa \mid D \triangle C \textit{ is finite}\}$.

Proof of (1). For the inclusion from right to left, if $D \triangle C$ is finite, then obviously, for any $X \in U$, $D \setminus X$ is finite, as $C \setminus X$ is finite, and $\mathrm{otp}(D) = \omega$. Thus, by Mathias' criterion for Příkrý genericity, D is a Příkrý sequence over V. Moreover, $\mathrm{V}[C] = \mathrm{V}[D]$.

For the other direction, suppose $D \in \mathcal{G}$. Since D is a Příkrý sequence over V and $D \in \mathrm{V}[C]$, it follows by the maximality of C that $D \setminus C$ is finite. However, since $C \in \mathrm{V}[D]$, it follows by the maximality of D that $C \setminus D$ is finite. Thus, $C \triangle D$ is finite. □

To prove (a), I must show that $<\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C]} \subseteq \mathrm{V}$.

Assume the contrary, and let $a \in <\!\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C]} \setminus \mathrm{V}$ be $\in$-minimal. So $a \subseteq \mathrm{V}$. Let A be an $\mathsf{OD}^{\mathrm{V}[C]}$ set such that $a \in A$ and $\overline{\overline{A}}^{\mathrm{V}[C]} = \bar{\kappa} < \kappa$. Let $A = \{x \mid \varphi(x, \alpha)\}^{\mathrm{V}[C]}$, for some ordinal α. Let $a = \dot{a}^G$, and let $p \in G$ be a condition which forces over V that $\dot{a}$ is contained in V, but not an element of V, and that $\varphi(\dot{a}, \check{\alpha})$ holds. Let us also assume that p forces that $\dot{a}$ is a subset of V_θ, for some sufficiently large ordinal θ, and moreover, that fixing a well-order R of V_θ, the order type of $\dot{a}$ under that well-order is the ordinal Ω. Let us write, for any subset b of V_θ, $b(\xi)$ for the ξ-th element of b in its enumeration according to R.

I will use the following notation: if D is a Příkrý sequence, then G_D denotes the $\mathbb{P}$-generic filter, i.e., the set of all conditions of the form $\langle s, T\rangle$ such that s is an initial segment of D and $D \setminus s \subseteq T$. I will also use the following fact.

(2) *For every condition $q \in \mathbb{P}$, there is a $D \in \mathcal{G}$ such that $q \in G_D$.*

Proof of (2)**.** Let $q = \langle s, T\rangle$. Since C is a Příkrý sequence, $e = C \setminus T$ is finite. Let $\alpha = \sup(\{\xi + 1 \mid \xi \in s\})$. Then set $D = s \cup (C \setminus (e \cup \alpha))$. Since $C \cap \alpha$ is finite, it follows that $D \triangle C$ is finite, so that $D \in \mathcal{G}$, by (1). In addition, since s is an initial segment of D and $D \setminus s \subseteq T$, it follows that $\langle s, T\rangle \in G_D$, as wished. □

One more basic fact on Příkrý forcing will prove useful.

(3) *Let $\dot{t}$ be a $\mathbb{P}$-name, let $q \in \mathbb{P}$ be a condition, and let Z be a set of size less than κ such that $q \Vdash \dot{t} \in \check{Z}$. Then there is a (unique) $z \in Z$ such that for some direct extension q' of q, $q' \Vdash \dot{t} = \check{z}$.*

Proof of (3). For each $z \in Z$, consider the statement $\varphi_z =$"$\dot{t} = \check{z}$". By the Příkrý property, for each such z, there is a direct extension $q_z \leq^* q$ deciding φ_z. Since all these conditions have the same first coordinate, and there are fewer than κ of them, they have a common extension, $q' \leq q_z$ for all $z \in Z$. It follows that there is a $z \in Z$ such that $q_z \Vdash \dot{t} = \check{z}$, or else q' would force that $\dot{t} \notin \check{Z}$, a contradiction. In addition, this z is unique, or else, q would both force that $\dot{t} = \check{z}$ and $\dot{t} = \check{z}'$, for distinct $z, z' \in Z$. □

(4) *If $q \leq p$ is a condition in $\mathbb{P}$ and $\xi < \Omega$, then the collection*

$$P_q = \{z \mid \exists r \leq q \; r \Vdash \dot{a}(\check{\xi}) = \check{z}\}$$

can have no more than $\bar{\kappa}$ members.

Proof of (4). Assume the contrary. For notational simplicity, let us assume $q = p$, for if not, then we may work with a different Příkrý sequence C' instead of C whose generic contains q.

