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Convergence Theorems in Interval-Valued Riemann–Lebesgue Integrability

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Abstract: We provide some limit theorems for sequences of Riemann–Lebesgue integrable functions. More precisely, Lebesgue-type convergence and Fatou theorems are established. Then, these results are extended to the case of Riemann–Lebesgue integrable interval-valued multifunctions.

Keywords: Riemann–Lebesgue integral; interval-valued (set) multifunction; non-additive set function; Lebesgue theorem; Fatou theorem

MSC: 28B20; 28C15; 49J53

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1. Introduction

The study of non-additive set functions and nonlinear integrals has received a wide recognition because of its applications in many domains such as: potential theory, subjective evaluation, optimization, economics, decision-making, data mining, artificial intelligence, and accident rate estimations (e.g., [1–14]). In the literature, there are several methods of integration for (multi)functions based on extensions of the Riemann and Lebesgue integrals. In this context, Kadets and Tseytlin [15] introduced the absolute Riemann–Lebesgue (RL) and unconditional Riemann–Lebesgue RL integrability, for Banach-valued functions with respect to countably additive measures. According to [15], in the finite measure space, the Bochner integrability implies RL integrability, which is stronger than RL integrability, which implies Pettis integrability. Contributions in this area were given in [16–23].

Interval analysis, as particular case of set-valued analysis, was introduced by Moore [24], motivated by its applications in computational mathematics (i.e., numerical analysis). The interval-valued multifunctions and multimeasures are involved in various applied sciences, for example in signal and image processing, since the discretization of a continuous signal causes different sources of uncertainty and ambiguity. As we can see in [25–31], the discretization of an analogue signal usually produces quantization errors, such as the round-off one. Following the case of real functions (see, e.g., [32,33]), this numerical discretization can be viewed as an approximation of a suitable sequence of interval-valued multifunctions G_n , which converges to a (multi)signal G corresponding to the original analogue one. The lack of continuity of interval-valued multifunctions G_n must be replaced by the notion of convergence under a suitable definition of integrals of interval-valued multifunctions.

We highlight that some convergence theorems of the RL integral for a sequence of interval-valued multifunctions $(G_n)_n$ with respect to an interval-valued multisubmeasure M were established in [34,35]. Furthermore, in [35], a generalized version of the monotone

convergence theorem [36,37] was proven for a pair of sequences (G_n, M_n) of interval-valued multifunctions and multisubmeasures.

In this paper, we prove new results regarding limit theorems for sequences of Riemann–Lebesgue integrable functions and interval-valued multifunctions. The paper is organized as follows: In Section 2, some basic concepts are introduced. In Section 3, we provide different properties for the RL integral of a real function with respect to a non-additive set function. Lebesgue-type convergence theorems and a Fatou-type theorem are established in Section 4. Finally, in Section 5, we obtain some convergence theorems for sequences of RL integrals in the interval-valued case.

2. Preliminaries

For every nonempty set A , let $\mathcal{P}(A)$ be the family of all subsets of A . Suppose S is a nonempty set and $\mathcal{C}$ a σ -algebra of subsets of S . Denote $\mathbb{N}^* = \{1, 2, 3, \dots\}$ and $\mathbb{R}_0^+ = [0, \infty)$. For every $E \subset S$, as usual, let $E^c = S \setminus E$, and let χ_E be the characteristic function of E .

Let $\mu : \mathcal{C} \rightarrow [0, \infty)$ be a set function, such that $\mu(\emptyset) = 0$. As in [22] (Definition 2.2) and in [34] (Definition 2), we introduce the definitions of monotonicity, subadditivity, countable additivity, and countable subadditivity and the definition of a submeasure in the sense of Drewnowski, which are usual in measure theory.

A set $A \in \mathcal{C}$ is said to be an atom of a set function $\mu : \mathcal{C} \rightarrow [0, \infty)$ if $\mu(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. Moreover, we recall the following:

Definition 1. A set function $\mu : \mathcal{C} \rightarrow [0, \infty)$ satisfies the property (σ) and the condition (E) if:

- (σ) If for every $\{E_n\}_n \subset \mathcal{C}$ with $\mu(E_n) = 0$, for every $n \in \mathbb{N}$, we have $\mu(\bigcup_{n=0}^{\infty} E_n) = 0$;
- (E) If for every double sequence $(A_n^m)_{n,m \in \mathbb{N}^*} \subset \mathcal{C}$, such that:
 - For every $m \in \mathbb{N}^*$, $A_n^m \searrow A^m$ ($n \rightarrow \infty$) and $\mu(\bigcup_{m=1}^{\infty} A^m) = 0$;
 - There exist two increasing sequences $(n_p)_p, (m_p)_p \subset \mathbb{N}$ such that $\lim_{k \rightarrow \infty} \mu(\bigcup_{p=k}^{\infty} A_{n_p}^{m_p}) = 0$.

The condition (σ) is a consequence of the countable subadditivity, and it will be needed in some of our results in Section 5. Instead, in Corollary 1.c) and Theorems 2–4, we need countable subadditivity, which cannot be replaced by the condition (σ) .

Observe that condition (E) was given, for example, in [38], in order to give sufficient and necessary conditions to obtain Egoroff's theorem for suitable non-additive measures. See also [39] for null additive set functions and related questions. An example of a set function that satisfies the condition (E) can be found in [38] (Example 3.3). In some of our results, we need [38] (Theorem 4.1-(1)), where the condition (E) ensures the fact that the convergence almost everywhere (ae) implies the convergence almost uniformly (au).

Non-additive set functions have applications in many areas, such as: subjective evaluation, decision-making, fuzzy logic, computer science, and data mining. That is why we were motivated to work with non-additivity.

Example 1.

- If $S = [0, 1]$, $\mathcal{C}$ is the Borel σ -algebra of S and m is the Lebesgue measure on $\mathcal{C}$, then $\mu_1 = m^2$ and $\mu_2 = m - m^2$ are not additive (in fact, μ_1 is superadditive and μ_2 is subadditive);
- Monotone set functions are used in decision-making. Thus, for $n \in \mathbb{N}^*$, let $S = \{1, \dots, n\}$ be a finite set of criteria, $\mathcal{C} = \mathcal{P}(S)$, and let $\mu : \mathcal{C} \rightarrow [0, 1]$ be monotone such that $\mu(\emptyset) = 0$ and $\mu(S) = 1$. For every set of criteria $A \in \mathcal{C}$, $\mu(A)$ represents the power of A to make the decision without the other criteria or the degree of importance of the criteria in A . If we consider another criterion to a set of $\mathcal{C}$, its importance increases, that is μ is monotone [40].

Definition 2. Given a set function $\mu : \mathcal{C} \rightarrow [0, \infty)$, we consider the following:

(2.i) The variation of μ is the set function $\bar{\mu} : \mathcal{P}(S) \rightarrow [0, \infty]$ defined by:

$$\bar{\mu}(E) = \sup \left\{ \sum_{i=1}^n \mu(E_i), \{E_i\}_{i=1}^n \subset \mathcal{C}, E_i \subseteq E, i \leq n \right\}.$$

If $\bar{\mu}(S) < \infty$, then μ is said to be of finite variation;

(2.ii) The semivariation of μ is the set function $\tilde{\mu} : \mathcal{P}(S) \rightarrow [0, \infty]$ defined by:

$$\tilde{\mu}(A) = \inf \{ \bar{\mu}(B) : A \subseteq B, B \in \mathcal{C} \}.$$

Moreover if $\mu_1 \leq \mu_2$, then $\tilde{\mu}_1 \leq \tilde{\mu}_2$.

