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Adopting Feynman–Kac Formula in Stochastic Differential Equations with (Sub-)Fractional Brownian Motion

Bodo Herzog [1,2,3]

[1] Economics Department, ESB Business School, Reutlingen University, 72762 Reutlingen, Germany; Bodo.Herzog@Reutlingen-University.de
[2] Reutlingen Research Institute (RRI), 72762 Reutlingen, Germany
[3] Institute of Finance and Economics (IFE), Reutlingen University, 72762 Reutlingen, Germany

Abstract: The aim of this work is to establish and generalize a relationship between fractional partial differential equations (fPDEs) and stochastic differential equations (SDEs) to a wider class of stochastic processes, including fractional Brownian motions $\{B_t^H, t \geq 0\}$ and sub-fractional Brownian motions $\{\xi_t^H, t \geq 0\}$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. We start by establishing the connection between a fPDE and SDE via the Feynman–Kac Theorem, which provides a stochastic representation of a general Cauchy problem. In hindsight, we extend this connection by assuming SDEs with fractional- and sub-fractional Brownian motions and prove the generalized Feynman–Kac formulas under a (sub-)fractional Brownian motion. An application of the theorem demonstrates, as a by-product, the solution of a fractional integral, which has relevance in probability theory.

Keywords: Cauchy problem; fractional-PDE; SDE; fractional Brownian motion; sub-fractional processes; Feynman–Kac formula; fractional calculus

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1. Introduction

Consider the Cauchy problem [1] of the following parabolic partial differential equation (PDE) on $\mathbb{R}^d$

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \kappa\frac{\partial^2}{\partial x^2}u(x,t) + \eta B^H(t), \qquad t \geq 0, x \in \mathbb{R}^d, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1}$$

where $u(x,t) \in C^{2,1}$, $u_0(x)$ is a bounded measurable function and $B^H(t)$ is a fractional Brownian motion (cf. Section 2). Without loss of generality, we assume that the parameter κ is constant. This second-order PDE has a stochastic representation for $\eta = 0$, according to the Feynman–Kac formula [2,3]. Indeed, we obtain

$$u(x_t, t) = \mathbb{E}_{x,t}[u_T(x)], \tag{2}$$

if x_t satisfies Equation (3) and the function $\sigma(x_t,t)$ is sufficiently integrable

$$dx_t = \mu(x_t,t)dt + \sigma(x_t,t)dB_t^H, \tag{3}$$

where B_t^H is a Brownian motion (BM) if the Hurst parameter is of $H = \frac{1}{2}$ [4–6]. Additionally, the problem of (1) has an intimate relationship to the fractional partial differential equation (fPDE) [7]:

$$\frac{\partial^{1/2}}{\partial t^{1/2}}u(x,t) = -\frac{\partial}{\partial x}u(x,t). \tag{4}$$

Note that this equation contains a fractional derivative in general or a semi-derivative in respect of time in special [8–13].

There is a large amount of the literature devoted to each issue of the Cauchy problem [6,14]. This research closes a gap by considering the linking relationships of (sub-)fractional Brownian motions as well as fPDEs. The Feynman–Kac formula (2) provides a unique weak solution to Equation (1). Different versions of the Feynman–Kac formula have been discovered for a variety of problems [15,16]. Some generalizations of the Feynman–Kac formula are discovered by Querdiane and Silva [17] and Hu et al. [18,19]. A Feynman–Kac formula also exists for Lévy processes by Nualart and Schoutens [20].

Advancements in stochastic differential equations and fractional partial differential equations to analyse complex systems are related to our research [21–24]. Furthermore, recent developments in fractional calculus contributed to a better understanding and further studies of the relationships between fractional PDEs and stochastic calculus [25–31]. However, we are concerned about the linkage of the Cauchy problem and the representation by a fPDE, as well as the Feynman–Kac formula. For the Cauchy problem, we generalize the stochastic representation of Feynman–Kac by utilizing fractional Brownian motion (fBM) with Hurst parameter $H > 1/2$.

In addition, the more recent literature looks at the idea of sub-fractional Brownian motion (sub-fBM). A sub-fBM is an intermediate between a Brownian motion and fractional Brownian motion. The existence and properties, such as long-range dependence, self-similarity and non-stationarity were introduced by Bojdecki et al. [32] and Tudor et al. [33,34]. Since the sub-fractional Brownian motion is not a martingale, methods of stochastic analysis are more sophisticated. However, several authors developed stochastic calculus and integration concepts for an fBM [25] and sub-fBM [35–37]. Recently, for a sub-fractional Brownian motion with Hurst parameters $H > \frac{1}{2}$, a maximal inequality was established according to the Burkholder–Davis–Gundy inequality for fractional Brownian motion [38]. It turns out that fBM and sub-fBM are adequate stochastic processes in scientific applications [13,39].

In this paper, our purpose is to construct and prove a general link of the Cauchy problem with the Feynman–Kac equation via Itô's formula for fBM and sub-fBM. Consequently, this paper links the solution of $u(x,t)$ defined by Equation (1) with the stochastic Feynman–Kac representation to a fractional Brownian motion $\{B_t^H\}$ and sub-fBM $\{\xi_t^H\}$. We prove the result and show the properties of (sub-)fractional processes in stochastic analysis. Note that, throughout this paper, we frequently assume $\frac{1}{2} < H < 1$.

