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A Dynamic Analysis of Randomly Oriented Functionally Graded Carbon Nanotubes/Fiber-Reinforced Composite Laminated Shells with Different Geometries

Ammar Melaibari ¹ , Ahmed Amine Daikh ^{2,3}, Muhammad Basha ¹, Ahmed Wagih ⁴, Ramzi Othman ¹, Khalid H. Almitani ¹, Mostafa A. Hamed ¹, Alaa Abdelrahman ⁴ and Mohamed A. Eltaher ^{1,4,*}

¹ Mechanical Engineering Department, Faculty of Engineering, King Abdulaziz University, Jeddah 80204, Saudi Arabia; aamelaibari@kau.edu.sa (A.M.); mabasha@kau.edu.sa (M.B.); rothman1@kau.edu.sa (R.O.); kalmettani@kau.edu.sa (K.H.A.); moshamed@kau.edu.sa (M.A.H.)

² Department of Technology, University Centre of Naama, Naama 45000, Algeria; daikh.ahmed.amine@gmail.com

³ Laboratoire d'Etude des Structures et de Mécanique des Matériaux, Département de Génie Civil, Faculté des Sciences et de la Technologie, Université Mustapha Stambouli, B.P. 305, R.P., Mascara 29000, Algeria

⁴ Mechanical Design and Production Department, Faculty of Engineering, Zagazig University, Zagazig 44519, Egypt; eng_awa2011@yahoo.com (A.W.); alaaabouahmed@gmail.com (A.A.)

* Correspondence: meltaher@kau.edu.sa or mmeltaher@zu.edu.eg; Tel.: +966-565-5186-13 or +20-1001-0895-61

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Abstract: The present study demonstrates the free vibration behavior of composite laminated shells reinforced by both randomly oriented single-walled carbon nanotubes (SWCNTs) and functionally graded fibers. The shell structures with different principal radii of curvature are considered, such as cylindrical, spherical, elliptical–paraboloid shell, hyperbolic–paraboloid shell, and plate. The volume fraction of the fibers has a linear variation along the shell thickness from layer to layer, while the volume fraction of CNTs is constant in all shell layers and uniformly distributed. The fiber-reinforced elements are distributed with three functions which are V-distribution, O-distribution, and X-distribution in addition to the uniform distribution. A numerical analysis was carried out systematically to validate the proposed solution. A new analytical solution is presented based on the Galerkin approach for shells and is exploited to illustrate the influence of some factors on the free vibration behavior of CNTs/fiber-reinforced composite (CNTs/FRC) laminated shells, including the distributions and volume fractions, various boundary conditions, and geometrical properties of the reinforcement materials. The proposed solution is shown to be an effective theoretical tool to analyze the free vibration response of shells.

Keywords: analytical solution; shell structures; different geometries; free vibration; CNTs/fiber-reinforced composite; higher-order shear deformation theory; Galerkin method

1. Introduction

A functionally graded material (FGM), which is an advanced composite material with a continuous gradation of materials through spatial directions, is broadly employed in various applications, ranging from macroscale (i.e., spacecraft, naval, nuclear structures, etc.) to micro/nano-scale electro-mechanical systems (MEMS/NEMS). Recently, the reinforcement of FGM by carbon nanotubes (CNTs) has been used in order to improve the mechanical, electrical, and thermal properties of composite structures due to the excellent properties of CNTs [1,2]. The characteristics of the CNTs/fiber-reinforced composite structure strongly depend on many factors such as the fractal contents of CNTs/fibers [3,4] and Fiber's geometry [5].

FG beams, plates, and shells have received substantial attention, and an extensive spectrum of beam and plate theories has been introduced, Boutahar et al. [6]. Shen and

Zhang [7] investigated the thermal buckling/postbuckling behavior for supported functionally graded carbon nanotubes-reinforced composite (FG-CNTRC) plates subjected to in-plane temperature variation. Zhu et al. [8] examined the static and vibration behaviors of FG-CNTRC plates using the finite element method with the first order shear deformation plate theory. Alibeigloo [9] developed the 3D thermoelasticity solution for an FG-CNTRC plate embedded in a piezoelectric sensor and actuator layers. Liew et al. [10] presented a review to identify and highlight topics relevant to FG-CNTRC and recent research works. Zhang et al. [11] developed a computational solution for the vibration response of FG-CNTRC thick plates resting on elastic foundations. Phung-Van et al. [12] presented the size-dependent impact on the nonlinear transient dynamic response of FG-CNTRC nonlocal nanoplates under a transverse uniform load in thermal environments. Daikh et al. [13] investigated the static response of simply supported FG-CNTRC nonlocal strain gradient nanobeams under various loading profiles using a hyperbolic higher shear deformation beam theory. He et al. [14] implemented the multi-parameter perturbation method to predict the static and stress distribution of FG thin circular piezoelectric plates. The mechanical response of piezoelectric FG plates via a simple first-order shear deformation theory was studied using the isogeometric analysis method [15] and the generalized finite difference method [16].

Daikh et al. [17] presented the influence of thickness stretching on the mechanical responses of FG-CNTRC nanoplates based on the nonlocal strain gradient theory. Esen et al. [18] studied analytically the vibration time response of an FG-CNTRC nanobeam under moving loads using the Navier Procedure. Employing the finite element method, Karamanli and Vo [19] analyzed the mechanical behavior of carbon nanotube-reinforced composites and graphene nanoplatelet-reinforced composite beams incorporating both normal and shear effects. Karamanli and Aydogdu [20] investigated the dynamic behavior of two directional functionally graded carbon nanotube-reinforced composite plates considering various boundary conditions. Boutahar et al. [6] illustrated the impact of thickness stretching on the bending vibratory behavior of thick FG beams using the refined hyperbolic function shear theory. Daikh et al. [21] examined the static response of sandwich FG nonlocal strain gradient nanoplates rested on variable Winkler elastic foundation based on a new quasi 3D hyperbolic shear theory. Daikh et al. [22] inspected the buckling stability and static response of axially FG-CNTRC plates with temperature-dependent material properties. Rostami and Mohammadimehr [23] determined the dynamic stability and bifurcation of an FG-CNTRC plate under lateral stochastic loads via the classical plate theory. Khadir et al. [24] exploited the four-unknowns quasi-3D theory to analyze the mechanical responses of FG-CNTRC nonlocal strain gradient nanoplates. Babaei et al. [25] studied the vibrational response of thermally pre-/post-buckled FG-CNTRC beams on a nonlinear elastic foundation. Duc and Minh [26] predicted the free vibration behavior of cracked FG-CNTRC plates using the phase field theory and the finite element method. Adhikari and Singh [27] illustrated the geometrical nonlinear dynamic response of the FG-CNTRC plate based on a novel shear strain function using the isogeometric finite element procedure.

For shell structures, Shen [28,29] studied the postbuckling of FG-CNTRC cylindrical shells in thermal environments under axial loads and pressure loads based on a higher order shear deformation theory with a von Kármán-type of kinematic nonlinearity by using the singular perturbation technique. Aragh et al. [30] exploited the 3D elasticity theory to evaluate the vibrations of FG-CNTRC cylindrical panels with two opposite edges simply supported via a semi-analytical solution procedure. Shen [31] studied the torsional post-buckling response of FG-CNTRC cylindrical shells in thermal environments. Mirzaei and Kiani [32] investigated the thermal buckling of temperature-dependent FG-CNTRC conical shells under the assumption of the first order shear deformation shell theory, Donnell kinematic assumptions, and the von Kármán type of geometrical nonlinearity. Thomas and Roy [33] examined the vibration analysis of FG-CNTRC Mindlin shell structures by using finite element modelling. The FG material properties were graded smoothly through the thickness. Pouresmaeli and Fazelzadeh [34] studied the frequency response of doubly

curved FG-CNTRC panels via the first-order shear deformation theory and Galerkin's method. Shojaee et al. [35] studied the vibration of FG-CNTRC skewed cylindrical panels using a transformed differential quadrature method. Tohidi et al. [36] presented the nonlinear dynamic buckling of an FG-CNTRC cylindrical shell via the modified strain gradient theory (the SGT and the Mori–Tanaka approach). Avramov et al. [37] illustrated analytically the self-sustained vibrations of FG-CNTRC shells under a supersonic flow using the linear piston theory. Aminipour et al. [38] presented the size-dependent wave propagation of FG doubly curved nonlocal nanoshells based on the higher order shear deformation theory.

