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Structural Stability on the Boundary Coefficient of the Thermoelastic Equations of Type III

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Abstract: This paper investigates the spatial behavior of the solutions of thermoelastic equations of type III in a semi-infinite cylinder by using the partial differential inequalities. By setting an arbitrary positive constant in the energy expression, the fast decay rate of the solutions is obtained. Based on the results of decay, the continuous dependence and the convergence results on the boundary coefficient are established by using the differential inequality technique and the energy analysis method. The main work of this paper is to extend the study of continuous dependence to a semi-infinite cylinder, which can be used as a reference for the study of other types of partial differential equations.

Keywords: spatial decay estimates; thermoelastic equations of type III; structural stability

1. Introduction

Since Hirsch and Smale [1] put forward the concept of structural stability in 1974, this type of structural stability research has attracted a lot of attention. Liu and Zheng [2] considered the exponential stability of the thermoelastic plate model

$$u_{tt} - h\Delta u_{tt} + \Delta^2 u + \alpha\Delta\theta = 0, \quad x \in \Omega, t > 0, \quad (1)$$

$$\theta_t - \mu\Delta\theta + \sigma\theta - \alpha\Delta u_t = 0, \quad x \in \Omega, t > 0, \quad (2)$$

$$u = \frac{\partial u}{\partial n} = 0, \quad x \in \Gamma_0, t > 0, \quad (3)$$

$$u = \Delta u + (1 - \mu_1)B_1 u + \alpha\theta, \quad x \in \Gamma_1, t > 0, \quad (4)$$

$$u = u_0(x, t), u_t(x, t) = u_1(x, t), \theta(x, 0) = \theta_0(x, t), \quad x \in \Omega, t > 0, \quad (5)$$

where Ω is a bounded region in \mathbb{R}^n with smooth boundary Γ and $\Gamma = \Gamma_0 \cup \Gamma_1, \Gamma_0 \cup \Gamma_1 \neq \emptyset$. $h, \alpha, \sigma, \mu, \mu_1$ are positive constants, and $u_0(x, t), u_1(x, t), \theta_0(x, t)$ are given functions. u and θ are unknown functions that represent the vertical deflection and the temperature of the plate, respectively.

In another paper [3], Avalos and Lesiecka proved that the solution of the Equations (1)–(5) decayed exponentially as $t \rightarrow \infty$ under the boundary conditions

$$u = (1 - k)\frac{\partial u}{\partial n} = 0, \quad x \in \Gamma, t > 0, \quad (6)$$

$$\frac{\partial \theta}{\partial n} + \lambda\theta = 0, \quad x \in \Gamma, t > 0, \quad (7)$$

$$k(u + (1 - \mu)B_1 u + \alpha\theta) = 0, \quad x \in \Gamma, t > 0. \quad (8)$$

Here, k is either 0 or 1. Meyvacı [4] obtained the continuous dependence on coefficients h and β , in the case of $\sigma = 0$ in Equations (1) and (2) under the boundary conditions

$$u = \frac{\partial u}{\partial n} = \theta = 0, \quad x \in \Gamma, t > 0,$$



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or

$$\begin{aligned} \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{n}} &= 0, \mathbf{x} \in \Gamma, t > 0, \\ \frac{\partial \theta}{\partial \mathbf{n}} + \beta \theta &= F(\mathbf{x}, t), \mathbf{x} \in \Gamma, t > 0, \end{aligned}$$

where $F(\mathbf{x}, t)$ is a known function. This type of boundary condition can be thought of as expressing Newton’s law of cooling with inhomogeneous outside temperature. β is the cooling coefficient. For more papers of the type, one can refer to [5–18].

However, these results above only considered the case of the bounded region. In recent years, structural stability on the solutions of partial differential equations in semi-infinite cylinders has also attracted some attention (see [19–24]). In this paper, we continue their work. We define a semi-infinite cylindrical pipe named R . The cylindrical pipe’s generator parallels to the x_3 -axis e.g.,

$$R = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 > 0 \right\},$$

where D is a bounded simply-connected region in (x_1, x_2) -plane with piecewise smooth boundary ∂D . We consider the following thermoelastic equations of type III

$$\mathbf{u}_{tt} - \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla(\operatorname{div} \mathbf{u}) + \alpha \nabla \theta = 0, \mathbf{x} \in R, t > 0, \tag{9}$$

$$\theta_{tt} - \kappa \Delta \theta - \delta \Delta \theta_t + \alpha \operatorname{div} \mathbf{u}_{tt} = 0, \mathbf{x} \in R, t > 0, \tag{10}$$

$$\mathbf{u}, \nabla \mathbf{u}, \theta, \nabla \theta \rightarrow 0, \text{ as } x_3 \rightarrow \infty, \tag{11}$$

with the initial-boundary conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_t(\mathbf{x}, 0) = 0, \mathbf{x} \in R, t > 0, \tag{12}$$

$$\theta(\mathbf{x}, 0) = \theta_t(\mathbf{x}, 0) = 0, \mathbf{x} \in R, t > 0, \tag{13}$$

$$\mathbf{u}(x_1, x_2, 0, t) = \mathbf{g}(x_1, x_2, t), \theta(x_1, x_2, 0, t) = h(x_1, x_2, t), (x_1, x_2) \in D, t > 0, \tag{14}$$

$$\mathbf{u} = 0, \frac{\partial \theta}{\partial \mathbf{n}} + \beta \theta = F(\mathbf{x}, t), \mathbf{x} \in \partial D \times \{x_3 \geq 0\}, t > 0, \tag{15}$$

where $\mu, \lambda, \delta, \kappa, \alpha$, and β are positive constants. By the end of the 20th century, Green and Naghdi [25–27] introduced three types of thermoelastic theories. They were respectively called thermoelasticity type I, type II, and type III based on different constitutive assumptions. Since then, thermoelastic equations of type III have attracted a lot of attention. Quintanilla [28] obtained the existence in thermoelasticity without energy dissipation. Ding and Zhou [29] proved the global existence and finite time blow-up for the solutions of the thermoelastic system with p-Laplacian. Zhang and Zuazua [30] have studied the long-time behavior of the solutions of the system. Quintanilla [31] proved that solutions of thermoelasticity of type III converged to solutions of the classical thermoelasticity and to the solution of thermoelasticity without energy dissipation. Quintanilla [32] obtained the structural stability results on the coupling coefficients and the external data in thermoelasticity of type III in a bounded domain. Yan et al. [33] further extended the convergence result of thermoelasticity of type III to a semi-infinite pipe, but in the lateral of the pipe, they assumed that the solutions satisfied

$$\mathbf{u} = 0, \theta = 0, \mathbf{x} \in \partial D \times \{x_3 \geq 0\}, t > 0.$$

This paper studies the structural stability on the coefficient β of system (9)–(15). Different from the continuous dependence on the initial data, the so-called structural stability studies the continuous dependence and convergence of the solutions of the equations on the coefficients in the equations, the parameters in the boundary conditions, and the equations themselves. In the process of model building, simplification, and numerical cal-

ulation, some errors will inevitably appear. Different from mistakes, the errors will not be completely avoided with the progress of measurement methods. Therefore, it is important for us to study the influence of these errors on the solution of the equations (see [34]). In this paper, we first derive the spatial decay bounds of the solutions by using the partial differential inequalities, and then, we study the effect of the coefficient β by using the total energy bounds obtained in the derivation of spatial decay. By setting an arbitrary positive constant, we also obtain the fast decay rate of the solutions. Obviously, our research is a generalization of [32,33]. Our innovation is to extend the study of continuous dependence to a semi-infinite cylinder. This type of study can be used as a reference for the study of other types of partial differential equations and has not received sufficient attention. Therefore, the research of this paper is very meaningful.

2. Preliminary

In this section, we give some preliminary work, which will be used frequently.

Lemma 1. Assume that $p > 0, \frac{1}{p} + \frac{1}{q} = 1, f \in L^p(\Omega), g \in L^q(\Omega)$, then

$$\int_{\Omega} fgdx \leq \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g|^q dx \right)^{\frac{1}{q}}.$$

This inequality is usually named as the Hölder inequality.

Lemma 2. Assume that $a, b > 0, \frac{1}{p} + \frac{1}{q} = 1$, then

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{1}{p} a + \frac{1}{q} b.$$

This inequality is usually named as the Young inequality.

Lemma 3. Assume that n is a positive integer and $x_i > 0 (i = 1, 2, \dots, n)$, then

$$\sqrt[n]{\prod_{i=1}^n x_i} \leq \frac{1}{n} \sum_{i=1}^n x_i.$$

This inequality is usually named as the arithmetic–geometric mean inequality.