Suppose S is a locally compact Hausdorff topological space. We denote by $\mathcal{K}$ the lattice of all compact subsets of S , $\mathcal{B}$ the Borel σ -algebra (i.e., the smallest σ -algebra containing $\mathcal{K}$) and $\mathcal{O}$ the class of all open sets. $\mu : \mathcal{B} \rightarrow \mathbb{R}_0^+$ is called *regular* if for every set $A \in \mathcal{B}$ and every $\varepsilon > 0$, there exist $K \in \mathcal{K}$ and $D \in \mathcal{O}$ such that $K \subseteq A \subseteq D$ and $\mu(D \setminus K) < \varepsilon$.

$(ck(\mathbb{R}), +, d_H)$ denotes the family of all nonempty convex compact subsets of $\mathbb{R}$ endowed with the Minkowski addition, the standard multiplication by scalars, and the Hausdorff–Pompeiu distance. It is a complete metric space [22,41]. By convention, $\{0\} = [0, 0]$. If $A = [x, y]$, then $\|A\| = \max\{|x|, |y|\}$. Moreover:

$$\begin{aligned} d_H([x, y], [w, z]) &= \max\{|x - w|, |y - z|\}, & \forall x, y, w, z \in \mathbb{R} \\ d_H([0, x], [0, y]) &= |y - x| & \forall x, y \in \mathbb{R}_0^+. \end{aligned}$$

In $ck(\mathbb{R}_0^+)$, we consider the following operations: multiplication $(\cdot)$, inclusion $(\subseteq)$, an order relation $(\preceq)$ weak interval order, and the suprema and infima $\vee, \wedge$, defined in the following way:

- (i) $[x, y] \cdot [w, s] = [xw, ys]$;
- (ii) $[x, y] \subseteq [w, s] \iff w \leq x \leq y \leq s$;
- (iii) $[x, y] \preceq [w, s] \iff x \leq w \text{ and } y \leq s$; (weak interval order);
- (iv) $[x, y] \wedge [w, s] = [\min\{x, w\}, \min\{y, s\}]$;
- (v) $[x, y] \vee [w, s] = [\max\{x, w\}, \max\{y, s\}]$.

There is no relation between the inclusion and the weak interval order on $ck(\mathbb{R}_0^+)$, but they coincide on $\{[0, x], x \in \mathbb{R}_0^+\}$. Moreover, if $(x_n)_n, (y_n)_n$ are two sequences of real numbers such that $0 \leq x_n \leq y_n$, for every $n \in \mathbb{N}$, we define:

- (vi) $\inf_n [x_n, y_n] = [\inf_n x_n, \inf_n y_n]$;
- (vii) $\sup_n [x_n, y_n] = [\sup_n x_n, \sup_n y_n]$;
- (viii) $\liminf_n [x_n, y_n] = [\liminf_n x_n, \liminf_n y_n]$.

Definition 3. Given two set functions $m_1, m_2 : \mathcal{C} \rightarrow \mathbb{R}_0^+$ with $m_1(\emptyset) = m_2(\emptyset) = 0$ and $m_1(A) \leq m_2(A)$ for every $A \in \mathcal{C}$, we call $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$ an interval-valued set multifunction if it is defined by:

$$\Gamma(A) = [m_1(A), m_2(A)], \text{ for every } A \in \mathcal{C}. \quad (1)$$

Let $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$. It is said that Γ is an interval-valued multisubmeasure if:

- (3.i)** $\Gamma(\emptyset) = \{0\}$;
- (3.ii)** $\Gamma(A) \preceq \Gamma(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$ (monotonicity assumption);
- (3.iii)** $\Gamma(A \cup B) \preceq \Gamma(A) + \Gamma(B)$ for every disjoint sets $A, B \in \mathcal{A}$ (subadditivity assumption).

A set $A \in \mathcal{C}$ is said to be an atom of an interval-valued set multifunction $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$ if $\{0\} \preceq \Gamma(A)$, $\{0\} \neq \Gamma(A)$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\Gamma(B) = \{0\}$ or $\Gamma(A \setminus B) = \{0\}$. Observe that $A \in \mathcal{C}$ is an atom of $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$, $\Gamma = [m_1, m_2]$, if and only if A is an atom of m_1 and m_2 .

We give now an example of an interval-valued set multifunction that arises in the theory of evidence.

Example 2. Assume that $(S, \mathcal{P}(S), \nu)$ is a probability space. In Dempster–Shafer’s [42] mathematical theory of evidence, the belief (m_1) and plausibility (m_2) functions are defined by a probability distribution $\nu : \mathcal{P}(S) \rightarrow [0, 1]$, with $\nu(\emptyset) = 0$ and $\sum_{B \subseteq S} \nu(B) = 1$. For every $B \subseteq S$, $m_1(B) = \sum_{C \subseteq B} \nu(C)$ and $m_2(B) = \sum_{C \cap B \neq \emptyset} \nu(C)$. m_1 and m_2 are non-additive set functions, which are the lower and upper bounds, respectively, for the family of all probability distributions $P : \mathcal{P}(S) \rightarrow [0, 1]$ ($\forall B \in \mathcal{P}(S), P(B) \in [m_1(B), m_2(B)]$).

The belief interval of B is the range defined by the minimum and maximum values, which could be assigned to B : $[m_1(B), m_2(B)]$. This interval probability representation contains the precise probability of a set of interest (in the classical sense). The probability is uniquely determined if $m_1(B) = m_2(B)$. We observe that in this case, which corresponds to the classical probability, all the probabilities P are uniquely determined for all subsets of S .

Let $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$ be an interval-valued set multifunction. As in the single-valued case, the variation of Γ is the set function $\bar{\Gamma} : \mathcal{P}(S) \rightarrow [0, +\infty]$ defined by:

$$\bar{\Gamma}(E) := \sup \left\{ \sum_{i=1}^n \|\Gamma(A_i)\|, \{A_i\}_{i=1}^n \subset \mathcal{A}, A_i \subseteq E, A_i \cap A_j = \emptyset, i \neq j \right\},$$

and analogously, Γ is said to be of finite variation if $\bar{\Gamma}(S) < \infty$.

In the same line, the semivariation of Γ is defined by $\tilde{\Gamma}(A) := \inf \{\bar{\Gamma}(B) : A \subset B, B \in \mathcal{C}\}$, for every $A \subset S$.

Remark 1. By [43] (Remark 3.6) the set measure given in Formula (1) is an interval-valued multi-submeasure if and only if m_1, m_2 are submeasures in the sense of Drewnowski. Moreover, according to [44] (Proposition 2.5 and Remark 3.3) Γ is monotone, countable subadditive, respectively, if and only if the set functions m_1 and m_2 are the same. Finally, Γ satisfies the property (σ) if and only if m_1 and m_2 have the same property.

