Article

The Generalized First- and Second-Price Auctions: Overbidding, Underbidding, and Optimal Reserve Price

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Abstract: The paper generalizes the models of the standard first- and second-price auctions (FPA and SPA) by using the Cobb-Douglas function to evaluate bidders' two conflicting bidding preferences for the probability of winning the item (C_{pwin}) and the profit conditional on winning (C_{profit}). Compared with the popular expected profit function, this function has the advantage of allowing the bidders to equally or differently value C_{pwin} and C_{profit} from the point of view of multiple criteria decision making, and then can capture bidders' bidding behavior in reality. Introduction of the Cobb-Douglas function into the FPA allows us to provide new interpretations for overbidding (underbidding) behavior and the dependence of the optimal reserve price on the number of bidders. Our new interpretations are all attributed to the bidders' different preferences for C_{pwin} and C_{profit}. In contrast, this introduction into the SPA does not matter. Our findings suggest that the seller should prefer the FPA (SPA) over the SPA (FPA) and set a lower (higher) optimal reserve price if he conjectures that the bidders have a stronger desire to win an auction (to obtain a profit). The above results help the seller to select between the two auctions and set a reserve price in practice.

Keywords: first-price auction; second-price auction; reserve price; Cobb-Douglas preference; overbidding and underbidding; multiple criteria decision making; order statistics

1. Introduction

It is well known that auctions have been applied to discover price and resource allocation in practice. In the standard auction model, when a bidder submits a bid, he has to care about both the probability of winning the item and the profit conditional on winning. It is assumed that a risk-neutral bidder's optimal bid is found by maximizing his expected profit. As indicated in [1], from both an economics perspective and a point of view of a multiple criteria decision making (MCDM) and followed by [2], the expected profit maximization that is utilized to find an equilibrium bid implicitly assumes that the two conflicting bidding criteria of "the probability of winning the item" (hereafter C_{pwin}) and "the profit conditional on winning" (hereafter C_{profit}) have the same importance to the bidder, and it cannot match real auctions because some bidders may prefer C_{pwin} to C_{profit}, whereas other bidders may do the opposite. This indicates that a bidder's different preference between C_{pwin} and C_{profit} might interpret overbidding and underbidding. Indeed, this is verified in Section 4. Particularly, Wang et al. [2] show some evidence to support why a bidder might have a different preference between C_{pwin} and C_{profit}. We find evidence associated with a reverse auction. Boughton [3] reports that in a survey of 126 construction contractors with six company objectives, the leading objective is to "maximize expected profit" (40 percent of respondents), but this is closely followed by "gaining or maintaining market share" (30 percent of respondents). Clearly, the secondary objective implies that some contractors prefer C_{pwin} to C_{profit}.

Because of the above-mentioned theoretical concern and evidence, this article introduces the bidders' Cobb-Douglas preference between the C_{pwin} and C_{profit} into the models

of the standard first- and second-price auctions (hereafter the standard FPA and SPA) with a reserve price r set by a risk-neutral seller. The resulting models are referred to as the generalized FPA and SPA models, respectively. They include the standard FPA and SPA models as special cases. According to [1,2], the Cobb-Douglas preference can capture the bidders' different or same preference between C_{pwin} and C_{profit}, which accords with bidders' responses to real auctions. Accordingly, the other motivation is to confirm whether the well-known results for the standard FPA and SPA are robust or not with the introduction of such a preference. We find that the introduction of such a preference into the standard SPA does not alter bidding behavior. Thus, the generalized SPA is equivalent to the standard SPA. In contrast, we find the introduction of such a preference into the standard FPA has significant effects on the bidders and the seller. Our first three aims are to focus on investigating implications of the introduction of such a preference into the standard FPA, as shown follows.

First, we want to study the implications for the resulting bidding behavior. Particularly, we want to give alternative reasons for underbidding and overbidding using the bidders' Cobb-Douglas preference between C_{pwin} and C_{profit} (i.e., two weights of α and β).

Second, we focus on implications for the optimal reserve prices. Specifically, we want to interpret why the optimal reserve prices are independent of or dependent on the number of bidders. We study whether the optimal reserve prices increase or decrease with the number of bidders and whether these monotonicities are related to the relative size of two weights. We are also interested in analyzing the effects of two weights on the optimal reserve prices.

Third, we analyze the effects of such preference (i.e., two weights) on the seller's (maximum) expected revenue. It is well known that the seller's (maximum) expected revenue increases with the number of bidders. We want to study whether such monotonicity is related to the relative size of two weights.

Lastly, we aim to compare the optimal reserve price and the seller's expected revenue across four kinds of auctions: the standard FPA and SPA and the generalized FPA and SPA. Particularly, with the introduction of such a preference we want to study whether the revenue equivalence principle still holds or not. Particularly, we are interested in the seller's preference over the generalized FPA and SPA.

Our article contributes to existing auction literature in the following four main aspects: (1) We provide alternative interpretations for overbidding and underbidding behavior by using the bidders' Cobb-Douglas preference between C_{pwin} and C_{profit} in the FPA. Specifically, overbidding and underbidding for all values in the FPA are attributed to bidders' greater (less) preference for C_{pwin} than C_{profit}, i.e., $\alpha > \beta$ ($\alpha < \beta$). (2) We generalize the necessary condition and the sufficient condition of the optimal reserve price in the FPA. The generalized necessary condition shows that the reason that the optimal reserve price is independent of the number of the bidders is due to the bidders' same preference between C_{pwin} and C_{profit}, i.e., $\alpha = \beta$. The optimal reserve price decreases (increases) with the number of bidders if $\alpha > \beta$ ($\alpha < \beta$). (3) We show that as the weight assigned to the C_{pwin} (C_{profit}) or α (β) increases, the optimal reserve price decreases (increases) and the seller's (maximum) expected revenue increases (decreases) in the FPA. (4) The generalized FPA (SPA) revenue-dominates the generalized SPA (FPA) if $\alpha > \beta$ ($\alpha < \beta$), while the revenue equivalence theorem holds if $\alpha = \beta$.

The remainder of this article is followed by a literature review in Section 2. Section 3 introduces our model and discusses its relationship with other auction models. Section 4 interprets the respective reasons why bidders overbid and underbid for all valuations in the FPA. Section 5 characterizes the optimal reserve price and interprets the respective reasons that the optimal reserve price is dependent on and independent of the number of bidders. Section 6 makes a comparative analysis by analyzing the respective effects of the number of bidders and two weights on the optimal reserve price and the seller's (maximum) expected revenue and investigates the seller's preference over the generalized

FPA and SPA. Section 7 concludes with the main management insights and directions for future research. All proofs are collected in the Appendix A.

2. Literature Review

This article is closely related to four strands of the literature.

2.1. Applications of Cobb-Douglas Functions

This paper is highly related to a growing body of literature, in which authors utilize Cobb-Douglas functions to model individual's utility and decision-making problems (e.g., [1,2,4–6]). Particularly, Liu and Wang [1] initially incorporate bidders' Cobb-Douglas preference between the two conflicting bidding criteria of C_{pwin} and C_{profit} into the models of the standard FPA and SPA. Wang et al. [2] extend the work of Liu and Wang [1] by considering both the commission rate and the reserve price. Ghatak and Jiang [5] assume that individuals have identical Cobb-Douglas utility functions over consumption and bequests which are conflicting. Corchón and Dahm [4] combine the probability of winning the prize and the quality of the prize in a Cobb-Douglas way to represent each contestant's utility. Tsai [6] uses the Cobb-Douglas utility function to capture a bank's preference between the like of higher equity returns and the dislike or disutility of higher equity risks. Similar to [1,2], we utilize the Cobb-Douglas utility function to capture bidders' two conflicting bidding preferences between C_{pwin} and C_{profit}.