The paper is organized as follows. Section 2 contains preliminaries on fractional calculus, particularly fractional Brownian motion. Thereafter, we examine sub-fractional stochastic processes and integration rules in Section 3. Here, we list the definitions and assumptions for the remainder of the article. In Section 4, we link the Cauchy problem to the Feynman–Kac formula with stochastic differential equations driven by fractional and sub-fractional Brownian motions. We state our theorems and prove our statements. In Section 5, we examine the Cauchy problem and the relationship to fractional partial differential equations (fPDE). Furthermore, we find a new fractional derivative and integral with relevance in probability theory. The conclusion is in Section 6.

2. Preliminaries

In the following section, we define preliminary concepts on fractional stochastic processes and fractional calculus.

2.1. Fractional Calculus

Since we deal with the Hurst parameter H, we need to know fractional calculus. Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left- and right-sided fractional integral of f of order α are defined for $x \in (a,b)$, respectively, as

$$ {}_aD_x^{-\alpha} f(x) = {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u) du \qquad -\infty \leq a \leq x, $$

and

$$ {}_xD_b^{-\alpha}f(x) = {}_xI_b^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (u-x)^{\alpha-1}f(u)du \qquad -\infty \leq x \leq b. $$

This is the fractional integral of Riemann–Liouville type. Similarly, the fractional left- and right-sided derivative, for $f \in I_a^{\alpha}(L^p)$ and $0 < \alpha < 1$, are defined by

$$ {}_aI_x^{-\alpha}f(x) = {}_aD_x^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dx}\right)\int_a^x (x-u)^{-\alpha}f(u)du \tag{5} $$

and

$$ {}_xI_b^{-\alpha}f(x) = {}_xD_b^{\alpha}f(x) = \frac{-1}{\Gamma(1-\alpha)}\left(\frac{d}{dx}\right)\int_x^b (u-x)^{-\alpha}f(u)du, \tag{6} $$

for all $x \in (a,b)$ and $I_a^{\alpha}(L^p)$ is the image of $L^p(a,b)$. It is easy to see that if $f \in I_a^1(L^1)$,

$$ {}_aD_x^{\alpha}{}_aD_x^{1-\alpha}f(x) = Df(x), \qquad {}_bD_x^{\alpha}{}_bD_x^{1-\alpha}f(x) = Df(x). \tag{7} $$

Note $D^{\alpha}f(x)$ exists for all $f \in C^{\beta}([a,b])$ if $\alpha < \beta$.

2.2. Fractional Stochastic Process

Mandelbrot and van Ness defined a fractional Brownian Motion (fBM), B_t^H, as a Brownian motion, $B(t)$, together with a Hurst parameter (or Hurst index) $H \in (0,1)$ in 1968 [8]. The new feature of fBM's is that the increments are interdependent. The latter property is defined as self-similarity. A self-similar process has invariance with respect to changes in timescale (scaling-invariance). Almost all other stochastic processes, such as the standard Brownian Motion or Lévy processes, likely have independent increments. They create the famous class of Markov processes. Empirically, there is ubiquitous evidence in science that fractional stochastic processes, for instance, spectral densities with a sharp peak, are related to the phenomena of long-range interdependence over time. Indeed, the observable presence of interdependence in many real-world applications calls for fractional stochastic processes.

Definition 1. *Let H be $0 < H < 1$ and B_0 an arbitrary real number. We call $B^H(t,\omega)$ a fractional Brownian Motion (fBM) with Hurst parameter H and starting value B_0 at time 0, such as*

(1) $B^H(0,\omega) = B_0$, *and;*

(2) $B^H(t,\omega) - B^H(0,\omega) = \frac{1}{\Gamma(H+\frac{1}{2})}\left[\int_{-\infty}^0[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]dB(s,\omega) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s,\omega)\right]$ *[Wyle fractional integral];*

(3) *[Or equivalently by the Riemann-Liouville fractional integral:* $B^H(t,\omega) - B^H(0,\omega) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}dB(s,\omega)$*].*

We immediately obtain the corollary.

Corollary 1. *For $H = \frac{1}{2}$ and $B_0 = 0$, we obtain a Brownian Motion $B(t,\omega) = B^{\frac{1}{2}}(t,\omega)$.*

Proof. If $H = \frac{1}{2}$, we obtain $B^{\frac{1}{2}}(t,\omega) - B^{\frac{1}{2}}(0,\omega) = \frac{1}{\Gamma(1)}\int_0^t dB(s,\omega) = B(t,\omega)$. □

For values of H, such as $0 < H < \frac{1}{2}$ or $\frac{1}{2} < H < 1$ the fBM $B^H(t,\omega)$ has different properties. If $0 < H < \frac{1}{2}$, we say that it has the property of short memory. Indeed, Mandelbrot and van Ness [8] shows that this range is associated with negative correlation. If $\frac{1}{2} < H < 1$, then the fBM has the property of long-memory or long-range dependence with time-persistence (Mandelbrot and van Ness [8]). Alternatively, we define a fractional Brownian motion by