The semivariation $\tilde{\Gamma}$ is a monotone set function with $\tilde{\Gamma}(\emptyset) = 0$ and $\tilde{\Gamma} = \widetilde{m_2}$, and it verifies $\tilde{\Gamma}(C) = \bar{\Gamma}(C)$, for every $C \in \mathcal{C}$.

We consider in the sequel an example of an interval-valued multifunction used in decision-making problems.

Example 3. Let $S = \{y_1, y_2, \dots, y_m\}$ be a set of criteria, and let $B_1, B_2, \dots, B_n \in \mathcal{P}(S)$ be sets (alternatives) of criteria. Then, the following multicriteria decision-making matrix is obtained:

$$\begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_m \\ B_1 & \left(\begin{array}{ccc} H_{B_1}(y_1) & \cdots & H_{B_1}(y_m) \\ H_{B_2}(y_1) & \cdots & H_{B_2}(y_m) \\ \vdots & \vdots & \vdots \\ H_{B_n}(y_1) & \cdots & H_{B_n}(y_m) \end{array} \right) \end{array} \end{array}$$

where $H_{B_i}(y_j) = [u_{B_i}(y_j), v_{B_i}(y_j)]$ represents the degree to which the alternative B_i satisfies the criterion y_j . Each of the intervals $H_{B_i}(y_j)$, $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$, provides the membership value of criteria y_j to the interval-valued fuzzy set B_i (see, for example, [45,46]).

Definition 4. Let $\mu : \mathcal{C} \rightarrow [0, \infty]$ be a set function with $\mu(\emptyset) = 0$ and g, g_n be scalar functions for every $n \in \mathbb{N}$. We say that:

- (g_n) converges μ -almost everywhere to g on S (in symbol $g_n \xrightarrow{\mu-ae} g$) if there exists $C \in \mathcal{C}$ with $\mu(C) = 0$ and $\lim_{n \rightarrow \infty} g_n(s) = g(s)$, for every $s \in S \setminus C$;
- (g_n) μ -converges to g on S (in symbol $g_n \xrightarrow{\mu} g$) if for every $\delta > 0$, $\lim_{n \rightarrow \infty} \mu(E_n(\delta)) = 0$, where:

$$E_n(\delta) = \{s \in S; |g_n(s) - g(s)| \geq \delta\} \in \mathcal{C}.$$

Definition 5. Let S be at least a countable set:

- (5.i) A measurable countable partition of S is a countable family of nonvoid sets $\Pi = \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ such that $\bigcup_{n \in \mathbb{N}} E_n = S$ with $E_i \cap E_j = \emptyset$, when $i \neq j$.
 $\mathcal{P}$ will be the set of all countable partitions of S , and $\mathcal{P}_E$ will be the set of countable partitions of $E \in S$;
- (5.ii) For every Π and $\Pi' \in \mathcal{P}$, it is said that Π' is finer than Π , (denoted by $\Pi' \geq \Pi$ or $\Pi \leq \Pi'$) if every set of Π' is included in some set of Π ;
- (5.iii) For every Π and $\Pi' \in \mathcal{P}$, $\Pi = \{E_n\}, \Pi' = \{C_m\}$, the common refinement of Π and Π' is the countable partition $\Pi \wedge \Pi' = \{E_n \cap C_m\}$.

3. Properties of the Riemann–Lebesgue Integral

In this section, we present some properties of the Riemann–Lebesgue integral. In the sequel, S is at least a countable set, $(X, \|\cdot\|)$ is a Banach space with its norm, and $m : \mathcal{C} \rightarrow [0, \infty)$ is a set function such that $m(\emptyset) = 0$.

Definition 6 ([15]). A vector function $g : S \rightarrow X$ is called absolutely (unconditionally, respectively) Riemann–Lebesgue integrable (on S) with respect to m , denoted ARL_m (URL_m , respectively) if there exists $x \in X$ such that for every $\varepsilon > 0$, there exists $\Pi_\varepsilon \in \mathcal{P}$, such that for every $\Pi \in \mathcal{P}$, $\Pi = (E_n)_{n \in \mathbb{N}}, \Pi \geq \Pi_\varepsilon$:

- g is bounded on every E_n , with $m(E_n) > 0$ and;
- For every $s_n \in E_n, n \in \mathbb{N}$, the series $\sum_{n=0}^{+\infty} g(s_n)m(E_n)$ is absolutely (unconditionally respectively) convergent and:

$$\left\| \sum_{n=0}^{+\infty} g(s_n)m(E_n) - x \right\| < \varepsilon.$$

With the symbol $\sigma_g(S, \Pi, s_n)$, we denote the vector $\sum_{n=0}^{+\infty} g(s_n)m(E_n)$. We call $x = \int_S g dm$ the Riemann–Lebesgue integral of g (on S) with respect to m . We denote by $ARL_m(S)$ ($URL_m(S)$, respectively) the set of all absolutely (unconditionally respectively) Riemann–Lebesgue integrable functions on S . Obviously, if x exists, then it is unique. According to the properties of [22], $ARL_m(S)$ and $URL_m(S)$ are linear spaces.

Remark 2. As said before, Kadets and Tseytlin [15] introduced the ARL-integral and the URL-integral for functions with values in a Banach space relative to a measure. They proved that if $(S, \mathcal{C}, m)$ is a finite measure space, then the following implications hold:

- g is Bochner integrable $\implies g$ is ARL-integrable $\implies g$ is URL-integrable $\implies g$ is Pettis integrable,

while in a separable Banach space X , the ARL-integrability coincides with the Bochner integrability and the URL-integrability coincides with the Pettis one. If X is finite-dimensional, then the ARL_m -integrability is equivalent to the URL_m -integrability. In this case, it is denoted by RL_m . In general, if g is ARL_m -integrable, then g is URL_m -integrable (see, for example, [15]).

If $(S, \mathcal{C}, m)$ is a σ -finite measure space, then the Birkhoff integrability coincides with the URL-integrability [18].

The notion of the RL-integrability for scalar functions is weaker than the notion of the Riemann integrability; in [18], the direct implication was proven together with an example showing that the opposite relation does not hold in general.

Finally, if m is monotone, countable subadditive, and of finite variation, then the Gould integrability and the RL-one are equivalent in the class of bounded and scalar functions (see, for example, [22] (Theorem 10)).

Theorem 1. Let $p \in \mathbb{N}^*$ be fixed. Let $E_i \in \mathcal{C}$, for every $i \leq p$, be such that $E_i \cap E_j = \emptyset$ if $i \neq j$. Let $g_i : S \rightarrow X$, $i \leq p$ a finite family of ARL_m -integrable functions on E_i . Then, $\sum_{i=1}^p g_i \chi_{E_i}$ is ARL_m -integrable on S and $\int_S \sum_{i=1}^p g_i \chi_{E_i} dm = \sum_{i=1}^p \int_{E_i} g_i dm$. The same is true for the URL_m -integrability.