2.2. Interpretations of Overbidding and Underbidding

Different interpretations for the overbidding, underbidding, or both phenomena have been given. As for the FPA, overbidding (underbidding) means that bidders tend to bid above (below) the risk-neutral Bayesian Nash equilibrium. Initially, overbidding phenomenon is explained by risk aversion within the expected utility framework in, for example, [7,8]. It is argued that risk aversion cannot be the sole factor and may well not be the most important factor behind overbidding (e.g., [9,10]). Along this line, a series of articles have provided alternative explanations, for example, by using a Star-shaped probability weighting function (hereafter PWF) [11], the anticipation of loser regret [12], and ambiguity aversion [10].

On the other hand, both overbidding and underbidding have been theoretically or experimentally rationalized. Some articles predict that bidders with higher values overbid, while those with lower values underbid, for example, by loss aversion [13], the inverse S-shaped PWF [14], the anticipated emotions [15], whereas other articles predict the opposite results experimentally (e.g., [16,17]) and theoretically (e.g., [18,19]).

In sharp contrast to the above papers, we find that both overbidding and underbidding for all valuations can be predicted by the bidders' Cobb-Douglas preference between C_{pwin} and C_{profit}. We show that overbidding (underbidding) for all valuations is due to the bidders' more (less) preference for C_{pwin} than C_{profit}. This is driven by the fact that a bidder with a more (less) preference for C_{pwin} than C_{profit} might be induced to bid more (less) aggressively compared to the same preference between them (As shown in Section 4, the equilibrium bidding strategy with the same preference reduces to the risk-neutral Bayesian Nash equilibrium). Thus, our interpretation is consistent with intuition but is different from the previous interpretations.

Few articles focus on the amounts of overbidding and underbidding. Roider and Schmitz [15] theoretically show that the amount of overbidding of a participating bidder is independent of the valuation in their generalized SPA model, and increases with the valuation in their generalized FPA model if the anticipated disappointment from losing the auction is positive. Kirchkamp and Reiß [17] find that the amount of underbidding depends on the seemingly innocuous parameters of the experimental setup. In contrast, we show that the amount of overbidding of a participating bidder increases with the weight assigned to C_{pwin} (i.e., α) and decreases with the weight assigned to C_{profit} (i.e., β) if $\alpha > \beta$,

the amount of underbidding decreases with α and increases with β if $\alpha < \beta$, and both of them depend on the bidder's valuation.

2.3. Optimal Reserve Prices

This article is mostly related to the auction literature that focuses on optimal reserve prices with the risk-neutral seller and bidders. It is well known that the optimal reserve price is independent of the number of bidders in the standard FPA (e.g., [20]). Rosenkranz and Schmitz [19] modify the standard FPA model by using the reference point (that equals the weighted average of the reserve price r and an exogenous variable x, i.e., $\lambda r + (1 - \lambda)x$ with $\lambda \geq 0$) to adjust bidder's utility and show that the optimal reserve price increases with (is independent of) the number of bidders if the reference point is (is not) affected by the reserve price (i.e., $\lambda > 0$ ($\lambda = 0$)). In contrast, Roider and Schmitz [15] extend the standard FPA model by utilizing a bidder's feelings of both the anticipated joy of winning the auction ($\varepsilon \geq 0$) and the anticipated disappointment from losing the auction ($\gamma \geq 0$) to adjust the bidder's utility and show that the optimal reserve price decreases with (is independent of) the number of bidders if $\gamma > 0$ ($\gamma = 0$), increases with ε and decreases with γ. More recently, Li [21] extends the standard FPA model by applying the exogenous reference point ρ for the price of the item to adjust a bidder's utility and shows that the optimal reserve price increases with ρ and is independent of the number of bidders when there is no reference effect in the bidder's utility. However, he neglects the monotonicity of the optimal reserve price in the number of bidders.

In sharp contrast with the above articles, we extend the standard FPA model by adopting bidders' Cobb-Douglas preference. We find that the reserve price is independent of (dependent on) the number of bidders in the generalized FPA if bidders have the same (different) preference between C_{pwin} and C_{profit}. We also find that whether the optimal reserve price increases or decreases with the number of bidders is due to bidders' less or more preference over C_{pwin}. Interestingly, the two weights play similar roles as the reference point ρ [21], the anticipated disappointment γ [15], and the weight λ [19].

2.4. The Revenue Equivalence Principle

This article is closely related to the auction literature on the revenue equivalence principle with the risk-neutral seller and bidders. Ahmad [18] shows that the first- and second-price auctions are revenue equivalent when naive bidders have dependent preferences and an exogenous reference point that is increasing. However, when the naive bidders are loss-averse, revenue equivalence is broken down. Wang and Liu [1] show that the FPA and SPA without a reserve price are revenue equivalent if bidders have the same preference between C_{pwin} and C_{profit}, and that the FPA (the SPA) revenue-dominates the SPA (the FPA) if the weight assigned to C_{pwin} is greater (less) the weight assigned to C_{profit}, while we obtain similar results with a reserve price. Roider and Schmitz [15] show the FPA and SPA with bidders' anticipated motions are also revenue equivalent. Rosenkranz and Schmitz [19] show the FPA and SPA with the reference point mentioned above are revenue equivalent. More recently, Wang, et al. [2] show that the revenue equivalence theorem fails in practical auctions with the commission rate, and Li [21] shows that the FPA (the SPA) revenue-dominates the SPA (the FPA) with loss-averse bidders, but the converse is true with gain-seeking bidders.

3. The Model

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, as well as the experimental conclusions that can be drawn.

A seller has a personal value $v_0 \in (L, H)$ for a single indivisible item to be sold. He sets a reserve price r within the interval $[L, H]$. The item is to be sold to one of $n \geq 2$ bidders through the generalized FPA or SPA auction with a reserve price r, where all the bidders simultaneously submit their sealed bids and the bidder with the highest

bid being greater than the reserve price r wins the auction and pays his bid price to the seller. Each bidder $i \in \{1, 2, \ldots, n\}$ has a private value for the object, v_i, which is unknown to the others. These values are independently and identically distributed on $[L, H]$ according to distribution function F, with a density function $f = F'$ that is strictly positive and continuously differentiable on $[L, H]$. In equilibrium, a bidder drops from bidding if his value v is less than r, and submits a bid of r if his value v is equal to r. By a participating bidder with a value v we mean $v \geq r$.

In the generalized FPA, the bidder is assumed to solve the following Cobb-Douglas utility maximization problem:

$$\max_b \ [\text{Pwin}(b)]^{\alpha}(v - b)^{\beta}, \ \alpha, \beta > 0, \ \alpha + \beta = 1, \tag{1}$$

where $\text{Pwin}(b)$ denotes the bidder's probability of winning at a bid b, α and β represent the bidder's preference between (or the two weights that the bidder places on) the two bidding criteria C_{pwin} and C_{profit}. Particularly, the bidder has more preference for C_{pwin} (C_{profit}) than C_{profit} (C_{pwin}) if he assigns more weight to C_{pwin} (C_{profit}) than to C_{profit} (C_{pwin}), i.e., $\alpha > \beta$ ($\alpha < \beta$), and has the same preference between C_{pwin} and C_{profit} if he weighs them equally, i.e., $\alpha = \beta$. All of these similar interpretations can also be found in [1,2], from both an economics perspective and from the point of view of MCDM.