Ahmadi et al. [39] studied the nonlinear vibration response of stiffened FG double-curved shallow shells exposed to thermal and nonlinear elastic environments under harmonic excitation using the perturbation methodology. Babaei et al. [40] developed a theoretical investigation for the frequency response of thermally pre/post buckled CNTRC pipes using a two-step perturbation method. Chakraborty and Dey [41] and Chakraborty et al. [42] developed a semi-analytical approach to explore the nonlinear stability characteristics of an FG-CNTRC cylindrical shell subjected to combined axial compressive loading and localized heating. Dai et al. [43] studied the mechanical behaviors of a 3D poroelasticity FG-GPLRC open shell resting on a non-polynomial viscoelastic substrate involving friction force and residual stresses. Fares et al. [44] developed a consistent layerwise/zigzag model to analyze the free vibrations of multilayered of FG-CNTRC conical shells using the Galerkin method. Fu et al. [45] investigated the vibration instability of FG-CNTRC laminated conical shells surrounded by elastic foundations using first order shear deformation. Yadav et al. [46] developed a semi-analytical solution for the nonlinear vibrations of FG-CNTRC circular cylindrical shells by a radial harmonic force and viscous structural damping. Mohammadi et al. [47] exploited an isogeometric Kirchhoff–Love shell in free and forced vibration analyses of sinusoidally corrugated FG-CNTRC panels. Cong et al. [48] studied the vibration and nonlinear dynamic responses of a temperature-dependent FG-CNTRC laminated double curved shallow shell with a positive and negative Poisson's ratio. Babaei [49] investigated analytically the nonlinear vibration and snap-buckling behaviors of FG-CNTRC arches using the perturbation method.

Based on the existing literature, the study of the free vibration behavior of composite laminated shells reinforced by randomly oriented SWCNTs remains unexplored, despite the interest for potential applications with different geometrical shapes such as cylindrical, spherical, elliptical–paraboloid, and hyperbolic–paraboloid shells. This motivated the present study. The volume fraction of fibers has a linear variation along the shell thickness from layer to layer, while the volume fraction of CNTs is constant in all shell layers and uniformly distributed. Three gradation functions, including V-distribution, O-distribution, and X-distribution, in addition to the uniform distribution, are considered. Analytical solutions with the Navier procedure are addressed in detail. The rest of the article will focus on the material and geometrical modelling, the governing equations of motion, and analytical solutions, validation, and parametric studies.

2. Material and Geometrical Modeling

2.1. Mechanical Properties and Geometrics

A rectangular multilayer shell in the Cartesian (x, y, z) coordinate system is shown in Figure 1. The proposed structure has a curved length and width $a \times b$, and thickness h . The principal radii of curvature of the middle plane in the x and y directions are R_x and R_y , respectively. Various forms are analyzed by varying the principal radius of curvature R_x and R_y (See Figure 2). The shell is reinforced by randomly oriented carbon nanotubes (CNTs) and long fibers. All shell layers have the same thickness. The volume fraction of the fibers has a linear variation along the shell thickness from layer to layer, while the volume fraction of CNTs is constant in all shell layers and uniformly distributed. Four different patterns of fiber distribution are created in this study, a uniform distribution UD and three functionally graded distributions FG-X, FG-V, and FG-O.

Figure 1. The material properties, geometry, and coordinate system of the shell, (a) Geometry and coordinates, (b) Gradation types of material.

Figure 2. Forms of various plate/shells.

The effective material properties of the CNTs/fiber-reinforced composite shell were obtained based on a micromechanical model as follows, [50,51]:

$$E_{11} = V_f E_{11}^f + V_{ef}^m E_{ef}^m \quad (1)$$

$$\frac{1}{E_{22}} = \frac{V_f}{E_{22}^f} + \frac{V_{ef}^m}{E_{ef}^m} - V_f V_{ef}^m \frac{\frac{v_f^2 E_{ef}^m}{E_{22}^f} + v_{ef}^{m2} E_{22}^f / E_{ef}^m - 2v_f v_{ef}^m}{V_f E_{22}^f + V_{ef}^m E_{ef}^m} \quad (2)$$

$$\frac{1}{G_{12}} = \frac{V_f}{G_{12}^f} + \frac{V_{ef}^m}{G_{ef}^m} \quad (3)$$

$$v_{12} = V_f v_f + V_{ef}^m v_{ef}^m \quad (4)$$

$$v_{21} = \frac{E_{22}}{E_{11}} v_{21} \quad (5)$$

$$\rho_{12} = V_f \rho_f + V_{ef}^m \rho_{ef}^m \quad (6)$$

Here E , G , ρ , and v are the Young's modulus, the shear modulus, the mass density, and Poisson's ratio, respectively. The subscripts/superscript f and eff and m denote the fibers, the effective material properties, and the matrix, respectively. V_f is the volume fraction of fibers and V_{ef}^m is the effective volume fraction of the matrix (Polymer/CNTs), where:

$$V_f = 1 - V_{ef}^m \quad (7)$$

In the case of the complete random orientation of CNTs throughout the polymer constituent, the composite is considered to be isotropic; therefore, the effective material properties of the mixture of CNTs and polymer can be expressed as:

$$E_{ef}^m = \frac{9KG}{3K + G} \quad (8)$$

$$\nu_{ef}^m = \frac{3K - 2G}{6K + 2G} \quad (9)$$

K and G are the effective bulk and shear moduli of the CNT/polymer composite and can be derived by the relations:

$$K = K_p + \frac{V_{cnt}(\delta_{cnt} - 3K_p\alpha_{cnt})}{3(V_p + V_{cnt}\alpha_{cnt})} \quad (10)$$

$$G = G_p + \frac{V_{cnt}(\eta_{cnt} - 2G_p\beta_{cnt})}{2(V_p + V_{cnt}\beta_{cnt})} \quad (11)$$

V_p and V_{cnt} are the volume fractions of the polymer and CNTs and are related by:

$$V_p = 1 - V_{cnt} \quad (12)$$

The other parameters appear in the above relations and have the following formulations:

$$\alpha_{cnt} = \frac{3(K_p + G_p) + k_{cnt} - l_{cnt}}{3(G_p + k_{cnt})} \quad (13)$$

$$\beta_{cnt} = \frac{1}{5} \left(\frac{4G_p + 2k_{cnt} + l_{cnt}}{3(G_p + k_{cnt})} + \frac{4G_p}{G_p + p_{cnt}} + \frac{2(G_p(3K_p + G_p) + G_p(3K_p + 7G_p))}{G_p(3K_p + G_p) + m_{cnt}(3K_p + 7G_p)} \right) \quad (14)$$

$$\delta_{cnt} = \frac{1}{3} \left(n_{cnt} + 2l_{cnt} + \frac{2(k_{cnt} + l_{cnt})(3K_p + 2G_p - l_{cnt})}{G_p + k_{cnt}} \right) \quad (15)$$

$$\eta_{cnt} = \frac{1}{5} \left(\frac{2}{3}(n_{cnt} - l_{cnt}) + \frac{8G_p p_{cnt}}{G_p + p_{cnt}} + \frac{8m_{cnt}G_m(3K_p + 4G_p)}{3K_m(m_{cnt} + G_p) + G_p(7m_{cnt} + G_p)} + \frac{2(k_{cnt} - l_{cnt})(2G_p + l_{cnt})}{3(G_p + k_{cnt})} \right) \quad (16)$$

Table 1 presents Hill's elastic moduli for (10, 10) single-walled carbon nanotubes for various chiral indices.

