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An Inventory Model for Non-Instantaneously Deteriorating Items with Nonlinear Stock-Dependent Demand, Hybrid Payment Scheme and Partially Backlogged Shortages

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Abstract: This research work presents an inventory model that involves non-instantaneous deterioration, nonlinear stock-dependent demand, and partially backlogged shortages by considering the length of the waiting time under a hybrid prepayment and cash-on-delivery scheme. The corresponding inventory problem is formulated as a nonlinear constraint optimization problem. The theoretical results for the unique optimal solution are presented, and eight special cases are also identified. Moreover, a salient theoretical result is provided: a certain condition where the optimal inventory policy may or may not involve deterioration. Finally, two numerical examples are provided using a sensitivity analysis to show the validity range of the inventory parameters.

Keywords: inventory; non-instantaneous deterioration; partial backlogging; stock-dependent demand

Citation: Khan, M.A.-A.; Shaikh, A.A.; Cárdenas-Barrón, L.E.; Mashud, A.H.M.; Treviño-Garza, G.; Céspedes-Mota, A. An Inventory Model for Non-Instantaneously Deteriorating Items with Nonlinear Stock-Dependent Demand, Hybrid Payment Scheme and Partially Backlogged Shortages. *Mathematics* **2022**, *10*, 434. <https://doi.org/10.3390/math10030434>

Academic Editors:
Zsolt Tibor Kosztyán, Zoltán Kovács and Frank Werner

Received: 13 October 2021

Accepted: 13 January 2022

Published: 29 January 2022

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1. Introduction

Harris [1] was the first researcher to design an economic order quantity (EOQ) inventory model by presenting the concept of inventory to encounter future demand by storing products in warehouses for an appropriate period of time. Notwithstanding, his inventory model incorporated many practical scenarios in simple forms, for instance, demand is constant and known, the quality level of the products during the storing period is uniform, products are delivered instantaneously after the order has been made, the payment is entirely dependent on the products delivery time, and products are always available to meet market demand. Nowadays, inventory management has become more much complicated because of the emergence of competitive market globalization, and hence, a substantial number of researchers in the inventory field have been developing several efficacious inventory models by taking more realistic assumptions that perfectly model the reality of businesses into consideration. A recent report by [2] indicates that about half of the total number of stored items in any US grocery industry are perishable while the remaining half consists of non-perishable foods and non-food items. Subsequently, the gross revenue of any grocery practitioner depends considerably upon how to manage these perishable items by increasing operational efficiency through the entire business with the help of proper purchasing coordination and by fulfilling the market demand on time. However, a plethora of perishable items (for instance, vegetables, fruits, milk, meat, among others) deteriorate during the storage time period due to their physical ingredients or due to other reasons. Additionally, for many other types of products (for instance, perfumes, radioactive materials, alcohol, among others), practitioners can observe the decay these items over their storage time durations. Due to the deterioration of these products, practitioners' profits

may be badly affected, and hence, the impact of deterioration must be considered in the inventory management of these items. Due to the original quality of the items (for instance, vegetables, fruits, milk, meat, among others), deterioration may not start at the moment when the products are received by the practitioner and might begin sometime after from the items have been received by the practitioner. This kind of phenomenon is termed as non-instantaneous deterioration. In general, customers always prefer to buy what they want from a place where a substantial number of items in perfect condition are stored. This study demonstrates the client inclinations to stock a huge amount product storage as a nonlinear stock-dependent function. In order to improve the operational efficiency of the inventory management for non-instantaneous deteriorating items, this research work outlines an inventory model with nonlinear stock-dependent demand and partial backlogged shortage with a hybrid advance and cash payment agreement. Under this agreement, for a product that is in high demand or a product that is in limited supply in markets, the retailer pays a fraction or the total of the purchase cost prior to receiving the delivery for the purpose of an on-time delivery.

The remaining portion of this research work is systematized as follows: Section 2 presents a literature review. Section 3 states the notation, description, and formulation of the inventory model as a nonlinear constraint optimization problem. Section 4 develops the solution procedure. Section 5 identifies some particular cases. Section 6 studies the impacts of the parameters of the advance payment scheme on the total cost. Section 7 solves some numerical examples to show the validity range of the inventory parameters. Finally, Section 8 provides conclusions and some opportunities for future research.

2. Literature Review

This section articulates the research gap and previous research contributions by describing existing studies related to this research work and then compares the studies in a tabular form.

Considering a constant deterioration rate, Ghare and Schrader [3] formulated an EOQ inventory model. After that, a plethora of inventory models were developed by several researchers by observing the characteristics of different deteriorating items to help practitioners reduce the losses incurred from the impact of deterioration efficaciously by maintaining the order size in a competent manner. Taleizadeh et al. [4] studied a vendor-managed inventory model for deteriorating items by adopting the Stackelberg approach. Some other correlated studies were conducted by Shaikh et al. [5], Tavakoli and Taleizadeh [6], Pando et al. [7], Khan et al. [8], Shaikh et al. [9], Khan et al. [10], Khan et al. [11], and Das et al. [12]. As a matter of fact, due to the original quality of the items (for instance, vegetables, fruits, milk, meat, among others), deterioration may not start from the moment when the products are received by the practitioner, and it might begin after some time the items have been received by the practitioner. This kind of phenomenon is termed as non-instantaneous deterioration. Musa and Sani [13] explored the impact of delayed deterioration on the inventory management policies of practitioners when they allowed a delay in the payment environment. Later, Sarkar and Sarkar [14] further investigated the consequences of delayed deterioration on the retailer's best stock policy when the demand is related to a linear form of the current stock amount. In this direction, it is worth referring to the following recent works: Tyagi et al. [15], Mashud et al. [16], Rastogi et al. [17], Khan et al. [18], Sundararajan et al. [19], and Sundararajan et al. [20].

According to Levin et al. [21], a large number of customers are attracted by the display of huge amounts of stock with lots of variety in super-shops, resulting in the market demand increasing. This is termed as stock-dependent demand. Valliathal and Uthayakumar [22] established an economic production quantity (EPQ) inventory model for time-reliant deteriorating goods with current stock-dependent market demand and partial backordering, and then they solved the problem by proposing a computational methodology. Later, Min et al. [23] extended Valliathal and Uthayakumar [22]'s production-inventory model by including the consequences incurred by delaying payments and solved

the problem mathematically by developing theoretical results. Pando et al. [24] analyzed another inventory management policy by considering the nonlinear stock-dependent market demand instead of the linear demand pattern when the carrying cost is proportional to power form of the current stock amount. Sarkar and Sarkar [14] further investigated the effect of delayed deterioration when the demand is related to a linear form of the current stock amount. Again, Pando et al. [25] extended the previous study by Pando et al. [24] by improving the carrying cost proportionally to the power form of both the current stock amount and the storage duration. Following that, Yang [26] described two inventory models on the basis of the terminal condition under power form of the current stock amount related to demand when the carrying cost is proportional to the nonlinear form of the current stock amount. Sarkar et al. [27] investigated a seasonal product related inventory model with preservation facilities and linearly stock-dependent item demand along with time-dependent partial backordering. Later, Pando et al. [7] considered the nonlinear stock amount-related consumption rate for decaying products with zero terminating scenario and obtained an optimal solution. Again, Pando et al. [28] and Pando et al. [29] further examined the power form of the stock amount-related consumption rate under the objective of optimizing the profit and cost ratio. Recently, Cárdenas-Barrón et al. [30] improved the inventory model developed by Yang [26] by allowing a delay in payments into two inventory models according to the terminal conditions. All of the aforementioned studies related to nonlinear stock-dependent client demand are formulated for non-deteriorating items, except for in a single study Pando et al. [7]. However, Pando et al. [7] considered the moment at which deterioration began as the moment at which the items began to be stored in the warehouse. Consequently, it is critical to make inventory management more robust and flexible to delay deteriorating items under nonlinear stock-dependent client demand by developing efficient and effective inventory models.

Most recently, during the global coronavirus pandemic, the stay-at-home orders have markedly stimulated online grocery shopping, i.e., e-shopping transactions that depend on advanced payment and cash-on-delivery. When online shopping, suppliers typically require a certain segment of the purchasing price in advance of delivery, after the order has been placed, and asks for the rest of the purchasing price when the order is delivered, i.e., cash-on-delivery. By receiving the advanced payment segment for the ordered goods, suppliers cannot only obtain assurance about the orders but can also earn interest from this segment. Relaxing the cash-on-delivery policy from typical inventory models, Zhang [31] introduced an advance payment strategy in the inventory management system for the first time. Connected to this, the researchers developed some noteworthy works, such as an EOQ inventory model, by allowing multiple prepayment payment opportunities (Taleizadeh et al. [32]); multiple prepayment payments opportunities for deteriorating items (Taleizadeh [33]); the inclusion of prepayment opportunities in the supply chain environment (Zhang et al. [34]), multiple prepayment models under capacity constraints (Khan et al. [18], Khan et al. [35] and Shaikh et al. [36]); price discount opportunities on the basis of full or partial prepayment (Tavakoli and Taleizadeh [6], and Khan et al. [11]); and multiple prepayment opportunities for a perishable item with a certain lifetime (Khan et al. [37]).

