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Algorithm for Two Generalized Nonexpansive Mappings in Uniformly Convex Spaces

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Abstract: In this paper, we study the common fixed-point problem for a pair of García-Falset mapping and (α, β)-generalized hybrid mapping in uniformly convex Banach spaces. For this purpose, we construct a modified three-step iteration by properly including together these two types of mappings into its formula. Under this modified iteration, a necessary and sufficient condition for the existence of a common fixed point as well as weak and strong convergence outcomes are phrased under some additional conditions.

Keywords: García-Falset mapping; condition (E); Suzuki mapping; condition (C); (α, β)-generalized hybrid mappings; common fixed point; iteration

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1. Introduction

Speaking of fixed-point theory, there is no doubt that so far it has been proven to be a rich and complex field, always generating various extensions and applicative results. Even today, it seems that this domain is far away from reaching an end when it comes to giving birth to new ideas or connecting the existing ones; for instance, in [1], new classes of mappings were analyzed on modular vector spaces, or in [2], an iteration process was extended in the same framework by iterating a modular class of nonexpansive mappings. Overall, the important contributions related to fixed-point theory can be summarized by the following three directions: the generalization of the working metric setting, the definition of more and more general classes of contractive operators and the elaboration of new iterative processes.

With regard to generalized classes of contractive operators, an intensively studied class of mappings that exceeds that of contractions is nonexpansive mappings (for some recent results, one might see, for example, [3,4]). Nevertheless, it did not take long for these operators themselves to undergo various generalizations. An important step in this direction was made by Suzuki [5] who introduced condition (C) on Banach spaces by imposing limitations regarding the pairs of elements that satisfy nonexpansiveness. The success of Suzuki's formulation mainly lies on all those extensions for which it served as a starting point: generalized α-nonexpansive mappings [6] (which includes α-nonexpansive mappings [7]), (α, β)-Suzuki-type generalized nonexpansive mappings [8], Reich–Suzuki type nonexpansive mappings [9], etc. A particular idea of generalizing mapping with property (C) was advanced by García-Falset et al. [10] who defined the concept of operators with property (E). This class of mappings seems to include most Suzuki-type applications and generated active research with respect to the idea of being surpassed by a wider condition (see, for example, Ref. [1] about condition (CDE), which is equivalent with condition (E) on Banach spaces). An adjacent direction that went into exceeding nonexpansiveness on Hilbert spaces was the definition of nonspreading mappings by

Kohsaka and Takahashi [11] (more precisely, studied in connection with firmly type nonexpansive mappings). In turn, this class was covered by hybrid mappings introduced by Takahashi [12] only for all nonexpansive, nonspreading and hybrid mappings to be later included by Kocourek et al. [13] into the class of (α, β)-generalized hybrid mappings. However, all these classes of operators are stronger than quasinonexpansive ones whenever a fixed point exists.

In a parallel direction, all these generalizations required more elaborated iterative processes to be designed as a consequence of the limitations caused by Picard iteration with respect to fixed-point approximations. Starting with the classical ones, such as Mann [14] or Ishikawa [15], we mention here the TTP iteration [16]—studied under nonexpansive mappings, Suzuki mappings [17], García-Falset mappings [18], etc.—S_n-iteration [19] initially developed for Berinde-type contractive mappings, and MCS iteration [20], constructed, again, for García-Falset mappings or $\mathcal{U}_n$-iteration [21] defined in connection to (α, β)-generalized hybrid mappings. Moreover, these iterative processes proved to be particularly flexible, as they showed their utility not only under the aspect of iterating nonlinear operators for reaching fixed points, but also in reinterpreting these fixed points as a solution for all sorts of problems: split problems [22], convex programming [23], approximating zeros of complex polynomials [21], signal processing [20,21], fractals and Julia sets [24–26].

Having this inspiring background, in this paper, we aim to extend the problem of fixed-point searches by developing a study regarding the approximation of the common fixed point for a pair of two distinct mappings. We chose to form the pair of operators by joining García-Falset and (α, β)-generalized hybrid mappings. The iterative procedure we use is an adapted version of $\mathcal{U}_n$-iteration made by intertwining both the García-Falset mapping and (α, β)-hybrid mapping into an iterative formula. By setting the general framework to a uniformly convex Banach space, we first phrase a necessary and sufficient condition for the existence of a common fixed point for the pair of mappings, under modified-$\mathcal{U}_n$ iteration. Afterwards, we give sufficient condition for weak and strong convergence of modified-$\mathcal{U}_n$ iteration to a common fixed point of a pair of García-Falset and (α, β)-generalized hybrid mappings.

