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Rota–Baxter (Co)algebra Equation Systems and Rota–Baxter Hopf Algebras

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Abstract: We introduce and discuss the notions of Rota–Baxter bialgebra equation systems and Rota–Baxter Hopf algebras. Then we construct a lot of examples based on Hopf quasigroups.

Keywords: Rota–Baxter Hopf algebras; Rota–Baxter bialgebra equation systems; Hopf quasigroups; two-side Hopf quasimodule bialgebras

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1. Introduction

Rota–Baxter algebras were introduced in [1] in the context of differential operators on commutative Banach algebras. At present, Rota–Baxter algebras have become a useful tool in many fields in mathematics and mathematical physics such as combinatorics, Loday type algebras, Pre-Lie algebras, Pre-Poisson algebras, multiple zeta values, quantum field theory and so on (cf. [2–11]).

Hopf quasigroups were introduced in [12] in order to capture the quasigroup feature of the 7-sphere and they are not required to be associative as a generalisation of Hopf algebras. Let H be a Hopf quasigroup. In [13], the authors obtained a braided monoidal category equivalence between the category of Yetter–Drinfeld quasimodules over H and the category of two-sided two-cosided Hopf quasimodules over H under some suitable assumption (cf. [14–18]).

Rota–Baxter bialgebras were introduced by Ma and Liu in [19]. As a continue of Ma and Liu's paper, in this article, we introduce and discuss the notions of Rota–Baxter bialgebra equation systems and Rota–Baxter Hopf algebras. We give a lot of examples of Rota–Baxter Hopf algebras based on Hopf quasigroups. Our ideal may be regarded as a guide for further development.

This paper is organized as follows. In Section 2, we recall and investigate some basic definitions and properties related to Hopf quasigroups and Rota–Baxter bialgebras.

In Section 3, we introduce and study the notions of Rota–Baxter (co)algebra equation systems. We give a construction of creating examples based on a two-side H -Hopf quasimodule bialgebra and Radford's admissible mapping system for Hopf algebras, respectively.

The final Section 4 is devoted to introduce and study the notion of a compatible Rota–Baxter bialgebra and construct a lot of examples.

Through the paper, we fix a ground field $\mathbb{F}$ of characteristic 0 and work on $\mathbb{F}$. We use the Sweedler's notation to express the coproduct of a coalgebra C as $\Delta(c) = \sum c_1 \otimes c_2$ (cf. [20]). For a left (resp. right) C -comodule coaction ρ^l (resp. ρ^r) on M , we write $\rho^l(m) = \sum m_{(-1)} \otimes m_0$ (resp. $\rho^r(m) = \sum m_0 \otimes m_{(1)}$) for any $m \in M$. Let U, V, W be vector spaces and $g : U \rightarrow V$ and $f : V \rightarrow W$ be two linear maps. Then we write fg simply for the composite $f \circ g$ from U to W . For any vector space V , we use id_V to denote the identity map from V to itself and always write id simply for id_V . We always write T for

the flipping map: $V \otimes V \longrightarrow V \otimes V, v \otimes w \mapsto w \otimes v$. Finally, the natural identification $V \otimes \mathbb{F} \cong V \cong \mathbb{F} \otimes V$ is assumed.

2. Preliminaries

In this section, some basic definitions and properties of Rota–Baxter bialgebras and Hopf quasigroups are recalled and investigated.

2.1. Algebras and Coalgebras

The following notions can be found in [21]. An algebra (A, ∇) is a vector space A equipped with a linear map $\nabla : A \otimes A \longrightarrow A$. The algebra (A, ∇) is called *associative* if $\nabla(id \otimes \nabla) = \nabla(\nabla \otimes id)$. It is customary to write $\nabla(x \otimes y) = xy, \forall x, y \in A$. A *unital algebra* (A, ∇, μ) is a vector space A equipped with two linear maps $\nabla : A \otimes A \longrightarrow A$ and $\mu : \mathbb{F} \longrightarrow A$ such that $\nabla(id \otimes \mu) = id = \nabla(\mu \otimes id)$. Generally, we write $1 \in A$ for $\mu(1_{\mathbb{F}})$.

Dually, a *coalgebra* (C, Δ) and a *bialgebra* (A, ∇, Δ) can be similarly given. In particular, a *Hopf algebra* is a unital counital associative coassociative bialgebra with an antipode (cf. [20]).

2.2. Rota–Baxter Bialgebras

Given $\lambda \in \mathbb{F}$. Recall from [9] (cf. [1,10,11]) that a quadruple (A, ∇, P, λ) is called a *Rota–Baxter algebra of weight λ* if (A, ∇) is an algebra and P is a linear endomorphism of A satisfying

$$\nabla(P \otimes P) = P \nabla[(id \otimes P) + (P \otimes id) + \lambda]. \quad (1)$$

The map P is called a *Rota–Baxter operator*. If $P^2 = P$, we say that (A, P) is *idempotent*. Sometimes, we also say that P is a solution to the Equation (1) on A with the weight λ .

Dually, given $\gamma \in \mathbb{F}$. Recall from [22] (cf. [9,19]) that a quadruple (C, Δ, Q, γ) is called a *Rota–Baxter coalgebra of weight γ* if (C, Δ) is a coalgebra and Q is a linear endomorphism of C such that

$$(Q \otimes Q)\Delta = [(id \otimes Q) + (Q \otimes id) + \gamma]\Delta Q. \quad (2)$$

The map Q is called a *Rota–Baxter operator*. If $Q^2 = Q$, then Q is called *idempotent*. We also say that Q is a solution to the Equation (2) on C with the weight γ .