Proof. Suppose $\varepsilon > 0$ is arbitrary. According to Definition 6 applied to g_i , for every $i \in \{1, 2, \dots, p\}$, there are $\Pi_\varepsilon^i \in \mathcal{P}_{E_i}$ such that for every $\Pi \in \mathcal{P}_{E_i}$, $\Pi \geq \Pi_\varepsilon^i$,

$$\left\| \sigma_{g_i}(E_i, \Pi, c_n) - \int_{E_i} g_i dm \right\| < \frac{\varepsilon}{p}. \quad (2)$$

Let $\Pi_\varepsilon = \cup_{i=1}^p \Pi_\varepsilon^i$ and $\Pi \in \mathcal{P}$, $\Pi \geq \Pi_\varepsilon$. We may write $\Pi = \cup_{i=1}^p \{D_n^i\}_{n \in \mathbb{N}^*}$, where $\{D_n^i\}_{n \in \mathbb{N}^*} \in \mathcal{P}_{E_i}$, $\{D_n^i\}_{n \in \mathbb{N}^*} \geq \Pi_\varepsilon^i$ and for every $i \leq p$. Let $c_n^i \in D_n^i$, for every $n \in \mathbb{N}^*$, $\forall i \in \{1, 2, \dots, p\}$, denoted by $\{t_n\}_{n \in \mathbb{N}^*}$. Then, by (2) it holds that:

$$\left\| \sigma_h(S, \Pi, t_n) - \sum_{i=1}^p \int_{E_i} g_i dm \right\| \leq \sum_{i=1}^p \left\| \sigma_{g_i}(E_i, \{D_n^i\}, c_n^i) - \int_{E_i} g_i dm \right\| < \varepsilon,$$

which shows that $\sum_{i=1}^p g_i \chi_{E_i} \in ARL_m(S)$ and $\int_S \sum_{i=1}^p g_i \chi_{E_i} dm = \sum_{i=1}^p \int_{E_i} g_i dm$. $\square$

If $m : \mathcal{C} \rightarrow [0, \infty)$ is a measure and $g = \sum_{i=1}^p a_i \chi_{E_i}$, then $g \in ARL_m(S)$ and $\int_S g dm = \sum_{i=1}^p a_i m(E_i)$. Moreover, easy consequences of Theorem 1 are the following:

Corollary 1.

(1.a) If $g_1, g_2 : S \rightarrow \mathbb{R}$ are measurable RL_m -integrable functions on S , then the same holds for $f = \max\{g_1, g_2\}$, and $h = \min\{g_1, g_2\}$. Moreover:

$$\int_S f dm = \int_E g_2 dm + \int_{E^c} g_1 dm, \quad \int_S h dm = \int_E g_1 dm + \int_{E^c} g_2 dm,$$

where $E = \{s \in S; g_1(s) \leq g_2(s)\}$;

(1.b) If $f : S \rightarrow \mathbb{R}$ is a measurable RL_m -integrable function on S , then $|f| \in RL_m(S)$;

(1.c) If $m : \mathcal{C} \rightarrow [0, \infty)$ is countable subadditive and $E \in \mathcal{C}$, with $\chi_E \in RL_m(S)$, then $\int_S \chi_E dm \geq m(E)$. Moreover, if m is countable additive, then $\int_S \chi_E dm = m(E)$.

Proof. (1.a) We write:

$$f = g_1 \cdot \chi_{E^c} + g_2 \cdot \chi_E, \quad h = g_1 \cdot \chi_E + g_2 \cdot \chi_{E^c}$$

and then, we apply Theorem 1.

(1.b) It holds that $|f| = f \cdot \chi_E + (-f) \chi_{E^c}$, where $E = \{s \in S; f(s) \geq 0\}$. By Theorem 1,

$|f| \in RL_m(S)$.

(1.c) For arbitrary $\varepsilon > 0$, there is $\Pi_\varepsilon \in \mathcal{P}$ such that for every $\Pi \in \mathcal{P}$, $\Pi = \{C_n\}_{n \in \mathbb{N}}$, $\Pi \geq \Pi_\varepsilon$, and every $s_n \in C_n$, $n \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} \chi_E(s_n)m(C_n)$ is absolutely convergent and:

$$\left| \sigma_{\chi_E}(S, \Pi, s_n) - \int_S \chi_E dm \right| < \varepsilon. \quad (3)$$

Let $\Pi = \{A_n, D_n\}_{n \in \mathbb{N}} \in \mathcal{P}$, $\Pi \geq \Pi_\varepsilon$ such that $E = \cup_{n=0}^{\infty} A_n$ and $E^c = \cup_{n=0}^{\infty} D_n$. Since m is countable subadditive, by (3), it follows that:

$$m(E) - \int_S \chi_E dm = m(E) - \sum_{n=0}^{\infty} m(A_n) + \sum_{n=0}^{\infty} m(A_n) - \int_S \chi_E dm < \varepsilon.$$

By the arbitrariness of ε , the conclusion follows. $\square$

In the sequel, we present some considerations regarding the integral function of m .

Remark 3. Let $m : \mathcal{C} \rightarrow [0, \infty)$ be a non-negative set function, with $m(\emptyset) = 0$ such that, for all $E \in \mathcal{C}$, $\chi_E \in RL_m(S)$, and denote $\mu_m(E) = \int_S \chi_E dm$. The following properties are satisfied:

(3.i) μ_m is finitely additive;

(3.ii) If $\mu_m = m$, then m is finitely additive. The set function μ_m is called the integral function of m .

According to [47] (Definition 1.1) and [48] (Definition 3.2) the next concept is introduced:

Definition 7. A set function $m : \mathcal{C} \rightarrow [0, \infty)$ is called RL-integrable if for all $E \in \mathcal{C}$, $\chi_E \in RL_m(S)$ and $\int_S \chi_E dm = m(E)$.

Example 4. The following set functions satisfy the Definition 7:

- Every measure $m : \mathcal{C} \rightarrow [0, \infty)$ is RL-integrable;
- If m is not countable additive, then m may not be RL-integrable, as we can see in the following examples: Suppose $S = \mathbb{N}$, $m : \mathcal{P}(S) \rightarrow [0, \infty)$, $m(A) = 1$ for every $A \in \mathcal{P}(S)$, $A \neq \emptyset$ and $m(\emptyset) = 0$ (m is not countable additive). If $B = \mathbb{N}^* \in \mathcal{P}(S)$, then $\chi_B \notin RL_m(S)$, while if $C = \{0, 1\} \in \mathcal{P}(S)$, then $\chi_C \in RL_m(S)$, but $\int_S \chi_C dm = 2 \neq 1 = m(C)$.

By [22,48] the following result holds, making possible the approach of the non-additive frame via the additive one.

Theorem 2. Suppose $m : \mathcal{C} \rightarrow [0, \infty)$ is RL-integrable, monotone, countable subadditive, and of finite variation and $f : S \rightarrow \mathbb{R}$ is bounded. Then, $f \in RL_m(S)$ if and only if $f \in RL_{\mu_m}(S)$. In this case, $\int_S f dm = \int_S f d\mu_m$.

Theorem 3. Let $m : \mathcal{C} \rightarrow [0, \infty)$ be a countable subadditive RL-integrable non-negative set function. If $g : S \rightarrow [0, \infty)$ is a measurable RL_m -integrable function and $\int_S g dm = 0$, then $g = 0$ m -ae.