2. Preliminaries

Let us begin by presenting some necessary notions and results that will be needed throughout this paper.

Definition 1 ([27])**.** *A normed vector space X is called uniformly convex if for each $\varepsilon \in (0, 2]$ there is $\delta > 0$ such that for $x, y \in X$, $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| \ge \varepsilon$ imply $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$.*

Recall that for C being a nonempty subset of a Banach space X, a mapping $T\colon C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. Moreover, if $F(T) \neq \emptyset$ ($F(T)$ denotes the set of fixed points of a mapping T) and $\|Tx - p\| \le \|x - p\|$, for all $x \in C$ and $p \in F(T)$, T is called quasinonexpansive. It is well known that if C is a nonempty, closed and convex subset of a Banach space X and $T\colon C \to C$ is quasinonexpansive, then $F(T)$ is closed and convex.

The following concepts refer to generalized classes of nonexpansive mappings on Banach spaces. We begin with condition (C) of Suzuki [5]:

Definition 2 ([5])**.** *Let C be a nonempty subset of a Banach space X and let T be a selfmap on C. Then T is said to satisfy condition (C) if*

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \quad \textit{implies} \quad \|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$.

Pant and Shukla [6] introduced the class of generalized α-nonexpansive mappings defined as follows:

Definition 3 ([6])**.** *Let C be a nonempty subset of a Banach space X. A mapping $T\colon C \to C$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0,1)$ such that*

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \textit{implies} \quad \|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|, \tag{1}$$

for all $x, y \in C$.

Obviously, when $\alpha = 0$, a generalized α-nonexpansive mapping reduces to a mapping satisfying condition (C), so the class of generalized α-nonexpansive mappings properly contains the class of Suzuki-type nonexpansive mappings.

We describe next the concept of generalized nonexpansive mappings which is due to García-Falset et al. [10], along with some properties of mappings satisfying condition (E).

Definition 4 ([10])**.** *Let C be a nonempty subset of a Banach space X and let $\mu \geq 1$. A mapping $T\colon C \to X$ is said to satisfy condition (E_μ) whenever the inequality*

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|, \tag{2}$$

holds true, for all $x, y \in C$. Moreover, we say that T satisfies condition (E) on C whenever T satisfies condition (E_μ) for a $\mu \geq 1$.

Proposition 1 ([10])**.** *Let $T\colon C \to X$ be a mapping which satisfies condition (E) on C. If T has some fixed point, then T is quasinonexpansive.*

By Lemma 5.2 in [6], if $T\colon C \to C$ is a generalized α-nonexpansive mapping, then it satisfies condition (E) on C; see [6] for a proof. Therefore, the class of generalized α-nonexpansive mappings is subordinated to the class of mappings satisfying condition (E). Some important examples provided in the next section will illustrate the more general nature of García-Falset mappings compared to generalized α-nonexpansive mappings.

The next property on mappings satisfying condition (E) requires the setting of a Banach space endowed with Opial's property [28].

Definition 5 ([28])**.** *A Banach space X is said to satisfy the Opial property if for each weakly convergent sequence $\{x_n\}$ in X with a weak limit x, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for all $y \in X$ with $y \neq x$.

Lemma 1 ([10])**.** *Let $T\colon C \to X$ be a mapping on a subset C of a Banach space X with the Opial property. Assume that T satisfies condition (E). If $\{x_n\}$ converges weakly to some $z \in C$ and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

We move further to the concept of (α, β)-generalized hybrid self mappings introduced by Kocourek et al. [13]. This class of operators is wider than the classes of nonexpansive mappings, nonspreading mappings [11] and hybrid mappings [12] in a Hilbert space, but remains stronger than quasinonexpansiveness. Although in [13], the study of (α, β)-generalized hybrid mappings was developed on Hilbert spaces, in this paper, we will extend the concept to Banach spaces.