Lemma 4. Assume that \mathbf{u} is a vector function in a bounded region Ω , then

$$\int_{\Omega} \operatorname{div} \mathbf{u} dx = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} dA,$$

where $\partial\Omega$ is the boundary surface of Ω and \mathbf{n} is the outward facing unit normal vector on $\partial\Omega$.

This inequality is usually named as the divergence theorem.

3. The Decay Results

We first give the notations

$$R_z = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq z > 0 \right\},$$

$$D_z = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = z > 0 \right\},$$

where z is a moving point on the coordinate axis x_3 and $0 \leq z < \infty$.

To get the decay result of the solutions to (9)–(15), using (9), we begin with the following identities

$$\int_0^t \int_{R_z} e^{-\omega\eta} \left[u_{i,\eta\eta\eta} - \mu \Delta u_{i,\eta} + (\mu + \lambda) u_{j,j\eta} + \alpha \theta_{,i\eta} \right] u_{i,\eta\eta} dx d\eta = 0, \tag{16}$$

$$\int_0^t \int_{R_z} e^{-\omega\eta} \left[\theta_{\eta\eta} - \kappa \Delta \theta - \delta \Delta \theta_{,\eta} + \alpha u_{i,i\eta\eta} \right] \theta_{,\eta} dx d\eta = 0, \tag{17}$$

where ω is a positive constant. In (16) and in the following, we use commas for derivation, repeated English subscripts for summation from 1 to 3, and repeated Greek subscripts for summation from 1 to 2, e.g., $u_{i,j}u_{i,j} = \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} \right)^2$, $u_{\alpha,\beta}u_{\alpha,\beta} = \sum_{\alpha,\beta=1}^2 \left(\frac{\partial u_\alpha}{\partial x_\beta} \right)^2$.

Using the divergence theorem and Equations (9)–(15) in (16) and (17), we have

$$\begin{aligned} & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[u_{i,tt}u_{i,tt} + \mu u_{i,jt}u_{i,jt} + (\mu + \lambda) (u_{i,it})^2 \right] dx \\ & + \frac{1}{2} \omega \int_0^t \int_{R_z} e^{-\omega\eta} \left[u_{i,\eta\eta}u_{i,\eta\eta} + \mu u_{i,j\eta}u_{i,j\eta} + (\mu + \lambda) (u_{i,i\eta})^2 \right] dx d\eta \\ & + \mu \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,3\eta}u_{i,\eta\eta} dAd\eta + (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega\eta} u_{j,j\eta}u_{3,\eta\eta} dAd\eta \\ & - \alpha \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}u_{3,\eta\eta} dAd\eta - \alpha \int_0^t \int_{R_z} e^{-\omega\eta} \theta_{,\eta}u_{i,i\eta\eta} dx d\eta = 0. \end{aligned} \tag{18}$$

and

$$\begin{aligned} & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[\theta_{,t}^2 + \kappa \theta_{,i}\theta_{,i} \right] dx + \int_0^t \int_{R_z} e^{-\omega\eta} \left[\frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i}\theta_{,i} + \delta \theta_{,i\eta}\theta_{,i\eta} \right] dx d\eta \\ & + \kappa \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3\eta}\theta_{,\eta} dAd\eta + \delta \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3\eta}\theta_{,\eta} dAd\eta \\ & + \frac{\beta\kappa}{2} e^{-\omega t} \int_z^\infty \int_{\partial D_\xi} \theta^2 ds d\xi + \frac{\beta\kappa\omega}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 ds d\xi d\eta \\ & - \kappa \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_{,\eta} F ds d\xi d\eta + \delta\beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_{,\eta}^2 ds d\xi d\eta \\ & - \delta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_{,\eta} F_{,\eta} ds d\xi d\eta + \alpha \int_0^t \int_{R_z} e^{-\omega\eta} \theta_{,\eta} u_{i,i\eta\eta} dx d\eta = 0. \end{aligned} \tag{19}$$

If we define

$$\begin{aligned} E(z, t) = & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[u_{i,tt}u_{i,tt} + \mu u_{i,jt}u_{i,jt} + (\mu + \lambda) (u_{i,it})^2 + \theta_{,t}^2 + \kappa \theta_{,i}\theta_{,i} \right] dx \\ & + \int_0^t \int_{R_z} e^{-\omega\eta} \left[\frac{1}{2} \omega u_{i,\eta\eta}u_{i,\eta\eta} + \frac{1}{2} \omega \mu u_{i,j\eta}u_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (u_{i,i\eta})^2 \right. \\ & + \frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i}\theta_{,i} + \delta \theta_{,i\eta}\theta_{,i\eta} \left. \right] dx d\eta \\ & + \frac{\beta\kappa\omega}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 ds d\xi d\eta + \frac{1}{2} \delta\beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_{,\eta}^2 ds d\xi d\eta, \end{aligned} \tag{20}$$

then from (18) and (19), we have

$$\begin{aligned}
 E(z, t) = & -\frac{1}{2}\delta\beta \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\xi d\eta - \frac{\beta\kappa}{2} e^{-\omega t} \int_z \int_{\partial D_\xi} \theta^2 ds d\xi \\
 & - \mu \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,3\eta} u_{i,\eta\eta} dAd\eta - (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega\eta} u_{j,i\eta} u_{3,\eta\eta} dAd\eta \\
 & - \alpha \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta} u_{3,\eta\eta} dAd\eta - \kappa \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3} \theta_{,\eta} dAd\eta - \delta \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3\eta} \theta_{,\eta} dAd\eta \quad (21) \\
 & + \kappa\beta \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta F ds d\xi d\eta + \delta\beta \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta F_\eta ds d\xi d\eta \\
 & \doteq \sum_{i=1}^9 A_i.
 \end{aligned}$$

From (20), we also have

$$\begin{aligned}
 -\frac{\partial}{\partial z} E(z, t) = & \frac{1}{2} e^{-\omega t} \int_{D_z} [u_{i,tt} u_{i,tt} + \mu u_{i,jt} u_{i,jt} + (\mu + \lambda) (u_{i,it})^2 + \theta_{,t}^2 + \kappa \theta_{,i} \theta_{,i}] dA \\
 & + \int_0^t \int_{D_z} e^{-\omega\eta} \left[\frac{1}{2} \omega u_{i,\eta\eta} u_{i,\eta\eta} + \frac{1}{2} \omega \mu u_{i,j\eta} u_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (u_{i,i\eta})^2 \right. \\
 & \left. + \frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i} \theta_{,i} + \delta \theta_{,i\eta} \theta_{,i\eta} \right] dAd\eta \\
 & + \frac{\beta\kappa\omega}{2} \int_0^t \int_{\partial D_z} e^{-\omega\eta} \theta^2 ds d\eta + \frac{1}{2} \delta \int_0^t \int_{\partial D_z} e^{-\omega\eta} \theta_\eta^2 ds d\eta. \quad (22)
 \end{aligned}$$

Using the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
 A_3 \leq & \mu \left[\int_0^t \int_{D_z} e^{-\omega\eta} u_{i,3\eta} u_{i,3\eta} dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,\eta\eta} u_{i,\eta\eta} dAd\eta \right]^{\frac{1}{2}} \\
 \leq & \frac{\sqrt{\mu}}{\omega} \left[\frac{1}{2} \omega \mu \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,3\eta} u_{i,3\eta} dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,\eta\eta} u_{i,\eta\eta} dAd\eta \right], \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 A_4 \leq & (\mu + \lambda) \left[\int_0^t \int_{D_z} e^{-\omega\eta} (u_{j,i\eta})^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\eta\eta}^2 dAd\eta \right]^{\frac{1}{2}} \\
 \leq & \frac{\sqrt{\mu + \lambda}}{\omega} \left[\frac{1}{2} \omega (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega\eta} (u_{j,i\eta})^2 dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\eta\eta}^2 dAd\eta \right], \quad (24)
 \end{aligned}$$

$$\begin{aligned}
 A_5 \leq & \alpha \left[\int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\eta\eta}^2 dAd\eta \right]^{\frac{1}{2}} \\
 \leq & \frac{\alpha}{\omega} \left[\frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\eta\eta}^2 dAd\eta \right], \quad (25)
 \end{aligned}$$

$$\begin{aligned}
 A_6 \leq & \kappa \left[\int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3}^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta \right] \\
 \leq & \frac{\sqrt{\kappa}}{\omega} \left[\frac{1}{2} \omega \kappa \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3}^2 dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta \right], \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 A_7 \leq & \delta \left[\int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3\eta}^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta \right] \\
 \leq & \frac{\sqrt{\delta}}{\sqrt{2}\omega} \left[\delta \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,3\eta}^2 dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} \theta_{,\eta}^2 dAd\eta \right], \quad (27)
 \end{aligned}$$