Practitioners are frequently confronted with two distinct situations during shortages, namely (i) backorders and (ii) sales opportunity when shortages appear due to uncertainty in the marketplaces. In fact, when shortages occur, the customers may wait for new products to arrive or may move to other available sources that are able to meet their requirements. When all of the customers wait for the new product they want to arrive, the situation is termed as complete backordering (Shaikh et al. [5], and San-José et al. [38], and San-José et al. [39]); moreover, when some customers wait for new products to arrive, the situation is defined as partial backordering. Many researchers have been studying partial backordering situations by assuming that a fixed portion of the customers wait for a backordered item, i.e., a constant backlogging rate (Yang [26]; Taleizadeh [33]; Singh et al. [40], Khan et al. [11], Khan et al. [35], and Cárdenas-Barrón et al. [30]). In fact, whether customers

wait for backorders or not depends on the duration of the waiting time. Hence, relaxing the concept of constant backlogging by waiting time, which is sensitive to the backlogging rate, some researchers have described several inventory policies (Sarkar and Sarkar [14]; Tyagi et al. [15]; Sarkar et al. [27]; Khan et al. [8], Khan et al. [37]; Shaikh et al. [36], and Panda et al. [28]). In addition, a comparison of the aforementioned studies and the proposed inventory model is presented in Table 1.

Table 1. A comparison of the inventory models.

Authors	EOQ/EPQ Inventory Model	Stock-Dependent Demand		Deterioration		Payment Scheme		Partial Backordering Rate	
		Linear	Nonlinear	Instantaneous	Non-Instantaneous	Advance	Cash on Delivery	Constant	Waiting Time Dependent
Shaikh et al. [5]	EOQ	✓			✓		✓		
Pando et al. [7]	EOQ		✓	✓			✓		
Khan et al. [11]	EOQ	✓		✓		✓		✓	
Sarkar and Sarkar [14]	EOQ	✓			✓		✓		✓
Tyagi et al. [15]	EOQ	✓			✓		✓		✓
Mashud et al. [16]	EOQ	✓			✓		✓	✓	
Valliathal and Uthayakumar [22]	EPQ	✓		✓			✓		✓
Min et al. [23]	EPQ	✓		✓			✓		
Pando et al. [24]	EOQ		✓				✓		
Pando et al. [25]	EOQ		✓				✓		
Yang [26]	EOQ		✓				✓	✓	
Sarkar et al. [27]	EOQ	✓		✓			✓		✓
Pando et al. [28]	EOQ		✓				✓		
Pando et al. [29]	EOQ		✓				✓		
Cárdenas-Barrón et al. [30]	EOQ		✓				✓	✓	
Alshanbari et al. [41]	EOQ			✓		✓	✓		✓
Rahman et al. [42]	EOQ	✓		✓		✓		✓	
This paper	EOQ		✓		✓	✓	✓		✓

Table 1 indicates that few works have explored the impacts of the nonlinear form of stock amount-related market demand on inventory policies and only a single work (Pando et al. [7]) in the literature has been conducted on decaying items under the nonlinear form of the stock amount-related consumption rate. In Pando et al. [7], deterioration commences as soon as the products are stored in the warehouse of the practitioner. However, they ignored the fact that a plethora of items (for instance, vegetables, fruits, milk, meat, fish, among others) has certain time intervals within the deterioration time span that do not commence immediately due to the original quality of the products. Moreover, Pando et al. [7] considered that products are delivered instantaneously after the practitioner has made the order and that the payment is entirely accomplished when the product is delivered. However, for a product that is in high or limited on the market, practitioners want to pay a fraction or total of the purchase cost prior to receiving the delivery for the purposes of having an on-time delivery. On the other hand, suppliers require a certain segment of the purchasing price after the product has been ordered and in advance of the rest of the segment that is paid when the order is delivered, i.e., cash-on-delivery, in order to obtain assurance about their orders. In addition, Pando et al. [7] did not take another practical scenario in marketplaces into consideration: backordering.

The salient findings of this research work can be abridged as follows: (i) the effects of the power form of the stock amount-related market demand on inventory policies for delayed or deteriorated items are investigated; (ii) a hybrid prepayment and cash-on-delivery payment scheme for the retailer is adopted; (iii) partial backordering on the basis of the length of the customer waiting time is incorporated; and (iv) a certain condition is provided to decide whether the optimal inventory policy involves deterioration. The combination of these four claims made by the present research work is unique in the inventory management literature.

3. Notation, Description and Formulation of the Inventory Model

This research work defines an inventory model for non-instantaneous deteriorating items with stock-dependent demand and partial-backlogged shortages with a hybrid payment system.

3.1. Notation

The following notation is used throughout the development of the inventory model:

Parameter	Units	Description
C_0	USD/order	replenishment cost
c_p	USD/unit	purchasing cost
c_h	USD/unit/unit of time	holding cost per unit per unit of time
c_b	USD/unit/unit of time	shortage cost per unit per unit of time
c_d	USD/unit/unit of time	deterioration cost per unit per unit of time
c_l	USD/unit/unit of time	opportunity cost per unit per unit of time
η	$\eta > 0$	scaling constant for demand rate
θ	$0 < \theta < 1$	deterioration rate
γ	$0 \leq \gamma < 1$	inventory level elasticity of demand rate
δ	$\delta \geq 0$	backlogging parameter
t_s	unit of time	time at which the inventory starts to deteriorate with a rate of θ
$\aleph$	integer value	number of installments to prepay
σ	unit of time	time interval to accomplish the prepayment
ω	%	portion of the purchase price to prepay
i_c	%/unit of time	interest charged for the loan
$I(t)$	units	inventory level at any time t where $0 \leq t \leq t_1 + t_2$
X	USD/cycle	the total cost per cycle
$TC(t_1, t_2)$	USD/unit of time	the total cost per unit of time
Dependent Decision variables		
S	units	maximum stock per cycle
R	units	maximum shortages level
Decision variables		
t_1	unit of time	time at which the inventory level becomes zero
t_2	unit of time	time duration at which the inventory level is negative

3.2. Description of the Inventory Model

Initially, a retailer places an order to a supplier following a hybrid advanced and cash payment scheme. According to this scheme, the order is made by giving the ω portion of the total purchase price with the help of $\aleph$ equal installments during σ time units, and when the order is received by the person paying, then the remaining $(1 - \omega)$ amount is paid instantaneously. The replenishment rate is deemed as infinite. This paper considers that the demand is a power function of the stock level at time t , then it is:

$D(t) = \begin{cases} \eta[I(t)]^\gamma, & \text{when } I(t) > 0 \\ \eta, & \text{when } I(t) \leq 0 \end{cases}$ where $\eta > 0$ and $0 \leq \gamma < 1$. Notice that when $I(t) > 0$, the demand is dependent on stock, and when $I(t) \leq 0$, demand is constant. This type of demand has been used previously by Pando et al. [7]; Pando et al. [25]; Yang [26]; and Cárdenas-Barron et al. [30]. In this inventory model, an infinite planning horizon is considered. It is well-known that product deterioration is a critical phenomenon in inventory management. Moreover, every deteriorating product has a fresh lifetime; after that time, it begins to deteriorate increasingly over time or constantly. Bearing its importance to inventory management, it is incorporated into the proposed inventory model, and the deterioration rate is considered as constant (Taleizadeh et al. [4]; Shaikh et al. [5]; Tavakoli and Taleizadeh [6]; Pando et al. [7]; and Sarkar and Sarkar [14]). In contrast, when there is no stock available in the retailer's warehouse, i.e., there is no deterioration during the shortage time. The $I_1(t)$ denotes the inventory level at any time $t \in [0, t_s]$ when deterioration has no effect on the product on the stock amount. $I_2(t)$ represents for the inventory level at any time $t \in [t_s, t_1]$ when there is product deterioration, while $I_3(t)$ represents the inventory level at any time $t \in [t_1, t_1 + t_2]$ when shortages have appeared. Due to the vagueness of the demand some time, it is difficult for the retailer to foresee how much stock needs to be preserved for the customers. Therefore, natural shortages are inevitable for variable demands. Moreover, it is important to satisfy the shortages more meticulously through proper management. In this inventory model, the backlogging rate depends on the customer waiting time, which is anticipated as $\frac{1}{1+\delta y}$, where y is the customer waiting time (Khan et al. [8], Sarkar et al. [27] and Khan et al. [37]).

Initially, the company places an order for a unique product with $S + R$ units by providing the $\omega c_p(S + R)$ amount, creating loans from a third party (i.e., a bank) through N equal installments during σ time units, and when the order is received, then the remaining $(1 - \omega)$ portion is paid at $t = 0$. The inventory level follows the pattern depicted in Figure 1.

Figure 1. Inventory system for non-instantaneous deterioration with partial backlogging.