The following remark connects our Model (1) with the other well-known auction models. It also distinguishes our Model (1) from them.

Remark 1. *The generalized FPA model (1) has the following three features.*

(i) First, Model (1) with $\alpha = \beta$ is equivalent to the FPA model, but it is different from the latter when $\alpha \neq \beta$. Clearly, the bidders are risk neutral as if they weigh C_{pwin} and C_{profit} equally.

(ii) Second, Model (1) with $\alpha \neq \beta$ is equivalent to an expected utility maximization model (EUMax) with a power utility function x^d with $d = \beta/\alpha$. However, Model (1) with $\alpha \neq \beta$ and the expected utility maximization model differ by meanings of the relevant parameters. The quotient of β and α (i.e., β/α) has the same meaning as d, where the degree of relative risk aversion (seeking) is given as $1 - d$ ($d - 1$) if $d < 1$ ($d > 1$), but α and β have the separate meanings mentioned above. Clearly, $\alpha > \beta$ ($\alpha < \beta$) is equivalent to $d < 1$ ($d > 1$). Thus, bidders have more preference for C_{pwin} (C_{profit}) than C_{profit} (C_{pwin}) as if they are CRRA (CRRS) bidders.

(iii) Third, Model (1) with $\alpha \neq \beta$ is equivalent to the probability weighting (PW) model with a power PWF of $[Pwin(b)]^c$ and $c = \alpha/\beta$. Specifically, it corresponds to a probability overweighting (underweighting) model when $\alpha < \beta$ ($\alpha > \beta$). However, Model (1) with $\alpha \neq \beta$ and the power PW model differ by meanings of the relevant parameters. The quotient of α and β (i.e., α/β) has the same meaning as c, but α and β have the separate meanings mentioned above. Bidders have more preference for C_{pwin} (C_{profit}) than C_{profit} (C_{pwin}) as if they underweight (overweight) $Pwin(b)$ because $\alpha > \beta$ ($\alpha < \beta$) is equivalent to $c > 1$ ($c < 1$).

It is worth mentioning that we mainly investigate the respective effects of the two weights of α and β on bidding behavior, properties of optimal reserve prices, and the seller's (maximum) expected revenue instead of the related effects of parameter $1 - d$ (or $d - 1$) in the EUMax model or parameter c in the PW model. Thus, all of the results obtained for our model (1) generalize and are beyond the related well-known results for the above three kinds of models because of Remark 1 and subsequent sections.

The following remark justifies our model (1) by the prospect theory [22].

Remark 2. *When a bidder submits a bid of b, he receives a revenue of $v - b$ with probability of $Pwin(b)$ and zero revenue with probability of $1 - Pwin(b)$. Thus, the bidder's bid of b yields a unique regular prospect $(v - b, Pwin(b); 0, 1 - Pwin(b))$ whose whole value is $\pi(Pwin(b))w(v - b) + \pi(1 - Pwin(b))w(0)$, where $\pi(x)$ is referred as the weighting function and increasing with $\pi(0) = 0$ and $\pi(1) = 1$, and $w(x)$ the value function. Now, if we assume that $\pi(x) = x^{\alpha}$ and $w(x) = x^{\beta}$ (for example, see [4]), then the whole value of the above prospect associated with the bidder's bid b is reduced to the objective function in (1).*

In a totally parallel method for proving Proposition 2.1 in [23] (or for proving Proposition 2 without a reserve price in [1]), we can show that in the generalized SPA, the weakly dominant strategy of a bidder with a valuation $v \geq r$ is to submit a bid that is equal to v, which is independent of the number of bidders and the two weights, and exactly the same as in the standard SPA. Thus, the generalized SPA is totally equivalent to the standard SPA for both the bidders and seller.

4. Overbidding and Underbidding

In an entirely similar way that Liu and Wang [1] obtain a bidder's symmetric equilibrium bid without a reserve price in their Proposition 1, we can derive a bidder's symmetric equilibrium bid function for the generalized FPA model (1), as follows:

$$b^I(v, r, n, \alpha, \beta) = v - \int_r^v \left[\frac{F(y)}{F(v)} \right]^{\alpha(n-1)/\beta} dy \text{ for } v \in [r, H] \tag{2}$$

Clearly, $b^I(v, r, n, \alpha, \beta)$ reduces to the well-known bid function in the standard FPA when $\alpha = \beta$ and $r = 0$ (for example, [23], p. 16), increases in both n and α, and decreases in β when $v \in (r, H]$ (Throughout this article, "increase" and "nondecrease" mean in strong and weak senses, respectively; "decrease" and "nonincrease" have similar meanings.). For simplicity, we drop some arguments in $b^I(v, r, n, \alpha, \beta)$ when we are not interested in them. For example, we rewrite $b^I(v, r, n, \alpha, \beta)$ as $b^I(v, r)$ when given n, α, and β. In addition, $b_v^I(v, r)$ and $b_r^I(v, r)$ denote the partial derivatives of $b^I(v, r)$ with respect to v and r, respectively.

Intuitively, when a bidder participates in the generalized FPA, his more (less) preference for C_{pwin} than C_{profit} might induce him to bid above (below) what he would bid with the same preference (i.e., the risk-neutral Nash equilibrium). Our first proposition confirms this intuition. Liu and Wang [1] have not talked about the overbidding and underbidding phenomena for the generalized FPA without a reserve price.

Proposition 1. *Consider the generalized FPA.*
(i) A bidder will overbid (underbid) for all $v > r$ if $\alpha > \beta$ ($\alpha < \beta$).
(ii) The amount of overbidding increases with α and decreases with β if $\alpha > \beta$, while the amount of underbidding decreases with α and increases with β if $\alpha < \beta$.

Remark 3. *The proof of Proposition 1 shows that both the amount of overbidding and the amount of underbidding depend on the valuations.*

5. Optimal Reserve Price

This section analyzes the implications of the bidders' Cobb-Douglas preference for optimal reserve prices. Let $V_I(r)$ denote the seller's equilibrium expected revenue as a function of the reserve price, and r_I be an optimal reserve price in the generalized FPA. It is well known that we have for $r \in [L, H]$

$$V_I(r) = \int_r^H b^I(v, r) dF^n(v) + v_0 F^n(r). \tag{3}$$

Since $b(r, r) = r$, differentiating (3) with respect to r yields

$$V_I'(r) = \int_r^H b_r^I(v, r) dF^n(v) - nF^{n-1}(r)f(r)(r - v_0). \tag{4}$$

Substituting $b_r^I(v, r) = [F(r)/F(v)]^{\alpha(n-1)/\beta}$ by (2) into (4) and simplifying yield

$$V_I'(r) = nF^{n-1}(r)f(r)[\Psi(r, m) - r + v_0], \tag{5}$$

where $\Psi(r, m) = \frac{1}{f(r)} \int_r^H \left[\frac{F(v)}{F(r)}\right]^m dF(v)$, $m = (1 - \alpha/\beta)(n - 1)$. Furthermore,

$$\Psi(r, m) = \begin{cases} \frac{1}{m+1} \frac{F^{-m}(r) - F(r)}{f(r)}, & \text{if } m \neq -1 \\ -\frac{F(r)\ln F(r)}{f(r)}, & \text{if } m = -1 \end{cases} \tag{6}$$

5.1. A Necessary Condition and a Sufficient Condition

Using Equations (5) and (6) and the following lemma we can obtain a necessary condition and a sufficient condition for the optimal reserve prices under the generalized FPA.