Definition 2. *A fractional Brownian Motion (fBM) is a centered Gaussian process $B^H(t)$ for $t \geq 0$ with the covariance function*

$$R^{fBM}(t,s) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}], \tag{8}$$

where $H \in (0,1)$ denotes the Hurst parameter*.*

Remark 1. *The covariance is trivially derived by starting with a standard Brownian motion and extending it with the Hurst index H, such as*

$$\begin{aligned} Var[B(t) - B(s)] = \mathbb{E}[(B(t) - B(s))^2] = |t-s| \\ \Leftrightarrow Var[B^H(t) - B^H(s)] = \mathbb{E}[(B^H(t) - B^H(s))^2] = |t-s|^{2H}, \end{aligned}$$

where, for $H = \frac{1}{2}$, we obtain the Brownian motion. The covariance is derived by the following steps

$$\begin{aligned} Cov[B^H(t)B^H(s)] &= \mathbb{E}[(B^H(t) - \mathbb{E}[B^H(t)])(B^H(s) - \mathbb{E}[B^H(s)])] = \mathbb{E}[B^H(t)B^H(s)] \\ &= \frac{1}{2}\left[\mathbb{E}[B^H(t)^2] + \mathbb{E}[B^H(s)^2] - \mathbb{E}[(B^H(t) - B^H(s))^2]\right] \\ &= \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]. \end{aligned}$$

Corollary 2. *The expectation of non-overlapping increments of an fBM is $\mathbb{E}[B^H(t) - B^H(s)] \neq 0$ and the variance is of $\mathbb{E}[(B^H(t) - B^H(s))^2] = |t-s|^{2H}$ for all $t, s \in \mathbb{R}$*

Proof. Let $t > s > 0$. The first part is

$$\begin{aligned} \mathbb{E}[(B^H(t) - B^H(s))(B^H(s) - B^H(0))] &= \mathbb{E}[B^H(t)B^H(s)] - \mathbb{E}[B^H(t)B^H(0)] - \\ &- \mathbb{E}[(B^H(s))^2] + \mathbb{E}[B^H(s)B^H(0)] \\ &= \frac{1}{2}[t^{2H} + s^{2H} - (t-s)^{2H}] - s^{2H} \\ &= \frac{1}{2}[t^{2H} - s^{2H} - (t-s)^{2H}] \neq 0. \end{aligned}$$

Thus, we can see that the expected increments are non-zero. Indeed, the increments are interdependent, contrary to Markov processes. The second part of the variance is

$$\begin{aligned} \mathbb{E}[(B^H(t) - B^H(s))^2] &= \mathbb{E}[(B^H(t) - B^H(s))(B^H(t) - B^H(s))] \\ &= \mathbb{E}[(B^H(t))^2] + \mathbb{E}[(B^H(s))^2] - 2\mathbb{E}[B^H(t)B^H(s)] \\ &= t^{2H} + s^{2H} - 2[\frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]] \\ &= |t-s|^{2H} \qquad \forall t, s \in \mathbb{R} \end{aligned}$$

□

Proposition 1. *A fractional Brownian Motion (fBM) has the following properties:*

(1) The fBM has stationary increments: $B_t^H - B_s^H \stackrel{dis.}{=} B_u^H - B_s^H$;
(2) The fBM is H-self-similar, such as $B^H(at) = a^H B^H(t)$;
(3) The fBM has dependence of increments for $H \neq \frac{1}{2}$.

Proof. Part (1): For $t_1 < t_2 < t_3 < t_4$, the equality of the covariance function implies that $Y := B^H(t_2) - B^H(t_1)$ has the same distribution as $X := B^H(t_4) - B^H(t_3)$. From above, we know

$$\mathbb{E}[(B^H(t_2) - B^H(t_1))^2] = (t_2 - t_1)^{2H} = (\Delta t)^{2H}$$
$$\mathbb{E}[(B^H(t_4) - B^H(t_3))^2] = (t_4 - t_3)^{2H} = (\Delta t)^{2H},$$

where $t_1 < t_2$ and $t_3 < t_4$ with $\Delta t = t_2 - t_1 = t_4 - t_3$. Hence, the incremental behavior at any point in the future is the same. Thus, we say that it has stationary increments.

Part (2): We show that $B^H(at) = a^H B^H(t)$. We utilize the definition,

$$\begin{aligned}\mathbb{E}[(B^H(at))^2] &= \frac{1}{2}[(at)^{2H} + (at)^{2H} - (at - at)^{2H}] = (at)^{2H} = a^{2H}t^{2H}\\ &= a^{2H}\mathbb{E}[(B^H(t))^2],\end{aligned}$$

hence, we obtain $(B^H(at))^2 = a^{2H}(B^H(t))^2$ and this equal to $B^H(at) = a^H B^H(t)$. The proof of part (3) is already in Corollary 2. □

2.3. Itô's Formula for Fractional Brownian Motion

A fractional Brownian motion is continuous but almost certainly not differentiable [8]. Hence, it is inconvenient that an fBM does not have a derivative or integral. Furthermore, the fBM is neither a martingale nor a semi-martinagle. Therefore, Itô calculus is not applicable to fractional Brownian Motions if $H \neq \frac{1}{2}$.