Recall from [19] that a *Rota–Baxter bialgebra of weight (λ, γ)* is a pentuple $(B, P, Q, \lambda, \gamma)$, where B is a bialgebra ((co)associative or non-(co)associative), (B, P, λ) is a Rota–Baxter algebra of weight λ and (B, Q, γ) is a Rota–Baxter coalgebra of weight γ .

2.3. Hopf Quasimodules

Recall from [12] that a *Hopf quasigroup* is a unital counital coassociative bialgebra $(H, \nabla, \mu, \Delta, \varepsilon)$ armed with a $\mathbb{F}$ -linear map $S : H \longrightarrow H$ such that

$$\sum S(h_1)(h_2 g) = \varepsilon(h)g = \sum h_1(S(h_2)g), \quad \sum (h g_1)S(g_2) = h\varepsilon(g) = \sum (hS(g_1))g_2 \quad (3)$$

for any $h, g \in H$.

The following notions and examples were investigated in [13] (cf. [14,15,23,24]).

Definition 1. Let H be a Hopf quasigroup. Let M be a unital right (resp. left) H -action φ_r (usually we denote it by $\triangleleft$) (resp. φ_l (usually we denote it by $\triangleright$)). We say that M is a right (resp. left) H -quasimodule if for any $h \in H, x \in M$,

$$\begin{aligned} \sum (x \triangleleft h_1) \triangleleft S(h_2) &= x\varepsilon(h) = \sum (x \triangleleft S(h_1)) \triangleleft h_2, \quad x \triangleleft 1 = x, \\ (\text{resp. } \sum S(h_1) \triangleright (h_2 \triangleright x) &= \varepsilon(h)x = \sum h_1 \triangleright (S(h_2) \triangleright x), \quad 1 \triangleright x = x). \end{aligned} \quad (4)$$

Furthermore, let M be a counital coassociative right (resp. left) H -comodule with coaction ρ^r (resp. ρ^l) in the sense of that satisfying

$$(id \otimes \varepsilon)\rho^r = id, (id \otimes \Delta)\rho^r = (\rho^r \otimes id)\rho^r \quad (\text{resp. } (\varepsilon \otimes id)\rho^l = id, (\Delta \otimes id)\rho^l = (id \otimes \rho^l)\rho^l).$$

We say that M is a right-right (resp. left-left) H -Hopf quasimodule if

$$\rho^r(x \triangleleft h) = \rho^r(x)\Delta(h) \quad (\text{resp. } \rho^l(h \triangleright x) = \Delta(h)\rho^l(x)) \quad (5)$$

for any $h \in H$ and $x \in M$.

Example 1. Let H be a Hopf quasigroup.

(1) Let M be a vector space. Then $H \otimes M$ is a left-left H -Hopf quasimodule with structures:

$$h \triangleright (a \otimes x) = ha \otimes x, \text{ and } \rho_{H \otimes M}^l(a \otimes x) = \sum a_1 \otimes (a_2 \otimes x)$$

for all $a, h \in H$ and $x \in M$. In particular $H \otimes H$ and H can be regarded as left-left H -Hopf quasimodules.

(2) Let M be a vector space. Then $M \otimes H$ is a right-right H -Hopf quasimodule with structures:

$$(x \otimes a) \triangleleft h = x \otimes ah, \text{ and } \rho_{M \otimes H}^r(x \otimes a) = \sum (x \otimes a_1) \otimes a_2$$

for all $a, h \in H$ and $x \in M$. In particular $H \otimes H$ and H can be regarded as right-right H -Hopf quasimodules.

The proofs of the following lemma and proposition are straightforward.

Lemma 1. Let M be a right and a left H -Hopf quasimodule. We define

$$P_R(x) = \sum x_0 \triangleleft S(x_{(1)}) \quad \text{and} \quad P_L(x) = \sum S(x_{(-1)}) \triangleright x_0$$

for any $x \in M$. Then we have, for any $x \in M$

- (1) $P_L^2 = P_L, \quad P_R^2 = P_R.$
- (2) $\rho^r(P_R(x)) = P_R(x) \otimes 1, \quad \rho^l(P_L(x)) = 1 \otimes P_L(x).$
- (3) $\sum x_{(-1)} \triangleright P_L(x_0) = x, \quad \sum P_R(x_0) \triangleleft x_{(1)} = x.$

Proposition 1. Let M be a right and a left H -Hopf quasimodule. If the following conditions hold:

$$g \triangleright (S(h) \triangleright x) = (g(S(h))) \triangleright x \quad \text{and} \quad (x \triangleleft S(g)) \triangleleft h = x \triangleleft (S(g)h), \quad (6)$$

for any $g, h \in H$ and $x \in M$, then we have

$$P_L(h \triangleright x) = \varepsilon(h)P_L(x) \quad \text{and} \quad P_R(x \triangleleft h) = P_R(x)\varepsilon(h).$$

Example 2. (1) If $(M, \triangleright, \rho^l)$ is a left-left H -Hopf quasimodule, then $(M, \blacktriangleright, \rho^l)$ is a left-left H -Hopf quasimodule with

$$h \blacktriangleright m = \sum (hm_{(-1)}) \triangleright P_L(m_0) \quad \text{for all } h \in H, m \in M.$$