Proposition 2. *Consider the generalized FPA. Let* $m = (1 - \alpha/\beta)(n - 1)$.
(i) (A necessary condition) r_I *exists within the open interval* (v_0, H) *and satisfies*

$$r_I = v_0 + \Psi(r_I, m), \tag{7}$$

or

$$r_I = \begin{cases} v_0 + \frac{1}{m+1} \frac{F^{-m}(r_I) - F(r_I)}{f(r_I)}, & \text{if } m \neq -1 \\ v_0 - \frac{F(r_I)\ln F(r_I)}{f(r_I)}, & \text{if } m = -1 \end{cases} \tag{8}$$

(ii) (A sufficient condition) there exists a unique solution $r_I \in (v_0, H)$ *to (7) that is the optimal reserve price if* $r - \Psi(r, m)$ *strictly increase in r over* $[L, H]$ *for any m.*

Remark 4. *Suppose that* $\alpha = \beta$. *Then, Model (1) reduces to the standard FPA model as indicated in Remark 1,* $m = 0$, *and* $\Psi(r, 0) = \frac{1 - F(r)}{f(r)}$. *Hence, from Proposition 2 our necessary condition (7) degenerates into the well-known necessary condition for the standard FPA model as expected (e.g., see Equation (2.12) in [23] and the equation in Proposition 4 in [20]):*

$$r_I = v_0 + \frac{1 - F(r_I)}{f(r_I)}$$

and our sufficient condition (i.e., $\Psi(r, 0)$ *nonincreases in* $[L, H]$*) becomes*

$$\frac{d}{dr}\left(r - \frac{1 - F(r)}{f(r)}\right) > 0$$

which is weaker than the well-known sufficient condition in the standard FPA model—the hazard rate function associated with the distribution F, $\lambda(v) = f(v)/(1 - F(v))$, *is nondecreasing (e.g., [23]).*

Let $V_{II}(r)$ denote the seller's equilibrium expected revenue as a function of the reserve price r, and r_{II} be an optimal reserve price in the generalized SPA. Since the weakly dominant strategy of a bidder with $v \geq r$ is to submit a bid that is equal to v, for $r \in [L, H]$ we have

$$V_{II}(r) = nF^{n-1}(r)(1 - F(r))r + n\int_r^H v(1 - F(v))dF^{n-1}(v) + v_0 F^n(r), \tag{9}$$

The Equation (9) directly follows from three events that occur in the auction: (i) the highest value $v_{[1]}$ is greater than the reserve price r and the second highest value $v_{[2]}$ is less than r, i.e., $v_{[1]} > r$ and $v_{[2]} < r$; (ii) $v_{[1]} > r$ and $v_{[2]} \geq r$; and (iii) $v_{[1]} \leq r$. When the first event occurs, i.e., there is only one bidder's value being greater than r and the other bidders' values are all less than r, the seller receives r with a probability of $nF^{(n-1)}(r)(1 - F(r))$, which leads to the first term in (9). When the second event occurs, the seller receives $v_{[2]}$ with the probability density function of $n(n - 1)F^{(n-2)}(v)(1 - F(v))f(v)$, which provides the second term. When the third event occurs, the auction fails and the seller keeps the item with a probability of $P\{v_{[1]} \leq r\} = F^{(n)}(r)$, which yields the third term.

Remark 5. *(i) Equation (9) is a special case of Equation (8) in [24]. However, Hu, et al. [24] have not interpreted how to obtain their Equation (8).*

(ii) To our best knowledge, we cannot find an interpretation of (9) in existing auction literature. Without the reserve price (i.e., r = 0), (9) reduces to $V_{II}(0) = n \int_0^H v(1 - F(v))dF^{n-1}(v)$, which is exactly the same as Equation (3.14) in [25].(p. 19). Menezes and Monteiro [25] provide the same interpretation of Equation (3.14) as we do for the second term in (9).

(iii) The necessary condition and the sufficient condition for the optimal reserve price r_{II} comes directly from Equations (11) and (12), and Proposition 5 for the risk averse bidders in [24].

5.2. Independence of and Dependence on the Number of Bidders

The next interesting result comes directly from Proposition 2(i).

Corollary 1. *Consider the generalized FPA. Let m = (1 − α/β)(n − 1). Then,*

(i) r_I is independent of the number of bidders if α = β (or equivalently m = 0) or α = n/(2n − 1) (or equivalently m = −1).

(ii) r_I is dependent on the number of bidders if α ≠ β (or equivalently m ≠ 0) and α ≠ n/(2n − 1) (or equivalently m ≠ −1).

Note that our model (1) for α = β is equivalent to the standard FPA model in the sense that both models give the identical equilibrium bid. Corollary 1 (i) shows that one of the reasons why the optimal reserve price r_I is independent of the number of bidders in the standard FPA model is due to the bidders' same weight being placed on C_{pwin} and C_{profit}. However, Part (ii) shows that the optimal reserve price r_I is dependent on the number of bidders when the two weights are different and the weight placed on C_{pwin} does not equal $n/(2n − 1)$.

6. Comparative Analysis

This section considers the respective effects of the parameters *n*, *α*, and *β* on the optimal reserve price, and the seller's (maximum) expected revenue under the generalized FPA. Let $V_I(r, n, α, β)$ and $r_I(n, α, β)$ be the seller's expected revenue and the optimal reserve price. For simplicity, we may drop some arguments in $V_I(r, n, α, β)$ and $r_I(r, n, α, β)$ when we are not interested in them. For example, we rewrite $V_I(v, r, n, α, β)$ as $V_I(r, n)$ and $r_I(n, α, β)$ as $r_I(n)$ when given α and β.

6.1. Effects on the Optimal Reserve Price

To investigate the effect of the number of *n* on the optimal reserve price for a fixed pair (α, β), we establish the following lemma.

Lemma 1. *Suppose that for any fixed positive number n, $h_n(x)$ is differentiable and quasi-concave on (c, d), and $h_n'(x) = p_n(x)q_n(x)$. Let x*(n) maximize $h_n(x)$ on (c, d). If $p_n(x) > 0$ holds on (c, d) for any fixed n, and $q_n(x)$ increases (decreases) in n for any fixed x∈ (c, d), then x*(n) < x*(n + 1) (x*(n) > x*(n + 1)).*

The next proposition shows that the relative importance of the two bidding criteria has significant effects on the monotonic properties of the optimal reserve price $r_I(n)$. It shows that $r_I(n)$ increases (decreases) with the number of bidders if the bidders have a less (more) preference for C_{pwin} than C_{profit}.

Proposition 3. *Consider the generalized FPA. Let m = (1 − α/β)(n − 1) and fix a pair (α, β). Then, $r_I(n) < r_I(n + 1)$ ($r_I(n) > r_I(n + 1)$) if α < β (α > β and α ≠ n/(2n − 1)).*

To investigate the effects of the two weights α and β on the optimal reserve price, we introduce the following lemma.

