mathematics

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Multiple Periodicity in a Predator–Prey Model with Prey Refuge

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Abstract: We consider a delayed prey–predator model incorporating a refuge with a non-monotone functional response. It is supposed that prey can live in the predatory region and prey refuge, respectively. Based on Mawhin's coincidence degree and nontrivial estimation techniques for a priori bounds of unknown solutions to the operator equation $Lv = \lambda Nv$, we prove the existence of multiple periodic solutions. Finally, an example demonstrates the feasibility of our main results.

Keywords: periodic solutions; prey refuge; non-monotone functional response

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Citation: Lu, W.; Xia, Y. Multiple Periodicity in a Predator–Prey Model with Prey Refuge. *Mathematics* **2022**, *10*, 421. https://doi.org/10.3390/math10030421

Academic Editor: Snezhana Hristova

Received: 11 December 2021
Accepted: 26 January 2022
Published: 28 January 2022

1. Introduction

The predator–prey model [1,2] generally takes the form of

$$\frac{dx_i(t)}{dt} = x_i(t)[r_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t)], \quad i = 1, 2, \cdots, n.$$

To this day, the Lotka–Volterra type system is one of the important themes in mathematical biology. Many scholars have made contributions to it (see, e.g., [3–21]). In the interaction between prey and predator, there always exists a phenomenon of prey refuge. In general, the entire prey population lives in two areas: the predatory region and the prey refuge. From a biological view, the prey refuge can exist without predators; therefore, it can help improve the population density of the prey. In addition, a prey refuge is an effective strategy to reduce predation in the evolution of prey population. For this reason, it was proposed by Gause et al. [22,23]. Moreover, Magalhães [24] considered the refuge effect on the dynamics of thrips prey and mite predators. Ghosh et al. [25] studied the influence on a predator–prey system of adding extra food for predators and incorporating a prey refuge. Xie [26] investigated a prey–predator model incorporating fractional-order factors and a prey refuge. When in a high-prey refuge ecological system, Sahoo et al. [27,28] observed that the possibility of predator extinction could be eliminated by providing additional food to the predator population. Motivated by these works, Jana et al. [29] considered the following prey–predator model with prey refuge:

$$\begin{cases} \frac{du}{dt} = r_1 u(1 - \frac{u}{k_1}) - \sigma_1 u + \sigma_2 v, \\ \frac{dv}{dt} = r_2 v(1 - \frac{v}{k_2}) + \sigma_1 u - \sigma_2 v - \frac{\alpha v w}{a+v}, \\ \frac{dw}{dt} = -\bar{d}w - \gamma w^2 + \frac{\beta v(t-\tau)w(t-\tau)}{a+v(t-\tau)}, \end{cases} \tag{1}$$

where u is the density of the prey in the refuge and v is the density the prey in the predatory region; w denotes the density of the predator in the predatory region; the intrinsic growth rate for u and v is denoted by r_1 and r_2, respectively; the prey migrating in the refuge to the predatory region is given by σ_1 and migrating from the predatory region to the refuge is denoted by σ_2; k_1 and k_2 are the environment carrying capacity for u and v, respectively; $\bar{d}$ and γ are the natural death rate and the density dependent mortality rate of predator, respectively; the rate of the predator consuming prey is denoted by β (suppose that $0 < \beta \leq \alpha$); τ is the delay; and a and α are two parameters in Holling type II functional response.

On the other hand, functional response is also an important factor affecting the predator–prey model. In [30], we proved that there exists at least one positive periodic solution for Jana's model with a Holling type II functional response. However, there exists a functional response unlike the Holling functional responses. Some experiments have indicated that it may occur at the microbial level: when nutrient concentrations reach a high level, they may inhibit specific growth rates; see [31]. Hence, the non-monotone functional response was considered and used to model the inhibitory effect at high concentrations [32–34]. In addition, in real-world applications, some researchers believe that predators living in the predatory region are classified by two fixed ages: one is mature predators, and the other is immature predators—the immature predator have no ability to attack prey; see [35–39].

Motivated by the works of Jana et al. [29] and Chen [34], in the present paper, we establish the following delayed stage-structured prey–predator model with a prey refuge and non-monotone functional response:

$$\begin{cases} \frac{dx}{dt} = r_1(t)x(t)(1-\frac{x(t)}{k_1(t)}) - \sigma_1(t)x(t) + \sigma_2(t)y(t), \\ \frac{dy}{dt} = r_2(t)y(t)(1-\frac{y(t)}{k_2(t)}) + \sigma_1(t)x(t) - \sigma_2(t)y(t) - \frac{\alpha(t)y(t)z_2(t)}{y^2(t)/m+y(t)+a}, \\ \frac{dz_1}{dt} = \frac{\beta(t)y(t)z_2(t)}{y^2(t)/m+y(t)+a} - \frac{\beta(t-\tau)y(t-\tau)z_2(t-\tau)}{y^2(t-\tau)/m+y(t-\tau)+a}e^{-\int_{t-\tau}^{t}k(s)ds} - k(t)z_1(t), \\ \frac{dz_2}{dt} = \frac{\beta(t-\tau)y(t-\tau)z_2(t-\tau)}{y^2(t-\tau)/m+y(t-\tau)+a}e^{-\int_{t-\tau}^{t}k(s)ds} - d_2(t)z_2(t), \end{cases} \tag{2}$$

where $z_1(t)$ and $z_2(t)$ are the density of the immature predator and mature predator at time t, respectively. $r_1(t)$, $r_2(t)$, $k_1(t)$, $k_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $\alpha(t)$, $\beta(t)$, $k(t)$, and $d_2(t)$ are all continuously positive periodic solutions with period ω. $k(t)$ and $d(t)$ are the death rate of the predator. Moreover, the non-monotone functional response is $\frac{\alpha y}{y^2/m+y+a}$. The term

$$\exp\{-\int_{t-\tau}^{t}k(s)ds\}\frac{y(t-\tau)z_2(t-\tau)}{y^2(t-\tau)/m+y(t-\tau)+a}$$

indicates the number of immature predators that were born at time $(t-\tau)$ that still survive at time t and then become mature predators. Moreover, the term $\int_{t-\tau}^{t}k(s)ds$ denotes the stage-structured degree of the immature predator; one can refer to Liu et al. (see [11] pp. 670–671).