$$\begin{aligned}
 A_8 &\leq \kappa\beta \left[\int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\xi d\eta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} F^2 ds d\xi d\eta \right]^{\frac{1}{2}} \\
 &\leq \frac{1}{4} \beta \delta \delta_1 \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\xi d\eta + \frac{\kappa^2 \beta}{\delta \delta_1} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} F^2 ds d\xi d\eta, \tag{28}
 \end{aligned}$$

$$A_9 \leq \frac{1}{4} \beta \delta \delta_2 \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\xi d\eta + \frac{\delta \beta}{\delta_2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} F_\eta^2 ds d\xi d\eta, \tag{29}$$

where δ_1, δ_2 are positive constants. Inserting (23)–(29) into (21), choosing $\delta_1 = \delta_2 = 1$ and combining (22), we have

$$E(z, t) \leq \frac{1}{m_1 \sqrt{\omega}} \left[-\frac{\partial}{\partial z} E(z, t) \right] + Q_1(z, t), \tag{30}$$

where $\frac{1}{m_1} = \max \left\{ \frac{\sqrt{\kappa}}{\sqrt{\omega}} + \frac{\sqrt{\delta}}{\sqrt{2}} + \frac{\beta}{\sqrt{\omega}}, \frac{\sqrt{\mu}}{\sqrt{\omega}} + \frac{\sqrt{\mu + \lambda}}{\sqrt{\omega}} + \frac{\beta}{\sqrt{\omega}} \right\}$, $Q_1(z, t) = \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \left[\frac{\kappa^2 \beta}{\delta} F^2 + \delta \beta F_\eta^2 \right] ds d\xi d\eta$.

From (30), we have

$$\frac{\partial}{\partial z} \left\{ E(z, t) e^{m_1 \sqrt{\omega} z} \right\} \leq m_1 \sqrt{\omega} Q_1(z, t) e^{m_1 \sqrt{\omega} z}. \tag{31}$$

Integrating (31) from 0 to z , we have

$$E(z, t) \leq E(0, t) e^{-m_1 \sqrt{\omega} z} + m_1 \sqrt{\omega} \int_0^z Q_1(\xi, t) e^{m_1 \sqrt{\omega}(\xi - z)} d\xi. \tag{32}$$

Combining (20) and (32), we can obtain the following theorem.

Theorem 1. Let (u, θ) be the solutions of Equations (9)–(15) with $F, F_\eta \in C(\partial D \times [0, \infty))$. Then, the following inequality

$$\begin{aligned}
 &\frac{1}{2} e^{-\omega t} \int_{R_z} \left[u_{i,tt} u_{i,tt} + \mu u_{i,jt} u_{i,jt} + (\mu + \lambda) (u_{i,it})^2 + \theta_{,t}^2 + \kappa \theta_{,i} \theta_{,i} \right] dx \\
 &+ \int_0^t \int_{R_z} e^{-\omega\eta} \left[\frac{1}{2} \omega u_{i,\eta\eta} u_{i,\eta\eta} + \frac{1}{2} \omega \mu u_{i,j\eta} u_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (u_{i,i\eta})^2 \right. \\
 &+ \left. \frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i} \theta_{,i} + \delta \theta_{,i\eta} \theta_{,i\eta} \right] dx d\eta \\
 &+ \frac{\beta \kappa \omega}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 ds d\xi d\eta + \frac{1}{2} \delta \beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\xi d\eta \\
 &\leq E(0, t) e^{-m_1 \sqrt{\omega} z} + m_1 \sqrt{\omega} \int_0^z Q_1(\xi, t) e^{m_1 \sqrt{\omega}(\xi - z)} d\xi
 \end{aligned}$$

holds.

Remark 1. Since $E(0, t) e^{-m_1 \sqrt{\omega} z} + \int_0^z Q_1(\xi, t) e^{m_1 \sqrt{\omega}(\xi - z)} d\xi \rightarrow 0$ as $z \rightarrow \infty$, we can conclude that the solutions of Equations (9)–(15) decay exponentially. Since ω is an arbitrary positive constant, the decay rate can be large enough.

Remark 2. Obviously, the decay bound in Theorem 1 depends on the total energy $E(0, t)$. To make the decay bound explicit, we have to derive the bound for $E(0, t)$. We write the result in the following theorem.

Theorem 2. Let (u, θ) be solutions of Equations (9)–(15) in R , then, for fixed t ,

$$\begin{aligned}
 E(0, t) &= \frac{1}{2}e^{-\omega t} \int_R \left[u_{i,tt}u_{i,tt} + \mu u_{i,jt}u_{i,jt} + (\mu + \lambda) \left(u_{i,it} \right)^2 + \theta_{,t}^2 + \kappa \theta_{,i}\theta_{,i} \right] dx \\
 &+ \int_0^t \int_R e^{-\omega \eta} \left[\frac{1}{2} \omega u_{i,\eta\eta}u_{i,\eta\eta} + \frac{1}{2} \omega \mu u_{i,j\eta}u_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) \left(u_{i,i\eta} \right)^2 \right. \\
 &+ \left. \frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i}\theta_{,i} + \delta \theta_{,i\eta}\theta_{,i\eta} \right] dx d\eta \\
 &+ \frac{\beta \kappa \omega}{2} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta^2 ds d\xi d\eta + \frac{1}{2} \delta \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta^2 ds d\xi d\eta \\
 &\leq 2Q_2(0, t),
 \end{aligned}$$

where $Q_2(0, t)$ is a positive function, which will be defined in (53).

Proof. We choose $z = 0$ in (20) and (21) to have

$$\begin{aligned}
 E(0, t) &= \frac{1}{2}e^{-\omega t} \int_R \left[u_{i,tt}u_{i,tt} + \mu u_{i,jt}u_{i,jt} + (\mu + \lambda) \left(u_{i,it} \right)^2 + \theta_{,t}^2 + \kappa \theta_{,i}\theta_{,i} \right] dx \\
 &+ \int_0^t \int_R e^{-\omega \eta} \left[\frac{1}{2} \omega u_{i,\eta\eta}u_{i,\eta\eta} + \frac{1}{2} \omega \mu u_{i,j\eta}u_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) \left(u_{i,i\eta} \right)^2 \right. \\
 &+ \left. \frac{1}{2} \omega \theta_{,\eta}^2 + \frac{1}{2} \omega \kappa \theta_{,i}\theta_{,i} + \delta \theta_{,i\eta}\theta_{,i\eta} \right] dx d\eta \\
 &+ \frac{\beta \kappa \omega}{2} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta^2 ds d\xi d\eta + \frac{1}{2} \delta \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta^2 ds d\xi d\eta,
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 E(0, t) &= -\frac{1}{2} \delta \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta^2 ds d\xi d\eta - \frac{\beta \kappa}{2} e^{-\omega t} \int_0^\infty \int_{\partial D_\xi} \theta^2 ds d\xi \\
 &- \mu \int_0^t \int_D e^{-\omega \eta} u_{i,3\eta}u_{i,\eta\eta} dAd\eta - (\mu + \lambda) \int_0^t \int_D e^{-\omega \eta} u_{j,\eta}u_{3,\eta\eta} dAd\eta \\
 &- \alpha \int_0^t \int_D e^{-\omega \eta} \theta_{,\eta}u_{3,\eta\eta} dAd\eta - \kappa \int_0^t \int_D e^{-\omega \eta} \theta_{,3\eta}dAd\eta - \delta \int_0^t \int_D e^{-\omega \eta} \theta_{,3\eta}\theta_{,\eta} dAd\eta \\
 &+ \kappa \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta F ds d\xi d\eta + \delta \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta F_\eta ds d\xi d\eta.
 \end{aligned} \tag{34}$$

□

Now, we define two new auxiliary functions

$$G_i(x_1, x_2, x_3, t) = g_i(x_1, x_2, t)e^{-\sigma_1 x_3}, H(x_1, x_2, x_3, t) = h(x_1, x_2, t)e^{-\sigma_2 x_3}, \tag{35}$$

where σ_1, σ_2 are positive constants. Obviously, G_i and H have the same boundary conditions with u_i and θ , respectively.