Let $\langle P_i \mid i < \bar{\kappa}^+\rangle$ be a partition of P_q, each P_i nonempty. Let A be a maximal antichain below q such that for every $r \in A$, there is an $i < \bar{\kappa}^+$ with $r \Vdash \dot{a}(\check{\xi}) \in \check{P}_i$. Let $D = \{s \mid \exists r \in A \quad s \leq r\}$ be the dense open subset below q that is generated by A.

Now, for every $i < \bar{\kappa}^+$, there is an $r_i \in A$ forcing that $\dot{a}(\check{\xi}) \in P_i$. By (2), there is a $D_i \in \mathcal{G}$ such that $r_i \in G_{D_i}$.

However, then, for $i < j < \bar{\kappa}^+$, we have that $\dot{a}^{G_{D_i}} \neq \dot{a}^{G_{D_j}}$. However, each G_{D_i} contains the condition p, so $\varphi(\dot{a}^{G_{D_i}}, \alpha)$ holds in $\mathrm{V}[G_{D_i}] = \mathrm{V}[C]$. However, the set $\{z \mid \mathrm{V}[C] \models \varphi(z, \alpha)\}$ has size $\bar{\kappa}$ in $\mathrm{V}[C]$, and $(\bar{\kappa}^+)^{\mathrm{V}}$ is still a cardinal in $\mathrm{V}[C]$, since $\mathbb{P}$ preserves cardinals. This is a contradiction. $\square$

In V, we will now recursively construct sequences $\langle p_i \mid i \leq \bar{\kappa}^+\rangle$, $\langle q_{i+1} \mid i < \bar{\kappa}^+\rangle$ and $\langle \xi_{i+1} \mid i < \bar{\kappa}^+\rangle$ with the following properties:

- for $i < j \leq \bar{\kappa}^+$, $p_j \leq^* p_i$ and $q_{i+1} \leq p_i$.
- for $i < \bar{\kappa}^+$, q_{i+1} and p_{i+1} decide the value of $\dot{a}(\check{\xi}_{i+1})$ in different ways.

We begin by setting $p_0 = p$. If $j \leq \bar{\kappa}^+$ is a limit ordinal and $\langle p_i \mid i < j\rangle$, $\langle q_{i+1} \mid i < j\rangle$ and $\langle \xi_{i+1} \mid i < j\rangle$ have been defined already, then let p_j be the limit of the sequence $\langle p_i \mid i < j\rangle$ (i.e., if $p_i = \langle s, T_i\rangle$ for $i < j$, then $p_j = \langle s, \bigcap_{i<j} T_i\rangle$). Now suppose $\langle p_i \mid i \leq j\rangle$, $\langle q_i \mid i \leq j,\ i \text{ a successor}\rangle$ and $\langle \xi_i \mid i \leq j,\ i \text{ a successor}\rangle$ have been defined. Since $p_j \leq p$, there must be a $\xi_{j+1} < \Omega$ such that p_j does not decide the value of $\dot{a}(\check{\xi}_{j+1})$ (or else p_j would force that $\dot{a} \in \mathrm{V}$). Since there are no more than $\bar{\kappa}$ many possible values, there is a direct extension p_{j+1} of p_j that decides it. Now let $q_{j+1} \leq p_j$ be some condition that decides $\dot{a}(\check{\xi}_{j+1})$ in a different way. This finishes the construction.

Now, for every successor ordinal $i < \bar{\kappa}^+$, let D_i be a Příkrý sequence such that $D_i \triangle C$ is finite, and such that, letting G_{D_i} be the generic filter corresponding to D_i, $q_i \in G_{D_i}$. Then for $i < j$, both successor ordinals, $\dot{a}^{G_{D_i}} \neq \dot{a}^{G_{D_j}}$, because q_i decides $\dot{a}(\check{\xi}_i)$ in a way that is different from the way p_i decides it, but since $q_j \leq p_{j-1} \leq p_i$, q_j decides $\dot{a}(\check{\xi}_i)$ the same way as p_i. Thus, $\dot{a}^{G_{D_i}}(\xi_i) \neq \dot{a}^{G_{D_j}}(\xi_i)$.

However, for each successor ordinal $i < \bar{\kappa}^+$, $\mathrm{V}[D_i] = \mathrm{V}[C]$, and since $p \in G_{D_i}$, $\mathrm{V}[C] \models \varphi(\dot{a}^{G_{D_i}}, \alpha)$, while there are only $\bar{\kappa}$ many b such that $\mathrm{V}[C] \models \varphi(b, \alpha)$. This is a contradiction.