Definition 6 ([13])**.** *Let X be a Banach space and let C be a nonempty closed convex subset of X. Then, a mapping $T\colon C \to C$ is called (α, β)-generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \textit{for all } x, y \in C. \tag{3}$$

Obviously, when $(\alpha, \beta) = (1, 0)$ in condition (3), T is nonexpansive. Moreover, a $(2, 1)$-hybrid mapping is nonspreading, and for the pair of parameters $(\alpha, \beta) = \left(\frac{3}{2}, \frac{1}{2}\right)$, we obtain the class of hybrid mappings.

The following feature regarding the class of (α, β)-generalized hybrid mappings refers to the property of demiclosedness. An outcome concerning the demiclosedness of (α, β)-hybrid mappings was originally proved using the setting of a Hilbert space in [13], Lemma 5.1. Nevertheless, this conclusion does not change at all when extending it to a Banach space that, in addition, is endowed with Opial's property. We shall present the proof here in order to make the exposition self-contained.

Lemma 2. *Let $T\colon C \to C$ be an (α, β)-generalized hybrid mapping on a subset C of a Banach space X with the Opial property. If $\{x_n\}$ converges weakly to some $z \in C$ and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Proof. We prove this statement by reductio ad absurdum. In this respect, suppose $z = Tz$. Since T is (α, β)-hybrid, let us start with the corresponding condition (3) which further can be written as

$$\begin{aligned} \alpha(\|Tx_n - Tz\| - \|x_n - Tz\|)(\|Tx_n - Tz\| + \|x_n - Tz\|) + \|x_n - Tz\|^2 \\ \leq \beta(\|Tx_n - z\| - \|x_n - z\|)(\|Tx_n - z\| + \|x_n - z\|) + \|x_n - z\|^2. \end{aligned} \tag{4}$$

Using the properties of the norm, we have

$$-\|x_n - Tx_n\| \leq \|Tx_n - Tz\| - \|x_n - Tz\| \leq \|x_n - Tx_n\|$$

and also

$$-\|x_n - Tx_n\| \leq \|Tx_n - z\| - \|x_n - z\| \leq \|x_n - Tx_n\|.$$

Keeping in mind that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, by taking the limit of the above two inequalities, we obtain

$$\lim_{n\to\infty}(\|Tx_n - Tz\| - \|x_n - Tz\|) = \lim_{n\to\infty}(\|Tx_n - z\| - \|x_n - z\|) = 0. \tag{5}$$

Before turning back to inequality (4), we shall point out two aspects regarding the boundedness of some terms. One one side, since X has the Opial's property, we have $\liminf_{n\to\infty}\|x_n - z\| < \infty$. Moreover,

$$0 \leq \|Tx_n - z\| + \|x_n - z\| \leq \|Tx_n - x_n\| + 2\|x_n - z\|,$$

therefore

$$\liminf_{n\to\infty}(\|Tx_n - z\| + \|x_n - z\|) < \infty. \tag{6}$$

On the other side, following a similar idea as above, we obtain

$$0 \leq \|Tx_n - Tz\| + \|x_n - Tz\| \leq \|Tx_n - x_n\| + 2\|x_n - z\| + 2\|z - Tz\|,$$

yielding

$$\liminf_{n\to\infty}(\|Tx_n - Tz\| + \|x_n - Tz\|) < \infty. \tag{7}$$

Now, applying $\liminf_{n\to\infty}$ to (4) and using the results provided by (5)–(7), we have

$$\liminf_{n\to\infty}\|x_n - Tz\|^2 \leq \liminf_{n\to\infty}\|x_n - z\|^2,$$

which obviously contradicts Opial's property. Therefore, $z = Tz$, implying that $I - T$ is demiclosed at zero. □

The following condition on a pair of operators (S, T) was defined by Fukhar-ud-din and Kahn [29]. It is worth mentioning that if $S = T$, the condition reduces to property (I) of Senter and Dotson [30].

Definition 7 ([29])**.** *Let C be a subset of normed space X. Two mappings $S, T\colon C \to C$ are said to satisy condition (A') if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, such that either $\|x - Sx\| \geq f(d(x,F))$ or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$.*

Definition 8 ([31])**.** *Let C be a nonempty subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, let*

$$r(x, \{x_n\}) = \limsup_{n\to\infty}\|x_n - x\|$$

denote the asymptotic radius of $\{x_n\}$ at x. The asymptotic radius of $\{x_n\}$ relative to C is the real number

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

and the asymptotic center of $\{x_n\}$ with respect to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

This definition is due to Edelstein [31] who also proved that for a nonempty, closed and convex subset of a uniformly convex Banach space and for each bounded sequence $\{x_n\}$, the set $A(C, \{x_n\})$ is a singleton.