Using Equation (9) and the divergence theorem, we have

$$\begin{aligned}
 &-\mu \int_0^t \int_D e^{-\omega\eta} u_{i,3\eta} u_{i,\eta\eta} dAd\eta - (\mu + \lambda) \int_0^t \int_D e^{-\omega\eta} u_{j,j\eta} u_{3,\eta\eta} dAd\eta \\
 &= -\mu \int_0^t \int_D e^{-\omega\eta} u_{i,3\eta} G_{i,\eta\eta} dAd\eta - (\mu + \lambda) \int_0^t \int_D e^{-\omega\eta} u_{j,j\eta} G_{3,\eta\eta} dAd\eta \\
 &= \mu \int_0^t \int_R e^{-\omega\eta} (u_{i,j\eta} G_{i,\eta\eta})_{,j} dx d\eta + (\mu + \lambda) \int_0^t \int_R e^{-\omega\eta} (u_{j,j\eta} G_{i,\eta\eta})_{,i} dx d\eta \tag{36} \\
 &= \mu \int_0^t \int_R e^{-\omega\eta} u_{i,j\eta} G_{i,j\eta\eta} dx d\eta + (\mu + \lambda) \int_0^t \int_R e^{-\omega\eta} u_{j,i\eta} G_{i,i\eta\eta} dx d\eta \\
 &+ \mu \int_0^t \int_R e^{-\omega\eta} [u_{i,\eta\eta\eta} + \alpha \theta_{,i\eta}] G_{i,\eta\eta} dx d\eta \\
 &= \mu \int_0^t \int_R e^{-\omega\eta} u_{i,j\eta} G_{i,j\eta\eta} dx d\eta + (\mu + \lambda) \int_0^t \int_R e^{-\omega\eta} u_{j,i\eta} G_{i,i\eta\eta} dx d\eta \\
 &+ e^{-\omega t} \int_R u_{i,tt} G_{i,tt} dx + \omega \int_0^t \int_R e^{-\omega\eta} u_{i,\eta\eta} G_{i,\eta\eta\eta} dx d\eta + \alpha \int_0^t \int_R e^{-\omega\eta} \theta_{,i\eta} G_{i,\eta\eta} dx d\eta \\
 &\doteq \sum_{i=1}^5 B_i.
 \end{aligned}$$

Using the Hölder inequality and the Young inequality in (36), we have

$$B_1 \leq \frac{1}{2} \mu \varepsilon_1 \int_0^t \int_R e^{-\omega\eta} u_{i,j\eta} u_{i,j\eta} dx d\eta + \frac{1}{2\varepsilon_1} \mu \int_0^t \int_R e^{-\omega\eta} G_{i,j\eta\eta} G_{i,j\eta\eta} dx d\eta, \tag{37}$$

$$B_2 \leq \frac{1}{2} (\mu + \lambda) \varepsilon_2 \int_0^t \int_R e^{-\omega\eta} (u_{j,j\eta})^2 dx d\eta + \frac{1}{2\varepsilon_2} (\mu + \lambda) \int_0^t \int_R e^{-\omega\eta} (G_{i,i\eta\eta})^2 dx d\eta, \tag{38}$$

$$B_3 \leq \frac{1}{2} \varepsilon_3 e^{-\omega t} \int_R u_{i,tt} u_{i,tt} dx + \frac{1}{2\varepsilon_3} e^{-\omega t} \int_R G_{i,tt} G_{i,tt} dx, \tag{39}$$

$$B_4 \leq \frac{1}{2} \varepsilon_4 \omega \int_0^t \int_R e^{-\omega\eta} u_{i,\eta\eta} u_{i,\eta\eta} dx d\eta + \frac{1}{2\varepsilon_4} \omega \int_0^t \int_R e^{-\omega\eta} G_{i,\eta\eta\eta} G_{i,\eta\eta\eta} dx d\eta, \tag{40}$$

$$B_5 \leq \frac{1}{2} \varepsilon_5 \alpha \int_0^t \int_R e^{-\omega\eta} \theta_{,i\eta} \theta_{,i\eta} dx d\eta + \frac{1}{2\varepsilon_5} \alpha \int_0^t \int_R e^{-\omega\eta} G_{i,\eta\eta} G_{i,\eta\eta} dx d\eta, \tag{41}$$

where $\varepsilon_i (i = 1, 2, \dots, 5)$ are positive constants. Using Equation (10) and the divergence theorem, we have

$$\begin{aligned}
 &-\kappa \int_0^t \int_D e^{-\omega\eta} \theta_{,3\theta,\eta} dAd\eta - \delta \int_0^t \int_D e^{-\omega\eta} \theta_{,3\eta} \theta_{,\eta} dAd\eta - \alpha \int_0^t \int_D e^{-\omega\eta} \theta_{,\eta} u_{3,\eta\eta} dAd\eta \\
 &= -\kappa \int_0^t \int_D e^{-\omega\eta} \theta_{,3} H_{,\eta} dAd\eta - \delta \int_0^t \int_D e^{-\omega\eta} \theta_{,3\eta} H_{,\eta} dAd\eta - \alpha \int_0^t \int_D e^{-\omega\eta} \theta_{,\eta} u_{3,\eta\eta} dAd\eta \\
 &= \kappa \int_0^t \int_R e^{-\omega\eta} (\theta_{,i} H_{,\eta})_{,i} dx d\eta + \delta \int_0^t \int_R e^{-\omega\eta} (\theta_{,i\eta} H_{,\eta})_{,i} dx d\eta \\
 &- \kappa \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} [-\beta\theta + F] H_{,\eta} ds d\zeta d\eta \\
 &- \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} [-\beta\theta_{,\eta} + F_{,\eta}] H_{,\eta} ds d\zeta d\eta - \alpha \int_0^t \int_D e^{-\omega\eta} \theta_{,\eta} u_{3,\eta\eta} dAd\eta \\
 &= e^{-\omega t} \int_R \theta_{,i} H_{,i} dx + \omega \int_0^t \int_R e^{-\omega\eta} \theta_{,\eta} H_{,\eta\eta} dx d\eta - \alpha \int_0^t \int_R e^{-\omega\eta} u_{i,\eta\eta} H_{,i\eta} dx d\eta \\
 &- \kappa \beta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} \theta H_{,\eta} ds d\zeta d\eta - \kappa \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} F H_{,\eta} ds d\zeta d\eta \\
 &+ \beta \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} \theta_{,\eta} H_{,\eta} ds d\zeta d\eta - \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} F_{,\eta} H_{,\eta} ds d\zeta d\eta
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^t \int_R e^{-\omega\eta} \theta_{,i} H_{,i\eta} dx d\eta + \delta \int_0^t \int_R e^{-\omega\eta} \theta_{,i\eta} H_{,i\eta} dx d\eta \\
 &\doteq \sum_{i=1}^9 C_i.
 \end{aligned} \tag{42}$$

Using the Hölder inequality and the Young inequality in (42), we have

$$C_1 \leq \frac{1}{2} \varepsilon_6 e^{-\omega t} \int_R \theta_{,i}^2 dx + \frac{1}{2\varepsilon_6} e^{-\omega t} \int_R H_{,i}^2 dx \tag{43}$$

$$C_2 \leq \frac{1}{2} \varepsilon_7 \omega \int_0^t \int_R e^{-\omega\eta} \theta_{,\eta}^2 dx d\eta + \frac{1}{2\varepsilon_7} \omega \int_0^t \int_R e^{-\omega\eta} H_{,\eta\eta}^2 dx d\eta, \tag{44}$$

$$C_3 \leq \frac{1}{2} \varepsilon_8 \alpha \int_0^t \int_R e^{-\omega\eta} u_{i,\eta\eta} u_{i,\eta\eta} dx d\eta + \frac{1}{2\varepsilon_8} \alpha \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta, \tag{45}$$

$$C_4 \leq \frac{1}{2} \kappa \beta \varepsilon_9 \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} \theta^2 ds d\zeta d\eta + \frac{1}{2\varepsilon_9} \kappa \beta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} H_\eta^2 ds d\zeta d\eta, \tag{46}$$

$$C_6 \leq \frac{1}{2} \varepsilon_{10} \beta \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} \theta_\eta^2 ds d\zeta d\eta + \frac{1}{2\varepsilon_{10}} \beta \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} H_\eta^2 ds d\zeta d\eta, \tag{47}$$

$$C_8 \leq \frac{1}{2} \varepsilon_{11} \int_0^t \int_R e^{-\omega\eta} \theta_{,i} \theta_{,i} dx d\eta + \frac{1}{2\varepsilon_{11}} \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta, \tag{48}$$

$$C_9 \leq \frac{1}{2} \varepsilon_{12} \delta \int_0^t \int_R e^{-\omega\eta} \theta_{,i\eta} \theta_{,i\eta} dx d\eta + \frac{1}{2\varepsilon_{12}} \delta \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta, \tag{49}$$

where ε_i ($i = 6, 7, \dots, 12$) are positive constants. For the last two terms on the right of (34), we can refer to the results that have been derived in (28) and (29), and we have the following inequalities

$$\begin{aligned}
 &\kappa \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta F ds d\zeta d\eta \\
 &\leq \frac{1}{4} \beta \delta \delta_1 \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\zeta d\eta + \frac{\kappa^2 \beta}{\delta \delta_1} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} F^2 ds d\zeta d\eta,
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 &\delta \beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta F_\eta ds d\zeta d\eta \\
 &\leq \frac{1}{4} \beta \delta \delta_2 \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 ds d\zeta d\eta + \frac{\delta \beta}{\delta_2} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} F_\eta^2 ds d\zeta d\eta.
 \end{aligned} \tag{51}$$