(2) If $(M, \triangleleft, \rho^r)$ is a right-right H -Hopf quasimodule, then $(M, \blacktriangleleft, \rho^r)$ is a right-right H -Hopf quasimodule with

$$m \blacktriangleleft h = \sum P_R(m_0) \triangleleft (m_{(1)}h) \quad \text{for all } h \in H, m \in M.$$

2.4. Yetter–Drinfeld Quasimodules

The following notion can be found in [23] (cf. [13,15]). Let H be a Hopf quasigroup. Let M be a counital coassociative left H -comodule with coaction ρ^l and a unital left H -

quasimodule with the action $\triangleright$. Then M is said to be a *left-left H -Yetter–Drinfeld quasimodule, or, left H - $\mathcal{YD}$ -quasimodule*, if the following conditions are satisfied:

$$\sum h_1 m_{(-1)} \otimes h_2 \triangleright m_0 = \sum (h_1 \triangleright m)_{(-1)} h_2 \otimes (h_1 \triangleright m)_0, \quad (7)$$

$$\sum (m_{(-1)} h) g \otimes m_0 = \sum m_{(-1)} (h g) \otimes m_0, \quad (8)$$

$$\sum h (m_{(-1)} g) \otimes m_0 = \sum (h m_{(-1)}) g \otimes m_0 \quad (9)$$

for any $h \in H$ and $m \in M$. The Equality (7) is equivalent to $\rho^l(h \triangleright m) = \sum (h_1 m_{(-1)}) S(h_3) \otimes h_2 \triangleright m_0$.

We will denote by ${}^H_H \mathcal{YDQ}$ the category of left-left Yetter–Drinfeld quasimodules over H with the obvious morphisms. Moreover, if we assume that M is a left H -module we say that M is a left-left Yetter–Drinfeld module over H . Obviously, the category ${}^H_H \mathcal{YDQ}$ of left-left Yetter–Drinfeld modules is a subcategory of ${}^H_H \mathcal{YDQ}$.

By [[23], Proposition 1.8], if the antipode S of H is bijective, then ${}^H_H \mathcal{YDQ}$ is a strict braided monoidal category with the braiding τ and its inverse τ^{-1} given by

$$\tau_{M,N}(m \otimes n) = \sum m_{(-1)} \triangleright n \otimes m_0 \quad \text{and} \quad \tau_{M,N}^{-1}(m \otimes n) = \sum n_0 \otimes S^{-1}(n_{(-1)}) \triangleright m$$

for any $m \in M, n \in N$ and $M, N \in {}^H_H \mathcal{YDQ}$.

Recall from [[13], Definition 3.7] that a Hopf quasigroup B is called a *coquasitriangular Hopf quasigroup* if there is a linear map $\sigma : B \otimes B \rightarrow \mathbb{F}$ such that σ is invertible and the following conditions hold, for any $x, y, z \in B$,

$$(QCT1) \quad \sigma(xy, z) = \sum \sigma(y, z_1) \sigma(x, z_2),$$

$$(QCT2) \quad \sigma(x, yz) = \sum \sigma(x_1, y) \sigma(x_2, z),$$

$$(QCT3) \quad \sum \sigma(x_1, y_1) y_2 x_2 = \sum x_1 y_1 \sigma(x_2, y_2).$$

Example 3. Let (B, σ) be a coquasitriangular Hopf quasigroup and M be a counital coassociative left B -comodule satisfying (8) and (9). We provide M a left H -quasimodule structure $a \triangleright x = \sum \sigma(a, x_{(-1)}) x_0$ for any $x \in M$ and $a \in B$. Then M is a left B - $\mathcal{YD}$ -quasimodule.

Furthermore, Equation (6) holds. In fact, for any $h, g \in B$ and $x \in M$

$$\begin{aligned} g \triangleright (S(h) \triangleright x) &= \sum \sigma(S(h), x_{(-1)}) \sigma(g, x_{0(-1)}) x_{00} \\ &= \sum \sigma(S(h), x_{(-1)1}) \sigma(g, x_{(-1)2}) x_0 \\ &\stackrel{(QCT1)}{=} \sum \sigma(g S(h), x_{(-1)}) x_0 \\ &= (g(S(h))) \triangleright x \end{aligned}$$

Similarly for the identity: $(x \triangleleft S(g)) \triangleleft h = x \triangleleft (S(g)h)$, where one provides M a right H -quasimodule structure $x \triangleleft a = \sum \sigma(x_{(1)}, a) x_0$ for any $x \in M$ and $a \in B$.

3. Rota–Baxter (Co)algebra Equation Systems

In this section, we introduce and study the notions of Rota–Baxter (co)algebra systems and give an approach to these equation systems. Our ideal may be regarded as a guide for further development.