Proof. For every $n \in \mathbb{N}$, denote $E_n = \{s \in S; g(s) \geq \frac{1}{n}\}$. Then, for every $n \in \mathbb{N}$, $E_n \in \mathcal{C}$ and according to Corollary 1.c) and [22] (Theorems 3 and 6) it holds that:

$$0 = \int_S g dm \geq \frac{1}{n} m(E_n) \geq 0.$$

Therefore, $m(E_n) = 0$, for every $n \in \mathbb{N}^*$. Let $V = \{s \in S; g(s) > 0\}$. Then, $V = \cup_{n=1}^{\infty} E_n \in \mathcal{C}$ and $m(V) = 0$, which concludes the proof. $\square$

Let $m : \mathcal{C} \rightarrow [0, \infty)$ be a non-negative set function, with $m(\emptyset) = 0$.
If $p \in [1, \infty)$ and $g : S \rightarrow \mathbb{R}$, with $|g|^p \in RL_m(S)$, we denote:

$$\|g\|_p = \left(\int_S |g|^p dm \right)^{\frac{1}{p}}.$$

Theorem 4. Let $m : \mathcal{C} \rightarrow [0, \infty)$ be a countable subadditive RL-integrable set function, and let $g, h : S \rightarrow \mathbb{R}$ be measurable functions:

(4.a) Let $p, q \in (1, \infty)$, with $p^{-1} + q^{-1} = 1$. If $g \cdot h \in RL_m(S)$, then:

$$\|g \cdot h\|_1 \leq \|g\|_p \cdot \|h\|_q \text{ (Hölder inequality);}$$

(4.b) Let $p \in [1, \infty)$. If $|g + h|^p, |g + h|^{q(p-1)}, |g|^p$, and $|h|^p$ are RL_m -integrable, then:

$$\|g + h\|_p \leq \|g\|_p + \|h\|_p \text{ (Minkowski inequality).}$$

Proof. (4.a) If $\int_S |g|^p dm = 0$ or $\int_S |h|^q dm = 0$, then according to Theorem 3, it follows $g \cdot h = 0$ m -a.e. In this case, the inequality of integrals is satisfied.

Consider $\int_S |g|^p dm > 0$ and $\int_S |h|^q dm > 0$. Applying [22] (Theorems 3 and 6) in the inequality

$$\frac{|g \cdot h|}{\left(\int_S |g|^p dm \right)^{\frac{1}{p}} \cdot \left(\int_S |h|^q dm \right)^{\frac{1}{q}}} \leq \frac{|g|^p}{p \int_S |g|^p dm} + \frac{|h|^q}{q \int_S |h|^q dm},$$

the conclusion is obtained;

(4.b) For $p = 1$, it follows easily by the triangular inequality of the modulus and applying again [22] (Theorems 3 and 6).

Suppose $p > 1$. By (4.a), it holds that:

$$\begin{aligned} \int_S |g + h|^p dm &\leq \left(\int_S |g + h|^{q(p-1)} dm \right)^{\frac{1}{q}} \cdot \left(\int_S |g|^p dm \right)^{\frac{1}{p}} + \\ &+ \left(\int_S |g + h|^{q(p-1)} dm \right)^{\frac{1}{q}} \cdot \left(\int_S |h|^p dm \right)^{\frac{1}{p}} = \\ &= \left(\int_S |g + h|^{q(p-1)} dm \right)^{\frac{1}{q}} \cdot (\|g\|_p + \|h\|_p). \end{aligned}$$

If $\int_S |g + h|^{q(p-1)} dm = 0$, then the conclusion is obvious.

If $\int_S |g + h|^{q(p-1)} dm > 0$, then dividing the above inequality by $\int_S |g + h|^{q(p-1)} dm$, we obtain the Minkowski inequality. $\square$

Remark 4. Let $p \in [1, \infty)$ and let $MRL_m(S) = \{g : 1, g \in RL_m(S) \text{ and } g \text{ be measurable}\}$. Then, the function $\|\cdot\|_p$ is a seminorm on the linear space $MRL_m(S)$.

4. Sequences of Riemann–Lebesgue Integrable Functions

In this section, we present different results regarding convergent sequences of Riemann–Lebesgue integrable functions. In the sequel, suppose $m : \mathcal{C} \rightarrow [0, \infty)$ is a non-negative set function, with $m(\emptyset) = 0$. We assumed also that $\overline{m}(S)$ is finite, unless otherwise specified.

Theorem 5. Let $g, g_n : S \rightarrow X$ such that $g, g_n \in ARL_m(S)$ for every $n \in \mathbb{N}$ and (g_n) is uniformly convergent to g . Then, $\lim_{n \rightarrow \infty} \int_S g_n dm = \int_S g dm$.

Proof. According to [22] (Theorems 3 and 5) we may write:

$$\left\| \int_S g_n dm - \int_S g dm \right\| \leq \left\| \int_S (g_n - g) dm \right\| \leq \sup_{s \in S} |g_n(s) - g(s)| \cdot \bar{m}(S),$$

whence the conclusion is obtained. $\square$

Theorem 6. Let $g : S \rightarrow \mathbb{R}$ be a bounded function, and for every $n \in \mathbb{N}$, let $g_n : S \rightarrow \mathbb{R}$ be such that (g_n) is uniformly bounded and $g_n \xrightarrow{\tilde{m}} g$. Then, $\lim_{n \rightarrow \infty} \int_S g_n dm = \int_S g dm$.

Proof. According to [22] (Proposition 1) $g, g_n, g_n - g \in RL_m(S)$, for every $n \in \mathbb{N}$. Let $\alpha \in (0, +\infty)$ such that:

$$\sup_{s \in S} |g(s)| < \alpha, \quad \sup_{s \in S, n \in \mathbb{N}} |g_n(s) - g(s)| < \alpha$$

Let $\varepsilon > 0$. By hypothesis, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\tilde{m}(B_n(\varepsilon/4\bar{m}(S))) < \varepsilon/4\alpha$, for every $n \geq n_0(\varepsilon)$, where $B_n(\delta) = \{s \in S; |g_n(s) - g(s)| \geq \delta\}$, for every $\delta \in (0, +\infty)$. Then, there is $A_n \in \mathcal{C}$ such that $B_n(\varepsilon/4\bar{m}(S)) \subset A_n$ and $\tilde{m}(A_n) = \bar{m}(A_n) < \varepsilon/4\alpha$. Using [22] (Theorem 3 and Corollary 1), for every $n \geq n_0(\varepsilon)$, it holds that:

$$\begin{aligned} \left| \int_S g_n dm - \int_S g dm \right| &\leq \left| \int_{A_n} (g_n - g) dm \right| + \left| \int_{A_n^c} (g_n - g) dm \right| \leq \\ &\leq \bar{m}(A_n) \cdot \sup_{s \in A_n} |g_n(s) - g(s)| + \bar{m}(A_n^c) \cdot \sup_{s \in A_n^c} |g_n(s) - g(s)| < \frac{\varepsilon}{4\alpha} \cdot \alpha + \alpha \cdot \frac{\varepsilon}{4\alpha} < \varepsilon, \end{aligned}$$

which shows that $\int_S g_n dm \rightarrow \int_S g dm$. $\square$

In the following theorem, we do not ask that m is of bounded variation.