Lemma 2. *([24]) For $c < d \leq \infty$ and $i = 1, 2$, let the functions h_i: $[c, d] \rightarrow R$ be differentiable and satisfy $h_1' < h_2'$ on (c, d). Let t_i maximize h_i on $[c, d]$. If $t_i \in (c, d)$ for $i = 1$ or $i = 2$, then $t_1 < t_2$.*

The following proposition shows that the more weight that the bidders place on the probability of winning (the profit conditional on winning), the lower (higher) the optimal reserve price $r_I(\alpha, \beta)$ that the seller should set. It could help the seller to set an appropriate reserve price.

Proposition 4. *Consider the generalized FPA. Then, $r_I(\alpha, \beta)$ decreases (increases) in $\alpha(\beta)$.*

6.2. Effects on the Seller's (Maximum) Expected Revenue

This subsection analyzes the respective effects of the number of bidders and the two weights on the seller's (maximum) expected revenue.

What happens to the seller's expected revenue under the generalized FPA if the number of bidders increases? Intuitively, the seller benefits from the more participating bidders. However, the answer is not clear by the expression of the seller's expected revenue (3) although the bidding strategy increases with the number of participants. The following proposition shows that the seller's expected revenue increases with the number of participants if the seller's personal value is less than or equal to the reserve price(Menezes and Monteiro [25] (p. 21) have shown this result under both the standard FPA and SPA auctions without a reserve price. Our proof is significantly different from theirs and shows that some specific manipulation is needed), and the seller's maximum expected revenue always increases with the number of bidders.

Proposition 5. *Consider the generalized FPA. Suppose that there are n participating bidders. Then*
(i) the seller's expected revenue increases with n if $r \in [v_0, H)$;
(ii) the seller's optimal expected revenue increases with n.

The following proposition shows that under the generalized FPA both the seller's expected revenue and maximum expected revenue increase with the weight assigned to the probability of winning and decrease with the weight assigned to the profit conditional on winning.

Proposition 6. *Consider the generalized FPA. Suppose that all the n bidders' values are greater than the reserve price. Then*
(i) the seller's expected revenue increase (decrease) in $\alpha(\beta)$.
(ii) the seller's optimal expected revenue increase (decrease) in $\alpha(\beta)$.

6.3. Comparisons across Four Kinds of Auctions

This subsection compares the optimal reserve price and the seller's expected revenue across the standard FPA and SPA, and the generalized FPA and SPA. Let $V_1(r)$ and $V_2(r)$ denote the seller's equilibrium expected revenues as a function of the reserve price, and r_1^* and r_2^* be the optimal reserve prices in the standard FPA and SPA, respectively. To investigate the effects of the bidders' preference on the optimal reserve price and the seller's expected revenue, we rewrite (3) as

$$V_I(r, \alpha, \beta) = \int_r^H b^I(v, r, \alpha, \beta) dF^n(v) + v_0 F^n(r) \tag{10}$$

Since $b^I(v, r, \alpha, \beta)$ is identical to the equilibrium bidding strategy in the standard FPA by (1) when $\alpha = \beta = 1/2$, we have $V_I(r, 1/2, 1/2) = V_1(r)$. Because the generalized SPA is equivalent to the standard SPA, as mentioned in Section 3, we have $V_2(r) = V_{II}(r)$. Furthermore, using the integration by part and repeated integral we can prove that $V_1(r) = V_2(r)$ (Menezes and Monteiro [25] (p. 20–21) have shown this result when there is no reserve

price. Our proof is significantly different from theirs and more difficult than theirs (See the Appendix A).). Thus,

$$V_I(r, 1/2, 1/2) = V_1(r) = V_2(r) = V_{II}(r), \qquad (11)$$

and

$$r_I(1/2, 1/2) = r_1^* = r_2^* = r_{II} \qquad (12)$$

Hence, the introduction of the bidders' Cobb-Douglas preference into the standard SPA has no effect on the optimal reserve price and the seller's expected revenue. However, it does have significant effects on the optimal reserve price and the seller's expected revenue for the standard FPA, as shown in the following Corollary.

The following proposition shows that with the presence of the bidders' different preference between the two bidding criteria, the results of optimal reserve prices and the seller's expected revenue comparisons between the generalized FPA and the generalized SPA are significantly different from the two standard auctions, and that the revenue equivalence theorem no longer holds when $\alpha \neq \beta$. Particularly, the seller should prefer the generalized FPA over the generalized SPA if he guesses that all the bidders are eager to win the auction. Otherwise, the seller should prefer the generalized SPA to the generalized FPA.

Proposition 7. *Consider the generalized FPA and SPA. Let $m = (1 - \alpha/\beta)(n - 1)$. If $\alpha > \beta$ ($\alpha = \beta$, $\alpha < \beta$) or $m > 0$ ($m = 0$, $m < 0$), then*
　　(i) $r_I(\alpha, \beta) < r_{II}$ (respectively, $r_I(\alpha, \beta) = r_{II}$, $r_I(\alpha, \beta) > r_{II}$);
　　(ii) $V_I(r, \alpha, \beta) > V_{II}(r)$ (respectively, $V_I(r, \alpha, \beta) = V_{II}(r)$, $V_I(r, \alpha, \beta) < V_{II}(r)$);
　　(iii) $V_I(r_I, \alpha, \beta) > V_{II}(r_{II})$ (respectively, $V_I(r_I, \alpha, \beta) = V_{II}(r_{II})$, $V_I(r_I, \alpha, \beta) < V_{II}(r_{II})$).

Corollary 2. *Consider the generalized FPA and the standard FPA. Let $m = (1 - \alpha/\beta)(n - 1)$. If $\alpha > \beta$ ($\alpha = \beta$, $\alpha < \beta$) or $m > 0$ ($m = 0$, $m < 0$), then,*
　　(i) $r_I(\alpha, \beta) < r_1^*$ (respectively, $r_I(\alpha, \beta) = r_1^*$, $r_I(\alpha, \beta) > r_1^*$);
　　(ii) $V_I(r, \alpha, \beta) > V_1(r)$ (respectively, $V_I(r, \alpha, \beta) = V_1(r)$, $V_I(r, \alpha, \beta) < V_1(r)$);
　　(iii) $V_I(r_I, \alpha, \beta) > V_1(r_1^*)$ (respectively, $V_I(r_I, \alpha, \beta) = V_1(r_1^*)$, $V_I(r_I, \alpha, \beta) < V_1(r_1^*)$).

7. Concluding Remarks

This article adopts the bidders' Cobb-Douglas preference to extend the standard FPA and SPA models, where the two powers α and β in the Cobb-Douglas utility function represent the relative importance of the two conflicting bidding criteria C_{pwin} and C_{profit} to the bidder or capture the bidder's same or different preference between C_{pwin} and C_{profit}. We analyze the resulting bidding behavior, optimal reserve prices, and the seller's (maximum) expected revenue. Interestingly, we have shown that in the presence of the bidders' different preferences between C_{pwin} and C_{profit}, our results for bidding behavior, optimal reserve prices, and the seller's expected revenue are significantly different from the standard FPA model. However, the prominent results for the standard SPA are robust.