Definition 2. Let (A, ∇) be an algebra and P a linear endomorphism of A . We say that P satisfies Rota–Baxter algebra equation system with regard to ∇ , if the one of the following equation systems is satisfied:

$$(RBa1) : \begin{cases} P \nabla[(P \otimes id) - id] = 0, \\ P \nabla(id \otimes P) - \nabla(P \otimes P) = 0. \end{cases} \quad (RBa2) : \begin{cases} P \nabla[(id \otimes P) - id] = 0, \\ P \nabla(P \otimes id) - \nabla(P \otimes P) = 0. \end{cases}$$

Dually, we have

Definition 3. Let (C, Δ) be a coalgebra and Q a linear endomorphism of C . We say that Q satisfies Rota–Baxter coalgebra equation system with regard to Δ , if the one of the following equation systems is satisfied:

$$(RBc1) : \begin{cases} [(Q \otimes id) - id]\Delta Q = 0, \\ (id \otimes Q)\Delta Q - (Q \otimes Q)\Delta = 0. \end{cases} \quad (RBc2) : \begin{cases} [(id \otimes Q) - id]\Delta Q = 0, \\ (Q \otimes id)\Delta Q - (Q \otimes Q)\Delta = 0. \end{cases}$$

For convenience's sake, we have

Lemma 2. Let (A, ∇) be an algebra and P a linear endomorphism of A . Let (C, Δ) be a coalgebra and Q a linear endomorphism of C . Let (B, ∇, Δ) be a bialgebra and R and L linear endomorphisms of B .

(1) If P is a solution to Rota–Baxter algebra equation system (RBa1) or (RBa2), then $(A, \nabla, P, -1)$ is a Rota–Baxter algebra of weight -1 .

(2) If Q is a solution to Rota–Baxter coalgebra equation system (RBc1) or (RBc2), then $(C, \Delta, Q, -1)$ is a Rota–Baxter coalgebra of weight -1 .

(3) If R is a solution to Rota–Baxter algebra equation system (RBa1) or (RBa2), and L a solution to Rota–Baxter coalgebra equation system (RBc1) or (RBc2), then the pentuple $(B, R, L, -1, -1)$ is a Rota–Baxter bialgebra of weight $(-1, -1)$.

Example 4. Let H be a Hopf quasigroup. In Example 1, we have a linear map $P_L = \varepsilon \otimes id_M$ from $H \otimes M$ to itself and $P_R = id_M \otimes \varepsilon$ from $M \otimes H$ to itself. In particular, we have two linear maps: $P_L = \varepsilon \otimes id_H$ and $P_R = id_H \otimes \varepsilon$ from $H \otimes H$ to itself. It is easy to check that $(H, P_L, -1)$ and $(H, P_R, -1)$ are both Rota–Baxter algebra of weight -1 and Rota–Baxter coalgebra of weight -1 .

Furthermore, $(H, P_R, P_L, -1, -1)$ and $(H, P_L, P_R, -1, -1)$ are Rota–Baxter bialgebras of weight $(-1, -1)$.

Definition 4. Let (H, ∇, Δ) be a Hopf quasigroup and let B be a bialgebra. We say that B is a two-side H -Hopf quasimodule bialgebra if $(B, \triangleright, \rho^l)$ is a left-left H -Hopf quasimodule and $(B, \triangleleft, \rho^r)$ is a right-right H -Hopf quasimodule as defined in Definition 1, such that

$$\rho^l(ab) = \rho^l(a)\rho^l(b), \quad \rho^r(ab) = \rho^r(a)\rho^r(b), \quad (10)$$

$$\Delta(h \triangleright a) = \Delta(h)\Delta(a), \quad \Delta(a \triangleleft g) = \Delta(a)\Delta(g), \quad (11)$$

$$(id \otimes \Delta)\rho^l = (\rho^l \otimes id)\Delta, \quad (\Delta \otimes id)\rho^r = (id \otimes \rho^r)\Delta, \quad (12)$$

$$\varphi_l(id \otimes \nabla) = \nabla(\varphi_l \otimes id), \quad \varphi_r(\nabla \otimes id) = \nabla(id \otimes \varphi_r), \quad (13)$$

for any $h, g \in H$ and $a, b \in B$.

Note that if H is a Hopf algebra, then B is called a two-side H -Hopf module bialgebra.

Theorem 1. Let H be a Hopf quasigroup and let M be a two-side H -Hopf quasimodule bialgebra. Set $P_R(x) = \sum x_0 \triangleleft S(x_{(1)})$ and $P_L(x) = \sum S(x_{(-1)}) \triangleright x_0$ for any $x \in M$. If Equation (6) holds, then $(M, P_R, P_L, -1, -1)$ and $(M, P_R, P_L, -1, -1)$ are Rota–Baxter bialgebras of weight $(-1, -1)$.

Proof. We show that P_R is a solution to Rota–Baxter algebra equation system (RBa2). For any $m, n \in M$, we have

$$\begin{aligned}
P_R(mP_R(n)) &= \sum (mP_R(n))_0 \triangleleft S((mP_R(n))_{(1)}) \\
&\stackrel{(10)}{=} \sum (m_0P_R(n)_0) \triangleleft S(m_{(1)}P_R(n)_{(1)}) \\
&\stackrel{\text{Lemma 1(2)}}{=} \sum (m_0P_R(n)) \triangleleft S(m_{(1)}) \\
&\stackrel{(13)}{=} \sum m_0[n_0 \triangleleft S(n_{(1)}) \triangleleft S(m_{(1)})] \\
&\stackrel{(6)}{=} \sum m_0[n_0 \triangleleft S(m_{(1)}n_{(1)})] \\
&\stackrel{(13)}{=} \sum (m_0n_0) \triangleleft S(m_{(1)}n_{(1)}) \\
&\stackrel{(10)}{=} P_R(mn)
\end{aligned}$$

and

$$\begin{aligned}
P_R(P_R(m)n) &= \sum (P_R(m)n)_0 \triangleleft S((P_R(m)n)_{(1)}) \\
&\stackrel{(10)}{=} \sum (P_R(m)_0n_0) \triangleleft S(P_R(m)_{(1)}n_{(1)}) \\
&\stackrel{\text{Lemma 1.3(2)}}{=} \sum (P_R(m)n_0) \triangleleft S(n_{(1)}) \\
&\stackrel{(13)}{=} P_R(m)P_R(n).
\end{aligned}$$

Therefore, we conclude that P_R is a solution to Rota–Baxter algebra equation system (RBa2). A similar computation for P_L gives rise to that P_L is a solution to Rota–Baxter algebra equation system (RBa1).