Theorem 7. Suppose $m : \mathcal{C} \rightarrow [0, +\infty)$ is countable subadditive. Let $p \in [1, +\infty)$, $g : S \rightarrow \mathbb{R}$, and for every $n \in \mathbb{N}$, let $g_n : S \rightarrow \mathbb{R}$ such that $|g_n - g|^p \in RL_m(E)$, for every $E \in \mathcal{C}$ and $\|g_n - g\|_p \rightarrow 0$. Then, $g_n \xrightarrow{\tilde{m}} g$.

Proof. Let $B_n(\delta) = \{s \in S; |g_n(s) - g(s)| \geq \delta\}$, for every $\delta \in (0, +\infty)$, $\forall n \in \mathbb{N}$, and let $\{A_k^n\}_{k=1}^{q_n} \subset \mathcal{C}$ be a family of pairwise disjoint sets such that $A_k^n \subset B_n(\delta)$, $\forall k \in \{1, \dots, q_n\}$. According to [22] (Corollary 1 and Theorem 6) and Corollary 1.c), we have:

$$\delta^p \sum_{k=1}^{q_n} m(A_k^n) \leq \sum_{k=1}^{q_n} \int_{A_k^n} |g_n - g|^p dm = \int_{\cup_{k=1}^{q_n} A_k^n} |g_n - g|^p dm \leq \|g_n - g\|_p^p.$$

This implies $\frac{1}{\delta^p} \|g_n - g\|_p^p \geq \bar{m}(B_n(\delta))$, and since $\|g_n - g\|_p \rightarrow 0$, the conclusion is obtained. $\square$

Theorem 8. Let $m : \mathcal{C} \rightarrow [0, +\infty)$ be a monotone set function such that $\tilde{m}$ satisfies the condition (E). Let $g : S \rightarrow \mathbb{R}$ be a bounded function, and for every $n \in \mathbb{N}$, let $g_n : S \rightarrow \mathbb{R}$ be such that (g_n) is uniformly bounded and $g_n \xrightarrow{m-ae} g$. Then, $\int_S g_n dm \rightarrow \int_S g dm$.

Proof. The following implications hold: $g_n \xrightarrow{m-ae} g \implies g_n \xrightarrow{\tilde{m}-ae} g$, which implies by [38] (Theorem 4.1-(1)) $g_n \xrightarrow{\tilde{m}-au} g \implies g_n \xrightarrow{\tilde{m}} g$.

The condition (E) is needed in order to apply [38] (Theorem 4.1-(1)). The conclusion now results from Theorem 6. $\square$

Theorem 9. Let $m : \mathcal{C} \rightarrow [0, +\infty)$ be a monotone set function such that $\tilde{m}$ satisfies the condition (E). For every $n \in \mathbb{N}$, let $g_n : S \rightarrow \mathbb{R}$ be such that (g_n) is uniformly bounded. Then:

$$\int_S (\liminf g_n) dm \leq \liminf \left(\int_S g_n dm \right).$$

Proof. Let $h_n = \inf_{k \geq n} g_k$, $n \in \mathbb{N}$, and $g = \liminf g_n$. Then, $h_n \xrightarrow{m-ae} g$, and according to Theorem 8, it holds that $\lim_{n \rightarrow \infty} \int_S h_n dm = \int_S g dm$. By [22] (Theorem 6) it is $\int_S h_n dm \leq \int_S g_n dm$ for every $n \in \mathbb{N}$, whence the conclusion is obtained. $\square$

In the following theorem, the following inequality is used. For every $\alpha, \beta \in \mathbb{R}$, $|\alpha - \beta|^p \leq 2^p(|\alpha|^p + |\beta|^p)$, for every $p \in [1, \infty)$.

Theorem 10. Let $p \in (1, +\infty)$ and $m : \mathcal{C} \rightarrow [0, +\infty)$ be a monotone set function such that $\tilde{m}$ satisfies the condition (E). Let $g : S \rightarrow \mathbb{R}$ be a bounded function, for every $n \in \mathbb{N}$; let $g_n : S \rightarrow \mathbb{R}$ such that (g_n) is pointwise convergent to g ; let $h_n = 2^{p-1}(|g_n|^p + |g|^p) - |g_n - g|^p$, $n \in \mathbb{N}$, such that (h_n) is uniformly bounded, $|g|^p, |g_n|^p, |g_n - g|^p, h_n, \inf_{k \geq n} h_k \in RL_m(S)$, for every $n \in \mathbb{N}$ and $\|g_n\|_p \rightarrow \|g\|_p$. Then, $\|g_n - g\|_p \rightarrow 0$.

Proof. By the previous formula and Theorem 9, we obtain:

$$2^p \int_S |g|^p dm \leq \liminf_{n \rightarrow \infty} \left(\int_S g_n dm \right) = 2^p \int_S |g|^p dm - \limsup_{n \rightarrow \infty} \left(\int_S |g_n - g|^p dm \right),$$

which concludes the proof. $\square$

5. Convergence Theorems for RL-Integrable Interval-Valued Multifunctions

In this section, we point out some limit theorems for sequences of Riemann–Lebesgue integrable interval-valued multifunctions. We recall from [34] the definition of the Riemann–Lebesgue integral of an interval-valued multifunction with respect to an interval-valued multifunction.

Definition 8. Let $H : S \rightarrow ck(\mathbb{R}_0^+)$, $H = [u, v]$, where $u, v : S \rightarrow \mathbb{R}_0^+$ and $u(s) \leq v(s)$, for every $s \in S$ and the interval-valued set multifunction Γ , given in Formula (1).

We say that $H : S \rightarrow ck(\mathbb{R}_0^+)$ is Riemann–Lebesgue integrable with respect to Γ (on S) (RL_Γ -integrable) if there exists $[c, d] \in ck(\mathbb{R}_0^+)$ such that for every $\varepsilon > 0$, there exists a countable partition Π_ε of S , so that for every partition $\Pi = \{(A_n, s_n)\}_{n \in \mathbb{N}}$ of S with $\Pi \geq \Pi_\varepsilon$, the series $\sigma_{H,\Gamma}(\Pi)$ is convergent and:

$$d_H(\sigma_{H,\Gamma}(\Pi), [c, d]) < \varepsilon. \quad (4)$$

Here, we denote $\sigma_{H,\Gamma}(\Pi) = \sum_{n=1}^{\infty} H(s_n) \Gamma(A_n) = \sum_{n=1}^{\infty} [u(s_n)m_1(A_n), v(s_n)m_2(A_n)]$.