To prove (b), assume that U is $<\kappa^+$-$\mathsf{HOD}^{\mathrm{V}[G]}$. I must show that $C \in {<}\kappa^+$-$\mathsf{HOD}^{\mathrm{V}[G]}$. Let A be $\mathsf{OD}^{\mathrm{V}[G]}$, of cardinality at most κ in $\mathrm{V}[G]$, such that $U \in A$. Recall that $\mathcal{G}$ was defined in $\mathrm{V}[C]$ to be the collection of all Příkrý sequences D over V with respect to U such that $\mathrm{V}[C] = \mathrm{V}[D]$. Formalizing this definition of $\mathcal{G}$ would require us to talk about V, which may or not be definable is a simple enough way. Instead, working in $\mathrm{V}[C]$, let me define a set D to be a *maximal* Příkrý sequence (with respect to U) if:

(i) $D \subseteq \kappa$, and $\mathrm{otp}(D) = \omega$.
(ii) For every $B \in U$, $D \setminus B$ is finite.
(iii) Call a set E satisfying (i) and (ii) a Příkrý sequence (with respect to U). Then for every Příkrý sequence E (with respect to U), $E \setminus D$ is finite.

It then follows that $\mathcal{G}$ is the set of all maximal Příkrý sequences (with respect to U). To see this, for the inclusion from left to right, assume that $D \in \mathcal{G}$. It then clearly satisfies (i) and (ii). In addition, since $\mathrm{V}[D] = \mathrm{V}[C]$ and D is maximal in $\mathrm{V}[D]$, it follows that D is also a maximal Příkrý sequence with respect to U. Vice versa, suppose that D satisfies (i)–(iii). D is then Příkrý-generic over V. To see that $\mathrm{V}[D] = \mathrm{V}[C]$, it suffices to show that $C \in \mathrm{V}[D]$. However, C is also a Příkrý sequence with respect to U, so by (iii), $C \setminus D$ is finite. On the other hand, $D \in \mathrm{V}[C]$, and C is maximal in $\mathrm{V}[C]$, so $D \setminus C$ is finite. Hence, $C \in \mathrm{V}[D]$, so that $\mathrm{V}[C] = \mathrm{V}[D]$, as claimed.

For every $W \in A$, we may now define

$$\mathcal{G}_W = \{D \subseteq \kappa \mid D \text{ is a maximal Příkrý sequence with respect to } W\}.$$

Note that the definition of $\mathcal{G}_W$ only uses W (and κ) as a parameter, and that $\mathcal{G}_U = \mathcal{G}$. Therefore, by (1), $\mathcal{G}_U$ has cardinality κ. Thus, by adding this requirement to the definition of A if necessary, we may assume that for every $W \in A$, the cardinality of $\mathcal{G}_W$ is κ. We then have that $C \in \bigcup_{W \in A} \mathcal{G}_W$, and this latter set is $\mathsf{OD}^{\mathrm{V}[G]}$ and has cardinality κ, making C $<\kappa^+$-$\mathsf{HOD}^{\mathrm{V}[G]}$, as claimed.

Finally, let me turn to (c). To be able to apply Lemma 4, note that $\mathbb{P}$ is cone homogeneous: given two conditions $p = \langle s_0, T_0 \rangle$ and $q = \langle s_1, T_1 \rangle$, let $T = T_0 \cap T_1$, $p' = \langle s_0, T \rangle$ and $q' = \langle s_1, T \rangle$. Then $p' \leq p$, $q' \leq q$ and $\mathbb{P}_{\leq p'}$ is isomorphic to $\mathbb{P}_{\leq q'}$ via the function that maps $\langle s, W \rangle \in \mathbb{P}_{\leq p'}$ to $\langle (s \setminus s_0) \cup s_1, W \rangle$. It is routine to check that this is an isomorphism between the cones. Lemma 4 now proves the lemma. □

Here is an application in a concrete scenario, producing a model whose least leap is the successor of a singular cardinal.

Theorem 16. *Assume* $\mathrm{V} = L[U]$, *where* U *is a normal ultrafilter on* κ. *Let* $\mathbb{P}$ *be the Příkrý forcing for* U, *and let* G *be* $\mathbb{P}$*-generic over* V. *Then*

$$L[U] = \mathsf{HOD}^{L[U][G]} = {<}\kappa\text{-}\mathsf{HOD}^{L[U][G]} \subsetneqq {<}\kappa^+\text{-}\mathsf{HOD}^{L[U][G]} = L[U][G].$$

In particular, $\Lambda_0 = \kappa^+$ *is the successor of a limit cardinal of countable cofinality in* $L[U][G]$.