It follows from Lemma 1(1) that (M, P_R) and (M, P_L) are idempotent Rota–Baxter algebras with the weight -1 .

We also compute, for any $a \in M$

$$\begin{aligned}
((id \otimes P_R)\Delta P_R)(a) &= \sum (id \otimes P_R)(\Delta(a_0 \triangleleft S(a_{(1)}))) \\
&\stackrel{(11)}{=} \sum a_{01} \triangleleft S(a_{(1)})_1 \otimes P_R(a_{02} \triangleleft S(a_{(1)})_2) \\
&\stackrel{\text{Proposition 1.4}}{=} \sum a_{01} \triangleleft S(a_{(1)}) \otimes P_R(a_{02}) \\
&= \sum a_{01} \triangleleft S(a_{(1)}) \otimes (a_{020} \triangleleft S(a_{02(1)})) \\
&\stackrel{(12)}{=} \sum (a_{001} \triangleleft S(a_{(1)})) \otimes (a_{002} \triangleleft S(a_{0(1)})) \\
&\stackrel{(11)}{=} \sum (a_0 \triangleleft S(a_{(1)}))_1 \otimes (a_0 \triangleleft S(a_{(1)}))_2 \\
&= \sum \Delta(a_0 \triangleleft S(a_{(1)})) = (\Delta P_R)(a)
\end{aligned}$$

and similarly one can obtain: $(P_R \otimes id)\Delta P_R = (P_R \otimes P_R)\Delta$. Therefore, P_R is a solution to Rota–Baxter coalgebra equation system (RBc2) and so (M, P_R) is an idempotent Rota–Baxter coalgebra with the weight -1 .

For P_L , one has

$$\begin{aligned}
 \Delta P_L(a) &= \sum \Delta(S(a_{(-1)}) \triangleright a_0) \\
 &\stackrel{(11)}{=} \sum (S(a_{(-1)2}) \triangleright a_{01}) \otimes (S(a_{(-1)1}) \triangleright a_{02}) \\
 &= \sum (S(a_{0(-1)}) \triangleright a_{001}) \otimes (S(a_{(-1)}) \triangleright a_{002}) \\
 &\stackrel{(12)}{=} \sum (S(a_{01(-1)}) \triangleright a_{010}) \otimes (S(a_{(-1)}) \triangleright a_{02}) \\
 &= \sum P_L(a_{01}) \otimes (S(a_{(-1)}) \triangleright a_{02}) \\
 &\stackrel{\text{Proposition 1.4}}{=} \sum P_L(S(a_{(-1)2}) \triangleright a_{01}) \otimes (S(a_{(-1)1}) \triangleright a_{02}) \\
 &\stackrel{(11)}{=} (P_L \otimes id) \sum \Delta(S(a_{(-1)}) \triangleright a_1) \\
 &= (P_L \otimes id) \Delta P_L(a)
 \end{aligned}$$

for any $a \in M$. Thus, P_L satisfies the first identity in (RBc1). Similarly for the second one in (RBc1) and so P_L is a solution to Rota–Baxter coalgebra equation system (RBc1).

Therefore, it follows from Lemma 2(3) and Lemma 1(1) that $(M, P_L, P_R, -1, -1)$ and $(M, P_R, P_L, -1, -1)$ are idempotent Rota–Baxter bialgebra with the weight $(-1, -1)$.

This completes the proof. $\square$

In what follows, we use the notion of Radford’s admissible mapping system for Hopf algebras ([25]) to construct Rota–Baxter bialgebra.

Proposition 2. Let B be a unital counital associative coassociative bialgebra and H a Hopf algebra with an antipode S . Suppose there are two bialgebra homomorphism $p : B \rightarrow H$ and $j : H \rightarrow B$ such that $p \circ j = id_H$. Define two linear maps $P_L : B \rightarrow B$ by $P_L(b) = \sum jSp(b_1)b_2$ and $P_R : B \rightarrow B$ by $P_R(b) = \sum b_1jSp(b_2)$ for any $b \in B$. Then $(B, P_L, P_R, -1, -1)$ and $(B, P_R, P_L, -1, -1)$ are Rota–Baxter bialgebras with the weight $(-1, -1)$. Furthermore, $P_L^2 = P_L$ and $P_R^2 = P_R$.

Proof. For this, we want show that P_L is a solution to the equation systems (Rba1) and (RBc1), and P_R is a solution to the equation systems (Rba2) and (RBc2).