Table 1. Hill's elastic moduli for (10, 10) single-walled carbon nanotubes [52].

k_{cnt} [GPa]	l_{cnt} [GPa]	m_{cnt} [GPa]	n_{cnt} [GPa]	p_{cnt} [GPa]
271	88	17	1089	442

2.2. Volume Fraction of the Reinforcement Fibers

The volume fraction of the fibers for various distribution patterns is expressed as the following relations:

Uniform distribution of the fibers UD_f :

$$V_f = V_{min}^f + \frac{(V_{max}^f - V_{min}^f)}{2} \quad (17)$$

Functionally graded fiber distribution $FG_f - X$:

$$V_f = V_{min}^f + (V_{max}^f - V_{min}^f) \frac{|2(k-1) - N - 1|}{N - 1} \quad (18)$$

Functionally graded fiber distribution $FG_f - O$:

$$V_f = V_{min}^f + (V_{max}^f - V_{min}^f) \left(1 - \frac{|2(k-1) - N - 1|}{N - 1}\right) \quad (19)$$

Functionally graded fiber distribution $FG_f - A$:

$$V_f = V_{min}^f + (V_{max}^f - V_{min}^f) \left(1 - \frac{k-1}{N-1}\right) \quad (20)$$

V_{max}^f and V_{min}^f are the maximum and minimum volume fractions of the fibers. N is the number of layers (odd number). The volume fractions along the thickness of the shell for various distribution patterns are plotted in Figure 3.

Figure 3. The volume-fractions of the fibers along the thickness of the shell ($V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

3. Mathematical Formulations

In the present work, we proposed a hyperbolic sine function shear deformation theory to define the governing equations for the free vibration problem of functionally

graded CNTs/fiber-reinforced composite laminated shell. The displacement field can be expressed as:

$$\begin{aligned} u(x, y, z, t) &= \left(1 + \frac{z}{R_x}\right) u_0 - z \frac{\partial w_0}{\partial x} + f(z) \psi_x \\ v(x, y, z, t) &= \left(1 + \frac{z}{R_y}\right) v_0 - z \frac{\partial w_0}{\partial y} + f(z) \psi_y \\ w(x, y, z, t) &= w_0 \end{aligned} \quad (21)$$

The displacements of the midplane of the composite plate are u_0 , v_0 , and w_0 , whereas ψ_x and ψ_y are the rotations of the transverse normal at the middle surface $z = 0$. The shape function $f(z)$ is proposed as hyperbolic sine, and it is given as:

$$f(z) = h \sinh\left(\frac{z}{h}\right) - \frac{3z^3}{2h^2} \quad (22)$$

where

$$g(z) = f'(z)$$

The nonzero strains can be determined from the displacement as:

$$\begin{aligned} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} &= \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} + f(z) \begin{Bmatrix} \varepsilon_{xx}^2 \\ \varepsilon_{yy}^2 \\ \gamma_{xy}^2 \end{Bmatrix} \\ \varepsilon_{zz} &= 0, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \frac{df(z)}{dz} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \end{aligned} \quad (23)$$

where:

$$\begin{aligned} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{w_0}{R_x} \\ \frac{\partial v_0}{\partial y} + \frac{w_0}{R_y} \\ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}, \\ \begin{Bmatrix} \varepsilon_{xx}^2 \\ \varepsilon_{yy}^2 \\ \gamma_{xy}^2 \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial \varphi_1}{\partial x} \\ \frac{\partial \varphi_2}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \varphi_x \\ \varphi_y \end{Bmatrix}. \end{aligned} \quad (24)$$

The stresses relations associated with the strains can be written as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11}^k & \bar{Q}_{12}^k & 0 & 0 & 0 \\ \bar{Q}_{12}^k & \bar{Q}_{22}^k & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{44}^k & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55}^k & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{66}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}^{(k)} \quad (25)$$

The transformed material constants $\bar{Q}_{ij}^k$ are expressed as:

$$\begin{aligned} \bar{Q}_{11}^k &= Q_{11} \cos^4 \theta_k + 2(Q_{12} + 2Q_{66}) \sin^2 \theta_k \cos^2 \theta_k + Q_{22} \sin^4 \theta_k \\ \bar{Q}_{12}^k &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta_k \cos^2 \theta_k + Q_{12} (\sin^4 \theta_k + \cos^4 \theta_k) \\ \bar{Q}_{22}^k &= Q_{11} \sin^4 \theta_k + 2(Q_{12} + 2Q_{66}) \sin^2 \theta_k \cos^2 \theta_k + Q_{22} \cos^4 \theta_k \\ \bar{Q}_{66}^k &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta_k \cos^2 \theta_k + Q_{66} (\sin^4 \theta_k + \cos^4 \theta_k) \\ \bar{Q}_{44}^k &= Q_{44} \cos^2 \theta_k + Q_{55} \sin^2 \theta_k \\ \bar{Q}_{55}^k &= Q_{55} \cos^2 \theta_k + Q_{44} \sin^2 \theta_k \end{aligned} \quad (26)$$

where θ_k is the lamination angle of the k th layer:

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, Q_{12} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}}, \\ Q_{44} = G_{23}, Q_{55} = G_{13}, Q_{66} = G_{12}$$

By integrating Equation (25), the stress relations, moment, and additional moment resultants can be obtained as:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [C] \\ [B] & [D] & [F] \\ [C] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \{\epsilon^0\} \\ \{\epsilon^1\} \\ \{\epsilon^2\} \end{Bmatrix} \quad (27)$$

$$\begin{Bmatrix} R_{yz} \\ R_{xz} \end{Bmatrix} = \begin{bmatrix} J_{44} & J_{45} \\ J_{45} & J_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \quad (28)$$

where:

$$\{N\} = \begin{Bmatrix} N_{xx} & N_{yy} & N_{xy} \end{Bmatrix}^T, \{M\} = \begin{Bmatrix} M_{xx} & M_{yy} & M_{xy} \end{Bmatrix}^T, \{P\} = \begin{Bmatrix} P_{xx} & P_{yy} & P_{xy} \end{Bmatrix}^T \\ \{\epsilon^0\} = \begin{Bmatrix} \epsilon_{xx}^0 & \epsilon_{yy}^0 & \gamma_{xy}^0 \end{Bmatrix}^T, \{\epsilon^1\} = \begin{Bmatrix} \epsilon_{xx}^1 & \epsilon_{yy}^1 & \gamma_{xy}^1 \end{Bmatrix}^T, \{\epsilon^2\} = \begin{Bmatrix} \epsilon_{xx}^2 & \epsilon_{yy}^2 & \gamma_{xy}^2 \end{Bmatrix}^T \quad (29)$$

The coefficients A_{ij} , B_{ij} , D_{ij} , C_{ij} , F_{ij} and H_{ij} are defined as:

$$\{A_{ij}, B_{ij}, D_{ij}, C_{ij}, F_{ij}, H_{ij}\} = \sum_{n=1}^n \int_{h_{n-1}}^{h_n} Q_{ij}^{(n)} \{1, z, z^2, f(z), zf(z), f(z)^2\} dz, (i, j = 1, 2, 6) \\ J_{ii} = \sum_{n=1}^n \int_{h_{n-1}}^{h_n} Q_{ii}^{(n)} \left[\frac{df(z)}{dz} \right]^2 dz, (i = 4, 5) \quad (30)$$

4. Variational Statements

Hamilton's principle is employed in this analysis to derive the equations of motion of the CNTs/fiber composite shell:

$$\delta \int_{t_2}^{t_1} \delta(U - T + V) dt = 0 \quad (31)$$

The variation of the strain energy of the composite shell can be expressed as:

$$\delta U_p = \frac{1}{2} \int_V \left[\sigma_{xx}^{(k)} \delta \epsilon_{xx} + \sigma_{yy}^{(k)} \delta \epsilon_{yy} + \sigma_{xy}^{(k)} \delta \gamma_{xy} + \sigma_{xz}^{(k)} \delta \gamma_{xz} + \sigma_{yz}^{(k)} \delta \gamma_{yz} \right] dV. \quad (32)$$