In terms of the number of creatures, the initial conditions are associated with

$$(x(t), y(t), z_1(t), z_2(t)) \in C_+ = C([-\tau, 0], \mathbb{R}_+^4), x(0) > 0, y(0) > 0, z_1(0) > 0, z_2(0) > 0.$$

It is well known that the global existence of periodic solutions is a very basic and important problem in the study of periodic population dynamics. Because periodic environments such as the seasonable effect are important factors, the existence of periodic solutions plays a similar role to a global equilibrium in an autonomous model. Thus, the aim of the present paper is to find some suitable conditions of the existence of positive periodic solutions for system (2). Based on Mawhin's coincidence degree theory and other nontrivial techniques, we prove the existence of multiple positive periodic solutions for system (2) in Section 2. In Section 3, we propose some examples to demonstrate the feasibility of our main results.

2. The Existence of Multiple Positive Periodic Solutions

Firstly, we separate the third equation of system (2) from the whole system and obtain the following subsystem:

$$\begin{cases} \dfrac{dx}{dt} = r_1(t)x(t)(1-\frac{x(t)}{k_1(t)}) - \sigma_1(t)x(t) + \sigma_2(t)y(t), \\ \dfrac{dy}{dt} = r_2(t)y(t)(1-\frac{y(t)}{k_2(t)}) + \sigma_1(t)x(t) - \sigma_2(t)y(t) - \dfrac{\alpha(t)y(t)z_2(t)}{y^2(t)/m + y(t) + a}, \\ \dfrac{dz_2}{dt} = -d_2(t)z_2(t) + \dfrac{\beta(t-\tau)y(t-\tau)z_2(t-\tau)}{y^2(t-\tau)/m + y(t-\tau) + a} e^{-\int_{t-\tau}^{t} k(s)ds}. \end{cases} \tag{3}$$

The initial values for system (3) are

$$(x(t), y(t), z_2(t)) \in C_+ = C([-\tau, 0], \mathbb{R}^3_+), x(0) > 0, y(0) > 0, z_2(0) > 0.$$

To obtain the multiple positive periodic solutions of system (3), we summarize the following lemmas.

Lemma 1. *Let $\Omega \in V$ be an open bounded set. Assume that L is a Fredholm operator of index zero and N is L-compact on $\bar{\Omega}$. Furthermore, if the following conditions are fulfilled*
(a) for each fixed $\lambda \in (0,1), v \in \partial\Omega \cap DomL, Lv \neq \lambda Nv$;
(b) for each fixed $v \in \partial\Omega \cap \ker L$, $QNv \neq 0$ and $\deg[JQN, \Omega \cap \ker L, 0] \neq 0$;
then the operator equation $Lv = Nv$ has at least one solution in $\bar{\Omega} \cap DomL$.

Note that operator L is said to be a Fredholm operator of index zero if $\dim\ker L = codim \Im L < \infty$ and $\Im L$ is closed in V. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : U \to U$ and $Q : V \to V$ such that $\Im P = \ker L$ and $\Im L = \ker Q = \Im(I - Q)$, where U, V are Banach spaces. For further symbolic meaning of the concepts in Lemma 2.1, one can refer to [40–42] for details.

Lemma 2 ([32,41])**.** *Assume that $x(t)$ is an ω-periodic function that is continuously differentiable. Then, there exists a $\hat{t} \in [0, \omega]$ such that*

$$|x(t)| \leq |x(\hat{t})| + \int_0^{\omega} |\dot{x}(s)|ds \quad or \quad |x(t)| \geq |x(\hat{t})| - \int_0^{\omega} |\dot{x}(s)|ds.$$

For the sake of convenience, we use notations as follows.

$\bar{\vartheta} = \frac{1}{\omega}\int_0^{\omega} \vartheta(s)ds, \qquad \vartheta^L = \min\limits_{t\in[0,\omega]} \vartheta(t), \qquad \vartheta^M = \max\limits_{t\in[0,\omega]} \vartheta(t),$

$l_{\pm} = \frac{1}{2d_2^L}\Big\{m(\beta^M e^{2\bar{\sigma}_2\omega} - d_2^L) \pm [m^2(\beta^M e^{2\bar{\sigma}_2\omega} - d_2^L)^2 - 4ma(d_2^L)^2]^{\frac{1}{2}}\Big\},$

$h_{\pm} = \frac{1}{2d_2^M e^{2\bar{\sigma}_2\omega}}\Big\{m(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega}) \pm [m^2(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega})^2 - 4ma(d_2^M e^{2\bar{\sigma}_2\omega})^2]^{\frac{1}{2}}\Big\},$

$b_1 = \frac{1}{2}(\frac{\bar{k}_1}{\bar{r}_1})\Big\{(\bar{r}_1 - \bar{\sigma}_1) + [(\bar{\sigma}_1 - \bar{r}_1)^2 + 4(\frac{\bar{r}_1}{\bar{k}_1})\bar{\sigma}_2 l_+ e^{2\bar{\sigma}_2\omega}]^{\frac{1}{2}}\Big\},$

$b_2 = (\frac{\bar{k}_1}{\bar{r}_1})(\bar{r}_1 - \bar{\sigma}_1), \quad b_3 = \frac{a}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r}_2}{\bar{k}_2})l_+ e^{2\bar{\sigma}_2\omega}\},$

$b_4 = \frac{1}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) + \bar{\sigma}_1 b_1 e^{2\bar{\sigma}_1\omega}\} \cdot f(\ln l_+ + 2\bar{\sigma}_2\omega),$

$b_5 = \frac{1}{2\bar{d}_2}\Big\{m(\bar{b} - \bar{d}_2) - [m^2(\bar{d}_2 - \bar{b})^2 - 4ma\bar{d}_2]^{\frac{1}{2}}\Big\},$

$b_6 = \frac{1}{2\bar{d}_2}\Big\{m(\bar{b} - \bar{d}_2) + [m^2(\bar{d}_2 - \bar{b})^2 - 4ma\bar{d}_2]^{\frac{1}{2}}\Big\},$