In fact, we have, for any $a, b \in B$

$$\begin{aligned}
 P_L(P_L(a)b) &= \sum P_L((jSp(a_1)a_2)b) \\
 &= \sum [jSp(b_1)(jSp(a_3)jS^2p(a_2))][jSp(a_1)a_4b_2] \\
 &= \sum jSp(b_1)((jSp(a_1)a_2)b_2) \\
 &= \sum [jSp(b_1)jSp(a_1)][a_2b_2] = P_L(ab).
 \end{aligned}$$

and

$$\begin{aligned}
 P_L(aP_L(b)) &= \sum P_L(a(jSp(b_1)b_2)) \\
 &= \sum [(jSp(b_3)jS^2p(b_2))jSp(a_1)][a_2(jSp(b_1)b_4)] \\
 &= \sum jSp(a_1)[a_2(jSp(b_1)b_2)] \\
 &= \sum jSp(a_1)(a_2P_L(b)) = P_L(a)P_L(b).
 \end{aligned}$$

This shows that P_L is a solution to Rota–Baxter algebra equation system (Rba1). For (RBc1), we have,

$$\begin{aligned}
 \sum P_L(P_L(a)_1) \otimes P_L(a)_2 &= \sum jSp((P_L(a)_1)_1)(P_L(a)_1)_2 \otimes P_L(a)_2 \\
 &= \sum jSp(((jSp(a_1)a_2)_1)_1)((jSp(a_1)a_2)_1)_2 \otimes (jSp(a_1)a_2)_2 \\
 &= \sum P_L(a_2) \otimes jSp(a_1)a_3 = \sum P_L(a)_1 \otimes P_L(a)_2.
 \end{aligned}$$

and

$$\begin{aligned}\sum P_L(a)_1 \otimes P_L(P_L(a)_2) &= \sum P_L(a)_1 \otimes jSp((P_L(a)_2)_1)(P_L(a)_2)_2 \\ &= \sum (jSp(a_1)a_2)_1 \otimes jSp(((jSp(a_1)a_2)_2)_1)((jSp(a_1)a_2)_2)_2 \\ &= \sum jSp(a_{11})a_{12} \otimes jSp(a_{21})a_{22} = \sum P_L(a_1) \otimes P_L(a_2).\end{aligned}$$

Similarly for P_R satisfying (RBa2) and (RBc2).

Therefore, it follows from Lemma 2.3(3) that $(B, P_L, P_R, -1, -1)$ and $(B, P_R, P_L, -1, -1)$ are Rota–Baxter bialgebras with the weight $(-1, -1)$.

Finally, we have

$$\begin{aligned}P_L^2(b) &= \sum jSp[(jSp(b_1)b_2)_1](jSp(b_1)b_2)_2 = \sum jSp(jSp(b_2)b_3)(jSp(b_1)b_4) \\ &= \sum (jSp(b_3)jSp(b_2))(jSp(b_1)b_4) = \sum jSp(b_1)b_2 = P_L(b)\end{aligned}$$

and similarly $P_R^2 = P_R$. $\square$

Proposition 3. Let A be a bialgebra. If P and Q satisfy Rota–Baxter Equation (co)algebra systems (RBa1) and (RBc1) (resp. (RBa2) and (RBc2)) such that $PQ = QP$, then the composite QP of P and Q satisfies Rota–Baxter (co)algebra systems (RBa1) and (RBc1) (resp. (RBa2) and (RBc2)).

Proof. For (RBa1), computing we have, for any $a, b \in A$

$$PQ(ab) = PQ(Q(a)b) = QP(Q(a)b) = QP(PQ(a)b) = PQ(PQ(a)b),$$

and

$$PQ(a)PQ(b) = P(Q(a)PQ(b)) = P(Q(a)QP(b)) = PQ(aQP(b)) = PQ(aPQ(b)).$$

Similar for (RBa2).

For (RBc1), one computes, for any $c \in C$

$$\begin{aligned}\sum (PQ)(c)_1 \otimes (PQ)(c)_2 &= \sum P((PQ)(c)_1) \otimes (PQ)(c)_2 \\ &= \sum PQP((QP)(c)_1) \otimes (QP)(c)_2 = \sum (PQ)((PQ)(c)_1) \otimes (PQ)(c)_2.\end{aligned}$$

and

$$\begin{aligned}\sum (PQ)(c)_1 \otimes (PQ)(c)_2 &= \sum P((QP)(c)_1) \otimes QP((QP)(c)_2) \\ &= \sum P((QP)(c)_1) \otimes PQ((QP)(c)_2) = \sum P^2(Q(c)_1) \otimes PQ(Q(c)_2) \\ &= \sum P(Q(c)_1) \otimes PQ(Q(c)_2) = \sum P(Q(c)_1) \otimes QP(Q(c)_2) \\ &= \sum (PQ)(c)_1 \otimes (PQ)((PQ)(c)_2).\end{aligned}$$

Similar for (RBc2).

This completes the proof. $\square$

We finish this section with the following result.

Proposition 4. Let $B = B_1 \oplus B_2$ be a direct sum of two bi-ideals B_1 and B_2 of a bialgebra B . Then for any P_L^i on B_i ($i = 1, 2$) (resp. P_R^i on B_i ($i = 1, 2$)) satisfying Rota–Baxter (co)algebra equation systems (RBa1) and (RBc1) (resp. (RBa2) and (RBc2)), the linear map $P_L : B \rightarrow B$ given by $P_L(x_1, x_2) = (P_L^1(x_1), P_L^2(x_2))$ (resp. $P_R : B \rightarrow B$ given by $P_R(x_1, x_2) = (P_R^1(x_1), P_R^2(x_2))$) for any $x_1 \in A_1, x_2 \in A_2$, solves Rota–Baxter (co)algebra equation systems (RBa1) and (RBc1) (resp. (RBa2) and (RBc2)).