We call $[c, d]$ the Riemann–Lebesgue integral of H relative to Γ (RL_Γ -integral), and we denote:

$$[c, d] = (RL_\Gamma) \int_S H d\Gamma.$$

Remark 5. As for the single-valued case, if it exists, the Riemann–Lebesgue integral is unique. We point out the form of the RL_Γ -integral in the following cases:

(5.i) Suppose $\Gamma = \{m\}$, where $m : \mathcal{C} \rightarrow \mathbb{R}_0^+$ is an arbitrary set function and H is as in Definition 8. Then:

$$(RL_\Gamma) \int_S H d\Gamma = \left[(RL_m) \int_S u dm, (RL_m) \int_S v dm \right];$$

(5.ii) If $\Gamma = [m_1, m_2]$ is given in (1) and $H = \{u\}$, where $u : S \rightarrow \mathbb{R}_0^+$, then:

$$({}^{RL}\Gamma) \int_S H d\Gamma = \left[({}^{RL}m_1) \int_S u dm_1, ({}^{RL}m_2) \int_S u dm_2 \right];$$

(5.iii) Suppose $H = [u, v]$ and $\Gamma = [m_1, m_2]$ are as in Definition 8. Then, H is RL_Γ -integrable on S if and only if u is RL_{m_1} -integrable and v is RL_{m_2} and:

$$({}^{RL}\Gamma) \int_S H d\Gamma = \left[({}^{RL}m_1) \int_S u dm_1, ({}^{RL}m_2) \int_S v dm_2 \right]. \quad (5)$$

Example 5. Let $S = \{s_n, n \in \mathbb{N}\}$ be a countable set, with $\{s_n\} \in \mathcal{C}$, for every $n \in \mathbb{N}$; let Γ be as in (1); let $H : S \rightarrow ck(\mathbb{R}_0^+)$, $H = [u, v]$ be such that the series $\sum_{n=1}^\infty u(s_n)m_1(\{s_n\})$, $\sum_{n=1}^\infty v(s_n)m_2(\{s_n\})$ are convergent in $\mathbb{R}$. Then, H is RL_Γ -integrable and:

$$({}^{RL}\Gamma) \int_S H d\Gamma = \left[\sum_{n=1}^\infty u(s_n)m_1(\{s_n\}), \sum_{n=1}^\infty v(s_n)m_2(\{s_n\}) \right]. \quad (6)$$

In the following, we provide some results regarding convergent sequences of Riemann–Lebesgue integrable interval-valued multifunctions. Firstly, we recall definition of convergence almost everywhere and convergence in the measure for interval-valued multimeasures.

Definition 9. Let $\mu : \mathcal{C} \rightarrow [0, \infty)$ be a set function with $\mu(\emptyset) = 0$, $H : S \rightarrow ck(\mathbb{R}_0^+)$ and a sequence of interval-valued multifunctions $H_n : S \rightarrow ck(\mathbb{R}_0^+)$, $\forall n \in \mathbb{N}$. It is said that:

- (9.i) (H_n) converges μ -almost everywhere to H on S (denoted by $H_n \xrightarrow{\mu-ae} H$) if there exists $C \in \mathcal{C}$ with $\mu(C) = 0$ and $\lim_{n \rightarrow \infty} d_H(H_n(s), H(s)) = 0, \forall s \in S \setminus C$;
- (9.ii) (H_n) μ -converges to H on S (denoted by $H_n \xrightarrow{\mu} H$) if for every $\delta > 0$, $\lim_{n \rightarrow \infty} \mu(B_n(\delta)) = 0$, where $B_n(\delta) = \{s \in S; d_H(H_n(s), H(s)) \geq \delta\} \in \mathcal{C}$.

Theorem 11. Let $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$ be as in (1), $\Gamma = [m_1, m_2]$, so that m_2 is of finite variation. Let $H = [u, v] : S \rightarrow ck(\mathbb{R}_0^+)$ be such that v is bounded, and for every $n \in \mathbb{N}$, let $H_n = [u_n, v_n] : S \rightarrow ck(\mathbb{R}_0^+)$ be such that $(v_n)_n$ is uniformly bounded and $H_n \xrightarrow{\tilde{\Gamma}} H$. Then:

$$\lim_{n \rightarrow \infty} d_H \left(({}^{RL}\Gamma) \int_S H_n d\Gamma, ({}^{RL}\Gamma) \int_S H d\Gamma \right) = 0.$$

Proof. By the properties of the semivariation, from $H_n \xrightarrow{\tilde{\Gamma}} H$, it follows that $u_n \xrightarrow{\tilde{m}_1} u$ and $v_n \xrightarrow{\tilde{m}_2} v$. Since (v_n) is uniformly bounded, it results that (u_n) is uniformly bounded as well. According to Remark 5, it holds that:

$$\begin{aligned} d_H \left(({}^{RL}\Gamma) \int_S H_n d\Gamma, ({}^{RL}\Gamma) \int_S H d\Gamma \right) &= \\ &= \max \left\{ \left| ({}^{RL}m_1) \int_S u_n dm_1 - ({}^{RL}m_1) \int_S u dm_1 \right|, \left| ({}^{RL}m_2) \int_S v_n dm_2 - ({}^{RL}m_2) \int_S v dm_2 \right| \right\}. \end{aligned}$$

Now, using Theorem 6 for (u_n) and (v_n) , the conclusion follows. $\square$

Theorem 12. Let $\mu : \mathcal{C} \rightarrow [0, \infty)$ be monotone and of finite variation, with $\bar{\mu}(S) > 0$, and $\tilde{\mu}$ satisfy the condition (E). Let $H : S \rightarrow ck(\mathbb{R}_0^+)$ be bounded, and for every $n \in \mathbb{N}$, $H_n = [u_n, v_n]$ such that $(v_n)_n$ is uniformly bounded and $H_n \xrightarrow{\mu-ae} H$, then:

$$\lim_{n \rightarrow \infty} d_H \left((RL_\mu) \int_S H_n d\mu, (RL_\mu) \int_S H d\mu \right) = 0.$$

Proof. Since $H_n \xrightarrow{\mu-ae} H$, it results that $u_n \xrightarrow{\mu-ae} u$ and $v_n \xrightarrow{\mu-ae} v$. Then, the conclusion follows by Theorem 8 and Remark 5. $\square$

Theorem 13. Let $\Gamma : \mathcal{C} \rightarrow ck(\mathbb{R}_0^+)$ be as in Formula (1), $\Gamma = [m_1, m_2]$, with m_1, m_2 monotone set functions satisfying (E) and m_2 of finite variation. Let $H = [u, v]$ be such that v is bounded, and for every $n \in \mathbb{N}$, $H_n = [u_n, v_n]$ such that $(v_n)_n$ is uniformly bounded and $H_n \xrightarrow{\tilde{\Gamma}-ae} H$, then:

$$\lim_{n \rightarrow \infty} d_H \left((RL_\Gamma) \int_S H_n d\Gamma, (RL_\Gamma) \int_S H d\Gamma \right) = 0.$$

Proof. According to Remark 1, the set function $\tilde{\Gamma}$ is monotone and $\tilde{\Gamma} = \tilde{m}_2$. Since $H_n \xrightarrow{\tilde{\Gamma}-ae} H$, it follows that $u_n \xrightarrow{\tilde{m}_1-ae} u$ and $v_n \xrightarrow{\tilde{m}_2-ae} v$. By Theorem 8, it results that $\lim_{n \rightarrow \infty} (RL_{m_1}) \int_S u_n dm_1 = (RL_{m_1}) \int_S u dm_1$ and $\lim_{n \rightarrow \infty} (RL_{m_2}) \int_S v_n dm_2 = (RL_{m_2}) \int_S v dm_2$. Then:

$$\begin{aligned} d_H \left((RL_\Gamma) \int_S H_n d\Gamma, (RL_\Gamma) \int_S H d\Gamma \right) &= \\ &= \max \{ |(RL_{m_1}) \int_S u_n dm_1 - (RL_{m_1}) \int_S u dm_1|, |(RL_{m_2}) \int_S v_n dm_2 - (RL_{m_2}) \int_S v dm_2| \} \rightarrow 0. \end{aligned}$$

$\square$

According to Theorem 9 and Remark 5, a Fatou-type theorem for sequences of RL-integrable interval-valued multifunctions holds.