$b_7 = \frac{1}{2}(\frac{\bar{k}_1}{\bar{r}_1})\Big\{(\bar{r}_1 - \bar{\sigma}_1) + [(\bar{r}_1 - \bar{\sigma}_1)^2 + 4\bar{\sigma}_2 b_5(\frac{\bar{r}_1}{\bar{k}_1})]^{\frac{1}{2}}\Big\},$

$b_8 = \frac{1}{2}(\frac{\bar{k}_1}{\bar{r}_1})\Big\{(\bar{r}_1 - \bar{\sigma}_1) + [(\bar{r}_1 - \bar{\sigma}_1)^2 + 4\bar{\sigma}_2 b_6(\frac{\bar{r}_1}{\bar{k}_1})]^{\frac{1}{2}}\Big\},$

$b_9 = \frac{1}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r}_2}{\bar{k}_2})b_5 + \bar{\sigma}_1\frac{b_7}{b_5}\} \cdot f(\ln b_5),$

$b_{10} = \frac{1}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r}_2}{\bar{k}_2})b_6 + \bar{\sigma}_1\frac{b_8}{b_6}\} \cdot f(\ln b_6).$

Furthermore, suppose that

$(H_1): \beta^M > d_2^L(1 + 2\sqrt{\frac{a}{m}})e^{\tau k^L - 2\bar{\sigma}_2\omega}$;

$(H_2): \bar{r}_2 > \bar{\sigma}_2 + (\frac{\bar{r_2}}{k_2})l_+ e^{2\bar{\sigma}_2\omega}$.

As mentioned in the last paragraph of the introduction, periodic solutions are of great importance. It is reasonable to seek conditions (that is, (H_1), (H_2)) under which the resulting periodic system would have positive periodic solutions. From a biological viewpoint, the assumption (H_1) implies that the rate of predator feeding on prey is affected by the death rate of predators, the time delay, and the migration of prey, while the assumption (H_2) implies that the intrinsic growth rate of prey in a predator region is influenced by prey migration, environmental carrying capacity, and predator predation.

Now, we are in a position to state our main theorem.

Theorem 1. *If (H_1) and (H_2) hold, then system (2) has at least two positive periodic solutions.*

Proof. We prove this theorem into two steps.

Step 1

We claim that there are at least two periodic solutions of subsystem (3). In fact, by the variables transformation

$$v_1(t) = \ln x(t), \qquad v_2(t) = \ln y(t), \qquad v_3(t) = \ln z_2(t),$$

then subsystem (3) reads

$$\begin{cases} \dot{v}_1(t) = r_1(t)(1 - \frac{e^{v_1(t)}}{k_1(t)}) + \sigma_2(t)e^{v_2(t)-v_1(t)} - \sigma_1(t), \\ \dot{v}_2(t) = r_2(t)(1 - \frac{e^{v_2(t)}}{k_2(t)}) + \sigma_1(t)e^{v_1(t)-v_2(t)} - \sigma_2(t) - \frac{\alpha(t)e^{v_3(t)}}{f(v_2(t))}, \\ \dot{v}_3(t) = \frac{\beta(t-\tau)e^{v_2(t-\tau)+v_3(t-\tau)-v_3(t)}}{f(v_2(t-\tau))} \cdot e^{-\int_{t-\tau}^{t} k(s)ds} - d_2(t), \end{cases}$$

where $f(v_2(t)) = \frac{(e^{v_2(t)})^2}{m} + e^{v_2(t)} + a$.

Define

$$V = U = \{v = (v_1, v_2, v_3) \in C(\mathbb{R}, \mathbb{R}^3) | v(t+\omega) = v(t)\},$$

then V, U are both Banach Spaces with the norm $||\cdot||$ as follows:

$$||v|| = \max_{t\in[0,\omega]} |v_1| + \max_{t\in[0,\omega]} |v_2| + \max_{t\in[0,\omega]} |v_3|, v = (v_1, v_2, v_3) \in V \ or \ U.$$

For any $v = (v_1, v_2, v_3) \in V$, it is easy to see that

$$r_1(t)(1 - \frac{e^{v_1(t)}}{k_1(t)}) + \sigma_2(t)e^{v_2(t)-v_1(t)} - \sigma_1(t) := \Theta_1(v, t),$$

$$r_2(t)(1 - \frac{e^{v_2(t)}}{k_2(t)}) + \sigma_1(t)e^{v_1(t)-v_2(t)} - \frac{\alpha(t)e^{v_3(t)}}{f(v_2(t))} - \sigma_2(t) := \Theta_2(v, t)$$

and

$$\frac{\beta(t-\tau)e^{v_2(t-\tau)+v_3(t-\tau)-v_3(t)}}{f(v_2(t))} e^{-\int_{t-\tau}^{t} k(s)ds} - d_2(t) := \Theta_3(v, t)$$

are all ω-periodic functions. Indeed,

$$\begin{aligned}\Theta_1(v(t+\omega),t+\omega) &= r_1(t+\omega)(1-\frac{e^{v_1(t+\omega)}}{k_1(t+\omega)})+\sigma_2(t+\omega)e^{v_2(t+\omega)-v_1(t+\omega)}-\sigma_1(t+\omega)\\ &= r_1(t)(1-\frac{e^{v_1(t)}}{k_1(t)})+\sigma_2(t)e^{v_2(t)-v_1(t)}-\sigma_1(t)\\ &= \Theta_1(v,t).\end{aligned}$$

Clearly, $\Theta_2(v,t),\Theta_3(v,t)$ are also periodic functions in a similar way. Set

$$L: DomL\bigcap V, \qquad L(v_1(t),v_2(t),v_3(t))=(\frac{dv_1(t)}{dt},\frac{dv_2(t)}{dt},\frac{dv_3(t)}{dt}),$$

where $DomL=\{(v_1,v_2,v_3)\in C(\mathbb{R},\mathbb{R}^3)\}$ and $N:V\to V$ is given by

$$N\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}\Theta_1(v,t)\\ \Theta_2(v,t)\\ \Theta_3(v,t)\end{pmatrix}.$$