For the generalized FPA we find that both the reason that the bidders tend to overbid and underbid for all valuations and the reason that the optimal reserve price is dependent on the number of bidders are all attributed to the bidders' different preferences between C_{pwin} and C_{profit}. Specifically, our results predict overbidding (underbidding) for all valuations (In the artwork market, there are many real examples of expensive prices at auctions. "Paradise" painted by Alibaba co-founder Jack Ma with his friend and well-known Chinese artist Zeng Fanzhi was sold at Sotheby's for about 42.2 million Hong Kong dollars (HKD; $5.4 million)—17 times more than the price estimated by the auction house—to Chinese businessman Qian Fenglei (see http://money.cnn.com/2015/10/04/technology/jack-ma-painting-charity-sothebys/, accessed on 22 January 2022). This implies that the overbidding phenomenon occurred in this real auction because the bidder wanted to hold the piece at any cost (i.e., the bidder has a greater preference for C_{pwin} than C_{profit}). According to Proposition 1, overbidding is predicted. Thus, our result can explain the

outcome of this real auction.) and show that the optimal reserve price decreases (increases) with the number of bidders if the bidders have a greater (less) preference for C_{pwin} than C_{profit}. The reason that the optimal reserve price is independent of the number of bidders is due to the bidders' same preference between C_{pwin} and C_{profit}. In contrast, the standard FPA theory cannot interpret overbidding or underbidding and establishes that the optimal reserve price is independent of the number of bidders. The bidders' preference between C_{pwin} and C_{profit} does not affect the well-known fact in the standard FPA theory that the seller's (maximum) expected revenue increases with the number of bidders. We show that the optimal reserve price decreases (increases) with the weight assigned to the C_{pwin} (C_{profit}). These results help the seller to set a reasonable reserve price. We find that both the seller's expected revenue and optimal expected revenue increase (decrease) with the weight assigned to the C_{pwin} (C_{profit}).

We show that the optimal reserve price is lower and the seller's (maximum) expected revenue is higher in the generalized FPA (SPA) than in the generalized SPA (FPA) if the bidders have a greater (less) preference for C_{pwin} than C_{profit}. In this case, the revenue equivalence theorem no longer holds. This finding suggests that the seller should prefer the FPA (SPA) over the SPA (FPA) if he thinks the bidders are more eager to win the auction (to obtain the profit conditional on winning). This helps the seller to select between the two auctions in practice.

Finally, we think that the Cobb-Douglas preference between the two conflicting bidding criteria C_{pwin} and C_{profit} might be applied to extend other kinds of auction models. Our related results could be useful for investigating econometrics of auctions and can be used to, for example, test bidders' preferences over C_{pwin} and C_{profit} by using real auction data or Monte Carlo simulations.

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Appendix A

Proof of Proposition 1. (i) By the increase (decrease) of $b^I(v, \alpha, \beta)$ in α (β) and $\alpha + \beta = 1$ we have

$$b^I(v, \alpha, \beta) > (<) b^I(v, 1/2, 1/2), \tag{A1}$$

if $\alpha > \beta$ ($\alpha < \beta$) or $\alpha > 1/2$ ($\alpha < 1/2$). Since $\max_b[\text{Pwin}(b)]^{1/2}(v - b)^{1/2}$ is equivalent to $\max_b \text{Pwin}(b)(v - b)$, $b^I(v, 1/2, 1/2)$ is also the risk-neutral Nash equilibrium in the standard FPA model. Thus, the proof is completed by (A1).

(ii) When $\alpha > \beta$, a bidder will overbid from (i), and by (1) the amount of overbidding is

$$b^I(v, \alpha, \beta) - b^I(v, 1/2, 1/2) = \int_r^v \left[\left(\frac{F(y)}{F(v)} \right)^{n-1} - \left(\frac{F(y)}{F(v)} \right)^{\alpha(n-1)/\beta} \right] dy,$$

which increases with α and decreases with β since $0 < F(y)/F(v) < 1$ for all $y \in [r, v]$. Similarly, we can prove the second part since the amount of underbidding is $b^I(v, 1/2, 1/2) - b^I(v, \alpha, \beta)$ when $\alpha < \beta$. □

Proof of Proposition 2. (i) Since $V_I(r)$ is continuous over a closed interval $[L, H]$ by (3), $r_I \in [L, H]$ exists. For any $r \in (L, v_0]$, the first term in (4) is positive since for $v \in (r, H)$, $b_r(v, r) > 0$ by (1), $F(v) > 0$, and $f(v) > 0$, and the second term is nonpositive. Hence, $V_I'(r) > 0$ on $(L, v_0]$, implying that $r_I \in (v_0, H]$. Since $\lim_{r \to H-0} V_I'(r) = -n(H - v_0)f(H) < 0$ by (4), $r_I \in (v_0, H)$. Thus, r_I must satisfy the first-order condition $V_I'(r^*) = 0$, i.e., (7) by (5).

(ii) Since $\Psi(r, m)$ nonincreases in r for any m, $\Psi(r, m) - r$ decreases in r for any m. Thus, there exists a unique solution $r_I \in (v_0, H)$ to (7). Differentiating (5) with respect to r yields

$$\begin{aligned} V_I''(r) = {} & n(n-1)F^{n-2}(r)(f(r))^2[\Psi(r,\,m) - r + v_0] \\ & + nF^{n-1}(r)f'(r)[\Psi(r,\,m) - r + v_0] \\ & + nF^{n-1}(r)f(r)[\partial\Psi(r,\,m)/\partial r - 1] \end{aligned}$$

Thus, by (7) we have $V_I''(r_I) = nf(r_I)F^{(n-1)}(r_I)[\partial\Psi(r_I, m)/\partial r - 1] < 0$ since $r - \Psi(r, m)$ strictly increase in r over $[L, H]$ for any m. Hence, the unique solution $r_I \in (v_0, H)$ to (7) is the optimal reserve price since (7) is equivalent to $V_I'(r_I) = 0$ by (5). $\square$

Proof of Lemma 1. Suppose that $q_n(x)$ increases in n for any fixed $x \in (c, d)$. Since $h_{n+1}'(x^*(n+1)) = 0$ and $p_{n+1}(x^*(n+1)) > 0$, $q_{n+1}(x^*(n+1)) = 0$. Thus, from $p_n(x^*(n+1)) > 0$ we have

$$h_n'(x^*(n+1)) = p_n(x^*(n+1))q_n(x^*(n+1)) < p_n(x^*(n+1))q_{n+1}(x^*(n+1)) = 0,$$

which implies $x^*(n) \neq x^*(n+1)$. Now, suppose that $x^*(n) > x^*(n+1)$. Since $h_n(x^*(n)) \geq h_n(x^*(n+1))$ and $h_n(x)$ is quasi-concave, we have $(x^*(n) - x^*(n+1))h_n'(x^*(n+1)) \geq 0$ or $h_n'(x^*(n+1)) \geq 0$ by $x^*(n) > x^*(n+1)$, contradicting $h_n'(x^*(n+1)) < 0$. Thus, we must have $x^*(n) < x^*(n+1)$.