3.3. Formulation of the Inventory Model

In the beginning, the inventory is declined due to customer consumption alone. However, after t_s units of time, the stock is not only depleted to satisfy customer demand but also due to deterioration and consequently, the inventory amount reaches zero at time $t = t_1$. Shortly after, shortages appear, and these are partially backlogged shortages with a

rate that depends upon the customer waiting time. Therefore, the inventory amount at any moment preserves the following differential equations:

$$\frac{dI_1(t)}{dt} = -\eta[I_1(t)]^\gamma \quad 0 \leq t \leq t_s \quad (1)$$

with the condition $I_1(0) = S$, and $I_1(t)$ is continuous at $t = t_s$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -\eta[I_2(t)]^\gamma \quad t_s < t \leq t_1 \quad (2)$$

with the subsidiary condition $I_2(t_1) = 0$, and $I_2(t)$ is continuous at $t = t_1$.

$$\frac{dI_3(t)}{dt} = -\frac{\eta}{1 + \delta(t_1 + t_2 - t)} \quad t_1 < t \leq t_1 + t_2 \quad (3)$$

with the auxiliary condition $I_3(t_1 + t_2) = -R$.

Utilizing the condition $I_1(0) = S$ from Equation (1), one has

$$I_1(t) = \left[S^{1-\gamma} - \eta t(1-\gamma) \right]^{\frac{1}{1-\gamma}} \quad 0 \leq t \leq t_s \quad (4)$$

Again, employing $I_2(t_1) = 0$, from Equation (2), one finds

$$I_2(t) = \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{1}{1-\gamma}} \left\{ e^{\theta(1-\gamma)(t_1-t)} - 1 \right\}^{\frac{1}{1-\gamma}} \quad t_s \leq t \leq t_1 \quad (5)$$

Using $I_3(t_1 + t_2) = -R$, from Equation (3), one has

$$I_3(t) = \frac{\eta}{\delta} \ln|1 + \delta(t_1 + t_2 - t)| - R \quad t_1 < t \leq t_1 + t_2 \quad (6)$$

Considering the continuity of the current inventory at $t = t_s$ and $t = t_1$, one has

$$S = [\eta t_s(1-\gamma) + \Delta_1]^{\frac{1}{1-\gamma}} \quad (7)$$

$$R = \frac{\eta}{\delta} \ln|1 + \delta t_2| \quad (8)$$

where $\Delta_1 = \frac{\eta}{\theta} \left\{ e^{\theta(1-\gamma)(t_1-t_s)} - 1 \right\}$.

The following costs are involved in the inventory model.

(a) The ordering cost per cycle is:

$$OC = C_0 \quad (9)$$

(b) The purchasing cost per cycle is:

$$PC = c_p(S + R) \quad (10)$$

(c) The loan cost per cycle from Figure 1 is: $LC = i_c \left[\left(\frac{\omega PC}{N} \right) \left(\frac{\sigma}{N} \right) (1 + 2 + \dots + N) \right]$

$$LC = \frac{i_c \omega \sigma (N+1) c_p (S + R)}{2N} \quad (11)$$

(d) The inventory holding cost per cycle is: $HC = c_h \left[\int_0^{t_s} I_1(t) dt + \int_{t_s}^{t_1} I_2(t) dt \right]$

$$HC = \frac{c_h}{\eta + \alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta + \alpha}{\alpha}} - \Delta_1^{\frac{\eta + \alpha}{\alpha}} + \{ \alpha(t_1 - t_s) \}^{\frac{\eta + \alpha}{\alpha}} \right] \quad (12)$$

where $\alpha = \eta(1-\gamma)$.

(e) The deterioration cost per cycle is: $DC = c_d \left[I_2(t_s) - \eta \int_{t_s}^{t_1} [I_2(t)]^\gamma dt \right]$

$$DC = c_d \left[\Delta_1^{\frac{1}{1-\gamma}} + (\gamma - 1) \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{1}{1-\gamma}} \right] \quad (13)$$

(f) The shortage cost per cycle is: $SC = -c_b \int_{t_1}^{t_1+t_2} I_3(t) dt$

$$SC = \frac{c_b \eta}{\delta} \left[t_2 - \frac{\ln|1 + \delta t_2|}{\delta} \right] \quad (14)$$

(g) The opportunity cost per cycle is: $OC = c_l \eta \int_{t_1}^{t_1+t_2} \left[1 - \frac{1}{1 + \delta(t_1+t_2-t)} \right] dt$

$$OC = c_l \eta \left[t_2 - \frac{\ln|1 + \delta t_2|}{\delta} \right] \quad (15)$$

Detailed calculations of HC and DC are given in Appendix A.

Therefore, the total inventory cost is determined as the sum of the ordering cost, purchasing cost, loan cost, holding cost, deterioration cost, shortage cost, and opportunity cost, that is, $X = C_0 + PC + LC + HC + DC + SC + OC$.

Hence, the total inventory cost per unit of time is

$$TC(t_1, t_2) = \frac{1}{t_1 + t_2} \left[C_0 + c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \left(\{ \eta t_s (1 - \gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right) \right. \\ \left. + \frac{c_h}{\eta + \alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta + \alpha}{\alpha}} - \Delta_1^{\frac{\eta + \alpha}{\alpha}} + \{ \alpha (t_1 - t_s) \}^{\frac{\eta + \alpha}{\alpha}} \right] \right. \\ \left. + c_d \left[\Delta_1^{\frac{1}{1-\gamma}} + (\gamma - 1) \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{1}{1-\gamma}} \right] + (c_l + \frac{c_b}{\delta}) \eta \left[t_2 - \frac{\ln|1 + \delta t_2|}{\delta} \right] \right] \quad (16)$$

where $\Delta_1 = \frac{\eta}{\theta} \left\{ e^{\theta(1-\gamma)(t_1-t_s)} - 1 \right\}$ and $\alpha = \eta(1 - \gamma)$.

Considering the total inventory cost, the nonlinear optimization problem is written as follows:

$$\begin{aligned} \text{Problem : Minimize } TC(t_1, t_2) &= \frac{X}{t_1 + t_2} \\ \text{Subject to } 0 < t_s &\leq t_1 \leq t_1 + t_2 \end{aligned} \quad (17)$$

4. Solution Procedure

The optimization problem given in (17) can be solved by the following solution procedure.

Computing the first and second order partial derivatives of $TC(t_1, t_2)$ with respect to t_1 and t_2 , one obtains

$$\frac{\partial TC(t_1, t_2)}{\partial t_1} = -\frac{X}{(t_1 + t_2)^2} + \frac{1}{t_1 + t_2} \frac{\partial X}{\partial t_1} \quad (18)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} = \frac{2X}{(t_1 + t_2)^3} - \frac{2}{(t_1 + t_2)^2} \frac{\partial X}{\partial t_1} + \frac{1}{t_1 + t_2} \frac{\partial^2 X}{\partial t_1^2} \quad (19)$$

$$\frac{\partial TC(t_1, t_2)}{\partial t_2} = -\frac{X}{(t_1 + t_2)^2} + \frac{1}{t_1 + t_2} \frac{\partial X}{\partial t_2} \quad (20)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_2^2} = \frac{2X}{(t_1 + t_2)^3} - \frac{2}{(t_1 + t_2)^2} \frac{\partial X}{\partial t_2} + \frac{1}{t_1 + t_2} \frac{\partial^2 X}{\partial t_2^2} \quad (21)$$

Now, the necessary conditions for optimizing $TC(t_1, t_2)$ are:

$$\frac{\partial TC(t_1, t_2)}{\partial t_1} = 0 \quad (22)$$

$$\frac{\partial TC(t_1, t_2)}{\partial t_2} = 0 \quad (23)$$

Using Equations (22) and (23), the reduced forms of Equations (18)–(21) can be obtained as follows:

$$X = (t_1 + t_2) \frac{\partial X}{\partial t_1} \quad (24)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} = \frac{1}{t_1 + t_2} \frac{\partial^2 X}{\partial t_1^2} \quad (25)$$

$$X = (t_1 + t_2) \frac{\partial X}{\partial t_2} \quad (26)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_2^2} = \frac{1}{t_1 + t_2} \frac{\partial^2 X}{\partial t_2^2} \quad (27)$$

Combining Equations (24) and (26), one writes

$$\frac{\partial X}{\partial t_1} = \frac{\partial X}{\partial t_2} \quad (28)$$

where $\frac{\partial X}{\partial t_1}$ and $\frac{\partial X}{\partial t_2}$ are computed as

$$\begin{aligned} \frac{\partial X}{\partial t_1} = & \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} + c_d \left[\frac{1}{1-\gamma} \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{\gamma}{1-\gamma}} \right] \\ & + \frac{c_h}{\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} + \alpha^{\frac{\eta+\alpha}{\alpha}} (t_1 - t_s)^{\frac{\eta}{\alpha}} \right], \end{aligned} \quad (29)$$

$$\frac{\partial X}{\partial t_2} = \frac{c_p \eta}{1 + \delta t_2} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} + \left(c_l + \frac{c_b}{\delta} \right) \eta \left(1 - \frac{1}{1 + \delta t_2} \right) \quad (30)$$

Based on the performed analysis, the following lemma is proposed:

Lemma 1. If $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \geq (c_l + \frac{c_b}{\delta})$, then the optimization problem given in (17) does not have an optimal solution.