Inserting (37)–(41) and (43)–(49) into (36) and (42) respectively, combining (33), (34), (50) and (51) and choosing

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_6 = \varepsilon_7 = \varepsilon_{10} = \delta_1 = \delta_2 = \varepsilon_{12} = \frac{1}{2}, \varepsilon_4 = \varepsilon_9 = \frac{1}{4}, \varepsilon_2 = \frac{\delta}{2\alpha}, \varepsilon_8 = \frac{\omega}{4\alpha}, \varepsilon_{11} = \frac{1}{2} \kappa \omega,$$

We have

$$E(0, t) \leq \frac{1}{2} E(0, t) + Q_2(0, t), \tag{52}$$

where

$$Q_2(0, t) = \frac{1}{2\varepsilon_1} \mu \int_0^t \int_R e^{-\omega\eta} G_{i,j\eta\eta} G_{i,j\eta\eta} dx d\eta + \frac{1}{2\varepsilon_2} (\mu + \lambda) \int_0^t \int_R e^{-\omega\eta} (G_{i,i\eta\eta})^2 dx d\eta$$

$$\begin{aligned}
 & + \frac{1}{2\varepsilon_3} e^{-\omega t} \int_R G_{i,tt} G_{i,tt} dx + \frac{1}{2\varepsilon_4} \omega \int_0^t \int_R e^{-\omega\eta} G_{i,\eta\eta\eta} G_{i,\eta\eta\eta} dx d\eta \\
 & + \frac{1}{2\varepsilon_5} \alpha \int_0^t \int_R e^{-\omega\eta} G_{i,\eta\eta} G_{i,\eta\eta} dx d\eta + \frac{1}{2\varepsilon_6} e^{-\omega t} \int_R H_{,t}^2 dx \\
 & + \frac{1}{2\varepsilon_7} \omega \int_0^t \int_R e^{-\omega\eta} H_{,\eta\eta}^2 dx d\eta + \frac{1}{2\varepsilon_8} \alpha \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta \\
 & + \frac{1}{2\varepsilon_9} \kappa\beta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} H_{,\eta}^2 ds d\zeta d\eta + \frac{1}{2\varepsilon_{10}} \beta\delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} H_{,\eta}^2 ds d\zeta d\eta \quad (53) \\
 & + \frac{1}{2\varepsilon_{11}} \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta + \frac{1}{2\varepsilon_{12}} \delta \int_0^t \int_R e^{-\omega\eta} H_{,i\eta} H_{,i\eta} dx d\eta \\
 & + \kappa \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} |FH_{,\eta}| ds d\zeta d\eta + \delta \int_0^t \int_0^\infty \int_{\partial D} e^{-\omega\eta} |F_{,\eta} H_{,\eta}| ds d\zeta d\eta \\
 & + \frac{\kappa^2 \beta}{\delta \delta_1} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} F^2 ds d\zeta d\eta + \frac{\delta \beta}{\delta_2} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} F_{,\eta}^2 ds d\zeta d\eta.
 \end{aligned}$$

Based on (33) and (52), we can obtain Theorem 2.

Combining Theorems 1 and 2, we can get the following theorem.

Theorem 3. Let (u, θ) be the solutions of Equations (9)–(15) with $F, F_{,\eta} \in C(\partial D \times [0, \infty))$. Then, the following inequality

$$\frac{\beta\kappa\omega}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 ds d\zeta d\eta + \frac{1}{2} \delta\beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_{,\eta}^2 ds d\zeta d\eta \leq Q_3(z, t) e^{-m_1 \sqrt{\omega} z},$$

where

$$Q_3(z, t) = 2Q_2(0, t) + m_1 \sqrt{\omega} \int_0^z Q_1(\xi, t) e^{m_1 \sqrt{\omega} \xi} d\xi.$$

4. Continuous Dependence on the Boundary Coefficient

We will use the results obtained in Section 2 to investigate the effect of the small change on the coefficient β in Equations (9)–(15). To do this, we let u^* and θ^* be the solutions of (9)–(15) with the boundary coefficient β replaced by the constant β^* and allow u and u^* to satisfy same conditions on the entrance D . It is worth noting that if u and u^* satisfy different boundary conditions at the entrance, our results are still valid because our problem is linear, and we can decompose and deal with the two effects, respectively. If we let v_i and Π be the differences between u_i, p and u_i^*, p^* , respectively, i.e.,

$$v = u - u^*, \quad \Pi = \theta - \theta^*, \quad \tilde{\beta} = \beta - \beta^*, \quad (54)$$

then v and Π satisfy the following equations

$$v_{tt} - \mu \Delta v + (\mu + \lambda) \nabla(\operatorname{div} v) + \alpha \nabla \Pi = 0, \quad x \in R, \quad t > 0, \quad (55)$$

$$\Pi_{tt} - \kappa \Delta \Pi - \delta \Delta \Pi_t + \alpha \operatorname{div} v_{tt} = 0, \quad x \in R, \quad t > 0, \quad (56)$$

$$v, \nabla v, \Pi, \nabla \Pi \rightarrow 0, \text{ as } x_3 \rightarrow \infty, \quad (57)$$

with the initial boundary conditions

$$v(x, 0) = v_t(x, 0) = 0, \quad x \in R, \quad t > 0, \quad (58)$$

$$\Pi(x, 0) = \Pi_t(x, 0) = 0, \quad x \in R, \quad t > 0, \quad (59)$$

$$v(x_1, x_2, 0, t) = 0, \theta(x_1, x_2, 0, t) = 0, \quad (x_1, x_2) \in D, \quad t > 0, \quad (60)$$

$$v = 0, \frac{\partial \Pi}{\partial n} + \tilde{\beta} \theta + \beta^* \Pi = 0, \quad x \in \partial D \times \{x_3 \geq 0\}, \quad t > 0. \quad (61)$$

We have the following theorem.

Theorem 4. Let (u, θ) be the solutions of Equations (9)–(15) and (u^*, θ^*) be the solutions of Equations (9)–(15) with $\beta = \beta^*$. The functions v and Π are defined in (54). If $F, F_\eta \in C(\partial D \times [0, \infty))$ and $\int_D f_3 dA = 0$, then

$$\begin{aligned} & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[v_{i,tt} v_{i,tt} + \mu v_{i,jt} v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_t^2 + \kappa \Pi_{,i} \Pi_{,i} \right] dx \\ & + \int_0^t \int_{R_z} e^{-\omega \eta} \left[\frac{1}{2} \omega v_{i,\eta\eta} v_{i,\eta} + \frac{1}{2} \omega \mu v_{i,j\eta} v_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda)(v_{i,i\eta})^2 + \frac{1}{2} \omega \Pi_{,\eta\eta}^2 \right. \\ & + \frac{1}{2} \omega \kappa \Pi_{,i} \Pi_{,i} + \delta \Pi_{,i\eta} \Pi_{,i\eta} \left. \right] dx d\eta + \frac{1}{2} e^{-\omega t} \beta^* \kappa \int_z^\infty \int_{\partial D_\xi} \Pi^2 ds d\xi \\ & + \frac{1}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\omega \beta^* \kappa \Pi^2 + \delta \beta^* \Pi_\eta^2 \right] ds d\xi d\eta \\ & \leq 2b_2 \tilde{\beta}^2 Q_2(0, t) e^{-b_1 \sqrt{\omega z}} + b_1 b_2 \sqrt{\omega} \tilde{\beta}^2 e^{-b_1 \sqrt{\omega z}} \int_0^z Q_3(\xi, t) e^{(b_1 - m_1) \sqrt{\omega} \xi} d\xi, \end{aligned}$$

where b_1, b_2 are positive constants. This demonstrates continuous dependence of (u, θ) on the parameter β .