However, stochastic calculus was developed with respect to fractional Brownian motion by [40] and the stochastic integral was introduced by [25]. The theory is a fractional extension of Itô-calculus, but limited to a Hurst index $H \in (1/2, 1)$. If $H > 1/2$ the fBM exhibits long-range dependence, which is a fundamental property in physics or finance.

By utilizing Wick calculus that has zero mean and explicit expressions for the second moment, we define the stochastic fractional integral, satisfying the property $\mathbb{E}[\int_0^t f(s)dB^H(s)] = 0$.

Suppose a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}^H)$, where the probability measure depends on H. Note that H is fixed by $H \in (1/2, 1)$. Let us define a kernel function $\phi(s,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi^{fBM}(s,t) := \phi(s,t) = H(2H-1)|s-t|^{2H-2}. \tag{9}$$

Further, the functions f and g belong to the Hilbert space L^2_ϕ if

$$|f|^2_\phi = \int_0^\infty \int_0^\infty f(s)f(t)\phi(s,t)dsdt < \infty, \tag{10}$$

with the inner product defined by

$$\langle f, g\rangle_\phi := \mathbb{E}\left[\int_0^\infty f(s)dB^H(s) \int_0^\infty g(t)dB^H(t)\right] = \int_0^\infty \int_0^\infty f(s)g(t)\phi(s,t)dsdt \tag{11}$$

This machinery leads to an analogue Itô formula for a fractional Brownian process. Already, Alòs et al. [41] proved this result under certain conditions for Itô's formula.

Theorem 1. *(Alòs et al., 2001). Let f be a function of class $C^2(\mathbb{R})$, satisfying the growth condition*

$$\max[|f(x)|, |f'(x)|, |f''(x)|] \leq ce^{(\lambda|x|^2)},$$

where c and λ are positive constants and $\lambda < \frac{1}{4}T^{-2H}$. Suppose that $B^H = \{B^H_t, t \in [0,T]\}$ is a zero mean continuous Gaussian process whose covariance function $R^{fBM}(t,s)$ is of the form *in Equation (8). Then, the process $F'(B_t^H)$ belongs to a Hilbert space and, for each $t \in [0, T]$, the following Itô's formula holds:*

$$f(B_T^H) = f(0) + \int_0^T f'(B_s^H)\delta B_s^H + \frac{1}{H}\int_0^T f''(B_s^H)s^{2H-1}ds. \tag{12}$$

However, we utilize a result by Duncan et al. [25], which is more convenient in our case. Here, is the Itô-Duncan theorem for a fractional Brownian motion:

Theorem 2. *(Duncan et al., 2000, Thm 4.1, p. 596). If $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function with bounded derivatives to order two, i.e., $f \in C^2$, then*

$$f(B_T^H) - f(B_0^H) = \int_0^T f'(B_s^H)dB_s^H + H\int_0^T s^{2H-1}f''(B_s^H)ds \qquad a.s.$$

Remark 2. *If $H = \frac{1}{2}$, we obtain, from Theorem 2, the usual Itô formula for a Brownian motion*

$$\begin{aligned} f(B^{\frac{1}{2}}(T)) = f(B_T) &= \int_0^T f'(B^{\frac{1}{2}}(s))dB^{\frac{1}{2}}(s) + \frac{1}{2}\int_0^T s^0 f''(B^{\frac{1}{2}}(s))ds \\ &= \int_0^T f'(B_s)dB_s + \frac{1}{2}\int_0^T f''(B_s)ds \end{aligned}$$

or in differential form

$$df(B_T) = f'(B_s)dB_s + \frac{1}{2}f''(B_s)ds. \tag{13}$$

Similarly, for a function with two parameters $f(t, B_t^H)$, a generalized rule exists according to Duncan et al. [25].

Theorem 3. *(Duncan et al., 2000, Thm 4.3, p. 596). Let $\eta_t = \int_0^t F_u dB_u^H$ for $t \in [0, T]$ and $(F_u, 0 \le u \le T)$ is a stochastic process in $\mathcal{L}(0, T)$. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function having the first continuous derivative in its first variable and the second continuous derivative in its second variable. Assume that these derivatives are bounded. Moreover, it is assumed that $\mathbb{E}\int_0^T |F_s D_s^\phi \eta_s|ds < \infty$ and $(f'(s, \eta_s)F_s, s \in [0, T])$ is in $\mathcal{L}(0, T)$. Then, for $0 \le t \le T$,*

$$\begin{aligned} f(t, \eta_t) =& f(0, 0) + \int_0^t \frac{\partial f(s, \eta_s)}{\partial s}ds + \int_0^t \frac{\partial f(s, \eta_s)}{\partial x}F_s dB_s^H \\ &+ \int_0^t \frac{\partial^2 f(s, \eta_s)}{\partial x^2}F_s D_s^\phi \eta_s ds \qquad a.s. \end{aligned}$$

this is equal to

$$df(t, \eta_t) = \frac{\partial f(t, \eta_t)}{\partial t} + \frac{\partial f(t, \eta_t)}{\partial x}F_t dB_t^H + \frac{\partial^2 f(t, \eta_t)}{\partial x^2}F_t D_t^\phi \eta_t dt,$$

where $D_s^\phi \eta_t = \int_0^t D_s^\phi F_u dB_u^H + \int_0^t F_u \phi(s, u)du$ a.s.