Proof. See Appendix B. $\square$

It follows from Equation (28) that

$$\begin{aligned} \frac{c_p \eta}{1 + \delta t_2} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} - (c_l + \frac{c_b}{\delta}) \eta \frac{1}{1 + \delta t_2} = & \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} \\ & - (c_l + \frac{c_b}{\delta}) \eta + c_d \left[\frac{1}{1-\gamma} \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{\gamma}{1-\gamma}} \right] \\ & + \frac{c_h}{\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} + \alpha^{\frac{\eta+\alpha}{\alpha}} (t_1 - t_s)^{\frac{\eta}{\alpha}} \right]. \end{aligned} \quad (31)$$

After performing some simplifications, from Equation (31), one has

$$t_2 = \frac{\eta}{\delta} \left[\frac{c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} - (c_l + \frac{c_b}{\delta})}{\Phi(t_1)} - \frac{1}{\eta} \right] \quad (32)$$

where

$$\begin{aligned} \Phi(t_1) = & \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} \\ & - (c_l + \frac{c_b}{\delta}) \eta + c_d \left[\frac{1}{1-\gamma} \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{\gamma}{1-\gamma}} \right] \\ & + \frac{c_h}{\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} + \alpha^{\frac{\eta+\alpha}{\alpha}} (t_1 - t_s)^{\frac{\eta}{\alpha}} \right]. \end{aligned} \quad (33)$$

Equation (32) reveals that t_2 is a function of t_1 . Now, the existence of the unique time at which the inventory level becomes zero, i.e., t_1 , is explored.

Performing differentiation with respect to t_1 on both sides of Equation (31), one obtains

$$\begin{aligned}
 -\frac{\left[c_p\left\{1+\frac{i_c\omega\sigma(\aleph+1)}{2\aleph}\right\}-(c_l+\frac{c_b}{\delta})\right]\eta\delta}{(1+\delta t_2)^2}\frac{dt_2}{dt_1} = c_d \left[\frac{1}{1-\gamma} \left\{ \frac{\gamma}{1-\gamma} \Delta_1^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right\} \right. \\
 \left. + \alpha^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{\gamma-1} (t_1 - t_s)^{\frac{2\gamma-1}{1-\gamma}} \right] \\
 + \frac{c_h}{\alpha} \left[\frac{\eta}{\alpha} (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right. \\
 \left. - \frac{\eta}{\alpha} \Delta_1^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} + \alpha^{\frac{\eta+\alpha}{\alpha}} \frac{\eta}{\alpha} (t_1 - t_s)^{\frac{\eta-\alpha}{\alpha}} \right] \\
 + \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} \left[\frac{\gamma}{1-\gamma} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 \right. \\
 \left. + \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right], \quad (34)
 \end{aligned}$$

where $\frac{\partial \Delta_1}{\partial t_1} = \eta(1-\gamma)e^{\theta(1-\gamma)(t_1-t_s)}$ and $\frac{\partial^2 \Delta_1}{\partial t_1^2} = \eta\theta(1-\gamma)^2 e^{\theta(1-\gamma)(t_1-t_s)}$.

Since $\frac{\partial^2 \Delta_1}{\partial t_1^2} > 0$, $\frac{\eta}{\alpha} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} - \Delta_1^{\frac{\eta-\alpha}{\alpha}} \right\} > 0$, and $\frac{\partial^2 \Delta_1}{\partial t_1^2} \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} - \Delta_1^{\frac{\eta}{\alpha}} \right\} > 0$, the expression on the right-hand side of Equation (34) is always positive. Consequently,

$$\frac{\left[c_p\left\{1+\frac{i_c\omega\sigma(\aleph+1)}{2\aleph}\right\}-(c_l+\frac{c_b}{\delta})\right]\eta\delta}{(1+\delta t_2)^2}\frac{dt_2}{dt_1} < 0 \quad (35)$$

Employing Equation (31) and accomplishing some simplifications, Equation (22) reduces to

$$\frac{1}{(t_1 + t_2)^2} \left[\begin{aligned} & (t_1 + t_2) \left[\frac{c_p\eta}{1+\delta t_2} \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} + (c_l + \frac{c_b}{\delta})\eta \left(1 - \frac{1}{1+\delta t_2} \right) \right] \\ & - C_0 - (c_l + \frac{c_b}{\delta})\eta \left[t_2 - \frac{\ln|1+\delta t_2|}{\delta} \right] - \frac{c_h}{\eta+\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta+\alpha}{\alpha}} - \Delta_1^{\frac{\eta+\alpha}{\alpha}} + \{ \alpha(t_1 - t_s) \}^{\frac{\eta+\alpha}{\alpha}} \right] \\ & - c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - (1-\gamma)\alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{1}{1-\gamma}} \right] - c_p \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} \left(\begin{aligned} & \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} \\ & + \frac{\eta}{\delta} \ln|1+\delta t_2| \end{aligned} \right) \end{aligned} \right] = 0 \quad (36)$$

For convenience, let us define the auxiliary function $\Psi(t_1)$ from Equation (36) as follows:

$$\begin{aligned}
 \Psi(t_1) = (t_1 + t_2) & \left[\frac{c_p\eta}{1+\delta t_2} \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} + (c_l + \frac{c_b}{\delta})\eta \left(1 - \frac{1}{1+\delta t_2} \right) \right] \\
 & - C_0 - (c_l + \frac{c_b}{\delta})\eta \left[t_2 - \frac{\ln|1+\delta t_2|}{\delta} \right] - \frac{c_h}{\eta+\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta+\alpha}{\alpha}} - \Delta_1^{\frac{\eta+\alpha}{\alpha}} + \{ \alpha(t_1 - t_s) \}^{\frac{\eta+\alpha}{\alpha}} \right] \\
 & - c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - (1-\gamma)\alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{1}{1-\gamma}} \right] - c_p \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} \left(\{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1+\delta t_2| \right), \quad (37)
 \end{aligned}$$

where $t_1 \in [t_s, \infty)$.

Differentiating $\Psi(t_1)$ with respect to t_1 , one obtains

$$\begin{aligned}
 \frac{d\Psi(t_1)}{dt_1} = & \left[\frac{c_p\eta}{1+\delta t_2} \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} + (c_l + \frac{c_b}{\delta})\eta \left(1 - \frac{1}{1+\delta t_2} \right) \right] - c_d \left[\frac{1}{1-\gamma} \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{\gamma}{1-\gamma}} \right] \\
 & - (t_1 + t_2) \frac{\eta\delta}{(1+\delta t_2)^2} \frac{dt_2}{dt_1} \left[c_p \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} - (c_l + \frac{c_b}{\delta}) \right] \\
 & - \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \frac{c_h}{\alpha} \left[(t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} + \alpha^{\frac{\eta+\alpha}{\alpha}} (t_1 - t_s)^{\frac{\eta}{\alpha}} \right]. \quad (38)
 \end{aligned}$$

Using the expression in Equation (31), the first order derivative of $\Psi(t_1)$ is expressed as

$$\frac{d\Psi(t_1)}{dt_1} = -(t_1 + t_2) \frac{\eta\delta}{(1+\delta t_2)^2} \frac{dt_2}{dt_1} \left[c_p \left\{ 1 + \frac{i_c\omega\sigma(\aleph+1)}{2\aleph} \right\} - (c_l + \frac{c_b}{\delta}) \right] > 0 \quad (39)$$

Equation (39) reveals that the auxiliary function $\Psi(t_1)$ strictly increases in $t_1 \in [t_s, \infty)$. In addition, at $t_1 = t_s$, from Equation (32), one has

$$t_2 = \frac{1}{\delta}(\zeta_1 - 1) \quad (40)$$

$$\text{where } \zeta_1 = \frac{c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} - (c_l + \frac{c_b}{\delta})}{c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} (\alpha t_s)^{\frac{\gamma}{1-\gamma}} - (c_l + \frac{c_b}{\delta}) + \frac{c_h}{\eta} (\alpha t_s)^{\frac{1}{1-\gamma}}} \quad (41)$$

Now, the expression of the auxiliary function $\Psi(t_1)$ at $t_1 = t_s$ is:

$$\begin{aligned} \Psi(t_s) = & \left\{ t_s + \frac{(\zeta_1 - 1)}{\delta} \right\} \left[\frac{c_p \eta}{\zeta_1} \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} + (c_l + \frac{c_b}{\delta}) \eta \left(1 - \frac{1}{\zeta_1} \right) \right] - C_0 - \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} \\ & - (c_l + \frac{c_b}{\delta}) \frac{\eta}{\delta} (\zeta_1 - 1 - \ln|\zeta_1|) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \left\{ (\alpha t_s)^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|\zeta_1| \right\} (= \Omega, \text{ say}) \end{aligned} \quad (42)$$

It is easy to show that when t_1 becomes larger, $\Psi(t_1)$ tends to be ∞ .