The variation of the kinetic energy of the composite shell at any moment is stated as:

$$\delta T = \frac{1}{2} \int_0^L \int_A \rho (\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) dA dx \quad (33)$$

$$\delta T = \int_V \left\{ I_0 (\dot{u}_0 \delta \dot{u}_0 + \dot{v}_0 \delta \dot{v}_0 + \dot{w}_0 \delta \dot{w}_0) + I_1 \left(\frac{\partial \dot{w}_0}{\partial x} \delta \dot{u}_0 + \frac{\partial \dot{w}_0}{\partial y} \delta \dot{v}_0 + \dot{u}_0 \frac{\partial \delta \dot{w}_0}{\partial x} + \dot{v}_0 \frac{\partial \delta \dot{w}_0}{\partial y} \right) \right. \\ \left. + I_2 \left(\frac{\partial \dot{w}_0}{\partial x} \frac{\partial \delta \dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial y} \frac{\partial \delta \dot{w}_0}{\partial y} \right) + I_3 \left(\dot{\phi}_x \delta \dot{u}_0 + \dot{u}_0 \delta \dot{\phi}_x + \dot{\phi}_y \delta \dot{v}_0 + \dot{v}_0 \delta \dot{\phi}_y \right) \right. \\ \left. + I_4 \left(\dot{\phi}_x \frac{\partial \delta \dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial x} \delta \dot{\phi}_x + \dot{\phi}_y \frac{\partial \delta \dot{w}_0}{\partial y} + \frac{\partial \dot{w}_0}{\partial y} \delta \dot{\phi}_y \right) + I_5 \left(\dot{\phi}_x \delta \dot{\phi}_x + \dot{\phi}_y \delta \dot{\phi}_y \right) \right\} dx dy dz \quad (34)$$

where:

$$\{I_0, I_1, I_2, I_3, I_4, I_5\} = \sum_{n=1}^k \rho(z) \{1, z, z^2, \phi(z), z\phi(z), (\phi(z))^2\} dz \quad (35)$$

By inserting Equations (32)–(34) into Equation (19), the equilibrium equations for a CNTs/fiber composite shell can be obtained as follows:

$$\begin{aligned}
& A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + \left(\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y} \right) \frac{\partial w_0}{\partial x} - B_{11} \frac{\partial^3 w_0}{\partial x^3} \\
& - (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x \partial y^2} + B_{11}^s \frac{\partial^2 \psi_x}{\partial x^2} + B_{66}^s \frac{\partial^2 \psi_x}{\partial y^2} + (B_{12}^s + B_{66}^s) \frac{\partial^2 \psi_y}{\partial x \partial y} \\
& = \left(I_0 + 2 \frac{I_1}{R_x} + \frac{I_3}{R_y^2} \right) \frac{\partial^2 u_0}{\partial t^2} - \left(I_1 + \frac{I_2}{R_x} \right) \frac{\partial^3 w_0}{\partial x \partial t^2} + \left(I_3 + \frac{I_4}{R_x} \right) \frac{\partial^2 \psi_x}{\partial t^2}
\end{aligned} \quad (36)$$

$$\begin{aligned}
& (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} - (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x^2 \partial y} - B_{22} \frac{\partial^3 w_0}{\partial y^3} \\
& + \left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y} \right) \frac{\partial w_0}{\partial x} + (B_{12}^s + B_{66}^s) \frac{\partial^2 \psi_x}{\partial x \partial y} + B_{22}^s \frac{\partial^2 \psi_y}{\partial y^2} + B_{66}^s \frac{\partial^2 \psi_y}{\partial x^2} \\
& = \left(I_0 + 2 \frac{I_1}{R_y} + \frac{I_2}{R_y^2} \right) \frac{\partial^2 v_0}{\partial t^2} - \left(I_1 + \frac{I_2}{R_y} \right) \frac{\partial^3 w_0}{\partial y \partial t^2} + \left(I_3 + \frac{I_4}{R_y} \right) \frac{\partial^2 \psi_y}{\partial t^2}
\end{aligned} \quad (37)$$

$$\begin{aligned}
& B_{11} \frac{\partial^3 u_0}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 u_0}{\partial x \partial y^2} - \left(\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y} \right) \frac{\partial u_0}{\partial x} + (B_{12} + 2B_{66}) \frac{\partial^3 v_0}{\partial x^2 \partial y} + B_{22} \frac{\partial^3 v_0}{\partial y^3} - \left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y} \right) \frac{\partial v_0}{\partial y} \\
& + \left(\frac{2B_{11}}{R_x} + \frac{2B_{12}}{R_y} \right) \frac{\partial^2 w_0}{\partial x^2} - D_{11} \frac{\partial^4 w_0}{\partial x^4} - (2D_{12} + 4D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} - D_{22} \frac{\partial^4 w_0}{\partial y^4} + \left(\frac{2B_{12}}{R_x} + \frac{2B_{22}}{R_y} \right) \frac{\partial^2 w_0}{\partial y^2} \\
& - \left(\frac{A_{11}}{R_x^2} + 2 \frac{A_{12}}{R_x R_y} + \frac{A_{22}}{R_y^2} \right) w_0 + D_{11}^s \frac{\partial^3 \psi_x}{\partial x^3} + (D_{12}^s + 2D_{66}^s) \frac{\partial^3 \psi_x}{\partial x \partial y^2} - \left(\frac{B_{11}^s}{R_x} + \frac{B_{12}^s}{R_y} \right) \frac{\partial \psi_x}{\partial x} \\
& + (D_{12}^s + 2D_{66}^s) \frac{\partial^3 \psi_y}{\partial x^2 \partial y} + D_{22}^s \frac{\partial^3 \psi_y}{\partial y^3} - \left(\frac{B_{12}^s}{R_x} + \frac{B_{22}^s}{R_y} \right) \frac{\partial \psi_y}{\partial y} = I_0 \frac{\partial^2 w_0}{\partial t^2} \\
& + \left(I_1 + \frac{I_2}{R_x} \right) \frac{\partial^3 u_0}{\partial x \partial t^2} + \left(I_1 + \frac{I_2}{R_y} \right) \frac{\partial^3 v_0}{\partial y \partial t^2} - I_2 \left(\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \frac{\partial^4 w_0}{\partial y^2 \partial t^2} \right) + I_4 \left(\frac{\partial^3 \psi_x}{\partial x \partial t^2} + \frac{\partial^3 \psi_y}{\partial y \partial t^2} \right)
\end{aligned} \quad (38)$$

$$\begin{aligned}
& B_{11}^s \frac{\partial^2 u_0}{\partial x^2} + B_{66}^s \frac{\partial^2 u_0}{\partial y^2} + (B_{12}^s + B_{66}^s) \frac{\partial^2 v_0}{\partial x \partial y} - D_{11}^s \frac{\partial^3 w_0}{\partial x^3} - (D_{12}^s + 2D_{66}^s) \frac{\partial^3 w_0}{\partial x \partial y^2} \\
& + \left(\frac{B_{11}^s}{R_x} + \frac{B_{12}^s}{R_y} \right) \frac{\partial w_0}{\partial x} + F_{11}^s \frac{\partial^2 \psi_x}{\partial x^2} + F_{66}^s \frac{\partial^2 \psi_x}{\partial y^2} - A_{44}^s \psi_x + (F_{12}^s + F_{66}^s) \frac{\partial^2 \psi_y}{\partial x \partial y} \\
& = \left(I_3 + \frac{I_4}{R_x} \right) \frac{\partial^2 u_0}{\partial t^2} - I_4 \frac{\partial^3 w_0}{\partial x \partial t^2} + I_5 \frac{\partial^2 \psi_x}{\partial t^2}
\end{aligned} \quad (39)$$

$$\begin{aligned}
& (B_{12}^s + B_{66}^s) \frac{\partial^2 u_0}{\partial x \partial y} + B_{11}^s \frac{\partial^2 v_0}{\partial y^2} + B_{66}^s \frac{\partial^2 v_0}{\partial x^2} - (D_{12}^s + 2D_{66}^s) \frac{\partial^3 w_0}{\partial x^2 \partial y} - D_{22}^s \frac{\partial^3 w_0}{\partial y^3} \\
& + \left(\frac{B_{12}^s}{R_x} + \frac{B_{22}^s}{R_y} \right) \frac{\partial w_0}{\partial x} + (F_{12}^s + F_{66}^s) \frac{\partial^2 \psi_x}{\partial x \partial y} + F_{11}^s \frac{\partial^2 \psi_y}{\partial y^2} + F_{66}^s \frac{\partial^2 \psi_y}{\partial x^2} - A_{44}^s \psi_y \\
& = \left(I_3 + \frac{I_4}{R_y} \right) \frac{\partial^2 v_0}{\partial t^2} - I_4 \frac{\partial^3 w_0}{\partial y \partial t^2} + I_5 \frac{\partial^2 \psi_y}{\partial t^2}
\end{aligned} \quad (40)$$

5. Analytical Solution

The aim of this work is to expand the use of analytical solutions to analyze the response of various structures such as shells by considering different boundary conditions, as shown in Table 2. Therefore, the Galerkin approach can provide accurate solutions. The expressions of generalized displacements can be expressed as:

$$\begin{aligned}
\{u_0, \psi_x\} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{U_{mn}, X_{mn}\} \frac{\partial X_m(x)}{\partial x} Y_n(y) e^{i\omega t} \\
\{v_0, \psi_y\} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{V_{mn}, Z_{mn}\} X_m(x) \frac{\partial Y_n(y)}{\partial y} e^{i\omega t} \\
w_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} X_m(x) Y_n(y) e^{i\omega t}
\end{aligned} \quad (41)$$

Table 2. The essential boundary conditions.