For the proof, we refer to Duncan et al. [25]. If $F(s) = a(s)$ is a deterministic function; then, the rule simplifies. Let $\eta_t = \int_0^t a_u dB_u^H$, where $a \in L_\phi^2$; then, we obtain

$$\begin{aligned} f(t, \eta_t) =& f(0, 0) + \int_0^t \frac{\partial f(s, \eta_s)}{\partial s}ds + \int_0^t \frac{\partial f(s, \eta_s)}{\partial x}a(s)dB_s^H \\ &+ \int_0^t \frac{\partial^2 f(s, \eta_s)}{\partial x^2}\int_0^s \phi(s, v)a(v)dvds \quad a.s. \end{aligned} \tag{14}$$

If $a_s \equiv 1$, then we obtain Itô's formula, such as in Theorem 2 and in Equation (13).

3. Sub-Fractional Stochastic Process

A sub-fractional Brownian motion (sub-fBM) is an intermediate between a Brownian motion and fractional Brownian motion. It is a more general, self-similar Gaussian process or a generalization of a fBM. The sub-fBM has the property of H-self-similarity and long-range dependence, such as the fBM, yet it does not have stationary increments [32].

It is well-established that a stochastic process is uniquely determined by its covariance function $\mathrm{Cov}(\xi_t^H, \xi_s^H)$. Thus, we define:

Definition 3. *A sub-fractional Brownian motion of Hurst parameter H is a centered mean zero Gaussian process $\xi^H = \{\xi_t^H, t \geq 0\}$ with covariance function*

$$R^{sfBM}(t,s) := \mathbb{E}[\xi_t^H \xi_s^H] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}], \tag{15}$$

where $\xi_0^H = 0$ and $\mathbb{E}[\xi_t^H] = 0$.

If $H = \frac{1}{2}$, it coincides with a Brownian motion on $\mathbb{R}_+$ with covariance $\mathrm{Cov}(\xi_t^H, \xi_s^H) = s \wedge t := \min[s,t]$. The process ξ_t^H has the following integral representation for $H > \frac{1}{2}$ (see [41]):

$$\xi_t^H = \int_0^t K^H(t,s) dW_s, \tag{16}$$

$$K^H(t,s) = c_H \left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du. \tag{17}$$

Hence, the sub-fractional Brownian motion has a kernel of

$$\phi^{sfBM}(s,t) = \frac{\partial^2 \mathrm{Cov}(\xi_t^H, \xi_s^H)}{\partial s \partial t} = H(2H-1)\Big[|s-t|^{2H-2} - (s+t)^{2H-2}\Big]. \tag{18}$$

Note that the kernel has similarities to the fBM, as in Equation (9). Next, we discuss the main properties of a sub-fBM:

Lemma 1. *Let ξ_t^H be a sub-fBM for all t. It has the following properties:*

(1) $\mathbb{E}[(\xi_t^H)^2] = (2 - 2^{2H-1}) t^{2H}$.
(2) $\mathbb{E}[(\xi_t^H - \xi_s^H)^2] = -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + (t-s)^{2H}$.
(3) *If $H \neq \frac{1}{2}$, then $\xi_t^H - \xi_s^H \overset{dis.}{\neq} \xi_u^H - \xi_s^H$, i.e., the increments are non-stationary.*

Proof. Part 1. Let $t = s$ in the covariance function $\mathrm{Cov}(\xi_t^H, \xi_s^H)$. We obtain $\mathrm{Cov}(\xi_t^H, \xi_t^H) = \mathbb{E}[\xi_t^{2H}] - (\mathbb{E}[\xi_t^H])^2 = \mathrm{Var}(\xi_t^H)$ and further we have $\mathrm{Var}(\xi_t^H) = \mathbb{E}[(\xi_t^H)^2]$ because ξ_t^H is Gaussian with mean zero. Thus, using the covariance function in Definition 3, we obtain

$$\mathbb{E}[(\xi_t^H)^2] = 2t^{2H} - \frac{1}{2}(2t)^{2H} = 2t^{2H} - \frac{1}{2}(2t)^{2H} = (2 - 2^{2H-1})t^{2H}.$$

Part 2. Given property 1, one immediately obtains

$$\begin{aligned}\mathbb{E}[(\xi_t^H - \xi_s^H)^2] &= (2 - 2^{2H-1})t^{2H} + (2 - 2^{2H-1})s^{2H} \\ &= -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + (t-s)^{2H}.\end{aligned}$$