Now, two cases for the optimal t_1 are recognized on the basis of the sign of Ω , i.e., $\Psi(t_s)$, as follows:

Case 1: When $\Omega < 0$, employing the intermediate value theorem, one can straightforwardly observe that Equation (22) represents a unique situation, say $\tilde{t}_1 \in (t_s, \infty)$, which is the unique optimal t_1 minimizing the total inventory cost per unit of time. Moreover, the corresponding optimal shortages duration, say $\tilde{t}_2$, is calculated from Equation (32). Now, the convexity of $TC(t_1, t_2)$ at the point $(\tilde{t}_1, \tilde{t}_2)$ is explored as follows:

Computing the second order partial derivatives of $TC(t_1, t_2)$ at the point $(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)$, one has

$$\begin{aligned} \frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} = & \frac{1}{\tilde{t}_1 + \tilde{t}_2} \left[\frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \left[\frac{\gamma}{1-\gamma} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 \right. \right. \\ & \left. \left. + \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right] \right. \\ & + \frac{c_h}{\alpha} \left[\frac{\eta}{\alpha} (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right. \\ & \left. \left. - \frac{\eta}{\alpha} \Delta_1^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} + \alpha^{\frac{\eta+\alpha}{\alpha}} \frac{\eta}{\alpha} (t_1 - t_s)^{\frac{\eta-\alpha}{\alpha}} \right] \right. \\ & \left. + c_d \left[\frac{1}{1-\gamma} \left\{ \frac{\gamma}{1-\gamma} \Delta_1^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right\} \right. \right. \\ & \left. \left. + \alpha^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{\gamma-1} (t_1 - t_s)^{\frac{2\gamma-1}{1-\gamma}} \right] \right] \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} \quad (43) \end{aligned}$$

Since $\frac{\partial^2 \Delta_1}{\partial t_1^2} > 0$, $\frac{\eta}{\alpha} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} - \Delta_1^{\frac{\eta-\alpha}{\alpha}} \right\} > 0$, and $\frac{\partial^2 \Delta_1}{\partial t_1^2} \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} - \Delta_1^{\frac{\eta}{\alpha}} \right\} > 0$, the expression on the right-hand side of Equation (43) is always positive. Consequently,

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} > 0 \quad (44)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_2^2} \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} = \frac{\eta \delta}{(\tilde{t}_1 + \tilde{t}_2)(1 + \delta \tilde{t}_2)^2} \left\{ \left(c_l + \frac{c_b}{\delta} \right) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \right\} \quad (45)$$

$$\frac{\partial^2 TC(t_1, t_2)}{\partial t_1 \partial t_2} \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} = \frac{\partial^2 TC(t_1, t_2)}{\partial t_2 \partial t_1} \Big|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} = 0 \quad (46)$$

Since $\left. \frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} \right|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} > 0$ and, from Equations (44)–(46), one can straightforwardly observe that $\left. \frac{\partial^2 TC(t_1, t_2)}{\partial t_1^2} \right|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} - \left[\left. \frac{\partial^2 TC(t_1, t_2)}{\partial t_1 \partial t_2} \right|_{(t_1, t_2) = (\tilde{t}_1, \tilde{t}_2)} \right]^2$ is only positive when $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < (c_l + \frac{c_b}{\delta})$.

Taking the above results into consideration, the following theorem can be proposed to achieve the optimal replenishment policy.

Theorem 1. *If $\Omega < 0$ and $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < (c_l + \frac{c_b}{\delta})$, then a unique $t_1^* = \tilde{t}_1$ and $t_2^* = \tilde{t}_2$ exist, where $\tilde{t}_1$ and $\tilde{t}_2$ satisfy Equations (22) and (32), respectively, and $TC(t_1, t_2)$ achieves the global minimum value at $(t_1^*, t_2^*) = (\tilde{t}_1, \tilde{t}_2)$.*

Case 2: When $\Omega \geq 0$, then the total inventory cost per unit of time is an increasing function for $t_1 \in [t_s, \infty)$, as $\Psi(t_1) > 0$ for all $t_1 \in (t_s, \infty)$. Consequently, the value of t_1 satisfying Equation (22) does not exist in this case, and hence, the unique optimal t_1 for minimizing the total cost is achieved at t_s . In this case, there only one decision variable exists, i.e., t_2 , and the corresponding nonlinear optimization problem becomes

$$\begin{aligned} \text{Problem : Minimize } \Pi(t_2) = TC(t_s, t_2) &= \frac{\tilde{X}}{t_s + t_2} \\ \text{Subject to } 0 < t_s = t_1 &\leq t_s + t_2 \end{aligned} \quad (47)$$

where

$$\begin{aligned} \tilde{X} = C_0 + c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} &\left\{ (\alpha t_s)^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right\} \\ &+ \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} + (c_l + \frac{c_b}{\delta}) \eta \left\{ t_2 - \frac{\ln|1 + \delta t_2|}{\delta} \right\}. \end{aligned}$$

The first order derivative of $\Pi(t_2)$ is

$$\Pi'(t_2) = \frac{1}{(t_s + t_2)^2} \left[-\tilde{X} + (t_s + t_2) \frac{d\tilde{X}}{dt_2} \right] \quad (48)$$

For notational convenience, let us define the auxiliary function $Z(t_2)$ from Equation (48) as follows:

$$Z(t_2) = -\tilde{X} + (t_s + t_2) \left[\frac{c_p \eta}{1 + \delta t_2} \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} + \left(c_l + \frac{c_b}{\delta} \right) \eta \left(1 - \frac{1}{1 + \delta t_2} \right) \right] \quad (49)$$

where $t_2 \geq 0$.

In addition, at $t_2 = 0$, the value of $Z(t_2)$ is

$$\begin{aligned} Z(0) = -C_0 - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} &(\alpha t_s)^{\frac{1}{1-\gamma}} - \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} \\ &+ t_s c_p \eta \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \end{aligned} \quad (50)$$

Approaching t_2 tends to ∞ , and one can straightforwardly observe that

$$\lim_{t_2 \rightarrow \infty} Z(t_2) = \infty \quad (51)$$

Differentiating $Z(t_2)$ with respect to t_2 , one has

$$\frac{dZ(t_2)}{dt_2} = (t_s + t_2) \frac{\eta \delta}{(1 + \delta t_2)^2} \left[\left(c_l + \frac{c_b}{\delta} \right) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \right] \quad (52)$$

To investigate the characteristics of Equation (48), let

$$\chi_1 = C_0 + c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} (\alpha t_s)^{\frac{1}{1-\gamma}} + \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}}$$

$$\text{and } \chi_2 = t_s c_p \eta \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\}.$$

Theorem 2.

- (a) If $\chi_1 = \chi_2$ and $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < (c_l + \frac{c_b}{\delta})$, then Equation (48) has a unique root at $t_2 = 0$.
- (b) If $\chi_1 > \chi_2$ and $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < (c_l + \frac{c_b}{\delta})$, then Equation (48) has a unique root of t_2 in $(0, \infty)$.
- (c) If $\chi_1 < \chi_2$, then Equation (48) has no real root of t_2 .

Proof.

- (a) When $\chi_1 = \chi_2$, then $t_2 = 0$ is a root of Equation (48). Moreover, if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} > 0$, then Equation (52) reveals that $Z(t_2)$ is strictly an increasing function of t_2 , and hence, $t_2 = 0$ is the unique root of Equation (48). On the other hand, if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \leq 0$, then Equation (52) shows that $Z(t_2)$ is either a strictly decreasing or constant function of t_2 in $(0, \infty)$, which contradicts the result of Equation (51).
- (b) If $\chi_1 > \chi_2$, then $Z(0) < 0$. When $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \leq 0$, then Equation (52) exposes the fact that $Z(t_2)$ is either a strictly decreasing or constant function of t_2 in $(0, \infty)$, and consequently, Equation (48) has no real root of t_2 . Again, if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} > 0$, then $Z(t_2)$ is strictly an increasing function of t_2 in $(0, \infty)$. Since $\lim_{t_2 \rightarrow \infty} Z(t_2) = \infty$, Equation (48) has a unique real root of t_2 in $(0, \infty)$.
- (c) Finally, when $\chi_1 < \chi_2$, one can observe from Equation (50) that $Z(0)$ is positive. As a result, Equation (48) has no real root of t_2 in $[0, \infty)$ when $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \geq 0$ because $Z(t_2)$ becomes either a strictly increasing or constant function of t_2 in $(0, \infty)$ in this case. On the other hand, if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < 0$, then the function $Z(t_2)$ is a strictly a decreasing function of t_2 in $(0, \infty)$, which opposes the result $\lim_{t_2 \rightarrow \infty} Z(t_2) = \infty$. $\square$

Theorem 3. If $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < (c_l + \frac{c_b}{\delta})$, then $\Pi(t_2)$ is strictly pseudo-concave in t_2 , and hence, a sole optimal t_2^* exists for which $\Pi(t_2)$ is minimized.