4. Compatible Rota–Baxter Bialgebras and Rota–Baxter Hopf Algebras

We begin with the notion of a compatible Rota–Baxter bialgebra by modifying the definition of a Rota–Baxter bialgebra given in [19] as follows.

Definition 5. A septuple $(B, \nabla, \Delta, P, Q, \lambda, \gamma)$ is called a compatible Rota–Baxter bialgebra of weight (λ, γ) if (B, ∇, Δ) is a bialgebra, (B, ∇, P, λ) is a Rota–Baxter algebra of weight λ , and (B, Δ, Q, γ) is a Rota–Baxter coalgebra of weight γ such that

$$(RBb1) : \begin{cases} \nabla(P \otimes P) - P\nabla(id \otimes P) = Q\nabla(Q \otimes id) + \lambda(Q\nabla), \\ (Q \otimes Q)\Delta - (id \otimes Q)\Delta Q = (P \otimes id)\Delta P + \gamma(\Delta P) \end{cases}$$

or

$$(RBb2) : \begin{cases} \nabla(P \otimes P) - P\nabla(P \otimes id) = Q\nabla(id \otimes Q) + \lambda(Q\nabla), \\ (Q \otimes Q)\Delta - (Q \otimes id)\Delta Q = (id \otimes P)\Delta P + \gamma(\Delta P) \end{cases}$$

As a matter of convenience, we have

Lemma 3. Let P and Q be two linear endomorphisms of a bialgebra (B, ∇, Δ) . Let P be a solution to Rota–Baxter algebra equation system (RBa1) and (RBa2), and let Q be a solution to Rota–Baxter coalgebra equation system (RBc1) and (RBc2). Then $(B, \nabla, \Delta, P, Q, -1, -1)$ and $(B, \nabla, \Delta, Q, P, -1, -1)$ are Rota–Baxter bialgebra of weight $(-1, -1)$.

Proposition 5. With the notations as in Proposition 2. Then $(B, \nabla, \Delta, P_L, P_R, -1, -1)$ and $(B, \nabla, \Delta, P_R, P_L, -1, -1)$ are compatible Rota–Baxter bialgebras of weight $(-1, -1)$.

Definition 6. An octuple $(H, \nabla, \Delta, S, P, Q, \lambda, \gamma)$ is called a Rota–Baxter Hopf algebra of weight (λ, γ) if the quadruple (H, ∇, Δ, S) is a Hopf algebra with antipode S and the septuple $(H, \nabla, \Delta, P, Q, \lambda, \gamma)$ is a compatible Rota–Baxter bialgebra, such that the antipode S of H is compatible with P and Q in the following sense that

$$S \circ P = P \circ S \quad \text{and} \quad S \circ Q = Q \circ S \quad (14)$$

Example 5. Let H be a Hopf algebra and A a braided Hopf algebra in the Yetter–Drinfeld module category ${}^H_H\mathcal{YD}$. Recall from [25] that the Radford’s biproduct $A \star H$ have the following Hopf algebra structures:

$$\begin{cases} (a \star h)(b \star g) = \sum a(h_1 \triangleright b) \star h_2 g, \\ 1_{A \star H} = 1_A \otimes 1_H, \quad \varepsilon_{A \star H} = \varepsilon_A \otimes \varepsilon_H, \\ \Delta_{A \star H}(a \star h) = \sum (a_1 \star a_{2(-1)} h_1) \otimes (a_{2(0)} \star h_2), \\ S_{A \star H}(a \star h) = \sum (1 \star S_H(a_{(-1)} h)) (S_A(a_0) \star 1) \end{cases}$$

for all $a, b \in A$ and $h, g \in H$. Then there are two bialgebra homomorphisms $p : A \star H \rightarrow H, a \star h \mapsto \varepsilon(a)h$ and $j : H \rightarrow A \star H, h \mapsto 1 \star h$ such that $p \circ j = id_H$. By Proposition 2, we can define two linear maps $P_L, P_R : A \star H \rightarrow A \star H$ by

$$P_L(a \star h) = \sum S_H(a_{(-1)2} h_2) \triangleright a_0 \star S_H(a_{(-1)1} h_1) h_3$$

and $P_R(a \star h) = a \star \varepsilon(h)1$. Therefore, P_L is a solution to Rota–Baxter algebra equation system (RBa1) and coalgebra equation system (RBc1), and P_R is a solution to Rota–Baxter algebra equation system (RBa2) and coalgebra equation system (RBc2).