Part 3. Let $s = 0$ and $t = h > 0$, then $\mathbb{E}[(\xi_h^H - \xi_0^H)^2] = \mathbb{E}[(\xi_h^H)^2] = (2 - 2^{2H-1})h^{2H}$ and we obtain

$$\begin{aligned}\mathbb{E}[(\xi^H_{t+h}-\xi^H_{s+h})^2] &= \mathbb{E}[(\xi^H_{2h}-\xi^H_h)^2]\\ &= \mathbb{E}[\xi^{2H}_{2h}]-2\mathbb{E}[\xi^H_{2h}]\mathbb{E}[\xi^H_h]+\mathbb{E}[\xi^{2H}_h]\\ &= (2-2^{2H-1})(2h)^{2H}+(2-2^{2H-1})h^{2H} =\\ &= [2-2^{2H-1}](2^{2H}+1)h^{2H}.\end{aligned}$$

The difference in both increments is

$$\Delta(H)=[2-2^{2H-1}]-[2-2^{2H-1}](2^{2H}+1)=-2^{2H}[2-2^{2H-1}],$$

where $\Delta(H) := \mathbb{E}[(\xi^H_h)^2]-\mathbb{E}[(\xi^H_{t+h}-\xi^H_{s+h})^2]$. For $\Delta(0)=-\frac{3}{2}$ and $\Delta(\frac{1}{2})=-2$ and $\Delta(1)=0$. This implies that $\mathbb{E}[(\xi^H_{2h}-\xi^H_h)^2] > \mathbb{E}[(\xi^H_t)^2]$ for all $H\in(0,1)$. Thus, the increments are non-stationary, such as $\xi^H_t-\xi^H_s \overset{\text{dis.}}{\neq} \xi^H_u-\xi^H_s$. □

Finally, we prove two differences of fBM and sub-fBM.

Proposition 2. *Let B^H_t be a fractional Brownian motion and ξ^H_t be a sub-fractional Brownian motion. For $H\in(\frac{1}{2},1)$ the following holds:*

(1) $\mathbb{E}[(\xi^H_t)^2] < \mathbb{E}[(B^H_t)^2]$;
(2) $R_{\xi^H_t}(s,t) \le R_{B^H_t}(s,t)$.

Proof. Part 1. For an fBM, we have $\mathrm{Var}[B^H_t]=|t|^{2H}$, and for the sub-fBM, we have $\mathrm{Var}[\xi^H_t]=(2-2^{2H-1})|t|^{2H}$. Hence, we obtain $0<(2H-1)\ln 2$ for $H>\frac{1}{2}$. For part 2, we show, under $s,t>0$, that

$$\begin{aligned}s^{2H}+t^{2H}-\frac{1}{2}[(s+t)^{2H}+|t-s|^{2H}] &\le \frac{1}{2}[|t|^{2H}+|s|^{2H}-|t-s|^{2H}]\\ s^{2H}+t^{2H} &\le (s+t)^{2H},\end{aligned}$$

where, only for $s=t=0$ or $s=0, t\neq 0$, we obtain equality. □

Itô's Formula for Sub-Fractional Brownian Motion

For a Hurst parameter $H>\frac{1}{2}$, the stochastic integral of a sub-fBM $\int_0^T f(t)d\xi^H_t$ exists. The following theorem holds and is proven by [42]:

Theorem 4. *Let ξ^H_t be a sub-fBM defined in Definition 3 with $H>\frac{1}{2}$ and a function $f\in L([0,T]^2,\phi^{sfBM}d\lambda_2)$, where λ_2 is a Lebesgue measure on $[0,T]^2$, where $\phi^{sfBM}(s,t)$ and $(s,t)\in[0,T]^2$. Then, there exists a constant $C_H>0$ such that*

$$\mathbb{E}\left[\int_0^T f(t)d\xi^H_t\right]^2 \le C_H\|f\|^2_{L^{1/H}([0,T],\lambda_1)}. \tag{19}$$

According to Yan et al. ([36], Theorem 3.2 on p. 139) Itô's formula under a sub-fBM can be computed as follows:

Theorem 5. *(Yan et al., 2011) Let $f\in C^2(\mathbb{R})$ and $H\in(\frac{1}{2},1)$. Then, we have*

$$f(\xi^H_t)=f(0)+\int_0^T f'(\xi^H_s)d\xi^H_s+H(2-2^{2H-1})\int_0^T f''(\xi^H_s)s^{2H-1}ds. \tag{20}$$

Details of the proof are given in ([36], pp. 139–140). The authors even extend Itô's formula to $d-$dimensional sub-fBM and convex functions $f:\xi^H_t\to\mathbb{R}$.