BCs.	Boundaries Parallel to the x -Axis	Boundaries Parallel to the y -Axis
Simply supported (S)	$u_0 = w_0 = \psi_x = M_{yy} = N_{yy} = 0$	$v_0 = w_0 = \psi_y = M_{xx} = N_{xx} = 0$
Clamped (C)	$u_0 = v_0 = w_0 = \psi_x = \psi_y = 0$	$u_0 = v_0 = w_0 = \psi_x = \psi_y = 0$

U_{mn} , V_{mn} , X_{mn} , and Z_{mn} are arbitrary parameters. The boundary conditions that can be imposed on all the four boundaries of the CNTs/F-RC shells are given in Table 2 as:

The functions $X_m(x)$ and $Y_n(y)$ that satisfy the above boundary conditions are given in Table 3.

Table 3. The admissible functions $X_m(x)$ and $Y_n(y)$.

BCs.	The Functions X_m and Y_n	
	$X_m(x)$	$Y_n(y)$
SSSS	$\sin(\alpha x)$	$\sin(\beta y)$
CCCC	$\sin^2(\alpha x)$	$\sin^2(\beta y)$
CSCS	$\sin(\alpha x)[\cos(\alpha x) - 1]$	$\sin(\beta y)[\cos(\beta y) - 1]$
CCSS	$\sin^2(\alpha x)$	$\sin(\beta y)$
CSSS	$\sin(\alpha x)[\cos(\alpha x) - 1]$	$\sin(\beta y)$
CCCS	$\sin^2(\alpha x)$	$\sin(\beta y)[\cos(\beta y) - 1]$

Where $\alpha = m\pi/a$, $\beta = n\pi/b$. m and n are mode numbers.

By substituting Equation (41) in Equations (36)–(40), one obtains:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{13} & K_{23} & K_{33} & K_{34} & K_{35} \\ K_{14} & K_{24} & K_{34} & K_{44} & K_{45} \\ K_{15} & K_{25} & K_{35} & K_{45} & K_{55} \end{bmatrix}, [M] = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{bmatrix} \quad (42)$$

where $[K]$ et $[M]$ are the rigidity matrix and mass matrix, respectively.

The elements K_{ij} and M_{ij} of the matrix $[K]$ and $[M]$ are given in Appendix A.

6. Results and Discussions

6.1. Verification Analysis

Firstly, to examine the accuracy and effectiveness of the developed model, the results for the free vibration of functionally graded shells were compared with those generated in the literature using various solution techniques (see Table 4). The materials used were Alumina (AlO_2) as the ceramic ($E_c = 380$ GPa, $\rho_c = 3800$ kg/m³) and Aluminium (Al) as the metal ($E_m = 70$ GPa, $\rho_c = 2707$ kg/m³). Poisson's ratio was taken as $\nu = 0.3$. It can be observed that the results computed by the proposed HSDT are absolutely identical to those generated in the literature.

Table 4. Comparison of the natural frequency parameter ($\tilde{\omega} = \omega h \sqrt{\rho_c/E_c}$) for simply supported Al/Al₂O₃ functionally graded square plates and doubly curved shells ($a = b = 10h$, $m = n = 1$).

	a/R_x	b/R_y	p	Alijani [53]	Matsunaga [54]	Chorfi [55]	Trinh [56]	Present
Plate	0	0	0	0.0597	0.0588	0.0577	0.0577	0.0577
			0.5	0.0506	0.0492	0.0490	0.0490	0.0490
			1	0.0456	0.0430	0.0442	0.0442	0.0442
			4	0.0396	0.0381	0.0383	0.0381	0.0380
			10	0.0380	0.0364	0.0366	0.0364	0.0363
Spherical	0.5	0.5	0	0.0779	0.0751	0.0762	0.0761	0.0753
			0.5	0.0676	0.0657	0.0664	0.0662	0.0653
			1	0.0617	0.0601	0.0607	0.0605	0.0595
			4	0.0519	0.0503	0.0509	0.0506	0.0496
			10	0.0482	0.0464	0.0471	0.0467	0.0459
Cylindrical	0.5	0	0	0.0648	0.0622	0.0629	0.0628	0.0622
			0.5	0.0553	0.0535	0.0540	0.0538	0.0533
			1	0.0501	0.0485	0.0490	0.0488	0.0482
			4	0.0430	0.0413	0.0419	0.0416	0.0410
			10	0.0408	0.0390	0.0395	0.0392	0.0387
Hyperbolic– paraboloidal	0.5	−0.5	0	0.0597	0.0563	0.0580	0.0577	0.0563
			0.5	0.0506	0.0479	0.0493	0.0490	0.0478
			1	0.0456	0.0432	0.0445	0.0442	0.0431
			4	0.0396	0.0372	0.0385	0.0381	0.0371
			10	0.0380	0.0355	0.0368	0.0364	0.0354

6.2. Parametric Stud

The analyzed composite shell was made of a mixture of the polymer, armchair (10, 10) single-walled CNTs, and long fibers as reinforcements. The material properties of the polymer were [57]: $E_p = 2.5$ GPa, $\rho_p = 1190$ kg/m³, and $\nu_p = 0.19$, while the material properties of the fiber were [58]: $E_{11}^f = 233.5$ GPa, $E_{22}^f = 23.1$ GPa, $G_{12}^f = 8.96$ GPa, $\rho_f = 1750$ kg/m³, and $\nu_f = 0.2$. The density of the CNTs was assumed to be equal to $\rho_f = 1400$ kg/m³. To standardize and simplify calculations, the normalized parameters for the vibration analyses of the CNTs/fiber shells are described using the following forms:

$$\bar{\omega} = \omega h \sqrt{\frac{\rho_m}{E_m}} \quad (43)$$

In the following analysis, a parametric study on the vibration of CNTs/F-RC shells is carried out. Two types of laminated shells are proposed: cross-ply laminates and unidirectional laminates. For the unidirectional laminates, the angle of the orientation θ was equal to 0° . For the cross-ply laminates, the angle of the orientation θ of each layer changed alternately ($\theta = 0^\circ$ or 90°), for example, in the case of five layers, the shell could be described as $[0^\circ/90^\circ/0^\circ/90^\circ/0^\circ]$ laminate. The effect of the number of layers N and the fiber distribution patterns on the dimensionless frequency of simply supported cross-ply F/CNT-RC shells for various curvature radii was examined as shown in Table 5. The maximum volume fraction was taken as $V_{max}^f = 30\%$, while the minimum is $V_{min}^f = 10\%$. The volume fraction of the CNTs was taken as $V_{cnt} = 1\%$. The same analysis is discussed on the unidirectional F/CNT-RC shells in Table 6.