Proof. For notational suitability, let us define

$$Z_1(t_2) = C_0 + c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \left\{ (\alpha t_s)^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right\} + \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} + (c_l + \frac{c_b}{\delta}) \eta \left\{ t_2 - \frac{\ln|1 + \delta t_2|}{\delta} \right\}, \quad (53)$$

$$Z_2(t_2) = t_s + t_2 > 0 \quad (54)$$

As a result, $\Pi(t_2)$ is repressed as follows: $\Pi(t_2) = \frac{Z_1(t_2)}{Z_2(t_2)}$. Moreover, $Z_1(t_2)$ is strictly positive as the sum of all of the inventory-associated costs. Taking the differentiation of $Z_1(t_2)$ two times with respect to t_2 , one finds

$$\frac{dZ_1(t_2)}{dt_2} = \frac{c_p \eta}{1 + \delta t_2} \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} + (c_l + \frac{c_b}{\delta}) \eta \left(1 - \frac{1}{1 + \delta t_2} \right) \quad (55)$$

$$\frac{d^2 Z_1(t_2)}{dt_2^2} = \frac{\eta \delta}{(1 + \delta t_2)^2} \left[\left(c_l + \frac{c_b}{\delta} \right) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} \right] \quad (56)$$

The second order derivative $\frac{d^2 Z_1(t_2)}{dt_2^2}$ is positive only when $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < \left(c_l + \frac{c_b}{\delta} \right)$. Therefore, $Z_1(t_2)$ is a differentiable and strictly convex in t_2 if $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < \left(c_l + \frac{c_b}{\delta} \right)$. Moreover, $Z_2(t_2) = t_s + t_2$ is a positive and affine function of t_2 . This implies that $\Pi(t_2)$ is a strictly pseudo-convex function in t_2 , and therefore, there a unique optimal solution of t_2^* exists. This completes the proof of the theorem. $\square$

Setting $\frac{d\Pi_1(t_2)}{dt_2}$, the necessary condition to achieve t_2^* is:

$$(t_s + t_2) \left\{ \frac{c_p \eta}{1 + \delta t_2} \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} + \left(c_l + \frac{c_b}{\delta} \right) \eta \left(1 - \frac{1}{1 + \delta t_2} \right) \right\} - \tilde{X} = 0 \quad (57)$$

Taking the above results into consideration, the following theorem can be proposed to achieve the optimal replenishment policy for $\Omega \geq 0$.

Theorem 4. If $\Omega \geq 0$, $\chi_1 \geq \chi_2$ and $c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph + 1)}{2\aleph} \right\} < \left(c_l + \frac{c_b}{\delta} \right)$, then $\Pi(t_2)$ is strictly pseudo-concave in t_2 , and hence, $TC(t_1, t_2)$ achieves the global minimum value at $t_1^* = t_s$ and t_2^* , which satisfies Equation (57).

Proof. The proof is immediate from Theorems 2 and 3. $\square$

5. Special Cases

The proposed inventory model involves the following inventory models as particular cases:

- (i) If the value of δ is chosen as 0, then the backlogging rate of the current inventory model becomes 1, that is, shortages are completely backlogged.
- (ii) If $\delta \rightarrow \infty$, one has $t_2 \approx 0$ from Equation (32), and hence, the current inventory model reduces to the inventory model without shortages.
- (iii) When $t_s = 0$ and $\delta = 0$, then the current inventory model becomes the inventory model with instantaneous deterioration and is fully backlogged.
- (iv) If $t_s = 0$ and $\delta \rightarrow \infty$, then one has $t_2 \approx 0$ from Equation (32), and therefore, the current inventory model transforms into the inventory model with instantaneous deterioration without shortages.
- (v) If $\omega = 1$, then the current inventory model involves a fully advance payment scheme. On the other hand, when $\omega = 0$ and $\gamma = 0$, then the present inventory model does not involve any advance payment policy under constant demand and hence involves a payment policy that is similar to the one seen in the classical EOQ inventory model.
- (vi) If $\aleph = 1$, then the present model includes a single installment opportunity for prepayment, whereas when $\aleph = 1$ and $\omega = 1$, then the present inventory model becomes a fully advance payment scheme with single installment instead of multiple installment opportunities.

6. Sensitivity Analysis

The impacts of the parameters of the advance payment scheme on the total cost per unit of time are examined in this section.

- (a) Calculating the derivative of $TC(t_1, t_2)$ with respect to $\aleph$, one has

$$\frac{dTC(t_1, t_2)}{dN} = -\frac{1}{t_1 + t_2} \left[\left(\frac{i_c \omega \sigma c_p}{2N^2} \right) \left(\{ \eta t_s (1 - \gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right) \right] < 0 \quad (58)$$

Equation (58) implies that increasing the number of installments to accomplish the prepayment decreases the total cost per unit of time.

(b) Taking the derivative of $TC(t_1, t_2)$ with respect to σ , one obtains

$$\frac{dTC(t_1, t_2)}{d\sigma} = \frac{1}{t_1 + t_2} \left[c_p \left\{ \frac{i_c \omega (N+1)}{2N} \right\} \left(\{ \eta t_s (1 - \gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right) \right] > 0 \quad (59)$$

It reveals that increasing the time duration for accomplishing prepayment opportunities increases the total cost per unit of time.

(c) By performing the first-order differentiation of $TC(t_1, t_2)$ with respect to ω , one obtains

$$\frac{dTC(t_1, t_2)}{d\omega} = \frac{1}{t_1 + t_2} \left[c_p \left\{ \frac{i_c \sigma (N+1)}{2N} \right\} \left(\{ \eta t_s (1 - \gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right) \right] > 0 \quad (60)$$

It follows that the total cost per unit of time increases when the portion of the total purchase price for accomplishing the prepayment scheme increases.

(d) By taking the derivative of $TC(t_1, t_2)$ with respect to i_c , one obtains

$$\frac{dTC(t_1, t_2)}{di_c} = \frac{1}{t_1 + t_2} \left[c_p \left\{ \frac{\omega \sigma (N+1)}{2N} \right\} \left(\{ \eta t_s (1 - \gamma) + \Delta_1 \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|1 + \delta t_2| \right) \right] > 0 \quad (61)$$

Therefore, Equation (61) exposes that the total cost per unit of time increases when the interest charging rate for the borrowed amounts increases.

7. Numerical Examples

To demonstrate the applicability of the inventory model, several numerical examples are solved in this section.

Example 1. The values of the input parameters for the example are from Pando et al. [25] and Khan et al. [37] with some additional data that were adopted in the present work. Let $C_0 = 10$, $c_p = 50$, $c_h = 0.5$, $c_b = 20$, $c_d = 50$, $c_l = 10$, $\eta = 1$, $\theta = 0.05$, $\gamma = 0.1$, $\delta = 0.1$, $t_s = 0.5$, $N = 3$, $\sigma = 5$, $\omega = 0.4$ and $i_c = 0.05$. The values of all of the parameters are in their appropriate units, and LINGO18.0 software was used to solve the example. Now,

$$\begin{aligned} \Omega = & \left\{ t_s + \frac{(\xi_1 - 1)}{\delta} \right\} \left[\frac{c_p \eta}{\xi_1} \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} + \left(c_l + \frac{c_b}{\delta} \right) \eta \left(1 - \frac{1}{\xi_1} \right) \right] \\ & - C_0 - \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} - \left(c_l + \frac{c_b}{\delta} \right) \frac{\eta}{\delta} (\xi_1 - 1 - \ln|\xi_1|) \\ & - c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} \left(\{ \eta t_s (1 - \gamma) \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|\xi_1| \right) = -10.95273 \end{aligned}$$

Since $c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} = 53.333$ and $(c_l + \frac{c_b}{\delta}) = 210$, one can observe that $c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} < (c_l + \frac{c_b}{\delta})$. Therefore, based on Theorem 1, the optimal time durations for positive and negative stock amounts are determined from Equations (22) and (32) and are given by $t_1^* = 1.1771$ and $t_2^* = 0.2718$. Moreover, the global minimum the total cost per unit of time is $TC^* = 57.4792$ (see Figure 2).

Figure 2. Graphical presentation of the convexity of TC against t_1 and t_2 when $\Omega < 0$.

Example 2. Let $C_0 = 10$, $c_p = 100$, $c_h = 15$, $c_b = 40$, $c_d = 100$, $c_l = 20$, $\eta = 1.2$, $\theta = 0.05$, $\gamma = 0.05$, $\delta = 0.4$, $t_s = 0.6$, $N = 3$, $\sigma = 5$, $\omega = 0.4$, and $i_c = 0.05$. The values of all of the parameters are in their appropriate units, and LINGO18.0 software was used to solve the example. In this case, the value of Ω is:

$$\begin{aligned} \Omega = & \left\{ t_s + \frac{(\xi_1 - 1)}{\delta} \right\} \left[\frac{c_p \eta}{\xi_1} \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} + (c_l + \frac{c_b}{\delta}) \eta \left(1 - \frac{1}{\xi_1} \right) \right] \\ & - C_0 - \frac{c_h}{\eta + \alpha} (t_s \alpha)^{\frac{\eta + \alpha}{\alpha}} - (c_l + \frac{c_b}{\delta}) \frac{\eta}{\delta} (\xi_1 - 1 - \ln|\xi_1|) \\ & - c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} \left(\{ \eta t_s (1 - \gamma) \}^{\frac{1}{1-\gamma}} + \frac{\eta}{\delta} \ln|\xi_1| \right) = 0.6191346 > 0 \end{aligned}$$

Since $\chi_1 = 84.4558$, $\chi_2 = 64$, $c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} = 106.6667$, and $(c_l + \frac{c_b}{\delta}) = 120$, one can observe that $\chi_1 > \chi_2$ and $c_p \left\{ 1 + \frac{i_c \omega \sigma (N+1)}{2N} \right\} < (c_l + \frac{c_b}{\delta})$. Consequently, according to Theorem 4, the optimal time duration for positive stock amounts is $t_1^* = t_s = 0.6$, and the optimal time duration for the negative stock amounts is obtained from Equation (52) and is provided by $t_2^* = 1.5487$. In addition, the global minimum of the total cost per unit of time is $TC^* = 134.1203$ (see Figure 3). Figure 3 reveals that the cost function $TC(t_1, t_2)$ is strictly increasing for $t_1 \in [t_s, \infty)$, and hence, $t_1^* = t_s$.