Define

$$P\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}=Q\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}\frac{1}{\omega}\int_0^\omega v_1(t)dt\\ \frac{1}{\omega}\int_0^\omega v_2(t)dt\\ \frac{1}{\omega}\int_0^\omega v_3(t)dt\end{pmatrix},\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}\in V=U.$$

It follows form the above definitions that $\ker L=\{v\in V|v=C_0,C_0\in\mathbb{R}^3\}$ and $ImL=\{u\in U|\int_0^\omega u(t)dt=0\}$. $\dim\ker L=\operatorname{codim} ImL=3<\infty$, $ImP=\ker L,\ker Q=ImL=Im(I-Q)$. It is easy to see that the inverse $K_p(u)=\int_0^t u(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^t u(s)dsdt$. Therefore,

$$QNv=\begin{pmatrix}\frac{1}{\omega}\int_0^\omega \Theta_1(v,t)dt\\ \frac{1}{\omega}\int_0^\omega \Theta_2(v,t)dt\\ \frac{1}{\omega}\int_0^\omega \Theta_3(v,t)dt\end{pmatrix}$$

and

$$K_p(I-Q)Nv=\int_0^t Nv(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^t Nv(s)dsdt-(\frac{t}{\omega}-\frac{1}{2})\int_0^\omega Nv(s)ds.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. It follows from the Arzela–Ascoli theorem [34,40] that N is L-compact on $\bar{\Omega}$ with any open-bounded set $\Omega\subset V$.

The next work is to find an appropriate open-bounded subset Ω for the application of the continuation theorem. Corresponding to $Lv=\lambda Nv$ for some $\lambda\in(0,1)$, we obtain

$$\begin{cases}\dot{v}_1(t)=\lambda[r_1(t)(1-\frac{e^{v_1(t)}}{k_1(t)})+\sigma_2(t)e^{v_2(t)-v_1(t)}-\sigma_1(t)],\\ \dot{v}_2(t)=\lambda[r_2(t)(1-\frac{e^{v_2(t)}}{k_2(t)})+\sigma_1(t)e^{v_1(t)-v_2(t)}-\sigma_2(t)-\frac{\alpha(t)e^{v_3(t)}}{f(v_2(t))}],\\ \dot{v}_3(t)=\lambda[\frac{\beta(t-\tau)e^{v_2(t-\tau)+v_3(t-\tau)-v_3(t)}}{f(v_2(t-\tau))}\cdot e^{-\int_{t-\tau}^t k(s)ds}-d_2(t)].\end{cases}\tag{4}$$

Suppose that $v=(v_1(t),v_2(t),v_3(t))^T\in V$ is a solution of Equation (4). Integrating Equation (4) over the interval $[0,\omega]$ leads to

$$\begin{cases}\bar{\sigma}_1\omega=\int_0^\omega[r_1(t)+\sigma_2(t)e^{v_2(t)-v_1(t)}-\frac{r_1(t)}{k_1(t)}e^{v_1(t)}]dt,\\ \bar{\sigma}_2\omega=\int_0^\omega[r_2(t)+\sigma_1(t)e^{v_1(t)-v_2(t)}-\frac{r_2(t)}{k_2(t)}e^{v_2(t)}-\frac{\alpha(t)e^{v_3(t)}}{f(v_2(t))}]dt,\\ \bar{d}_2\omega=\int_0^\omega[\frac{\beta(t-\tau)e^{v_2(t-\tau)+v_3(t-\tau)-v_3(t)}}{f(v_2(t-\tau))}\cdot e^{-\int_{t-\tau}^t k(s)ds}]dt.\end{cases}\tag{5}$$

From the first equation of (4) and (5), we have

$$\begin{aligned}\int_0^{\omega}|\dot{v}_1(t)|dt &= \lambda\int_0^{\omega}\left|r_1(t)(1-\frac{e^{v_1(t)}}{k_1(t)})+\sigma_2(t)e^{v_2(t)-v_1(t)}-\sigma_1(t)\right|dt\\ &< \int_0^{\omega}\left|r_1(t)-\frac{r_1(t)}{k_1(t)}e^{v_1(t)}+\sigma_2(t)e^{v_2(t)-v_1(t)}\right|dt+\int_0^{\omega}|\sigma_1(t)|dt\\ &< \bar{\sigma}_1\omega+\bar{\sigma}_1\omega\\ &= 2\bar{\sigma}_1\omega,\end{aligned}$$

that is,

$$\int_0^{\omega}|\dot{v}_1(t)|dt < 2\bar{\sigma}_1\omega. \tag{6}$$

Similarly, from the second equation of (4), (5) and from the third equation of (4), (5), the following inequalities hold:

$$\int_0^{\omega}|\dot{v}_2(t)|dt < 2\bar{\sigma}_2\omega, \tag{7}$$

$$\int_0^{\omega}|\dot{v}_3(t)|dt < 2\bar{d}_2\omega. \tag{8}$$

Combining Lemma 2 with these inequalities, we next construct the upper and lower bounds of subsystem (3). Since $(v_1(t), v_2(t), v_3(t)) \in V$, there exist $\eta_i, \xi_i \in [0,\omega]$ such that

$$v_i(\eta_i) = \max_{t\in[0,\omega]} v_i(t), \quad v_i(\xi_i) = \min_{t\in[0,\omega]} v_i(t), \quad i = 1,2,3.$$

Multiplying the third equation of (4) by $e^{v_3(t)}$ and integrating over $[0,\omega]$, we obtain

$$\int_0^{\omega}\left[\frac{\beta(t-\tau)e^{v_2(t-\tau)+v_3(t-\tau)}}{f(v_2(t-\tau))}\cdot e^{-\int_{t-\tau}^{t}k(s)ds}-d_2(t)e^{v_3(t)}\right]dt = 0,$$

that is,

$$\int_0^{\omega}d_2(t)e^{v_3(t)}dt = \int_0^{\omega}\beta(t)\frac{e^{v_2(t)}e^{v_3(t)}}{f(v_2(t))}\cdot e^{-\int_t^{t+\tau}k(s)ds}dt. \tag{9}$$