Proof. We began from the following identities

$$\int_0^t \int_{R_z} e^{-\omega \eta} \left[v_{i,\eta\eta\eta} - \mu \Delta v_{i,\eta} + (\mu + \lambda) v_{j,j\eta} + \alpha \Pi_{,i\eta} \right] v_{i,\eta\eta} dx d\eta = 0, \tag{62}$$

$$\int_0^t \int_{R_z} e^{-\omega \eta} \left[\Pi_{,\eta\eta} - \kappa \Delta \Pi - \delta \Delta \Pi_\eta + \alpha v_{i,i\eta\eta} \right] \Pi_{,\eta} dx d\eta = 0. \tag{63}$$

□

Applying the divergence theorem and using (57)–(61), we have from (62) and (63)

$$\begin{aligned} & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[v_{i,tt} v_{i,tt} + \mu v_{i,jt} v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 \right] dx \\ & + \frac{1}{2} \omega \int_0^t \int_{R_z} e^{-\omega \eta} \left[v_{i,\eta\eta} v_{i,\eta\eta} + \mu v_{i,j\eta} v_{i,j\eta} + (\mu + \lambda)(v_{i,i\eta})^2 \right] dx d\eta \\ & = -\mu \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,3\eta} v_{i,\eta\eta} dAd\eta - (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega \eta} v_{j,j\eta} v_{3,\eta\eta} dAd\eta \\ & - \alpha \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_\eta v_{3,\eta\eta} dAd\eta + \alpha \int_0^t \int_{R_z} e^{-\omega \eta} \Pi_\eta v_{i,i\eta\eta} dx d\eta, \end{aligned} \tag{64}$$

and

$$\begin{aligned} & \frac{1}{2} e^{-\omega t} \int_{R_z} \left[\Pi_t^2 + \kappa \Pi_{,i} \Pi_{,i} \right] dx + \frac{1}{2} e^{-\omega t} \beta^* \kappa \int_z^\infty \int_{\partial D_\xi} \Pi^2 ds d\xi \\ & + \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\frac{1}{2} \omega \beta^* \kappa \Pi^2 + \delta \beta^* \Pi_\eta^2 \right] ds d\xi d\eta \\ & + \int_0^t \int_{R_z} e^{-\omega \eta} \left[\frac{1}{2} \omega \Pi_{,\eta\eta}^2 + \frac{1}{2} \omega \kappa \Pi_{,i} \Pi_{,i} + \delta \Pi_{,i\eta} \Pi_{,i\eta} \right] dx d\eta \\ & = -\kappa \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3} \Pi_\eta dAd\eta - \delta \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3\eta} \Pi_\eta dAd\eta \\ & - \tilde{\beta} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\kappa \Pi_\eta \theta + \delta \Pi_\eta \theta_\eta \right] ds d\xi d\eta \\ & - \alpha \int_0^t \int_{R_z} e^{-\omega \eta} \Pi_\eta v_{i,i\eta\eta} dx d\eta. \end{aligned} \tag{65}$$

Now, if we define

$$\begin{aligned} \Gamma(z, t) = & \frac{1}{2}e^{-\omega t} \int_{R_z} \left[v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i} \Pi_{,i} \right] dx \\ & + \int_0^t \int_{R_z} e^{-\omega \eta} \left[\frac{1}{2} \omega v_{i,\eta\eta} v_{i,\eta\eta} + \frac{1}{2} \omega \mu v_{i,j\eta} v_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (v_{i,i\eta})^2 + \frac{1}{2} \omega \Pi_{,\eta\eta}^2 \right. \\ & + \frac{1}{2} \omega \kappa \Pi_{,i} \Pi_{,i} + \delta \Pi_{,i\eta} \Pi_{,i\eta} \left. \right] dx d\eta + \frac{1}{2} e^{-\omega t} \beta^* \kappa \int_z^\infty \int_{\partial D_\xi} \Pi^2 ds d\xi \\ & + \frac{1}{2} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\omega \beta^* \kappa \Pi^2 + \delta \beta^* \Pi_{,\eta}^2 \right] ds d\xi d\eta, \end{aligned} \tag{66}$$

then we have

$$\begin{aligned} -\frac{\partial}{\partial z} \Gamma(z, t) = & \frac{1}{2} e^{-\omega t} \int_{D_z} \left[v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i} \Pi_{,i} \right] dA \\ & + \int_0^t \int_{D_z} e^{-\omega \eta} \left[\frac{1}{2} \omega v_{i,\eta\eta} v_{i,\eta\eta} + \frac{1}{2} \omega \mu v_{i,j\eta} v_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (v_{i,i\eta})^2 + \frac{1}{2} \omega \Pi_{,\eta\eta}^2 \right. \\ & + \frac{1}{2} \omega \kappa \Pi_{,i} \Pi_{,i} + \delta \Pi_{,i\eta} \Pi_{,i\eta} \left. \right] dA d\eta + \frac{1}{2} e^{-\omega t} \beta^* \kappa \int_{\partial D_z} \Pi^2 ds \\ & + \frac{1}{2} \int_0^t \int_{\partial D_z} e^{-\omega \eta} \left[\omega \beta^* \kappa \Pi^2 + \delta \beta^* \Pi_{,\eta}^2 \right] ds d\eta. \end{aligned} \tag{67}$$

Combining (64) and (65), we have

$$\begin{aligned} \Gamma(z, t) = & -\frac{1}{2} \delta \beta^* \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta - \mu \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,3\eta} v_{i,\eta\eta} dA d\eta \\ & - (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega \eta} v_{j,j\eta} v_{3,\eta\eta} dA d\eta - \alpha \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,\eta} v_{3,\eta\eta} dA d\eta \\ & - \kappa \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3} \Pi_{,\eta} dA d\eta - \delta \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3\eta} \Pi_{,\eta} dA d\eta \\ & - \tilde{\beta} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\kappa \Pi_{,\eta} \theta + \delta \Pi_{,\eta} \theta_{,\eta} \right] ds d\xi d\eta \\ & \doteq -\frac{1}{2} \delta \beta^* \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta + \sum_{i=1}^6 I_i. \end{aligned} \tag{68}$$

By using the Hölder inequality and the arithmetic–geometric mean inequality, we have

$$\begin{aligned} I_1 \leq & \mu \left[\int_0^t \int_{D_z} e^{-\omega \eta} v_{i,3\eta} v_{i,3\eta} dA d\eta \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,\eta\eta} v_{i,\eta\eta} dA d\eta \right]^{\frac{1}{2}} \\ \leq & \frac{\sqrt{\mu}}{\omega} \left[\frac{1}{2} \omega \mu \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,3\eta} v_{i,3\eta} dA d\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,\eta\eta} v_{i,\eta\eta} dA d\eta \right], \end{aligned} \tag{69}$$

$$\begin{aligned} I_2 \leq & (\mu + \lambda) \left[\int_0^t \int_{D_z} e^{-\omega \eta} v_{3,\eta\eta}^2 dA d\eta \int_0^t \int_{D_z} e^{-\omega \eta} (v_{j,j\eta})^2 dA d\eta \right]^{\frac{1}{2}} \\ \leq & \frac{\sqrt{\mu + \lambda}}{\omega} \left[\frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega \eta} v_{3,\eta\eta}^2 dA d\eta + \frac{1}{2} \omega (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega \eta} (v_{j,j\eta})^2 dA d\eta \right], \end{aligned} \tag{70}$$

$$\begin{aligned} I_3 \leq & \alpha \left[\int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,\eta}^2 dA d\eta \int_0^t \int_{D_z} e^{-\omega \eta} v_{3,\eta\eta}^2 dA d\eta \right]^{\frac{1}{2}} \\ \leq & \frac{\alpha}{\omega} \left[\frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,\eta}^2 dA d\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega \eta} v_{3,\eta\eta}^2 dA d\eta \right], \end{aligned} \tag{71}$$

$$\begin{aligned}
 I_4 &\leq \kappa \left[\int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,3}^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,\eta}^2 dAd\eta \right]^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{\kappa}}{\omega} \left[\frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,3}^2 dAd\eta + \frac{1}{2} \omega \kappa \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,\eta}^2 dAd\eta \right], \tag{72}
 \end{aligned}$$

$$\begin{aligned}
 I_5 &\leq \delta \left[\int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,3\eta}^2 dAd\eta \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,\eta}^2 dAd\eta \right]^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{\delta}}{\sqrt{\omega}} \left[\delta \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,3\eta}^2 dAd\eta + \frac{1}{2} \omega \int_0^t \int_{D_z} e^{-\omega\eta} \Pi_{,\eta}^2 dAd\eta \right], \tag{73}
 \end{aligned}$$

and

$$\begin{aligned}
 I_6 &\leq \kappa \tilde{\beta} \left[\int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \Pi_\eta^2 dsd\xi d\eta \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 dsd\xi d\eta \right]^{\frac{1}{2}} \\
 &+ \delta \tilde{\beta} \left[\int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \Pi_\eta^2 dsd\xi d\eta \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 dsd\xi d\eta \right]^{\frac{1}{2}} \tag{74} \\
 &\leq \frac{1}{2} \delta \beta^* \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \Pi_\eta^2 dsd\xi d\eta \\
 &+ \frac{\kappa^2 \tilde{\beta}^2}{\delta \beta^*} \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 dsd\xi d\eta + \frac{\delta \tilde{\beta}^2}{\beta^*} \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 dsd\xi d\eta.
 \end{aligned}$$