Table 5. The effect of the number of layers and fiber distribution patterns on the dimensionless frequency of cross-ply CNTs/F-RC shells for various curvature radii (SSSS, $b/a = 1$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

	R_x/a	R_y/b	N	UD	FG-X	FG-O	FG-V
Plate	inf	inf	3	0.1896	0.1862	0.1750	0.1779
			7	0.1900	0.1840	0.1734	0.1783
			11	0.1900	0.1927	0.1823	0.1881
			15	0.1901	0.2040	0.1934	0.1984
			19	0.1901	0.1976	0.1821	0.1893
Cylindrical shell	5	inf	3	0.1917	0.1883	0.1778	0.1789
			7	0.1921	0.1862	0.1762	0.1792
			11	0.1922	0.1948	0.1848	0.1888
			15	0.1922	0.2059	0.1959	0.1988
			19	0.1922	0.1997	0.1844	0.1899
Spherical shell	5	5	3	0.1986	0.1953	0.1856	0.1855
			7	0.1991	0.1933	0.1842	0.1854
			11	0.1992	0.2016	0.1924	0.1947
			15	0.1992	0.2125	0.2031	0.2041
			19	0.1992	0.2064	0.1917	0.1955
Elliptical–paraboloid shell	5	7.5	3	0.1959	0.1925	0.1825	0.1828
			7	0.1964	0.1905	0.1811	0.1829
			11	0.1964	0.1989	0.1894	0.1923
			15	0.1964	0.2098	0.2003	0.2019
			19	0.1964	0.2038	0.1888	0.1932
Hyperbolic–paraboloidal shell	5	−5	3	0.1888	0.1855	0.1744	0.1769
			7	0.1892	0.1834	0.1728	0.1776
			11	0.1893	0.1920	0.1816	0.1873
			15	0.1893	0.2032	0.1927	0.1975
			19	0.1893	0.1969	0.1814	0.1885

Table 6. The effect of the number of layers and fiber distribution patterns on the dimensionless frequency of unidirectional F/CNT-RC shells for various curvature radii (SSSS, $b/a = 1$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

	R_x/a	R_y/b	N	UD	FG-X	FG-O	FG-V
Plate	inf	inf	3	0.1894	0.1858	0.1748	0.1776
			7	0.1894	0.1834	0.1735	0.1773
			11	0.1894	0.1919	0.1818	0.1868
			15	0.1894	0.2029	0.1928	0.1968
			19	0.1894	0.1967	0.1817	0.1879
Cylindrical shell	5	inf	3	0.1913	0.1877	0.1771	0.1790
			7	0.1913	0.1854	0.1754	0.1787
			11	0.1913	0.1937	0.1840	0.1881
			15	0.1913	0.2046	0.1948	0.1979
			19	0.1913	0.1985	0.1838	0.1891
Spherical shell	5	5	3	0.1979	0.1941	0.1841	0.1855
			7	0.1979	0.1916	0.1822	0.1849
			11	0.1979	0.1995	0.1901	0.1939
			15	0.1979	0.2102	0.2007	0.2035
			19	0.1979	0.2048	0.1906	0.1955
Elliptical–paraboloid shell	5	7.5	3	0.1953	0.1915	0.1813	0.1829
			7	0.1953	0.1891	0.1795	0.1824
			11	0.1953	0.1972	0.1877	0.1916
			15	0.1953	0.2079	0.1983	0.2012
			19	0.1953	0.2023	0.1879	0.1929
Hyperbolic–paraboloidal shell	5	−5	3	0.1887	0.1852	0.1743	0.1768
			7	0.1887	0.1831	0.1728	0.1768
			11	0.1887	0.1917	0.1817	0.1864
			15	0.1887	0.2026	0.1926	0.1960
			19	0.1887	0.1959	0.1810	0.1867

Table 7 shows the impact of the volume fraction and the fiber distribution patterns on the dimensionless frequency of simply supported cross-ply F/CNT-RC laminated shells. The number of layers was fixed at nine (9) layers. It is clear from this table that the change in the volume fraction of the CNTs had a significant influence on the free vibration behaviour of the shells.

Table 7. The effect of the volume fraction of CNTs and fiber distribution patterns on the dimensionless frequency of cross-ply F/CNT-RC shells (SSSS, $b/a = 1$, $N = 9$, $a/h = 10$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

Type of Shells	Fiber Distribution Pattern				
	V_{cnt}	FRC UD	FRC FG-X	FRC FG-O	FRC FG-V
Plate	0.0%	0.1176	0.1288	0.1033	0.1140
	0.2%	0.1387	0.1508	0.1246	0.1358
	0.4%	0.1543	0.1658	0.1414	0.1521
	0.6%	0.1676	0.1781	0.1560	0.1659
	0.8%	0.1794	0.1888	0.1691	0.1780
	1.0%	0.1900	0.1984	0.1811	0.1891
Cylindrical shell	0.0%	0.1181	0.1292	0.1039	0.1140
	0.2%	0.1398	0.1517	0.1258	0.1358
	0.4%	0.1557	0.1671	0.1430	0.1522
	0.6%	0.1693	0.1797	0.1578	0.1661
	0.8%	0.1813	0.1906	0.1712	0.1785
	1.0%	0.1922	0.2004	0.1834	0.1896

Table 7. Cont.

Type of Shells	Fiber Distribution Pattern				
	V_{cnt}	FRC UD	FRC FG-X	FRC FG-O	FRC FG-V
Spherical shell	0.0%	0.1200	0.1310	0.1062	0.1163
	0.2%	0.1434	0.1551	0.1298	0.1389
	0.4%	0.1605	0.1715	0.1481	0.1561
	0.6%	0.1749	0.1850	0.1639	0.1707
	0.8%	0.1877	0.1967	0.1779	0.1836
	1.0%	0.1992	0.2071	0.1907	0.1953
Elliptical–paraboloid shell	0.0%	0.1193	0.1303	0.1053	0.1154
	0.2%	0.1419	0.1538	0.1283	0.1377
	0.4%	0.1586	0.1698	0.1461	0.1545
	0.6%	0.1727	0.1829	0.1615	0.1688
	0.8%	0.1852	0.1943	0.1753	0.1815
	1.0%	0.1964	0.2045	0.1878	0.1929
Hyperbolic–paraboloidal shell	0.0%	0.1171	0.1283	0.1029	0.1125
	0.2%	0.1382	0.1502	0.1241	0.1348
	0.4%	0.1537	0.1651	0.1408	0.1511
	0.6%	0.1669	0.1774	0.1554	0.1649
	0.8%	0.1786	0.1888	0.1684	0.1771
	1.0%	0.1893	0.1976	0.1804	0.1881

The effect of the volume fraction and the distribution patterns of the fibers on the dimensionless frequency of simply supported cross-ply F/CNT-RC shells is demonstrated in Table 8. The volume fraction of the CNTs was proposed as $V_{cnt} = 1\%$, whereas the minimal volume fraction of the fibers was $V_{min}^f = 0.05$. The increase in the fiber volume fraction further reinforced the composite plate; therefore, the dimensionless frequencies increased.

Table 8. The effect of the volume fraction and distribution patterns of the fibers on the dimensionless frequency of cross-ply F/CNT-RC shells (SSSS, $b/a = 1$, $a/h = 10$, $N = 9$, $V_{cnt} = 0.01$, $V_{min}^f = 0.05$).

Type of Shells	Fiber Distribution Pattern				
	V_F	FRC UD	FRC FG-X	FRC FG-O	FRC FG-V
Plate	5%	0.1735	0.1735	0.1735	0.1735
	10%	0.1765	0.1790	0.1740	0.1765
	20%	0.1822	0.1891	0.1751	0.18158
	30%	0.1875	0.1982	0.1760	0.1859
Cylindrical shell	5%	0.1758	0.1758	0.1758	0.1758
	10%	0.1788	0.1812	0.1763	0.1782
	20%	0.1844	0.1912	0.1774	0.1823
	30%	0.1897	0.2002	0.1784	0.1860
Spherical shell	5%	0.1831	0.1831	0.1831	0.1831
	10%	0.1861	0.1884	0.1837	0.1850
	20%	0.1916	0.1982	0.1848	0.1883
	30%	0.1968	0.2069	0.1859	0.1912
Elliptical–paraboloid shell	5%	0.1802	0.1802	0.1802	0.1802
	10%	0.1832	0.1856	0.1808	0.1822
	20%	0.1888	0.1954	0.1819	0.1858
	30%	0.1940	0.2042	0.1829	0.1890
Hyperbolic–paraboloidal shell	0.0%	0.1728	0.1728	0.1728	0.1728
	0.2%	0.1758	0.1783	0.1733	0.1757
	0.8%	0.1815	0.1884	0.1743	0.1808
	1.0%	0.1868	0.1974	0.1753	0.1850

Table 9 presents the influence of the fiber distribution patterns on the dimensionless frequency of cross-ply F/CNT-RC shells for various boundary conditions. The fully clamped plate (CCCC) had the highest frequencies, while the lowest frequencies were for the simply supported shells, wherever the fiber distribution pattern was.