Suppose that $q_n(x)$ decreases in n for any fixed $x \in (c, d)$. Since $h_n'(x^*(n)) = 0$ and $p_n(x^*(n)) > 0$, $q_n(x^*(n)) = 0$. Furthermore, by $p_{n+1}(x^*(n)) > 0$ we have

$$h_{n+1}'(x^*(n)) = p_{n+1}(x^*(n))q_{n+1}(x^*(n)) < p_{n+1}(x^*(n))q_n(x^*(n)) = 0,$$

which implies $x^*(n+1) \neq x^*(n)$. Now, suppose that $x^*(n) < x^*(n+1)$. Since $h_{n+1}(x^*(n+1)) \geq h_{n+1}(x^*(n))$ and $h_{n+1}(x)$ is quasi-concave, we have $(x^*(n+1) - x^*(n))h_{n+1}'(x^*(n)) \geq 0$ or $h_{n+1}'(x^*(n)) \geq 0$ by $x^*(n) < x^*(n+1)$, contradicting $h_{n+1}'(x^*(n)) < 0$. Thus, we must have $x^*(n) > x^*(n+1)$. $\square$

Proof of Proposition 3. In the proof, we treat n as if it were a continuous variable. When $\alpha < \beta$, or $\alpha > \beta$ and $\alpha \neq n/(2n-1)$, we have $m \neq 0$ and $m \neq -1$, implying that the optimal reserve price depends on the number of the bidders by Corollary 1(ii). To analyze the effect of n on $r^*(n)$, we rewrite (5) as

$$\partial V(r, n)/\partial r = nF^{(n-1)}(r)\Phi_n(r),$$

where

$$\Phi_n(r) = \int_r^H \left[\frac{F(v)}{F(r)}\right]^m dF(v) - f(r)(r - v_0)$$

Since $F(v) > F(r)$ and $f(v) > 0$ for $v \in (r, H]$,

$$\frac{\partial \Phi_n(r)}{\partial n} = \int_r^H (1 - \alpha/\beta)\left[\frac{F(v)}{F(r)}\right]^m \ln\frac{F(v)}{F(r)} dF(v) > (<)0,$$

and then $\Phi_n(r)$ increases (decreases) in n for any fixed r if $\alpha < \beta$ ($\alpha > \beta$ and $\alpha \neq n/(2n-1)$). By Proposition 2(i) both $r_I(n)$ and $r_I(n+1)$ are within the interval (v_0, H). Since $nF^{(n-1)}(r) > 0$ for $r \in (v_0, H)$ for any fixed n, the conclusion follows from applying Lemma 1 to the function $V(r, n)$. $\square$

Proof of Proposition 4. Substituting $b_r{}^I(v, r) = [F(r)/F(v)]^{\alpha(n-1)/\beta}$ into (4) yields

$$\frac{\partial V_I(r, \alpha, \beta)}{\partial r} = n \int_r^H \left[\frac{F^{n-1}(r)}{F^{n-1}(v)} \right]^{\alpha/\beta} F^{n-1}(v) dF(v) - n F^{n-1}(r) f(r)(r - v_0) \tag{A2}$$

Consider two pairs of (α_1, β_1) and (α_2, β_2) with $\alpha_1 + \beta_1 = 1$, $\alpha_2 + \beta_2 = 1$, and $0 < \alpha_1 < \alpha_2 < 1$. Then, we have $\alpha_1/\beta_1 < \alpha_2/\beta_2$. Hence, $[F(r)/F(v)]^{\alpha_1/\beta_1} > [F(r)/F(v)]^{\alpha_2/\beta_2}$ for $v \in (r, H)$ since $0 < F(r)/F(v) < 1$ for $v \in (r, H)$. Thus, by (A2) we have

$$\frac{\partial V_I(r, \alpha_2, \beta_2)}{\partial r} - \frac{\partial V_I(r, \alpha_1, \beta_1)}{\partial r} = n \int_r^H F^{n-1}(v) \left[\left(\frac{F^{n-1}(r)}{F^{n-1}(v)} \right)^{\alpha_2/\beta_2} - \left(\frac{F^{n-1}(r)}{F^{n-1}(v)} \right)^{\alpha_1/\beta_1} \right] dF(v) < 0.$$

Since $r_I(\alpha_i, \beta_i) \in (v_0, H)$ for $i = 1, 2$ by Proposition 2, we conclude that $r_I(\alpha_2, \beta_2) < r_I(\alpha_1, \beta_1)$ by applying Lemma 2 to $V_I(r, \alpha_1, \beta_1)$ and $V_I(r, \alpha_2, \beta_2)$. Similarly, we can show that $r_I(\alpha, \beta)$ increases in β. $\square$

Proof of Proposition 5. (i) Since $b^I(v, r, n)$ increases in n by (2), i.e., $b^I(v, r, n) < b^I(v, r, n + 1)$, from (3) we have

$$\begin{aligned} V_I(r, n) &< \int_r^H b^I(v, r, n + 1) dF^n(v) + v_0 F^n(r) \\ &= b^I(H, r, n + 1) - \int_r^H F^n(v) b_v^I(v, r, n + 1) dv + F^n(r)(v_0 - r) \end{aligned} \tag{A3}$$

where the equality follows from integration by part, $F(H) = 1$, and $b^I(r, r, n + 1) = r$. Using integration by part, we obtain

$$\begin{aligned} V_I(r, n + 1) &= \int_r^H b^I(v, r, n + 1) dF^{n+1}(v) + v_0 F^{n+1}(r) \\ &= b^I(H, r, n + 1) - \int_r^H F^{n+1}(v) b_v^I(v, r, n + 1) dv + F^{n+1}(r)(v_0 - r) \end{aligned} \tag{A4}$$

Subtracting (A3) from (A4), we have

$$\begin{aligned} V_I(r, n + 1) - V_I(r, n) &> \int_r^H (F^n(v) - F^{n+1}(v)) b_v^I(v, r, n + 1) dv \\ &\quad + (F^{n+1}(r) - F^n(r))(v_0 - r) \\ &> 0 \end{aligned}$$

since $F^n(v) > F^{n+1}(v)$ and $b_v^I(v, r, n + 1) > 0$ for $r < v < H$ by (2), $F^{n+1}(r) \leq F^n(r)$, and $v_0 \leq r$.

(ii) The proof follows from

$$V_I(r_I(n), n) < V_I(r_I(n), n + 1) \leq V_I(r_I(n + 1), n + 1),$$

where the first inequality holds by part (i) and $r_I(n) \in (v_0, H)$, and the second inequality is true since $r_I(n + 1)$ is the optimal reserve price given $n + 1$ bidders. $\square$

Proof of Proposition 6. (i) Since $b^I(v, r, \alpha, \beta)$ increases in α and decreases in β by (2), so does the seller's expected revenue $V_I(r, \alpha, \beta)$ by (3).

(ii) With (α_1, β_1) and (α_2, β_2) such that $\alpha_1 + \beta_1 = 1$, $\alpha_2 + \beta_2 = 1$, and $0 < \alpha_1 < \alpha_2 < 1$ (or equivalently, $0 < \beta_2 < \beta_1 < 1$), we have $V_I(r_I(\alpha_1, \beta_1), \alpha_1, \beta_1) < V_I(r_I(\alpha_1, \beta_1), \alpha_2, \beta_2) \leq V_I(r_I(\alpha_2, \beta_2), \alpha_2, \beta_2)$ by part (i). This completes the proof. $\square$

Proof of $V_1(r) = V_2(r)$. Substituting $\alpha = \beta = 1/2$ into (1) and simplifying lead to

$$b^I(v, r, 1/2, 1/2) = \frac{r F^{n-1}(r)}{F^{n-1}(v)} + \frac{1}{F^{n-1}(v)} \int_r^v y \, dF^{n-1}(y)$$

Substituting the above equation into (3) yields

$$V_1(r) = V_I(r, 1/2, 1/2)$$
$$= \int_r^H \left[r\frac{F^{n-1}(r)}{F^{n-1}(v)} + \frac{1}{F^{n-1}(v)}\int_r^v ydF^{n-1}(y)\right]dF^n(v) + v_0F^n(r) \tag{A5}$$