Inserting (69)–(74) into (68), we have

$$\begin{aligned}
 \Gamma(z, t) &\leq \frac{1}{b_1 \sqrt{\omega}} \left[-\frac{\partial}{\partial z} \Gamma(z, t) \right] + \max \left\{ \frac{2\kappa}{\omega \beta \beta^* \delta}, \frac{2}{\beta \beta^*} \right\} \left[\frac{\kappa \beta \omega \tilde{\beta}^2}{2} \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 dsd\xi d\eta \right. \\
 &\left. + \frac{\delta \beta \tilde{\beta}^2}{2} \int_0^t \int_z \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 dsd\xi d\eta \right], \tag{75}
 \end{aligned}$$

where

$$\frac{1}{b_1} = \sqrt{\frac{\mu}{\omega}} + \sqrt{\frac{\mu + \lambda}{\omega}} + \sqrt{\frac{\alpha}{\omega}} + \sqrt{\frac{\kappa}{\omega}} + \sqrt{\delta}.$$

Using the Theorem 3 in (75), we obtain

$$\Gamma(z, t) \leq \frac{1}{b_1 \sqrt{\omega}} \left[-\frac{\partial}{\partial z} \Gamma(z, t) \right] + b_2 \tilde{\beta}^2 Q_3(z, t) e^{-m_1 \sqrt{\omega} z}, \tag{76}$$

where $b_2 = \max \left\{ \frac{2\kappa}{\omega \beta \beta^* \delta} \right\}$. From (76) it follows that

$$\frac{\partial}{\partial z} \left[\Gamma(z, t) e^{b_1 \sqrt{\omega} z} \right] \leq b_1 b_2 \sqrt{\omega} \tilde{\beta}^2 Q_3(z, t) e^{(b_1 - m_1) \sqrt{\omega} z}. \tag{77}$$

Integrating (77) from 0 to z , we have

$$\Gamma(z, t) \leq \Gamma(0, t) e^{-b_1 \sqrt{\omega} z} + b_1 b_2 \sqrt{\omega} \tilde{\beta}^2 e^{-b_1 \sqrt{\omega} z} \int_0^z Q_3(\xi, t) e^{(b_1 - m_1) \sqrt{\omega} \xi} d\xi. \tag{78}$$

To obtain Theorem 4, we have to derive bound for $\Gamma(0, t)$. To do this, we choose $z = 0$ in (66) and (68) and use the boundary conditions (59)–(61) in (68) to have

$$\begin{aligned} \Gamma(0, t) = & \frac{1}{2}e^{-\omega t} \int_R [v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i} \Pi_{,i}] dx \\ & + \int_0^t \int_R e^{-\omega \eta} \left[\frac{1}{2} \omega v_{i,\eta\eta} v_{i,\eta} + \frac{1}{2} \omega \mu v_{i,j\eta} v_{i,j\eta} + \frac{1}{2} \omega (\mu + \lambda) (v_{i,i\eta})^2 + \frac{1}{2} \omega \Pi_{,\eta\eta}^2 \right. \\ & + \left. \frac{1}{2} \omega \kappa \Pi_{,i} \Pi_{,i} + \delta \Pi_{,i\eta} \Pi_{,i\eta} \right] dx d\eta + \frac{1}{2} e^{-\omega t} \beta^* \kappa \int_0^\infty \int_{\partial D_\xi} \Pi^2 ds d\xi \\ & + \frac{1}{2} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} [\omega \beta^* \kappa \Pi^2 + \delta \beta^* \Pi_{,\eta}^2] ds d\xi d\eta, \end{aligned} \tag{79}$$

and

$$\begin{aligned} \Gamma(0, t) = & -\frac{1}{2} \delta \beta^* \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta \\ & - \tilde{\beta} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} [\kappa \Pi_{,\eta} \theta + \delta \Pi_{,\eta} \theta_{,\eta}] ds d\xi d\eta. \end{aligned} \tag{80}$$

Choosing $z = 0$ in (74) and using the Theorem 2, we have

$$\begin{aligned} & -\tilde{\beta} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} [\kappa \Pi_{,\eta} \theta + \delta \Pi_{,\eta} \theta_{,\eta}] ds d\xi d\eta \\ & \leq \frac{1}{2} \delta \beta^* \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta \\ & + \frac{\kappa^2 \tilde{\beta}^2}{\delta \beta^*} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta^2 ds d\xi d\eta + \frac{\delta \tilde{\beta}^2}{\beta^*} \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_{,\eta}^2 ds d\xi d\eta \\ & \leq -\frac{1}{2} \delta \beta^* \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta + 2b_2 \tilde{\beta}^2 Q_2(0, t). \end{aligned} \tag{81}$$

Inserting (81) into (80), we have

$$\Gamma(0, t) \leq 2b_2 \tilde{\beta}^2 Q_2(0, t). \tag{82}$$

Combining (66), (78) and (82), we can obtain the Theorem 4.

5. Convergence on the Boundary Coefficient

In this section, we derive the convergence result on the boundary coefficient which is different from the continuous dependence result. We let u^* and θ^* be the solutions of (9)–(15) with the boundary coefficient $\beta = 0$ and v and Π are also defined as (54). So, v and Π also satisfy (55)–(60), but the condition (61) can be replaced by

$$v = 0, \frac{\partial \Pi}{\partial n} + \beta \theta = 0, x \in \partial D \times \{x_3 \geq 0\}, t > 0. \tag{83}$$

To get our main result, we will use the following lemma.

Lemma 5 (see [35]). *Let D be a bounded star region in \mathbb{R}^2 . If $w \in C^1(D)$, then*

$$\int_{\partial D} w^2 ds \leq \frac{2}{p_0} \int_D w^2 dA + \frac{2d}{p_0} \left[\int_D w^2 dA \int_D w_{,i} w_{,i} dA \right]^{\frac{1}{2}}, \tag{84}$$

where $p_0 = \min_{\partial D} (x \cdot n)$, $d = \max_{\bar{D}} |x|$.

We define

$$\begin{aligned}
 F(z, t) = & \frac{1}{2}e^{-\omega t} \int_{R_z} \left[v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i}\Pi_{,i} \right] dx \\
 & + \int_0^t \int_{R_z} e^{-\omega \eta} \left[\frac{1}{2}\omega v_{i,\eta\eta}v_{i,\eta\eta} + \frac{1}{2}\omega \mu v_{i,j\eta}v_{i,j\eta} + \frac{1}{2}\omega(\mu + \lambda)(v_{i,i\eta})^2 + \frac{1}{2}\omega \Pi_{,\eta}^2 \right. \\
 & \left. + \frac{1}{2}\omega \kappa \Pi_{,i}\Pi_{,i} + \delta \Pi_{,i\eta}\Pi_{,i\eta} \right] dx d\eta, \tag{85}
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 -\frac{\partial}{\partial z} F(z, t) = & \frac{1}{2}e^{-\omega t} \int_{D_z} \left[v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i}\Pi_{,i} \right] dA \\
 & + \int_0^t \int_{D_z} e^{-\omega \eta} \left[\frac{1}{2}\omega v_{i,\eta\eta}v_{i,\eta\eta} + \frac{1}{2}\omega \mu v_{i,j\eta}v_{i,j\eta} + \frac{1}{2}\omega(\mu + \lambda)(v_{i,i\eta})^2 + \frac{1}{2}\omega \Pi_{,\eta}^2 \right. \\
 & \left. + \frac{1}{2}\omega \kappa \Pi_{,i}\Pi_{,i} + \delta \Pi_{,i\eta}\Pi_{,i\eta} \right] dA d\eta. \tag{86}
 \end{aligned}$$

Taking calculations similar to those in Section 3, we can get

$$\begin{aligned}
 F(z, t) = & -\mu \int_0^t \int_{D_z} e^{-\omega \eta} v_{i,3\eta}v_{i,\eta\eta} dA d\eta \\
 & - (\mu + \lambda) \int_0^t \int_{D_z} e^{-\omega \eta} v_{j,\eta}v_{3,\eta\eta} dA d\eta - \alpha \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,\eta}v_{3,\eta\eta} dA d\eta \\
 & - \kappa \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3}\Pi_{,\eta} dA d\eta - \delta \int_0^t \int_{D_z} e^{-\omega \eta} \Pi_{,3\eta}\Pi_{,\eta} dA d\eta \\
 & - \beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\kappa \Pi_{,\eta}\theta + \delta \Pi_{,\eta}\theta_\eta \right] ds d\xi d\eta. \tag{87}
 \end{aligned}$$

By using the Hölder inequality, the arithmetic–geometric mean inequality, we have

$$\begin{aligned}
 -\beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \left[\kappa \Pi_{,\eta}\theta + \delta \Pi_{,\eta}\theta_\eta \right] ds d\xi d\eta \\
 \leq \kappa \beta \left[\int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta^2 ds d\xi d\eta \right]^{\frac{1}{2}} \\
 + \delta \beta \left[\int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta^2 ds d\xi d\eta \right]^{\frac{1}{2}} \\
 \leq \frac{1}{2}\epsilon_1 \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \Pi_{,\eta}^2 ds d\xi d\eta \\
 + \frac{\kappa^2 \beta^2}{\epsilon_1} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta^2 ds d\xi d\eta + \frac{\delta^2 \beta^2}{\epsilon_1} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega \eta} \theta_\eta^2 ds d\xi d\eta, \tag{88}
 \end{aligned}$$

where ϵ_1 is a positive constant.