Table 9. The effect of different boundary conditions and fiber distribution patterns on the dimensionless frequency of cross-ply F/CNT-RC shells ($b/a = 1$, $N = 9$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

Type of Shells	BCs.	Fiber Distribution Pattern			
		FRC UD	FRC FG-X	FRC FG-O	FRC FG-V
Plate	SSSS	0.1900	0.1984	0.1811	0.1891
	CCCC	0.3599	0.3778	0.3398	0.3572
	CCSS	0.2978	0.3162	0.2767	0.2950
	CSCS	0.3409	0.3567	0.3237	0.3389
	CSSS	0.2821	0.2989	0.2634	0.2797
	CCCS	0.2386	0.2581	0.2155	0.2353
Cylindrical shell	SSSS	0.1922	0.2004	0.1834	0.1896
	CCCC	0.3661	0.3839	0.3461	0.3633
	CCSS	0.3053	0.3235	0.2844	0.3023
	CSCS	0.3449	0.3607	0.3277	0.3421
	CSSS	0.2866	0.3033	0.2681	0.2837
	CCCS	0.2484	0.2676	0.2258	0.2452
Spherical shell	SSSS	0.1992	0.2071	0.1907	0.1953
	CCCC	0.3729	0.3904	0.3533	0.3706
	CCSS	0.3083	0.3264	0.2877	0.3042
	CSCS	0.3501	0.3656	0.3332	0.3470
	CSSS	0.2911	0.3075	0.2729	0.2868
	CCCS	0.2611	0.2794	0.2398	0.2584
Elliptical-paraboloid shell	SSSS	0.1964	0.2045	0.1878	0.1929
	CCCC	0.3694	0.3871	0.3496	0.3669
	CCSS	0.3069	0.3251	0.2862	0.3032
	CSCS	0.3476	0.3632	0.3305	0.3446
	CSSS	0.2892	0.3057	0.2708	0.2853
	CCCS	0.2546	0.2734	0.2327	0.2517
Hyperbolic-paraboloidal shell	SSSS	0.1893	0.1976	0.1804	0.1881
	CCCC	0.3703	0.3879	0.3506	0.3671
	CCSS	0.3056	0.3238	0.2848	0.3038
	CSCS	0.3469	0.3625	0.3298	0.3443
	CSSS	0.2865	0.3031	0.2680	0.2851
	CCCS	0.2566	0.2752	0.2349	0.2532

The action of the geometric parameters a/h and b/a on the dimensionless frequency of simply supported cross-ply F/CNT-RC shells for various curvature radii is tabulated in Table 10. It is observed that the frequencies decreased when the thickness ratio a/h was decreased and the aspect ratio b/a was increased.

Figure 4 shows the the impact of the number of layers “ N ” on the vibration frequencies of simply supported CNTs/F-RC shells using different fiber distribution patterns. Two types of shells were analyzed: cross-ply fibers and unidirectional fibers. The number of layers “ N ” changed from 3 to 19. In the case of cross-ply fibers, the increase in the number of layers “ N ” led to an increment in the dimensionless frequencies. The FG-X fiber distribution shells had the highest frequencies because of their excellent rigidity. In the case of unidirectional fibers, contradictory results were obtained. Precisely, the dimensionless frequencies increased with the increase in the number of layers for the patterns FG-O and FG-B, while the opposite effect was seen for the FG-X patterns. As is known, to increase the rigidity of the structure, it is obligatory to increase the reinforcement materials at the

superior and the inferior sheets. On the other hand, the increase in the number of layers meant that the thickness of each layer decreased; therefore, the thickness of the strongest layer (superior and inferior layers) decreased, and this can explain the reduction in the rigidity of the plate. For the case of unidirectional fibers with uniform distribution UD, the number of layers did not have any influence.

Table 10. The effect of geometric parameters a/h and b/a on the dimensionless frequency of cross-ply F/CNT-RC shells for various curvature radii (SSSS, $N = 9$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

Type of Shells	b/a	a/h			
		5	10	20	30
Plate	0.5	1.4689	0.4559	0.1231	0.0556
	1	0.7077	0.1984	0.0513	0.0230
	2	0.5230	0.1437	0.0369	0.0165
	3	0.4952	0.1358	0.0348	0.0156
Cylindrical shell	0.5	1.4715	0.4587	0.1260	0.0584
	1	0.7092	0.2004	0.0535	0.0251
	2	0.5226	0.1440	0.0374	0.0170
	3	0.4944	0.1356	0.0350	0.0157
Spherical shell	0.5	1.4749	0.4620	0.1294	0.0617
	1	0.7164	0.2071	0.0597	0.0307
	2	0.5276	0.1486	0.0417	0.0210
	3	0.4973	0.1383	0.0375	0.0181
Elliptical–paraboloid shell	0.5	1.4741	0.4609	0.1282	0.0605
	1	0.7137	0.2045	0.0572	0.0286
	2	0.5255	0.1466	0.0399	0.0193
	3	0.4960	0.1371	0.0363	0.0170
Hyperbolic–paraboloidal shell	0.5	1.4652	0.4552	0.1235	0.0562
	1	0.7049	0.1976	0.0511	0.0229
	2	0.5219	0.1438	0.0374	0.0170
	3	0.4947	0.1361	0.0354	0.0162

Figure 4. The effect of the number of layers “ N ”: (a) Cross-ply fibers (b) Unidirectional fibers (Plate, SSSS, $b/a = 1$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

Figure 5 shows the variation of dimensionless frequencies in the function of the radii of curvature (R_x/a). The radii of curvature R_y/b is fixed at ∞ , 5, and -5 . In general, the increase in the radii of curvature R_x/a led to a critical decrement in frequencies for the values ($R_x/a \leq 5$), and then, the frequencies continued to decrease in the cylindrical and the elliptical-paraboloid shells ($R/b = \infty$, 5) slightly, and they increased in the case of the hyperbolic-paraboloid shell ($R/b = -5$).

Figure 5. The effect of the radius of curvature R/a on the dimensionless frequency of various shell types (Cross-ply, SSSS, $N = 9$, $b/a = 1$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

In Figure 6, the influences of the volume fraction of fibers and CNTs on the vibration frequencies of simply supported FG-X CNTs/F-RC plate are plotted. The minimal volume fraction of the fibers was $V_{min}^f = 0.05$. It is evident that the increase in the volume fraction of the reinforcement led to an increment in plate stiffness; therefore, the natural frequencies increased. Comparing the two reinforcement materials, the increase in the volume fraction of the CNTs had a more important effect than the volume fraction of the fibers.

Figure 6. The effect of the volume fraction of fibers and CNTs (Plate, Cross-ply, SSSS, $N = 9$, $b/a = 1$, $a/h = 10$, $V_{min}^f = 0.05$).

The effect of both the aspect ratio b/a and the thickness ratio a/h on the dimensionless frequency of a spherical shell ($R_x/a = R_y/b = 5$) for various boundary conditions is demonstrated in Figure 7. The increase in the aspect ratio led to a decrement in the dimensionless frequencies regardless of the boundary condition type. For the thickness ratio a/h effect, the frequencies decreased critically for values $a/h \leq 10$. The fully clamped shells had the highest values of frequency, while the lowest values were for the simply supported shells.

Figure 7. The effect of the geometry parameters b/a and a/h on the dimensionless frequency of a spherical shell for various boundary conditions ($N = 9$, $V_{cnt} = 0.01$, $V_{max}^f = 0.1$, $V_{min}^f = 0.3$). (a) inplane ratio b/a , (b) slenderness ratio a/h .

Finally, in Figure 8, the effect of the modes of vibration “ m ” and “ n ” on the dimensionless frequency of a spherical shell is plotted. We can see that the dimensionless frequency is influenced by mode shapes m and n , where the frequency increased with the increase in the vibrational mode shapes.