Clearly, we have

$$\Delta_1 = \int_r^H \frac{rF^{n-1}(r)}{F^{n-1}(v)}dF^n(v) = nF^{n-1}(r)(1 - F(r))r, \tag{A6}$$

and

$$\Delta_2 = \int_r^H \left[\frac{1}{F^{n-1}(v)}\int_r^v ydF^{n-1}(y)\right]dF^n(v)$$
$$= n\int_r^H \left[\int_r^v (n-1)yF^{n-2}(y)f(y)f(v)dy\right]dv$$
$$= n\int_r^H \left[\int_y^H (n-1)yF^{n-2}(y)f(y)f(v)dv\right]dy \tag{A7}$$
$$= n\int_r^H (n-1)yF^{n-2}(y)f(y)(1 - F(y))dy$$
$$= n\int_r^H y(1 - F(y))dF^{n-1}(y)$$

where the second equality holds by changing the order of integration. Observing (A5)–(A7), we have $V_1(r) = \Delta_1 + \Delta_2 + v_0F^n(r)$, which is equal to $V_{II}(r) = V_2(r)$ by (9). This completes the proof. □

Proof of Proposition 7. (i) Clearly, $\alpha > \beta$ ($\alpha = \beta$, $\alpha < \beta$) is equivalent to $m > 0$ ($m = 0$, $m < 0$). Since $r_I(\alpha, \beta)$ decreases (increases) in $\alpha(\beta)$ by Proposition 4,

$$r_I(\alpha, \beta) < r_I(1/2, 1/2) \text{ (respectively, } r_I(\alpha, \beta) = r_I(1/2, 1/2), r_I(\alpha, \beta) > r_I(1/2, 1/2),$$

holds when $\alpha > \beta$ ($\alpha = \beta$, $\alpha < \beta$), or equivalently, $\alpha > 1/2$ ($\alpha = 1/2$, $\alpha < 1/2$). This and $r_I(1/2, 1/2) = r_{II}$ by (12) imply the conclusion.

(ii) By Proposition 1, when $\alpha > \beta$ ($\alpha = \beta$, $\alpha < \beta$) we have $b(v, r, \alpha, \beta) > $ (respectively, =, <) $b(v, r, 1/2, 1/2)$, implying that $V_I(r_I, \alpha, \beta) > $ (respectively, =, <) $V_I(r_I, 1/2, 1/2) = V_{II}(r)$ by (10) and (11).

(iii) It directly follows from (ii). □

References

1. Liu, S.; Wang, M. Sealed-bid auctions based on Cobb-Douglas utility function. *Eco. Lett.* **2010**, *107*, 1–3. [CrossRef]
2. Wang, M.; Hu, Y.; Dang, C.; Wang, S. A non-expected utility model and its application in practical auctions. *Appl. Econ.* **2019**, *52*, 2931–2944. [CrossRef]
3. Boughton, P.D. The competitive bidding process: Beyond probability models. *Ind. Mark. Manage.* **1987**, *16*, 87–94. [CrossRef]
4. Corchón, L.; Dahm, M. Welfare maximizing contest success functions when the planner cannot commit. *J. Math. Econ.* **2011**, *47*, 309–317. [CrossRef]
5. Ghatak, M.; Jiang, N.H. A simple model of inequality, occupational choice, and development. *J. Dev. Econ.* **2002**, *69*, 205–226. [CrossRef]
6. Tsai, J.Y. Bank interest margin management based on a path-dependent Cobb-Douglas utility framework. *Econ. Model.* **2013**, *35*, 751–762. [CrossRef]
7. Cox, J.C.; Smith, V.L.; Walker, J.M. Tests of a heterogeneous bidders theory of first price auctions. *Eco. Lett.* **1983**, *12*, 207–212. [CrossRef]
8. Cox, J.C.; Smith, V.L.; Walker, J.M. Theory and Individual Behavior of First-Price Auctions. *J. Risk Uncertain.* **1988**, *1*, 61–99. [CrossRef]
9. Kagel, J.H.; Roth, A.E. Theory and misbehavior of first-price auctions: Comment. *Am. Econ. Rev.* **1992**, *82*, 1379–1391. Available online: https://www.jstor.org/stable/2117486 (accessed on 12 December 2021).
10. Salo, A.A.; Weber, M. Ambiguity Aversion in First-Price Sealed-Bid Auctions. *J. Risk Uncertain.* **1995**, *11*, 123–137. [CrossRef]
11. Armantier, O.; Treich, N. Star-shaped probability weighting functions and overbidding in first-price auctions. *Eco. Lett.* **2009**, *104*, 83–85. [CrossRef]

12. Filiz-Ozbay, E.; Ozbay, E.Y. Auctions with Anticipated Regret: Theory and Experiment. *Am. Econ. Rev.* **2007**, *97*, 1407–1418. Available online: https://www.jstor.org/stable/30034098 (accessed on 12 December 2021). [CrossRef]

13. Lange, A.; Ratan, A. Multi-dimensional reference-dependent preferences in sealed-bid auctions—How (most) laboratory experiments differ from the field. *Games Econ. Behav.* **2010**, *68*, 634–645. [CrossRef]

14. Keskin, K. Inverse S-shaped probability weighting functions in first-price sealed-bid auctions. *Rev. Econ. Des.* **2016**, *20*, 57–67. [CrossRef]

15. Roider, A.; Schmitz, P.W. Auctions with Anticipated Emotions: Overbidding, Underbidding, and Optimal Reserve Prices. *Scand. J. Econ.* **2012**, *114*, 808–830. [CrossRef]

16. Ivanova-Stenzel, R.; Sonsino, D. Comparative Study of One-Bid Versus Two-Bid Auctions. *J. Econ. Behav. Organ.* **2004**, *54*, 561–583. [CrossRef]

17. Kirchkamp, O.; Reiß, J.P. Heterogeneous bids in auctions with rational and boundedly rational bidders: Theory and experiment. *Int. J. Game Theory* **2019**, *48*, 1001–1031. [CrossRef]

18. Ahmad, F.H. Endogenous price expectations as reference points in auctions. *J. Econ. Behav. Organ.* **2015**, *112*, 46–63. [CrossRef]

19. Rosenkranz, S.; Schmitz, P.W. Reserve Prices in Auctions as Reference Points. *Econ. J.* **2007**, *117*, 637–653. [CrossRef]

20. Riley, J.; Samuelson, W. Optimal auctions. *Am. Econ. Rev.* **1981**, *71*, 381–392. [CrossRef]

21. Li, Y. Optimal reserve prices in sealed-bid auctions with reference effects. *Int. J. Ind. Organ.* **2020**, *71*, 102624. [CrossRef]

22. Kahneman, D.; Tversky, A. Prospect theory: An analysis of decision under risk. *Econometrica* **1979**, *47*, 263–291. [CrossRef]

23. Krishna, V. *Auction Theory*, 2nd ed.; Academic Press; Elsevier Inc.: Waltham, MA, USA, 2010; pp. 13–23.

24. Hu, A.; Matthews, S.A.; Zou, L. Risk aversion and optimal reserve prices in first- and second-price auctions. *J. Econ. Theory* **2010**, *145*, 1188–1202. [CrossRef]

25. Menezes, F.; Monteiro, P.K. *An Introduction to Auction Theory*; Oxford University Press Inc.: New York, NY, USA, 2005; pp. 19–21.