Proof. In $L[U][G]$, $L[U]$ is definable (it is the core model), and U is the unique normal measure on κ in $L[U]$, so U is definable in $L[U][G]$. Thus, the assumptions of Theorem 15(b) and (c) are satisfied, yielding that ${<}\kappa\text{-}\mathsf{HOD}^{L[U][G]} \subseteq {<}\kappa\text{-}\mathsf{HOD}^{L[U]}$ and letting C be the Příkrý sequence corresponding to G, $C \in {<}\kappa^+\text{-}\mathsf{HOD}^{L[U][G]}$. However, $L[U] = \mathsf{HOD}^{L[U]} \subseteq \mathsf{HOD}^{L[U][G]}$, since $L[U]$ is definable in $L[U][G]$; see Proposition 9. Therefore, we have that $L[U] = \mathsf{HOD}^{L[U][G]} = {<}\kappa\text{-}\mathsf{HOD}^{L[U][G]}$. The rest follows because $C \in {<}\kappa^+\text{-}\mathsf{HOD}^{L[U][G]}$, as this implies that $L[U][G] = L[U][C] \subseteq {<}\kappa^+\text{-}\mathsf{HOD}^{L[U][G]} \subseteq L[U][G]$. □

Finally, here is another simple application, paralleling Theorem 14 and producing a model with a singular limit leap whose successor cardinal is also a leap.

Theorem 17. *If* ZFC *is consistent with a measurable cardinal, then* ZFC *is also consistent with the existence of a singular limit leap of countable cofinality, whose cardinal successor is a leap.*

Proof. Working in a model of ZFC with a measurable cardinal, we can form the canonical inner model $L[U]$ and work there, so let us assume that $\mathrm{V} = L[U]$. Let κ be the measurable cardinal (so U is the unique normal ultrafilter on κ). Let $\mathbb{P}_U$ be the Příkrý forcing with respect to U, and let C be a Příkrý sequence with respect to U.

In $\mathrm{V}[C]$, let $\mathbb{P}$ be the reverse Easton iteration employing $\mathrm{Add}(\gamma, 1)$ at stage γ when $\gamma < \kappa$ is an *uncountable* regular cardinal, and using trivial forcing at all other stages.

(1) $C \in {<}\kappa^+\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]} \setminus {<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$.

To see that $C \in {<}\kappa^+\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$, recall that C is one of κ many maximal Příkrý sequences with respect to U in $\mathrm{V}[C]$, by Theorem 15. The forcing $\mathbb{P}$ is ${<}\omega_1$-closed in $\mathrm{V}[C]$, so does not add any countable sequences of ordinals, and preserves cardinals. Moreover, U is still definable in $\mathrm{V}[C][G]$. Hence, C is still ${<}\kappa^+\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$, using the same definition.

To see that C does not belong to ${<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$, suppose towards a contradiction that it did. Let $A = \{x \mid \varphi(x, \alpha)\}^{\mathrm{V}[C][G]}$ be $\mathsf{OD}^{\mathrm{V}[C][G]}$ with $C \in A$ and $\overline{\overline{A}}^{\mathrm{V}[C][G]} < \kappa$. Arguing in $\mathrm{V}[C]$, since $\mathbb{P}$ is weakly homogeneous in $\mathrm{V}[C]$, we know that for every x, either $\Vdash_{\mathbb{P}} \varphi(\check{x}, \check{\alpha})$ or $\Vdash_{\mathbb{P}} \neg\varphi(\check{x}, \check{\alpha})$. Therefore, define, still in $\mathrm{V}[C]$:

$$\tilde{A} = \{x \mid \ \Vdash_{\mathbb{P}} \varphi(\check{x}, \check{\alpha})\}.$$

Then $\tilde{A} = A \cap \mathrm{V}[C] \subseteq A$, and in particular, $C \in \tilde{A}$. We have that

$$\overline{\overline{\tilde{A}}}^{\mathrm{V}[C]} = \overline{\overline{\tilde{A}}}^{\mathrm{V}[C][G]} \leq \overline{\overline{A}}^{\mathrm{V}[C][G]} < \kappa.$$

Moreover, since $\mathbb{P}$ is $\mathsf{OD}^{\mathrm{V}[C]}$, $\tilde{A}$ is $\mathsf{OD}^{\mathrm{V}[C]}$. Taken together, this shows that $C \in {<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C]}$. However, this contradicts Theorem 15 (a), which says that ${<}\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[C]} \subseteq \mathrm{V}$.