This implies that

$$\begin{aligned}d_2^L\int_0^{\omega}e^{v_3(t)}dt &\le \int_0^{\omega}d_2(t)e^{v_3(t)}dt = \int_0^{\omega}\beta(t)\frac{e^{v_2(t)}e^{v_3(t)}}{f(v_2(t))}\cdot e^{-\int_t^{t+\tau}k(s)ds}dt\\ &\le \beta^M e^{-\tau k^L}\cdot\frac{e^{v_2(\eta_2)}}{f(v_2(\xi_2))}\int_0^{\omega}e^{v_3(t)}dt;\end{aligned}$$

therefore, we have

$$d_2^L \le \beta^M e^{-\tau k^L}\frac{e^{v_2(\eta_2)}}{f(v_2(\xi_2))},$$

or

$$v_2(\eta_2) \ge \ln\frac{d_2^L e^{\tau k^L}}{\beta^M}f(v_2(\xi_2)). \tag{10}$$

It follows from (7), (10) and Lemma 2.2 that

$$v_2(t) \ge v_2(\eta_2) - \int_0^{\omega}|\dot{v}_2(t)|dt > \ln\frac{d_2^L e^{\tau k^L}}{\beta^M}f(v_2(\xi_2)) - 2\bar{\sigma}_2\omega.$$

In particular,

$$v_2(\xi_2) > \ln \frac{d_2^L e^{\tau k^L}}{\beta^M} f(v_2(\xi_2)) - 2\bar{\sigma}_2\omega,$$

or

$$d_2^L (e^{v_2(\xi_2)})^2 - m(\beta^M e^{2\bar{\sigma}_2\omega - \tau k^L} - d_2^L) e^{v_2(\xi_2)} + mad_2^L < 0. \tag{11}$$

Therefore, in view of (H_1), we obtain $\Delta = m^2(\beta^M e^{2\bar{\sigma}_2\omega - \tau k^L} - d_2^L)^2 - 4ma(d_2^L)^2 > 0$. Now, let Equation (11) equal 0; we see that

$$l_{\pm} = \frac{1}{2d_2^L} \Big\{ m(\beta^M e^{2\bar{\sigma}_2\omega} - d_2^L) \pm [m^2(\beta^M e^{2\bar{\sigma}_2\omega} - d_2^L)^2 - 4ma(d_2^L)^2]^{\frac{1}{2}} \Big\}.$$

Hence, $\ln l_- < v_2(\xi_2) < \ln l_+$.
Similarly, from Equation (9), we have

$$\beta^L e^{-\tau k^M} \frac{e^{v_2(\xi_2)}}{f(v_2(\eta_2))} \le d_2^M,$$

that is,

$$v_2(\xi_2) \le \ln \frac{d_2^M e^{\tau k^M}}{\beta^L} f(v_2(\eta_2)). \tag{12}$$

It follows from (7), (12) and Lemma 2.2 that

$$v_2(t) \le v_2(\xi_2) + \int_0^\omega |\dot{v}_2(t)| dt < \ln \frac{d_2^M e^{\tau k^M}}{\beta^L} f(v_2(\eta_2)) + 2\bar{\sigma}_2\omega,$$

which implies that

$$v_2(\eta_2) \le \ln \frac{d_2^M e^{\tau k^M}}{\beta^L} f(v_2(\eta_2)) + 2\bar{\sigma}_2\omega.$$

This can be rewritten as

$$d_2^M e^{2\bar{\sigma}_2\omega} (e^{v_2(\eta_2)})^2 - m(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega}) e^{v_2(\eta_2)} + mad_2^M e^{2\bar{\sigma}_2\omega} \ge 0. \tag{13}$$

In view of (H_1), we obtain

$$\Delta = m^2(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega})^2 - 4ma(d_2^M e^{2\bar{\sigma}_2\omega})^2 > 0.$$

Let Equation (13) equal 0; we see that

$$h_{\pm} = \frac{1}{2d_2^M e^{2\bar{\sigma}_2\omega}} \Big\{ m(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega}) \pm [m^2(\beta^L e^{-\tau k^M} - d_2^M e^{2\bar{\sigma}_2\omega})^2 - 4ma(d_2^M e^{2\bar{\sigma}_2\omega})^2]^{\frac{1}{2}} \Big\}.$$

Therefore, $\ln h_- > v_2(\xi_2), v_2(\eta_2) > \ln h_+$.
Clearly,

$$v_2(t) \le v_2(\xi_2) + \int_0^\omega |\dot{v}_2(t)| dt < \ln l_+ + 2\bar{\sigma}_2\omega := B_{21},$$

$$v_2(t) \ge v_2(\eta_2) - \int_0^\omega |\dot{v}_2(t)| dt > \ln h_+ - 2\bar{\sigma}_2\omega := B_{22}.$$

Hence, we take

$$\max_{t \in [0,\omega]} |v_2(t)| < \max\{B_{21}, B_{22}\} := B_2.$$

From the first equation of (5), we write

$$\bar{\sigma}_1\omega \le \bar{r}_1\omega + \frac{\bar{\sigma}_2\omega e^{v_2(\eta_2)}}{e^{v_1(\xi_1)}} - (\frac{\bar{r_1}}{k_1})\omega e^{v_1(\xi_1)},$$

that is,

$$(\frac{\bar{r_1}}{k_1})(e^{v_1(\xi_1)})^2+(\bar{\sigma}_1-\bar{r}_1)e^{v_1(\xi_1)}-\bar{\sigma}_2 l_+ e^{2\bar{\sigma}_2\omega}\leq 0,$$

then, $\Delta=(\bar{\sigma}_1-\bar{r}_1)^2+4(\frac{\bar{r_1}}{k_1})\bar{\sigma}_2 l_+ e^{2\bar{\sigma}_2\omega}>0$.