Figure 3. Graphical presentation of the convexity of TC against t_1 and t_2 when $\Omega \geq 0$.

Example 3. The solutions of the special cases mentioned in Section 5 are investigated with the same data from Example 1 and the corresponding conditions for the cases. The computational results are summarized in Table 2.

Table 2. Optimal solutions for the special cases.

Special Case	t_1^*	t_2^*	TC^*
(i) when $\delta = 0$	1.1856	0.2119	57.5717
(ii) when $\delta \rightarrow \infty$	1.22	0	57.9451
(iii) when $t_s = 0$ and $\delta = 0$	1.0833	0.2889	59.112
(iv) when $t_s = 0$ and $\delta \rightarrow \infty$	1.1481	0	59.8604
(v) when $\omega = 1$	1.1292	0.2553	62.1095
(vi) when $\omega = 0$ and $\gamma = 0$	1.7639	0.4864	57.4215
(vii) when $\aleph = 1$	1.1606	0.2666	59.025
(viii) when $\aleph = 1$ and $\omega = 1$	1.0928	0.2396	65.9521

Example 4. By adopting Example 1 in this example, the consequence of estimating the parameters of the optimal results of t_1 , t_2 and total cost TC is explored. The percentage of variations in the optimal results are taken as measures of the analysis, increasing and decreasing the parameters by -20% to $+20\%$. These results are obtained by altering a single parameter value at a time and by keeping the rest of the parameters values unchanged. The outcomes of the analysis are presented in Table 3. The * denotes the optimal solution.

Table 3. Consequence of changing the parameters of the proposed inventory model.

Parameter	% Changes of Parameters	% Changes in TC^*	% Changes in			
			S^*	R^*	t_1^*	t_2^*
δ	−20	0.04	0.19	−5.04	0.17	−5.35
	−10	0.02	0.10	−2.59	0.09	−2.75
	10	−0.02	−0.10	2.73	−0.09	2.91
	20	−0.04	−0.21	5.61	−0.19	5.99
γ	−20	1.53	9.99	21.62	6.76	21.98
	−10	0.79	4.85	11.16	3.25	11.33
	10	−0.85	−4.60	−11.88	−3.02	−12.02
	20	−1.75	−8.97	−24.52	−5.85	−24.77
C_0	−20	−2.61	−12.90	−36.56	−11.48	−36.88
	−10	−1.25	−6.26	−17.52	−5.54	−17.71
	10	1.16	5.95	16.31	5.21	16.57
	20	2.24	11.63	31.64	10.16	32.20
θ	−20	−0.27	4.79	−3.83	4.40	−3.88
	−10	−0.13	2.28	−1.85	2.10	−1.88
	10	0.12	−2.09	1.75	−1.93	1.77
	20	0.24	−4.02	3.41	−3.71	3.45
c_b	−20	−0.22	−1.13	30.68	−0.99	31.22
	−10	−0.10	−0.50	13.28	−0.44	13.48
	10	0.08	0.40	−10.48	0.35	−10.61
	20	0.14	0.73	−18.96	0.64	−19.17
c_h	−20	−0.07	0.59	−1.02	0.52	−1.03
	−10	−0.04	0.30	−0.51	0.26	−0.51
	10	0.04	−0.29	0.50	−0.26	0.51
	20	0.07	−0.59	1.01	−0.52	1.02

Table 3. Cont.

Parameter	% Changes of Parameters	% Changes in TC^*	% Changes in			
			S^*	R^*	t_1^*	t_2^*
c_p	−20	−17.31	11.85	9.91	10.35	10.06
	−10	−8.63	5.54	5.42	4.86	5.50
	10	8.59	−4.90	−6.42	−4.33	−6.50
	20	17.14	−9.27	−13.92	−8.22	−14.08
c_l	−20	−0.01	−0.04	1.19	−0.04	1.20
	−10	−0.004	−0.02	0.59	−0.02	0.60
	10	0.004	0.02	−0.58	0.02	−0.59
	20	0.01	0.04	−1.16	0.04	−1.17
η	−20	−13.10	−70.15	78.60	−57.52	127.00
	−10	−	−	−	−	−
	10	12.29	−20.49	42.26	−25.65	29.84
	20	23.14	−37.42	64.34	−44.91	37.63
t_s	−20	0.41	−2.24	5.76	5.84	−2.24
	−10	0.20	−1.15	2.79	2.83	−1.14
	10	−0.19	1.22	−2.63	−2.66	1.19
	20	−0.36	2.49	−5.09	−5.16	2.44

From Table 3, the following interpretations are given:

- (i) The total cost (TC) is decreased; consequently, with the increase in the inventory level elasticity parameter (γ), the total stock (S), maximum shortage (R), and the time where the stock becomes zero (t_1) sharply fall. This same tendency is also identified in the shortage period (t_2).
- (ii) When the value of the backlogging parameter (δ) increases, the total cost of the system (TC) declines as well as the stock amount (S). In contrast, the value of the shortage amount (R) intensifies; contrasting observations are noticed at point (t_1), where shortages are started. This reveals that an increase in the backlogging parameter triggers the customer demand; as a result, the stock is consumed quickly; consequently, it decreases the time (t_1) at which the shortages commence. The duration of the shortage period (t_2) increase significantly simultaneously as the backlogging parameter (δ) increases.
- (iii) It is observed that an intensification of the ordering cost triggers the value of the stock (S), shortage (R), and the time (t_1), resulting in stock becoming zero. This means that the retailer has much more time to sell their own products without any interruptions (i.e., shortages). It also affects the total cost (TC) positively. This is a positive sign for the retailer, as the ordering cost neutralizes the holding cost of the system. However, an increase in the holding cost (c_h) results in a significant increase in total cost (TC), as the practitioner has to hold the products for a long time before they can be sold.
- (iv) It can be concluded that an upsurge in the purchase cost badly affects the total cost (TC) because the retailer has to buy goods at a high cost. Thence, the retailer reduces the capacity to purchase products, affecting the stock (S) and shortage amount (R).
- (v) As the value of the lost sale cost per unit (c_l) increases, the total cost (TC) decreases, and it has a significant effect on the shortage amount (R), where it diminishes as the lost sale cost increases. The length where the (t_1) shortage commences is less sensitive with regard to the lost sale cost per unit (c_l), while it is moderately sensitive with respect to the rest of the parameters. It should also be noted that the investment in

shortage cost (c_b) intensifies the total cost (TC) as well as increases the amount of shortages (R).

- (vi) The total cost (TC) upsurges as the rate of deterioration increases (θ); consequently it reduces the stock (S) in the retailer's warehouse. This is exhibited by the fact that an intensification in the deterioration rate diminishes the on-hand inventory of the retailer, as deterioration is considered as the obsolescence or decay of products. A massive effect is noted with the increase in the scaling factor of the demand rate (η). When it increases, the total cost (TC) and the stock (S) significantly increases resulting in some of losses in business for the retailer.
- (vii) When the deterioration free time (t_s) increases, the total cost (TC) decreases. Nonetheless, the practitioner's stock rises at the same time because during this period, there is no deterioration, so the stock only depletes due to customer demand. Moreover, a higher fresh item period reduces the number of shortages (R) and, consequently, the duration of the shortages (t_2) as well. In contrast, a proliferation in the fresh item period prolongs the shortage-free duration (t_1), which provides more flexibility to the retailer to sell his products according to market demand. As a result, the retailer can maintain the products' original quality for a longer period of time by providing a better holding environment.

8. Conclusions

This research studies an inventory model that considers the effect of delayed deterioration under nonlinear stock-dependent market demand and partial backlogged shortages with respect to the length of the customer waiting time. In the inventory procedure, demand is modeled as a power function of the inventory level when the inventory level is positive while it is constant during shortage periods. The inventory model was formulated as a nonlinear optimization problem, which was solved mathematically. The convexity was proven mathematically as well as numerically. A certain condition was found for the existence of the optimal solution to the problem. Moreover, a salient theoretical result was obtained that guarantees whether the optimal inventory policy involves deterioration or not. The executed analysis points out that a proliferation in the fresh-item period prolongs the shortage-free duration, which provides more flexibility to the inventory manager to sell his/her products according to market demand. This result has a direct influence on the inventory policy to reduce the cost of inventory management. The total cost increases as the deterioration rate increases because it consequently reduces the stock in the retailer's warehouse. This exhibits the fact that an intensification in the deterioration rate diminishes the on-hand inventory of the retailer, as deterioration is considered the obsolescence or decay of products.

In this research work, an optimal policy for an economic order quantity inventory model was derived under the following limitations:

- (i) The proposed inventory model was derived based on deteriorating products, nonlinear stock-dependent demand, and partially backlogged shortages. However, preservation technology was not applied to reduce the rate of deterioration.
- (ii) Advanced payment with an installment facility was considered for the development of this inventory model. Other facilities such as delay in payment, all unit discount facility, among others are not considered here.