4. Linking Cauchy via Feynman–Kac to SDEs with fBM and Sub-fBM

Next, we derive the link between the Cauchy problem (1) and the stochastic representation according to Feynman–Kac by Equation (2). Consider a stochastic process x_s on the time interval $[t, T]$ as the solution to the SDE in Equation (3). Next, use the Dynkin operator or Fokker-Planck operator $\mathcal{A}$ defined by

$$\mathcal{A} = \mu(x,s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x,s)\frac{\partial^2}{\partial x^2}. \tag{21}$$

We may write the Cauchy problem (1) as

$$\begin{aligned} \frac{\partial u(x,s)}{\partial s} + \mathcal{A}u(x,s) &= 0, \\ u(x,T) &= u_T(x). \end{aligned} \tag{22}$$

Cauchy Problem and Feynman–Kac

Applying Itô's lemma to $u(x,s)$. We obtain

$$\int_t^T du(x_s,s)ds = \int_t^T \left[\frac{\partial u(x_s,s)}{\partial s} + \mathcal{A}u(x_s,s)\right]ds + \int_t^T \sigma(x_s,s)\frac{\partial u(x_s,s)}{\partial x_s}dB_s. \tag{23}$$

After integration, we obtain

$$u(x_T,T) - u(x_t,t) = \int_t^T \left[\frac{\partial u(x_s,s)}{\partial s} + \mathcal{A}u(x_s,s)\right]ds + \int_t^T \sigma(x_s,s)\frac{\partial u(x_s,s)}{\partial x_s}dB_s. \tag{24}$$

Since, by assumption $u(x,t)$ satisfies Equation (22), the time integral ds in the last line of Equation (23) will vanish. Furthermore, if the process $\sigma(x_s,s)\frac{\partial u(x_s,s)}{\partial x_s}$ is sufficiently integrable, and after taking the expectation, the stochastic integral will vanish. Finally, considering the initial and boundary condition, such as $u(x,T) = u_T(x)$, we obtain the stochastic representation of the Cauchy problem (1) using the Feynman–Kac Formula (2) [2,3]:

$$u(x_t,t) = \mathbb{E}_{x,t}[u_T(x)]. \tag{25}$$

Theorem 6. *The stochastic representation of the Cauchy problem (1) under a generalized fractional Brownian Motion, B_t^H, with $H \in (\frac{1}{2}, 1)$, under the assumptions above, follows*

$$u(x_t,t) = \mathbb{E}_{x,t}\left[u_T(x) - \int_t^T \frac{\partial^2 u(x_t,t)}{\partial x_t^2}\left[\int_0^t Hf''(B_v^H)v^{2H-1}dv\right]ds\right], \tag{26}$$

and this simplifies under the conditions in Equation (14) to

$$u(x_t,t) = \mathbb{E}_{x,t}\left[u_T(x) - \int_t^T \frac{\partial^2 u(x_t,t)}{\partial x_t^2}\left[\int_0^t H(2H-1)|t-v|^{2H-2}a(v)dv\right]ds\right], \tag{27}$$

if $x_t \in C^2$ and $\sigma(x_t,s)$ is independent of x_t. Note, for $H = \frac{1}{2}$, we obtain (2).

Proof. Consider $u(x_t,t)$ as solution of the Cauchy problem (1) under a generalized fractional Brownian Motion, B_t^H, with $H \in (\frac{1}{2}, 1)$. Applying Theorem 2 on $u(x,s)$, we obtain

$$\begin{aligned} \int_t^T du(x_s,s)ds = &\int_t^T \left[\frac{\partial u(x_s,s)}{\partial s} + \mathcal{A}u(x_s,s)\right]ds + \int_t^T \sigma(x_s,s)\frac{\partial u(x_s,s)}{\partial x_s}dB_s + \\ &+ \int_t^T \frac{\partial^2 f(x_s,s)}{\partial x_s^2}\left[\int_0^t H(2H-1)|t-v|^{2H-2}a(v)dv\right]ds \end{aligned}$$

After integration and under the assumption that $u(x,t)$ satisfies Equation (22). The time integrals will vanish. Given $x_t \in C^2$ and a deterministic σ, we obtain, after taking the expectation and the property that the stochastic integral vanishes, the stochastic representation as follows:

$$u(x_t,t) = \mathbb{E}_{x,t}\left[u_T(x) - \int_t^T \frac{\partial^2 u(x_t,t)}{\partial x_t^2}\left[\int_0^t H(2H-1)|t-v|^{2H-2}a(v)dv\right]ds\right]. \tag{28}$$

If $H = \frac{1}{2}$, the stochastic representation simplifies to the well-known Feynman–Kac formula $u(x_t,t) = \mathbb{E}_{x,t}[u_T(x)]$. $\square$

Next, we state the Feynman–Kac formula for our Cauchy problem (1), given a sub-fractional Brownian motion.

Theorem 7. *The stochastic representation of the Cauchy problem (1) under a sub-fractional Brownian Motion, ξ_t^H, with $H \in (\frac{1}{2},1)$ is*

$$u(x_t,t) = \mathbb{E}_{x,t}\left[u_T(x) - \int_t^T \frac{\partial^2 u(x_t,t)}{\partial x_t^2}\left[\int_0^t H(2-2^{2H-1})f''(\xi_v^H)v^{2H-1}dv\right]ds\right], \tag{29}$$

if $x_t \in C^2$. Note, for $H = \frac{1}{2}$, we obtain the same as in Theorem 6.

The proof follows an equal argument as above in the proof of Theorem 6.