Therefore, we know that

$$v_1(\xi_1)\leq \ln\left\{\frac{1}{2}(\frac{\bar{k_1}}{r_1})\{(\bar{r}_1-\bar{\sigma}_1)+[(\bar{\sigma}_1-\bar{r}_1)^2+4(\frac{\bar{r_1}}{k_1})\bar{\sigma}_2 l_+ e^{2\bar{\sigma}_2\omega}]^{\frac{1}{2}}\}\right\}=\ln b_1. \tag{14}$$

It follows from (6), (14), and Lemma 2.2 that

$$v_1(t)\leq v_1(\xi_1)+\int_0^\omega |\dot{v}_1(t)|dt<\ln b_1+2\bar{\sigma}_1\omega:=B_{11}.$$

In a similar way, we obtain the following inequality from the first equation of (5),

$$\bar{\sigma}_1\omega\geq \bar{r}_1\omega-(\frac{\bar{r_1}}{k_1})\omega e^{v_1(\eta_1)},$$

which is

$$v_1(\eta_1)\geq \ln\{(\frac{\bar{k_1}}{r_1})(\bar{r}_1-\bar{\sigma}_1)\}=\ln b_2. \tag{15}$$

From (6), (15) and Lemma 2.2, we have

$$v_1(t)\geq v_1(\eta_1)-\int_0^\omega |\dot{v}_1(t)|dt>\ln b_2-2\bar{\sigma}_1\omega:=B_{12}.$$

Therefore, we take

$$\max_{t\in[0,\omega]}|v_1(t)|<\max\{B_{11},B_{12}\}:=B_1.$$

From the second equation of (5), we obtain

$$\bar{\sigma}_2\omega\geq \bar{r}_2\omega+\bar{\sigma}_1\omega\frac{e^{v_1(\xi_1)}}{e^{v_2(\eta_2)}}-\frac{\bar{\alpha}\omega e^{v_3(\eta_3)}}{f(v_2(\xi_2))}-(\frac{\bar{r_2}}{k_2})\omega e^{v_2(\eta_2)},$$

which implies that

$$\bar{\sigma}_2\geq \bar{r}_2-\frac{\bar{\alpha}e^{v_3(\eta_3)}}{a}-(\frac{\bar{r_2}}{k_2})l_+e^{2\bar{\sigma}_2\omega}.$$

In view of (H_2), we see that

$$v_3(\eta_3)\geq \ln\left\{\frac{a}{\bar{\alpha}}\{(\bar{r}_2-\bar{\sigma}_2)-(\frac{\bar{r_2}}{k_2})l_+e^{2\bar{\sigma}_2\omega}\}\right\}=\ln b_3. \tag{16}$$

It follows from (8), (16) and Lemma 2.2 that

$$v_3(t)\geq v_3(\eta_3)-\int_0^\omega |\dot{v}_3(t)|dt>\ln b_3-2\bar{d}_2\omega:=B_{31}.$$

In a similar way, from the second equation of (5), we have

$$v_3(\xi_3)\leq \ln\left\{\frac{1}{\bar{\alpha}}\{(\bar{r}_2-\bar{\sigma}_2)+\bar{\sigma}_1 b_1 e^{2\bar{\sigma}_1\omega}\}\cdot f(\ln l_+ +2\bar{\sigma}_2\omega)\right\}=\ln b_4. \tag{17}$$

It follows from (8), (17) and Lemma 2.2 that

$$v_3(t)\leq v_3(\xi_3)+\int_0^\omega |\dot{v}_3(t)|dt<\ln b_4+2\bar{d}_2\omega:=B_{32}.$$

Then we take

$$\max_{t\in[0,\omega]} |v_3(t)| < \max\{B_{31}, B_{32}\} := B_3.$$

Now, we consider QNv with $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Note that

$$\begin{aligned} QN(v_1, v_2, v_3) =& [(\bar{r}_1 - \bar{\sigma}_1) - (\frac{\bar{r_1}}{k_1})e^{v_1(t)} + \bar{\sigma}_2\frac{e^{v_2(t)}}{e^{v_1(t)}}, \\ & (\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r_2}}{k_2})e^{v_2(t)} + \bar{\sigma}_1\frac{e^{v_1(t)}}{e^{v_2(t)}} - \frac{\bar{\alpha}e^{v_3(t)}}{f(v_2(t))}, -\bar{d}_2 + \frac{\bar{b}e^{v_2(t)}}{f(v_2(t))}], \end{aligned}$$

where $\bar{b} = \frac{1}{\omega}\int_0^{\omega}\alpha(t)e^{-\int_t^{t+\tau}\beta(s)ds}dt$.

From (H_1) and (H_2), we know that the equation $QN(v_1, v_2, v_3) = 0$ has two different solutions: $\tilde{v}, \hat{v}$. Thus, we have the following formula:

(1) Since $\bar{d}_2 + \dfrac{\bar{b}e^{v_2(t)}}{f(v_2(t))} = 0$, we have

$$v_{21} = \ln\left\{\frac{1}{2\bar{d}_2}\{m(\bar{b} - \bar{d}_2) - [m^2(\bar{d}_2 - \bar{b})^2 - 4ma\bar{d}_2]^{\frac{1}{2}}\}\right\} = \ln b_5,$$

$$v_{22} = \ln\left\{\frac{1}{2\bar{d}_2}\{m(\bar{b} - \bar{d}_2) + [m^2(\bar{d}_2 - \bar{b})^2 - 4ma\bar{d}_2]^{\frac{1}{2}}\}\right\} = \ln b_6;$$

(2) Since $(\bar{r}_1 - \bar{\sigma}_1) - (\frac{\bar{r_1}}{k_1})e^{v_1(t)} + \bar{\sigma}_2\frac{e^{v_2(t)}}{e^{v_1(t)}} = 0$, we have

$$v_{11} = \ln\left\{\frac{1}{2}(\frac{\bar{k_1}}{r_1})\{(\bar{r}_1 - \bar{\sigma}_1) + [(\bar{r}_1 - \bar{\sigma}_1)^2 + 4\bar{\sigma}_2 b_5(\frac{\bar{r_1}}{k_1})]^{\frac{1}{2}}\}\right\} = \ln b_7,$$

$$v_{12} = \ln\left\{\frac{1}{2}(\frac{\bar{k_1}}{r_1})\{(\bar{r}_1 - \bar{\sigma}_1) + [(\bar{r}_1 - \bar{\sigma}_1)^2 + 4\bar{\sigma}_2 b_6(\frac{\bar{r_1}}{k_1})]^{\frac{1}{2}}\}\right\} = \ln b_8;$$