By the arithmetic–geometric mean inequality and Lemma 5, we have

$$\begin{aligned}
 \int_{\partial D_\xi} \Pi_{,\eta}^2 ds & \leq \frac{2}{p_0} \int_{D_\xi} \Pi_{,\eta}^2 dA + \frac{2d}{p_0} \left[\int_{D_\xi} \Pi_{,\eta}^2 dA \int_{D_\xi} \Pi_{,i\eta}\Pi_{,i\eta} dA \right]^{\frac{1}{2}} \\
 & \leq \frac{2+d}{p_0} \int_{D_\xi} \Pi_{,\eta}^2 dA + \frac{d}{p_0} \int_{D_\xi} \Pi_{,i\eta}\Pi_{,i\eta} dA. \tag{89}
 \end{aligned}$$

Inserting (89) into (88), we have

$$\begin{aligned}
 &-\beta \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} [\kappa\Pi_\eta\theta + \delta\Pi_\eta\theta_\eta] dsd\xi d\eta \\
 &\leq \frac{2+d}{2p_0} \epsilon_1 \int_0^t \int_{R_z} e^{-\omega\eta} \Pi_\eta^2 dx d\eta + \frac{d}{2p_0} \epsilon_1 \int_0^t \int_{R_z} e^{-\omega\eta} \Pi_{,i\eta} \Pi_{,i\eta} dx d\eta \\
 &+ \frac{\kappa^2\beta^2}{\epsilon_1} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta^2 dsd\xi d\eta + \frac{\delta^2\beta^2}{\epsilon_1} \int_0^t \int_z^\infty \int_{\partial D_\xi} e^{-\omega\eta} \theta_\eta^2 dsd\xi d\eta.
 \end{aligned} \tag{90}$$

Inserting (69)–(73) and (90) into (87), choosing $\epsilon_1 \leq \min\{\frac{p_0\omega}{2(2+d)}, \frac{p_0\delta}{2d}\}$ and using the Theorem 3, we have

$$F(z, t) \leq \frac{1}{2}F(z, t) + \frac{1}{b_1\sqrt{\omega}} \left[-\frac{\partial}{\partial z} \Gamma(z, t) \right] + b_3\beta Q_3(z, t)e^{-m_1\sqrt{\omega}z},$$

or

$$F(z, t) \leq \frac{2}{b_1\sqrt{\omega}} \left[-\frac{\partial}{\partial z} \Gamma(z, t) \right] + 2b_3\beta Q_3(z, t)e^{-m_1\sqrt{\omega}z}, \tag{91}$$

where $b_3 = \max\{\frac{2\kappa}{\omega\epsilon_1}, \frac{\delta}{\epsilon_1}\}$. Integrating (91) from 0 to z , we have

$$F(z, t) \leq F(0, t)e^{-\frac{b_1\sqrt{\omega}}{2}z} + b_1b_3\sqrt{\omega}\beta e^{-\frac{b_1\sqrt{\omega}}{2}z} \int_0^z Q_3(\xi, t)e^{(\frac{b_1}{2}-m_1)\sqrt{\omega}\xi} d\xi. \tag{92}$$

To bound $F(0, t)$, choosing $z = 0$ in (85) and (87) and in view of (59), (60) and (83), we have

$$\begin{aligned}
 F(0, t) &= \frac{1}{2}e^{-\omega t} \int_R [v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,i}^2 + \kappa\Pi_{,i}\Pi_{,i}] dx \\
 &+ \int_0^t \int_R e^{-\omega\eta} \left[\frac{1}{2}\omega v_{i,\eta\eta}v_{i,\eta} + \frac{1}{2}\omega\mu v_{i,j\eta}v_{i,j\eta} + \frac{1}{2}\omega(\mu + \lambda)(v_{i,i\eta})^2 + \frac{1}{2}\omega\Pi_{,\eta\eta}^2 \right. \\
 &\left. + \frac{1}{2}\omega\kappa\Pi_{,i}\Pi_{,i} + \delta\Pi_{,i\eta}\Pi_{,i\eta} \right] dx d\eta,
 \end{aligned} \tag{93}$$

and

$$F(0, t) = -\beta \int_0^t \int_0^\infty \int_{\partial D_\xi} e^{-\omega\eta} [\kappa\Pi_\eta\theta + \delta\Pi_\eta\theta_\eta] dsd\xi d\eta. \tag{94}$$

In view of (90) and (93), and the Theorem 2, we have

$$F(0, t) \leq \frac{1}{2}F(0, t) + 2b_3Q_2(0, t)\beta,$$

or

$$F(0, t) \leq 4b_3Q_2(0, t)\beta. \tag{95}$$

Inserting (95) into (92) and in view of (86), we can have the following theorem.

Theorem 5. Let (\mathbf{u}, θ) be the solutions of Equations (9)–(15) and (\mathbf{u}^*, θ^*) be the solutions of Equations (9)–(15) with $\beta = 0$. The functions v and Π are defined in (54). If $F, F_\eta \in C(\partial D \times [0, \infty))$ and $\int_D f_3 dA = 0$, then

$$(\mathbf{u}, \theta) \rightarrow (\mathbf{u}^*, \theta^*), \text{ as } \beta \rightarrow 0.$$

Specifically,

$$\begin{aligned} & \frac{1}{2}e^{-\omega t} \int_{R_z} \left[v_{i,tt}v_{i,tt} + \mu v_{i,jt}v_{i,jt} + (\mu + \lambda)(v_{i,it})^2 + \Pi_{,t}^2 + \kappa \Pi_{,i}\Pi_{,i} \right] dx \\ & + \int_0^t \int_{R_z} e^{-\omega \eta} \left[\frac{1}{2}\omega v_{i,\eta\eta}v_{i,\eta} + \frac{1}{2}\omega \mu v_{i,j\eta}v_{i,j\eta} + \frac{1}{2}\omega(\mu + \lambda)(v_{i,i\eta})^2 + \frac{1}{2}\omega \Pi_{,\eta\eta}^2 \right. \\ & \left. + \frac{1}{2}\omega \kappa \Pi_{,i}\Pi_{,i} + \delta \Pi_{,i\eta}\Pi_{,i\eta} \right] dx d\eta \\ & \leq 4b_3 Q_2(0, t) \beta e^{-\frac{b_1\sqrt{\omega}}{2}z} + b_1 b_3 \sqrt{\omega} \beta e^{-\frac{b_1\sqrt{\omega}}{2}z} \int_0^z Q_3(\xi, t) e^{(\frac{b_1}{2} - m_1)\sqrt{\omega}\xi} d\xi, \end{aligned}$$

where b_1, b_3 are positive constants. This demonstrates convergence of (u, θ) on the parameter β .

Theorem 5 shows that when $z \rightarrow 0$, it will not have a significant impact on the solution of the system of equations. It shows the stability of the equations.

6. Conclusions

In this paper, Equations (9)–(15) are reconsidered in a new semi-infinite cylinder. The structural stability of the solution is obtained by using the differential inequality technique and energy analysis method. In a two-dimensional pipe, Payne and Schaefer [36] obtained Phragmén–Lindelöf alternative results of biharmonic equation. As far as we know, there are a few results in this type of three-dimensional cylinder region. Therefore, it is very interesting to replace the pipe R by

$$\left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D_{x_3}, x_3 > 0 \right\},$$

where D_{x_3} can be defined as

$$\left\{ (x_1, x_2, x_3) \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = x_3^\gamma, x_3 > m > 0, 0 < \gamma \leq 1 \right\}.$$

On the other hand, if the boundary conditions (13)–(15) are replaced by (3), (4) and (6)–(8), how to obtain the continuous dependence of Equations (9)–(15) is also a interesting topic.

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