Furthermore, one can directly show that $S_{A \star H} \circ P_L = P_R \circ S_{A \star H}$ as follows:

$$\begin{aligned}
 (S_{A \star H} \circ P_L)(a \star h) &= \sum S_{A \star H}[(1 \star S_H(a_{(-1)}h_1))(a_0 \star h_2)] \\
 &= \sum S_{A \star H}(a_0 \star h_2)S_{A \star H}(1 \star S_H(a_{(-1)}h_1)) \\
 &= \sum (1 \star S_H(a_{0(-1)}h_2))(S_A(a_{00}) \star 1)(1 \star S_H[S_H(a_{(-1)}h_1)]) \\
 &= \sum (1 \star S_H(a_{0(-1)}h_2))(S_A(a_{00}) \star S_H[S_H(a_{(-1)}h_1)]) \\
 &= \sum S_H(a_{0(-1)}h_2)_1 \triangleright S_A(a_{00}) \star S_H(a_{0(-1)}h_2)_2 S_H[S_H(a_{(-1)}h_1)] \\
 &= \sum S_H(a_{(-1)}h)_1 \triangleright S_A(a_{00} \star S_H(a_{(-1)}h)_2 S_H[(S_H(a_{(-1)}h)_3]) \\
 &= \sum S_H(a_{(-1)}h) \triangleright S(a_0) \star 1 \\
 &= \sum P_R[(1 \star S_H(a_{(-1)}h))(S_A(a_0) \star 1)] \\
 &= (P_R \circ S_{A \star H})(a \star h).
 \end{aligned}$$

Furthermore, similar for $P_L \circ S_{A \star H} = S_{A \star H} \circ P_R$. Thus, P_L and P_R satisfy the Condition (14).

Therefore, $(A \star H, \star, \Delta_{A \star H}, S_{A \star H}, P_L, P_R, -1, -1)$ is a Rota–Baxter Hopf algebra of weight $(-1, -1)$.

Proposition 6. Let (H, ∇, Δ) be a Hopf algebra with antipode S . Let A be a Hopf algebra with antipode S . If A is a two-side H -Hopf module bialgebra such that, for $h, g \in H, a \in A$

$$S(a \triangleleft h) = S(h) \triangleright S(a), \quad S(g \triangleright a) = S(a) \triangleleft S(g), \quad (15)$$

$$\rho^l(S(a)) = (S \otimes S)T\rho^r(a), \quad \rho^r(S(a)) = (S \otimes S)T\rho^l(a), \quad (16)$$

then $(H, \nabla, \Delta, S, P_L, P_R, -1, -1)$ is a Rota–Baxter Hopf algebra of weight $(-1, -1)$.

Proof. By Theorem 1, we only show that $P_R \circ S = S \circ P_L$ and $P_L \circ S = S \circ P_R$. In fact, we have

$$\begin{aligned}
 (P_R \circ S)(a) &= \sum S(a)_0 \triangleleft S(S(a)_{(1)}) \\
 &\stackrel{(16)}{=} \sum S(a_0) \triangleleft S[S(a_{(-1)})] \\
 &\stackrel{(15)}{=} \sum S[S(a_{(-1)}) \triangleright a_0] = (S \circ P_L)(a)
 \end{aligned}$$

and similarly for $P_L \circ S = S \circ P_R$.

This completes the proof. $\square$

Theorem 2. With the notations as in Proposition 2. If B is a Hopf algebra with antipode S , then $(B, \nabla, \Delta, P_L, P_R, -1, -1)$ is a Rota–Baxter Hopf algebra of weight $(-1, -1)$.

Proof. By Proposition 5, we only show that $P_R \circ S = S \circ P_L$ and $P_L \circ S = S \circ P_R$. Actually, we have

$$\begin{aligned}
 (S \circ P_L)(b) &= \sum S[jSp(b_1)b_2] \\
 &= \sum S(b_2)SjSp(b_1) = \sum S(b_2)jSpS(b_1) \\
 &= \sum S(b)_1jSpS(b)_2 = (P_R \circ S)(b)
 \end{aligned}$$

and similarly for $P_L \circ S = S \circ P_R$. $\square$

Example 6. With the notations as in Proposition 2. Then B can be made into a two-side H -Hopf module bialgebra with the following H -module and H -comodule structures, for any $h \in H$ and $b \in B$

$$\begin{aligned}
 h \triangleright b &= j(h)b, \quad \rho^l : B \longrightarrow H \otimes B, \quad b \mapsto \sum p(b_1) \otimes b_2, \\
 b \triangleleft h &= bj(h), \quad \rho^r : B \longrightarrow B \otimes H, \quad b \mapsto \sum b_1 \otimes p(b_2).
 \end{aligned}$$

It is easy to check that the conditions: (10)–(13) are satisfied.

Equation (6) also holds. In order to check Equation (15), we have

$$S(h \triangleright b) = S[j(h)b] = S(b)Sj(h) = S(b)jS(h) = S(b) \triangleleft S(h)$$

and similarly for $S(h \triangleright b) = S(b) \triangleleft S(h)$.

Doing calculation one has

$$\rho^l(S(b)) = \sum pS(b_2) \otimes S(b_1) = \sum (S \otimes S)(p(b_2) \otimes b_1) = (S \otimes S)T\rho^r(b)$$

and $\rho^r(S(b)) = (S \otimes S)T\rho^l(b)$. This proves Equation (16).

Then we have two linear maps $P_L, P_R : B \longrightarrow B$ by

$$P_L(b) = \sum S(b_{(-1)}) \triangleright b_0 = \sum Sp(b_1) \triangleright b_2 = \sum jSp(b_1)b_2$$

and

$$P_R(b) = \sum b_0 \triangleleft S(b_{(1)}) = \sum b_1 \triangleleft Sp(b_2) = \sum b_1 jSp(b_2)$$

for any $b \in B$.

It follows from Proposition 6 that we also can obtain Theorem 2.

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