5. Cauchy Problem and Fractional-PDE

Next, we demonstrate the direct linkage for the Cauchy-problem (1) to the fPDE in Equation (4). In step one, we compute the Laplace transform of the right-hand side of the heat equation:

$$\begin{aligned}
\mathfrak{L}[u_t(x,t)] = \mathfrak{L}\left[\frac{\partial u(x,t)}{\partial t}\right] &= \int_0^\infty e^{-st}\frac{\partial u(x,t)}{\partial t}dt \\
&= -u_0(x) + s\bar{u}(x,t) \\
&= s\bar{u}(x,t),
\end{aligned}$$

where $\bar{u}(x,t) := \mathfrak{L}[u(x,t)]$. Thus, we obtain

$$\begin{aligned}
\mathfrak{L}\left[\frac{\partial}{\partial x^2}u(x,t)\right] &= s\bar{u}(x,t) \\
\frac{\partial}{\partial x^2}\mathfrak{L}[u(x,t)] &= s\bar{u}(x,t) \\
\frac{\partial}{\partial x^2}\bar{u}(x,t) &= s\bar{u}(x,t).
\end{aligned}$$

This is a second-order ordinary differential equation in the $x-$variable. The solution is $\bar{u}(x,t) = c * e^{-\sqrt{s}x}$ for some constant c. Determining the constant by the second-derivative $\bar{u}_{xx} = c * se^{-\sqrt{s}x}$ shows that $c = 1$. In step two, we compute the first-derivative of the solution

$$\begin{aligned}
\frac{\partial}{\partial x}\bar{u}(x,t) &= -\sqrt{s}e^{-\sqrt{s}x} \\
\frac{\partial}{\partial x}\bar{u}(x,t) &= -\sqrt{s}\bar{u}(x,t).
\end{aligned}$$

This is a first-order partial differential equation of the Laplace-transform $\bar{u}(x,t)$. Finally, compute the inverse Laplace transform and obtain the fPDE in Equation (4) by

$$\frac{\partial}{\partial x}u(x,t) = -\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t). \tag{30}$$

Indeed, the inverse Laplace transform of the semi-derivative on the right-hand side is as follows:

$$-\mathfrak{L}\left[\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t)\right] = u_0(x) - s^{\frac{1}{2}}\bar{u}(x,t) = -s^{\frac{1}{2}}\bar{u}(x,t) = -\sqrt{s}\bar{u}(x,t).$$

From the fractional representation of the Cauchy problem (1), we find the following fractional derivatives and integrals in relation to the normal distribution:

Proposition 3. *Consider that the solution of the Cauchy problem (1) is of $u(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$, which represents the normal probability density function $N'(x)$ for a constant t. Thus, the solution of the fPDE (4) implies the following fractional derivative and integral:*

(a) $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}u(x,t) = D_t^{\frac{1}{2}}u(x,t) = \frac{1}{\sqrt{2\pi t}}\frac{x}{t}e^{-\frac{x^2}{2t}}$.

(b) *For $\alpha = \frac{1}{2}$, we find $I^{\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}u(x,t)dt = N'(x)$, where $N'(x)$ is the density of the normal probability distribution in regard to x, or $N'(x) = n(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$.*

Proof. Part (a): given $u(x,t)$, it follows from Equation (30) that the semi-derivative with respect to time t is equal to $\frac{\partial}{\partial x}u(x,t)$. Computing the partial derivative of $u(x,t)$ with respect to x is $u_x(x,t) = \frac{\partial u(x,t)}{\partial x} = \frac{1}{\sqrt{2\pi t}}\frac{x}{t}e^{-\frac{x^2}{2t}}$.

Part (b): In order to explicitly evaluate the fractional derivative, we utilize the linearity of both operators. Using operator calculus, we see that

$$D_t^{\frac{1}{2}}u(x,t) = D_t^1 D_t^{-\frac{1}{2}}u(x,t) = D_t^1 I_t^{\frac{1}{2}}u(x,t).$$

Thus, the first-derivative of the semi-integral of $I_t^{\frac{1}{2}}u(x,t)$ with respect to t must be equal to $u_x(x,t)$. Hence, the semi-integral

$$I_t^{\frac{1}{2}}u(x,t) = \frac{1}{\Gamma(\frac{1}{2})}\int_{-\infty}^{x}(x-t)^{\alpha-1}u(x,t)dt = N'(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}},$$

consequently, the first-derivative of $N'(x)$ is of $\frac{dN'(x)}{dx} = N''(x) = \frac{1}{\sqrt{2\pi t}}\frac{x}{t}e^{-\frac{x^2}{2t}}$. The final term solves the fPDE in Equation (30). Thus, the fractional integral for $\alpha = \frac{1}{2}$ must be equal to the probability density function $N'(x)$ in order to satisfy the fPDE in Equation (30). □

6. Conclusions

This article studies the relationships of the Cauchy problem (1) and relates them to fractional partial-differential equations, as well as to the stochastic representations by the Feynman–Kac formula with a generalized fractional and sub-fractional Brownian motion with Hurst parameter $H > 1/2$. In addition, we find fractional derivatives and integrals in relation to the Gaussian probability function by utilizing the novel insight into the linkage of the Cauchy problem and fPDE. This vantage point is of importance in probability theory, fractional calculus and stochastic theory. In future research, we intend to extend our theorems to Hurst parameters $H < 1/2$ and the stochastic Cauchy problem under a sub-fBM.

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