In the future, on the one hand, the inventory model can be expanded for various kinds of variable demands that are dependent on the displayed stock-level, time, quantity discount, etc. On the other hand, the inventory model can also be generalized by including single-level trade credit or two-level credit policies. Finally, one can also explore this inventory model in fuzzy and interval environments. Due to the high nonlinearity of the objective function, soft computing techniques, metaheuristic algorithms, and uncertainty techniques can be applied in order to solve the proposed inventory model.

Author Contributions: Conceptualization, M.A.-A.K., A.A.S., L.E.C.-B., G.T.-G. and A.C.-M.; data curation, M.A.-A.K., A.A.S. and L.E.C.-B.; formal analysis, M.A.-A.K., A.A.S., L.E.C.-B., A.H.M.M., G.T.-G. and A.C.-M.; investigation, M.A.-A.K., A.A.S., L.E.C.-B., A.H.M.M., G.T.-G. and A.C.-M.; methodology, M.A.-A.K., A.A.S., L.E.C.-B., A.H.M.M., G.T.-G. and A.C.-M.; supervision, L.E.C.-B.; validation, M.A.-A.K., L.E.C.-B. and A.H.M.M.; writing—original draft, M.A.-A.K., A.A.S. and A.H.M.M.; writing—review and editing, L.E.C.-B., G.T.-G. and A.C.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data are contained in the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Holding cost (HC): $HC = c_h \left[\int_0^{t_s} I_1(t) dt + \int_{t_s}^{t_1} I_2(t) dt \right]$

$$\begin{aligned} \text{Now, } \int_0^{t_s} I_1(t) dt &= \int_0^{t_s} [\eta t_s(1-\gamma) + \Delta_1 - \eta t(1-\gamma)]^{\frac{1}{1-\gamma}} dt \\ &= \int_0^{t_s} [\alpha(t_s - t) + \Delta_1]^{\frac{1}{1-\gamma}} dt, \text{ where } \alpha = \eta(1-\gamma) \\ &= \frac{1}{\eta+\alpha} \left[(t_s\alpha + \Delta_1)^{\frac{\eta+\alpha}{\alpha}} - \Delta_1^{\frac{\eta+\alpha}{\alpha}} \right] \end{aligned}$$

$$\begin{aligned} \text{Again, } \int_{t_s}^{t_1} I_2(t) dt &= \int_{t_s}^{t_1} \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{1}{1-\gamma}} \left\{ e^{\theta(1-\gamma)(t_1-t)} - 1 \right\}^{\frac{1}{1-\gamma}} dt \\ &= \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{1}{1-\gamma}} \int_{t_s}^{t_1} \left\{ 1 + \frac{\theta\alpha(t_1-t)}{\eta} - 1 \right\}^{\frac{1}{1-\gamma}} dt \\ &= \alpha^{\frac{\eta}{\alpha}} \int_{t_s}^{t_1} (t_1 - t)^{\frac{\eta}{\alpha}} dt = \frac{1}{\eta+\alpha} \{ \alpha(t_1 - t_s) \}^{\frac{\eta+\alpha}{\alpha}} \end{aligned}$$

$$\text{Therefore, } HC = \frac{c_h}{\eta+\alpha} \left[(t_1\alpha + \Delta_1)^{\frac{\eta+\alpha}{\alpha}} - \Delta_1^{\frac{\eta+\alpha}{\alpha}} + \{ \alpha(t_1 - t_s) \}^{\frac{\eta+\alpha}{\alpha}} \right].$$

Deterioration cost (DC): $DC = c_d \left[I_2(t_s) - \eta \int_{t_s}^{t_1} [I_2(t)]^{\gamma} dt \right]$

$$\begin{aligned} DC &= c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{\gamma}{1-\gamma}} \int_{t_s}^{t_1} \left\{ e^{\theta(1-\gamma)(t_1-t)} - 1 \right\}^{\frac{\gamma}{1-\gamma}} dt \right] \\ &= c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{\gamma}{1-\gamma}} \int_{t_s}^{t_1} \left\{ e^{\frac{\theta\alpha(t_1-t)}{\eta}} - 1 \right\}^{\frac{\gamma}{1-\gamma}} dt \right] \end{aligned}$$

$$DC \approx c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - \eta^{\frac{1}{1-\gamma}} \theta^{-\frac{\gamma}{1-\gamma}} \int_{t_s}^{t_1} \left\{ 1 + \frac{\theta\alpha(t_1-t)}{\eta} - 1 \right\}^{\frac{\gamma}{1-\gamma}} dt \right]$$

$$DC = c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - \alpha^{\frac{\gamma}{1-\gamma}} \int_{t_s}^{t_1} (t_1 - t)^{\frac{\gamma}{1-\gamma}} dt \right] = c_d \left[\Delta_1^{\frac{1}{1-\gamma}} - (1-\gamma)\alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{1}{1-\gamma}} \right]$$

Appendix B

From Equation (29), one has

$$\begin{aligned} \frac{\partial X}{\partial t_1} &= \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \left\{ \eta t_s(1-\gamma) + \Delta_1 \right\}^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} + c_d \left[\frac{1}{1-\gamma} \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial \Delta_1}{\partial t_1} - \alpha^{\frac{\gamma}{1-\gamma}} (t_1 - t_s)^{\frac{\gamma}{1-\gamma}} \right] \\ &\quad + \frac{c_h}{\alpha} \left[(t_s\alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial \Delta_1}{\partial t_1} + \alpha^{\frac{\eta+\alpha}{\alpha}} (t_1 - t_s)^{\frac{\eta}{\alpha}} \right] \end{aligned} \quad (A1)$$

The expression on the right-hand side of (A1) only involves the decision variable t_1 where $t_1 \in [t_s, \infty)$. Now at $t_1 = t_s$,

$$\left. \frac{\partial X}{\partial t_1} \right|_{t_1=t_s} = c_p \eta \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} (\alpha t_s)^{\frac{\gamma}{1-\gamma}} + c_h (t_s \alpha)^{\frac{\eta}{\alpha}} > 0 \quad (A2)$$

Moreover,

$$\begin{aligned} \frac{d}{dt_1} \left(\frac{\partial X}{\partial t_1} \right) = & \frac{c_p}{1-\gamma} \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \left[\frac{\gamma}{1-\gamma} \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + \{ \eta t_s (1-\gamma) + \Delta_1 \}^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right] \\ & + c_d \left[\frac{1}{1-\gamma} \left\{ \frac{\gamma}{1-\gamma} \Delta_1^{\frac{2\gamma-1}{1-\gamma}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + \Delta_1^{\frac{\gamma}{1-\gamma}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right\} + \alpha^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{\gamma-1} (t_1 - t_s)^{\frac{2\gamma-1}{1-\gamma}} \right] \\ & + \frac{c_h}{\alpha} \left[\frac{\eta}{\alpha} (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 + (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} - \frac{\eta}{\alpha} \Delta_1^{\frac{\eta-\alpha}{\alpha}} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 - \Delta_1^{\frac{\eta}{\alpha}} \frac{\partial^2 \Delta_1}{\partial t_1^2} \right. \\ & \left. + \alpha^{\frac{\eta+\alpha}{\alpha}} \frac{\eta}{\alpha} (t_1 - t_s)^{\frac{\eta-\alpha}{\alpha}} \right], \end{aligned} \quad (A3)$$

where $\frac{\partial \Delta_1}{\partial t_1} = \eta(1-\gamma)e^{\theta(1-\gamma)(t_1-t_s)}$ and $\frac{\partial^2 \Delta_1}{\partial t_1^2} = \eta\theta(1-\gamma)^2 e^{\theta(1-\gamma)(t_1-t_s)}$.

Since $\frac{\partial^2 \Delta_1}{\partial t_1^2} > 0$, $\frac{\eta}{\alpha} \left(\frac{\partial \Delta_1}{\partial t_1} \right)^2 \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta-\alpha}{\alpha}} - \Delta_1^{\frac{\eta-\alpha}{\alpha}} \right\} > 0$, and $\frac{\partial^2 \Delta_1}{\partial t_1^2} \left\{ (t_s \alpha + \Delta_1)^{\frac{\eta}{\alpha}} - \Delta_1^{\frac{\eta}{\alpha}} \right\} > 0$, the expression on the right-hand side of Equation (A3) is always positive. Therefore, $\frac{\partial X}{\partial t_1} > 0$ for all $t_1 \in [t_s, \infty)$. Combining Equations (28) and (30) and then by performing some simplifications one can write

$$\frac{1}{1 + \delta t_2} < \frac{(c_l + \frac{c_b}{\delta})}{\left[(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \right]} \quad (A4)$$

Since the left-hand side of the inequality (B4) is always positive, the inequality (A4) will be true only if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} > 0$. Otherwise, if $(c_l + \frac{c_b}{\delta}) - c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\} \leq 0$, then one can find $t_2 < 0$, which contradicts the assumption $t_2 \geq 0$. Consequently, if $(c_l + \frac{c_b}{\delta}) \leq c_p \left\{ 1 + \frac{i_c \omega \sigma (\aleph+1)}{2\aleph} \right\}$, then there is no optimal solution for $TC(t_1, t_2)$.

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