Theorem 14. Let $\mu : \mathcal{C} \rightarrow [0, \infty)$ be monotone and of finite variation, with $\bar{\mu}(S) > 0$ and $\tilde{\mu}$ satisfy the condition (E). For every $n \in \mathbb{N}$, let $H_n = [u_n, v_n]$ such that $(v_n)_n$ is uniformly bounded. Then:

$$(RL_\mu) \int_S (\liminf_n H_n) d\mu \preceq \liminf_n ((RL_\mu) \int_S H_n d\mu).$$

In the following, our aim was to establish convergence results on atoms. Suppose S is a locally compact Hausdorff topological space. We denote by $\mathcal{K}$ the lattice of all compact subsets of S , $\mathcal{B}$ the Borel σ -algebra (i.e., the smallest σ -algebra containing $\mathcal{K}$), and $\mathcal{O}$ the class of all open sets.

Definition 10. $\Gamma : \mathcal{B} \rightarrow ck(\mathbb{R}_0^+)$ is said to be regular if for every set $A \in \mathcal{B}$ and every $\varepsilon > 0$, there exist $K \in \mathcal{L}$ and $D \in \mathcal{O}$ such that $K \subseteq A \subseteq D$ and $\|\Gamma(D \setminus K)\| < \varepsilon$.

We note that Γ is regular if and only if m_2 is regular.

Theorem 15. Let $\Gamma : \mathcal{B} \rightarrow ck(\mathbb{R}_0^+)$ be an interval-valued multisubmeasure as in Formula (1), regular, of finite variation, and satisfying property (σ) , and let $H : S \rightarrow ck(\mathbb{R}_0^+)$ be bounded. If $B \in \mathcal{B}$ is an atom of Γ , then H is RL_Γ -integrable on B and $(RL_\Gamma) \int_B H d\Gamma = H(b) \cdot \Gamma(\{b\})$, where $b \in B$ is the single point resulting from [49] (Corollary 4.7).

Proof. Firstly, we prove that the point $b \in B$ resulting from [49] (Corollary 4.7), is unique for Γ . Because Γ is an interval-valued multisubmeasure, then the set functions m_1 and m_2

is null-additive. Furthermore, by the regularity of Γ , it follows that m_1 and m_2 are regular as well.

Suppose $B \in \mathcal{B}$ is an atom of Γ . Then, B is an atom of m_1 and m_2 . According to [49] (Corollary 4.7), for $m_i, i \in \{1, 2\}$, there exists a unique point $b_i, i \in \{1, 2\}$ such that $m_i(\{b_i\}) = m_i(B)$ and $m_i(B \setminus \{b_i\}) = 0$, for $i \in \{1, 2\}$. We prove that $b_1 = b_2$. Let us suppose that $b_1 \neq b_2$. Since $b_1, b_2 \in B$, then $\{b_1\} \subset B \setminus \{b_2\}$. The monotonicity of m_2 implies $m_2(\{b_1\}) \leq m_2(B \setminus \{b_2\}) = 0$. The inequality $m_1 \leq m_2$ leads us to $m_1(\{b_1\}) = 0$, but $m_1(\{b_1\}) = m_1(B) > 0$, and in this way, we obtain a contradiction. Therefore, there is only one point $b \in B$ such that $m_i(\{b\}) = m_i(B)$ and $m_i(B \setminus \{b\}) = 0$, for $i \in \{1, 2\}$.

Since H is RL_Γ -integrable, then u is RL_{m_1} -integrable and v is RL_{m_2} -integrable. According to [22] (Theorem 11), u, v are Gould integrable in the sense of [50], and moreover:

$$({}^{RL_{m_1}}) \int_B u dm_1 = (G) \int_B u dm_1, \quad ({}^{RL_{m_2}}) \int_B v dm_2 = (G) \int_B v dm_2,$$

where $(G) \int_B u dm_1, (G) \int_B v dm_2$ are the Gould integrals of u, v , respectively. Applying now [20] (Theorem 3) and Remark 5, we have $({}^{RL_\Gamma}) \int_B H d\Gamma = (G) \int_B H d\Gamma = H(b)\Gamma(\{b\})$.

□

Theorem 16. Let $\Gamma : \mathcal{B} \rightarrow ck(\mathbb{R}_0^+)$ be an interval-valued multisubmeasure as in (1), regular of finite variation, and satisfying the property (σ) . Let $H : S \rightarrow ck(\mathbb{R}_0^+)$ be bounded, and for every $n \in \mathbb{N}$, let $H_n = [u_n, v_n]$ be such that $(v_n)_n$ is uniformly bounded. If $B \in \mathcal{B}$ is an atom of Γ and $H_n(b) \xrightarrow{d_H} H(b)$, where $b \in B$ is the single point resulting from Theorem 15, then:

$$\lim_{n \rightarrow \infty} d_H \left(({}^{RL_\Gamma}) \int_B H_n d\Gamma, ({}^{RL_\Gamma}) \int_B H d\Gamma \right) = 0.$$

Proof. By Theorem 15, there exists a unique point $b \in B$ such that:

$$\Gamma(B \setminus \{b\}) = \{0\}, \quad ({}^{RL_\Gamma}) \int_B H d\Gamma = H(b) \cdot \Gamma(B).$$

Similarly, for every $n \in \mathbb{N}$, there is a unique $b_n \in B$ such that:

$$\Gamma(B \setminus \{b_n\}) = \{0\}, \quad ({}^{RL_\Gamma}) \int_B H_n d\Gamma = H_n(b_n) \cdot \Gamma(B).$$

If there exists $n_0 \in \mathbb{N}$ such that $b_{n_0} \neq b$, this means that $\{b_{n_0}\} \subset B \setminus \{b\}$, and by the monotonicity of Γ , it follows that:

$$\Gamma(\{b_{n_0}\}) \preceq \Gamma(B \setminus \{b\}) = \{0\};$$

however, this is not possible since $\Gamma(\{b_{n_0}\}) = \Gamma(B) \neq \{0\}$. Therefore, for every $n \in \mathbb{N}$, $b_n = b$. Then:

$$d_H \left(({}^{RL_\Gamma}) \int_B H_n d\Gamma, ({}^{RL_\Gamma}) \int_B H d\Gamma \right) \leq d_H(H_n(b), H(b)) \cdot \bar{\Gamma}(B) \longrightarrow 0, \quad \text{for } n \rightarrow \infty.$$

□

6. Conclusions

Some new results regarding limit theorems for sequences of Riemann–Lebesgue integrable functions and Riemann–Lebesgue integrable interval-valued multifunctions were presented. Different properties of the RL integral of a real function with respect to a non-additive set function were established. Furthermore, Lebesgue-type convergence theorems and a Fatou-type theorem were obtained in the real case and in the interval-

valued case. The study of other inequalities and the space L^∞ for the Riemann–Lebesgue integrable (multi)functions will be our next research.

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