Figure 8. The effect of the modes of vibration “ m ” and “ n ” on the dimensionless frequency of a spherical shell ($R_x = R_y = 5$, SSSS, $N = 9$, $b/a = 1$, $a/h = 10$, $V_{cnt} = 0.01$, $V_{max}^f = 0.3$, $V_{min}^f = 0.1$).

7. Conclusions

The free vibration behavior of composite laminated plates and shells reinforced by both randomly oriented (10, 10) single-walled carbon nanotubes (SWCNTs) and functionally graded fibers is presented in this work. Four distribution patterns of fiber reinforcement including UD-distribution, V-distribution, O-distribution, and X-distribution were analyzed. The problem was tackled theoretically based on the Galerkin technique and accounting for different boundary conditions. A parametric study was performed systematically to check for the effect of some significant factors on the free vibration response of CNTs/F-RC laminated shells, namely the fiber reinforcement patterns and the volume fraction of CNTs, together with the geometric parameters of the shells. The results showed that the present solution is in close agreement with the other available solutions in the literature. Based on the parametric investigation, the following concluding remarks are revealed:

- It seems that the dimensionless frequencies increased for an increased volume fraction of CNTs/fibers because of a global augmentation in the stiffness of the CNTs/F-RC laminated shell;
- Because of the considerable reinforcement in the superior and inferior layers, the distribution pattern of FG-X fibers produced the highest values of frequencies;
- The variation of the geometric parameters such as aspect ratio, thickness ratio, and radii of curvature had a significant effect on the vibration response of the proposed structure;
- The analytical solution proposed in this work could represent valid benchmarks for engineers and researchers for the purposes of the practical design of shell structures.

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Appendix A

Rigidity matrix elements;

$$K_{11} = A_{11} \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy + A_{66} \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{12} = (A_{12} + A_{66}) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{13} = \left(\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y} \right) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy - B_{11} \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy - (B_{12} + 2B_{66}) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{14} = B_{11}^s \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy + B_{66}^s \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{14} = B_{11}^s \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy + B_{66}^s \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{15} = (B_{12}^s + B_{66}^s) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$K_{21} = (A_{12} + A_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$K_{22} = A_{22} \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy + A_{66} \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$K_{23} = \left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy - B_{22} \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy - (B_{12} + 2B_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$K_{23} = \left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy - B_{22} \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$- (B_{12} + 2B_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$\begin{aligned}
K_{24} &= (B_{12}^s + B_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
K_{25} &= B_{22}^s \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy + B_{66}^s \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
K_{31} &= -\left(\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y}\right) \int_0^a \int_0^b \frac{\partial^2 X_n}{\partial x^2} Y_n X_m Y_n dx dy + B_{11} \int_0^a \int_0^b \frac{\partial^4 X_n}{\partial x^4} Y_n X_m Y_n dx dy + (B_{12} + 2B_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
K_{32} &= -\left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y}\right) \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy + B_{22} \int_0^a \int_0^b X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n dx dy \\
&\quad + (B_{12} + 2B_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
K_{33} &= 2\left(\frac{B_{11}}{R_x} + \frac{B_{12}}{R_y}\right) \int_0^a \int_0^b \frac{\partial^2 X_n}{\partial x^2} Y_n X_m Y_n dx dy + 2\left(\frac{B_{12}}{R_x} + \frac{B_{22}}{R_y}\right) \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
&\quad - \left(\frac{A_{11}}{R_x^2} + 2\frac{A_{12}}{R_x R_y} + \frac{A_{22}}{R_y^2}\right) \int_0^a \int_0^b X_m Y_n X_m Y_n dx dy - D_{11} \int_0^a \int_0^b \frac{\partial^4 X_m}{\partial x^4} Y_n X_m Y_n dx dy \\
&\quad - D_{22} \int_0^a \int_0^b X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n dx dy - 2(D_{12} + 2D_{66}) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
K_{34} &= -\left(\frac{B_{11}^s}{R_x} + \frac{B_{12}^s}{R_y}\right) \int_0^a \int_0^b \frac{\partial^2 X_n}{\partial x^2} Y_n X_m Y_n dx dy + D_{11}^s \int_0^a \int_0^b \frac{\partial^4 X_m}{\partial x^4} Y_n X_m Y_n dx dy \\
&\quad + (D_{12}^s + 2D_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
K_{35} &= -\left(\frac{B_{12}^s}{R_x} + \frac{B_{22}^s}{R_y}\right) \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy + D_{22}^s \int_0^a \int_0^b X_m \frac{\partial^4 Y_n}{\partial y^4} X_m Y_n dx dy + \\
&\quad (D_{12}^s + 2D_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
K_{41} &= B_{11}^s \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy + B_{66}^s \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy \\
K_{42} &= (B_{12}^s + B_{66}^s) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy \\
K_{43} &= \left(\frac{B_{11}^s}{R_x} + \frac{B_{12}^s}{R_y}\right) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy - D_{11}^s \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy - \\
&\quad (D_{12}^s + 2D_{66}^s) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy \\
K_{44} &= F_{11}^s \int_0^a \int_0^b \frac{\partial^3 X_m}{\partial x^3} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy + F_{66}^s \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy - A_{44}^s \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy \\
K_{45} &= (F_{12}^s + F_{66}^s) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} \frac{\partial^2 Y_n}{\partial y^2} \frac{\partial X_m}{\partial x} Y_n dx dy \\
K_{51} &= (B_{12}^s + B_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy
\end{aligned}$$

$$\begin{aligned}
K_{52} &= B_{22}^s \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy + B_{66}^s \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
K_{53} &= \left(\frac{B_{12}^s}{R_x} + \frac{B_{22}^s}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy - D_{22}^s \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy - \\
&\quad (D_{12}^s + 2D_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
K_{54} &= (F_{12}^s + F_{66}^s) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
K_{55} &= F_{22}^s \int_0^a \int_0^b X_m \frac{\partial^3 Y_n}{\partial y^3} X_m \frac{\partial Y_n}{\partial y} dx dy + F_{66}^s \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy - A_{55}^s \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy
\end{aligned}$$

Mass matrix elements;

$$\begin{aligned}
M_{11} &= \left(I_0 + 2 \frac{I_1}{R_x} + \frac{I_3}{R_y^2} \right) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy \\
M_{13} &= - \left(I_1 + \frac{I_2}{R_x} \right) - \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy \\
M_{14} &= \left(I_3 + \frac{I_4}{R_x} \right) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy \\
M_{22} &= \left(I_0 + 2 \frac{I_1}{R_y} + \frac{I_2}{R_y^2} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
M_{23} &= - \left(I_1 + \frac{I_2}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
M_{25} &= - \left(I_3 + \frac{I_4}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy \\
M_{31} &= \left(I_1 + \frac{I_2}{R_x} \right) \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n dx dy \\
M_{32} &= \left(I_1 + \frac{I_2}{R_y} \right) I_1 \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \\
M_{33} &= I_0 \int_0^a \int_0^b X_m Y_n X_m Y_n dx dy - I_2 \left(\int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n dx dy + \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy \right) \\
M_{34} &= -I_4 \int_0^a \int_0^b \frac{\partial^2 X_m}{\partial x^2} Y_n X_m Y_n dx dy \\
M_{34} &= -I_4 \int_0^a \int_0^b X_m \frac{\partial^2 Y_n}{\partial y^2} X_m Y_n dx dy
\end{aligned}$$

$$M_{41} = \left(I_3 + \frac{I_4}{R_x} \right) \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$M_{43} = -I_4 \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$M_{44} = I_5 \int_0^a \int_0^b \frac{\partial X_m}{\partial x} Y_n \frac{\partial X_m}{\partial x} Y_n dx dy$$

$$M_{52} = \left(I_3 + \frac{I_4}{R_y} \right) \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$M_{53} = -I_4 \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$M_{55} = -I_5 \int_0^a \int_0^b X_m \frac{\partial Y_n}{\partial y} X_m \frac{\partial Y_n}{\partial y} dx dy$$

$$M_{12} = M_{15} = M_{21} = M_{24} = M_{42} = M_{45} = M_{51} = M_{54} = 0$$

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