(3) Since $(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r_2}}{k_2})e^{v_2(t)} + \bar{\sigma}_1\frac{e^{v_1(t)}}{e^{v_2(t)}} - \frac{\bar{\alpha}e^{v_3(t)}}{f(v_2(t))} = 0$, we have

$$v_{31} = \ln\left\{\frac{1}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r_2}}{k_2})b_5 + \bar{\sigma}_1\frac{b_7}{b_5}\}\cdot f(\ln b_5)\right\} = \ln b_9,$$

$$v_{32} = \ln\left\{\frac{1}{\bar{\alpha}}\{(\bar{r}_2 - \bar{\sigma}_2) - (\frac{\bar{r_2}}{k_2})b_6 + \bar{\sigma}_1\frac{b_8}{b_6}\}\cdot f(\ln b_6)\right\} = \ln b_{10}.$$

Define

$$\tilde{v} = (\ln b_7, \ln b_5, \ln b_9), \qquad \hat{v} = (\ln b_8, \ln b_6, \ln b_{10}).$$

We choose $C_1, C_3 > 0$, such that

$$C_1 > \max_{t\in[0,\omega]}\{|\ln b_7|, |\ln b_8|\}, \quad C_3 > \max_{t\in[0,\omega]}\{|\ln b_9|, |\ln b_{10}|\}.$$

Let

$$\Omega_1 = \left\{ v = (v_1, v_2, v_3) \in V \left| \begin{array}{c} \max|v_1(t)| < B_1 + C_1 \\ v_2(t) \in (\ln l_-, \ln h_-) \\ \max|v_3(t)| < B_3 + C_3 \end{array} \right. \right\},$$

$$\Omega_2 = \left\{ v = (v_1, v_2, v_3) \in V \left| \begin{array}{c} \max|v_1(t)| < B_1 + C_1 \\ v_2(t) \in (B_{22}, B_{21}) \\ \max|v_3(t)| < B_3 + C_3 \end{array} \right. \right\}.$$

Then, both Ω_1 and Ω_2 are bounded open subsets of V, and $\tilde{v} \in \Omega_1, \hat{v} \in \Omega_2$. It is easily noticed that $\Omega_1 \bigcap \Omega_2 = \varnothing$ and Ω_i satisfies the requirement (a) in Lemma 1 for $i = 1, 2$. In addition, $QNv \neq 0$ for $v \in \partial\Omega \bigcap \ker L = \partial\Omega \bigcap \mathbb{R}^3$. A direct computation gives

$$\deg\{JQN, \Omega_i \bigcap \ker L, 0\} = (-1)^{i+1} \neq 0.$$

Therefore, system (3) has at least two ω-periodic solutions v^*, v^+. Let $x^*(t) = e^{v_1^*(t)}$, $y^*(t) = e^{v_2^*(t)}$, $z_2^*(t) = e^{v_3^*(t)}$ and $x^+(t) = e^{v_1^+(t)}$, $y^+(t) = e^{v_2^+(t)}$, $z_2^+(t) = e^{v_3^+(t)}$. Then, by Equation (4), $(x^*(t), y^*(t), z_2^*(t))$ and $(x^+(t), y^+(t), z_2^+(t))$ are two different positive ω-periodic solutions of (3).

Step 2

We now claim that the third equation of (2) has two ω-periodic solutions associated with the obtained solutions. Let

$$h(t) = -k(t),$$

$$g(t) = \frac{\beta(t)y(t)z_2(t)}{y^2(t)/m + y(t) + a} - \frac{\beta(t-\tau)y(t-\tau)z_2(t-\tau)}{y^2(t-\tau)/m + y(t-\tau) + a} \cdot e^{-\int_{t-\tau}^{t} k(s)ds}.$$

Then, the third equation of (2) is given by

$$\frac{dz_1}{dt} = h(t)z_1(t) + g(t). \tag{18}$$

In fact, we see that

$$h^*(t+\omega) = h^*(t),$$

$$\begin{aligned} g^*(t+\omega) =& \frac{\beta(t+\omega)y^*(t+\omega)z_2^*(t+\omega)}{{y^*}^2(t+\omega)/m + y^*(t+\omega) + a} \\ &- \frac{\beta(t+\omega-\tau)y^*(t+\omega-\tau)z_2^*(t+\omega-\tau)}{{y^*}^2(t+\omega-\tau)/m + y^*(t+\omega-\tau) + a} e^{-\int_{t+\omega-\tau}^{t+\omega} k(s)ds} \\ =& \frac{\beta(t)y^*(t)z_2^*(t)}{{y^*}^2(t)/m + y^*(t) + a} - \frac{\beta(t-\tau)y^*(t-\tau)z_2^*(t-\tau)}{{y^*}^2(t-\tau)/m + y^*(t-\tau) + a} e^{-\int_{t-\tau}^{t} k(s)ds} \\ =& g^*(t). \end{aligned}$$

Similarly,

$$h^+(t+\omega) = h^+(t), \qquad g^+(t+\omega) = g^+(t).$$

Since $h(t)$ is negative and $\bar{h} < 0$, the linear system $\frac{dz_1}{dt} = h(t)z_1(t)$ admits exponential dichotomy. Hence,

$$z_1^*(t) = \int_{-\infty}^{t} e^{\int_s^t \alpha(\sigma)d\sigma} g^*(s)ds, \quad z_1^+(t) = \int_{-\infty}^{t} e^{\int_s^t \alpha(\sigma)d\sigma} g^+(s)ds.$$

Consequently, $(x^+(t), y^+(t), z_1^+(t), z_2^+(t))$ and $(x^*(t), y^*(t), z_1^*(t), z_2^*(t))$ are two different ω periodic solutions of system(2). $\square$