Not least, we recall below a lemma which will be instrumental in the development of our outcomes.

Lemma 3 ([32])**.** *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$ and $\limsup_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

We now have sufficient preliminary results, so we are ready to introduce our main outcomes.

3. Examples

Before moving on to the main results section, we give below some examples of operators belonging to the previously mentioned classes of mappings. The purpose is to emphasize the relationships that can be established between these classes of operators.

We begin by illustrating that, indeed, García-Falset's condition holds as a generalization of multiple classes of Suzuki-type mappings. Due to their successful formulation, we chose generalized α-nonexpansive mappings as the comparison class. In this regard, we consider first a mapping on $\mathbb{R}$ endowed with the usual metric.

Example 1. *Let $C = [0,1]$ be endowed with the usual metric on $\mathbb{R}$, and consider the mapping*

$$T\colon C \to C, \quad Tx = \begin{cases} \dfrac{x}{2}, & x \in [0,1) \\ \dfrac{4}{5}, & x = 1 \end{cases}$$

Our aim is to prove that T is not a generalized α-nonexpansive mapping but it satisfies condition (E) of García-Falset.

Proof. We first prove that T does not satisfy the generalized α-nonexpansive condition provided by relation (1). Indeed, if we choose $x = \dfrac{4}{5}$ and $y = 1$, by direct computation, we obtain

$$\frac{1}{2}|x - Tx| = \frac{1}{5} = |x - y|,$$

while

$$\alpha|Tx - y| + \alpha|Ty - x| + (1 - 2\alpha)|x - y| = \frac{1}{5}(\alpha + 1) < \frac{2}{5} = |Tx - Ty|$$

so the required implication does not hold. On the other side, if we choose the admissible parameter $\mu = 4$, the mapping will prove to have condition (E). In this respect, we shall analyze the following cases:

Case 1: Let $x, y \in [0,1]$, such that $|x - Tx| = \dfrac{1}{2}x$. Thus, it follows

$$|x - Ty| = \left|x - \frac{y}{2}\right| \le x + \frac{1}{2}y \le \frac{3}{2} + \frac{1}{2}|x - y| \le 3|x - Tx| + |x - y|,$$

so, for this case, T satisfies condition (E_3).

Case 2: Let $x = 1$ and $y \in [0,1)$, which leads to $|x - Tx| = \dfrac{1}{5}$. Evaluating condition (E) for this case, we find

$$|x - Ty| = 1 - \frac{y}{2} \le \mu\frac{1}{5} + 1 - y,$$

or, equivalently, $y \le \mu\dfrac{2}{5}$, so here, T has (E_3)-property also.

Case 3: Let $x \in [0,1]$ and $y = 1$ for which condition (E) becomes

$$|x - Ty| = \left|x - \frac{4}{5}\right| \le \mu\frac{x}{2} + 1 - x.$$

If $x \ge \dfrac{4}{5}$, then the above inequality can be equivalently written as $2x \le \dfrac{x}{2}\mu + \dfrac{9}{5}$, so T has the (E_4)-property. If, however, $x \le \dfrac{4}{5}$, it is easy to notice that T has the (E_1)-property.

Taking the maximum value of parameter μ, we conclude that, indeed, T has the (E_4)-property, so overall, García-Falset's condition is satisfied. □

For the same purpose of illustrating that condition (E) is wider than generalized α-nonexpansiveness, we are moving toward an infinite dimensional space and define a García-Falset mapping on the space of essentially Lebesgue measurable functions.