(2) Let $\gamma < \kappa$ be regular and uncountable. Then

$$G(\gamma) \in {<}\gamma^{++}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]} \setminus {<}\gamma^{+}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}.$$

Viewing $G(\gamma)$ as a subset of γ as usual, it follows that it belongs to $\mathcal{P}(\gamma)^{\mathrm{V}[C][G]}$, an $\mathsf{OD}^{\mathrm{V}[C][G]}$ set of cardinality less than γ^{++} in $\mathrm{V}[C][G]$, showing that $G(\gamma) \in {<}\gamma^{++}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$.

To see that $G(\gamma)$ does not belong to ${<}\gamma^{+}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G]}$, assume it did. As before, it would follow that $G(\gamma) \in {<}\gamma^{+}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G\upharpoonright(\gamma+1)]}$ by Corollary 4, and further, by Theorem 11, that $G(\gamma) \in {<}\gamma^{+}\text{-}\mathsf{HOD}^{\mathrm{V}[C][G\upharpoonright\gamma]}$, which is absurd. This proves (2).

Now, in $\mathrm{V}[C][G]$, κ has countable cofinality, and by (2), κ is a limit of leaps, and by (1), κ^{+} is a leap, as wished. $\square$

4. Questions and Directions for Future Research

Blurry definability is a new topic, and as such, there are many directions future research could take.

Some research has been undertaken and some deep results have been obtained on ${<}\omega_1$-HOD, the "hereditarily nontypical" sets, by Kanovei and Lyubetsky. Some of their techniques might be usable at other stages of the blurry definability hierarchy. An intriguing set of results concerns the status of the *axiom of choice* in ${<}\omega_1$-HOD. Specifically, in [6] (Lemma 10.2), they show that in the forcing extension by their forcing $\mathbb{P}^{<\omega}$, outlined in Section 3.4, ${<}\omega_1$-HOD does not satisfy the axiom of choice. On the other hand, they provide in [5] a different construction of a model in which ${<}\omega_1$-HOD is situated strictly between HOD and V, but this time it *does* satisfy AC. It would be interesting to see whether one can control the structure of leaps, while also controlling whether or not ${<}\kappa$-HOD satisfies the axiom of choice. Of course, there is the limitation that at a limit leap κ, ${<}\kappa$-HOD cannot satisfy AC. Given the nature of the forcing notions used by Kanovei and Lyubetsky, this would probably entail analyzing forcing notions whose conditions are perfect trees on some uncountable regular cardinal κ, as was done in [28], for example, and it might involve techniques from the area of generalized Baire spaces.

Another topic of interest is the status of *gaps* between leaps. I did not verify that successors of limit cardinals below the length of the iteration to produce the model of Theorem 12 are not leaps there, but I suspect that an argument such as the one establishing part (c) of the theorem, together with the techniques employed in the proof of Theorem 11 would show that - this is something that should be checked. Taking this further, one could try to construct models where as many consecutive cardinals as possible are leaps. One naive approach to constructing a model where this is true up to $\aleph_{\omega+2}$ would be to use the method of Theorem 17 but use iterated collapse forcings (as in [29], for example) between the elements of the Příkrý sequence to arrange that the formerly measurable cardinal becomes $\aleph_\omega$. Analyzing the effect of collapse forcing itself on blurry definability would be an interesting first step.

Relating to the use of Cohen forcing to produce leaps, the reason $\mathrm{Add}(\kappa, 1)$ introduced a leap at κ^{++} was that GCH held in the models under consideration. Therefore, it would be interesting to have *failures of* GCH, say at κ, but still have κ^{++}, or even κ^{+} be a leap.

Another question concerns the use of *large cardinals* to produce models in which the successor cardinal of a limit leap is a leap: are large cardinals necessary for this? Can the presence of a core model be used, via covering, to rule out the occurrence of such a situation?

Furthermore, it seems as though the features of Příkrý forcing used to prove the salient facts around blurry definability in its forcing extensions, Theorem 15, are shared by Magidor forcing; see [30]. Therefore, it would seem promising to attempt to use it to produce a model with a singular limit leap of *higher cofinality*, whose cardinal successor is a leap.

Additionally, it seems that it should be possible to iterate adding Cohen sets past a measurable cardinal and much more, preserving measurability, using lifting techniques, and therefore producing models with many singular limit leaps whose successor cardinals are leaps. I do not foresee any problems in performing a length On iteration, to produce models with a *proper class of leaps*, etc.

Finally, and this may be a leap, but let me close with the question whether it is consistent that *every* uncountable cardinal is a leap.

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