3. Example

Corresponding to system (2), we give an example as follows:

$$\begin{cases} \frac{dx}{dt} &= (8+\sin t)x(t)(1 - \frac{x(t)}{11+\sin t}) - (0.2+\sin t)x(t) + (0.15+\sin t)y(t), \\ \frac{dy}{dt} &= (12+\sin t)y(t)(1 - \frac{y(t)}{15+\sin t}) + (0.2+\sin t)x(t) - (0.15+\sin t)y(t) - \frac{(6+\sin t)y(t)z_2(t)}{3y^2(t)+y(t)+1/4}, \\ \frac{dz_1}{dt} &= \frac{(3+\sin t)y(t)z_2(t)}{3y^2(t)+y(t)+1/4} - \frac{(3+\sin(t-0.08))y(t-0.08)z_2(t-0.08)}{3y^2(t-0.08)+y(t-0.08)+1/4} e^{\int_{t-0.08}^{t}(0.3+\sin s)ds} - (0.3+\sin t)z_1(t), \\ \frac{dz_2}{dt} &= \frac{(3+\sin(t-0.08))y(t-0.08)z_2(t-0.08)}{3y^2(t-0.08)+y(t-0.08)+1/4} e^{\int_{t-0.08}^{t}(0.3+\sin s)ds} - (0.0635+\sin t)z_2(t), \end{cases} \tag{19}$$

where $r_1(t) = 8 + \sin t$ and $r_2(t) = 12 + \sin t$, $k_1(t) = 11 + \sin t$ and $k_2(t) = 15 + \sin t$, $\sigma_1(t) = 0.2 + \sin t$, $\sigma_2(t) = 0.15 + \sin t$, $\alpha(t) = 6 + \sin t$ and $\beta(t) = 3 + \sin t$. $\frac{y(t)}{3y^2(t)+y(t)+1/4}$ here is the non-monotone functional response, which reflects the capture ability of the mature predator at time t. The term

$$\exp\{\int_{t-\tau}^{t} k(s)ds\} \frac{y(t-\tau)z_2(t-\tau)}{\frac{y^2(t-\tau)}{m} + y(t-\tau) + a} = \exp\{\int_{t-0.08}^{t} (1+\sin s)ds\} \frac{y(t-0.08)z_2(t-0.08)}{\frac{y^2(t-0.08)}{1/3} + y(t-0.08) + 1/4}$$

stands for the number of immature predators born at time $(t-0.08)$ that still survive at time t and become mature predators.

Taking the initial values $x(0) = 0.5$, $y(0) = 0.5$, $z_1(0) = 8$, and $z_2(0) = 5$, the periodic solution is shown in Figure 1.

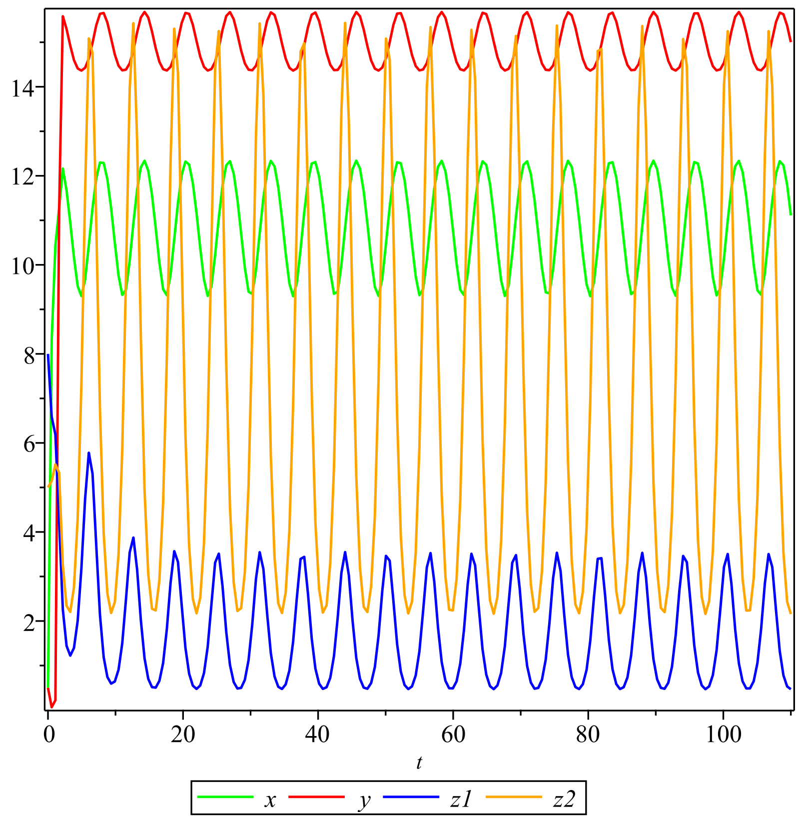

Figure 1. The periodic solution.

4. Conclusions

In this paper, a delayed prey–predator model incorporating a refuge with a non-monotone functional response is considered. It is assumed that prey can live in the predatory region and the prey refuge, respectively. From the biological point, a prey refuge can help to improve the population density of the prey, and it is an effective strategy to reduce predation in the evolution of prey population. Based on the method of Mawhin's coincidence degree theory and non-trivial estimation techniques for a priori bounds of unknown solutions to the operator equation $Lv = \lambda Nv$, we obtain some interesting and novel sufficient conditions for the existence of multiple periodic solutions of the prey–predator model. However, the limitation of this method is that we cannot determine the specific number of periodic solutions. However, predictably, for different biological models, we can give a lower bound on the number of periodic solutions. In addition, we believe that, in this paper, the occurrence of two periodic solutions is influenced by non-monotonic functional response. These two periodic solutions are generated by the system for parameters in different parameter ranges, but the periods are the same. Namely, in the bounded open subset Ω_1, system (2) presentws a positive periodic solution; in another bounded open subset Ω_2, system (2) also has a positive periodic solution with the same period.

Author Contributions: Methodology, W.L. and Y.X.; software and figure, W.L.; validation, W.L. and Y.X.; formal analysis, W.L. and Y.X.; investigation, W.L. and Y.X.; resources, W.L. and Y.X.; writing—original draft preparation, W.L. and Y.X.; writing—review and editing, W.L. and Y.X.; supervision, Y.X. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China under Grant (No. 11931016).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this article.

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