Example 2. *Consider the Banach space $X = L^\infty(\mathbb{R})$ of all essentially bounded Lebesgue measurable functions, endowed with the essential supremum norm*

$$\|f\|_\infty = \operatorname{ess\,sup}|f|_{\mathbb{R}} = \inf\{M > 0 \,:\, |f(x)| \le M \text{ a.e. on } \mathbb{R}\}.$$

Let $C = \{f\colon \mathbb{R} \to [0,1] \,:\, f(x) = f(0), \forall x \leq 0 \text{ a.e.}\}$ *and define*

$$T\colon C \to C, \quad Tf(x) = \begin{cases} f(x), & x > 0 \\ \frac{1}{3}f(0), & x \leq 0, f(0) \neq 1 \\ \frac{2}{3}, & x \leq 0, f(0) = 1. \end{cases}$$

Again, we aim to prove that T *is not generalized* α*-nonexpansive but it satisfies condition* (E) *of García-Falset.*

Proof. We have

$$\|f - Tg\|_\infty = \max\{|f(0) - Tg(0)|, \operatorname{ess\,sup}_{(0,\infty)}|f(x) - Tg(x)|\},$$

$$\|f - Tf\|_\infty = |f(0) - Tf(0)|,$$

and

$$\|f - g\|_\infty = \max\{|f(0) - g(0)|, \operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)|\}.$$

Substituting them in inequality (2), condition (E) becomes

$$\begin{aligned} &\max\{|f(0) - Tg(0)|, \operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)|\} \\ &\quad \leq \mu|f(0) - Tf(0)| + \max\{|f(0) - g(0)|, \operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)|\}. \end{aligned} \tag{8}$$

We shall split the analysis into four cases, as follows:
Case 1: Suppose

$$|f(0) - Tg(0)| \geq \operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)|$$

and also

$$|f(0) - g(0)| \geq \operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)|.$$

Thus, condition (E) reduces to

$$|f(0) - Tg(0)| \leq \mu|f(0) - Tf(0)| + |f(0) - g(0)|. \tag{9}$$

(a) If $f(0) \in [0,1)$ and $g(0) \in [0,1)$, then $|f(0) - Tf(0)| = \frac{2}{3}f(0)$ and

$$|f(0) - Tf(0)| = \left|f(0) - \frac{1}{3}g(0)\right| \leq \frac{4}{3}f(0) + |f(0) - g(0)| = 2|f(0) - Tf(0)| + |f(0) - g(0)|,$$

so T has the (E_2)-property.

(b) If $f(0) = 1$ and $g(0) \in [0,1]$, then $|f(0) - Tf(0)| = \frac{1}{3}$ and $|f(0) - Tg(0)| = 1\frac{1}{3}g(0)$, which, considering relation (9), yields $g(0) \leq \frac{1}{2}\mu$, so T satisfies condition (E_2).

(c) Finally, if $f(0) \in [0,1)$ and $g(0) = 1$, then $|f(0) - Tf(0)| = \frac{2}{3}f(0)$ and $|f(0) - Tg(0)| = \left|f(0) - \frac{2}{3}\right|$. It is easy to check that, for $f(0) \geq \frac{2}{3}$, relation (9) implies $f(0) \leq \frac{1}{3}\mu + \frac{5}{6}$, so T satisfies condition (E_3), while for $x \leq \frac{2}{3}$ it satisfies condition (E_1). Therefore, for **Case 1**, condition (E) is fulfilled.

Case 2: Assume now that

$$\operatorname{ess\,sup}_{(0,\infty)}|f(x) - g(x)| \geq |f(0) - Tg(0)|$$

and also that

$$\operatorname{ess\,sup}_{0,\infty}|f(x)-g(x)| \geq |f(0)-g(0)|.$$

Thus, inequality (8) is further written as

$$\operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)| \leq \mu|f(0)-Tf(0)| + \operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)|$$

which is obviously true for any $\mu \geq 1$.

Case 3: Assume next that $|f(0)-g(0)| \leq \operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)| \leq |f(0)-Tg(0)|$. It is actually easy to notice that this case reduces to **Case 1** as previously analyzed, so the desired result follows.

Case 4: Finally, suppose $|f(0)-Tg(0)| \leq \operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)| \leq |f(0)-g(0)|$. This case is in fact included in **Case 2**, so condition (E) is again satisfied for any $\mu \geq 1$.

All four previously analyzed cases allow us to state that inequality (4) is overall satisfied, and T is indeed a García-Falset mapping.

In order to prove that T is not generalized α-nonexpansive, let us consider **Case 1**, with the additional assumptions that $|Tf(0)-Tg(0)| \geq \operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)|$ and $|Tf(0)-g(0)| \geq \operatorname{ess\,sup}_{(0,\infty)}|f(x)-g(x)|$. If we choose $f(0) = \frac{2}{3}$ and $g(0) = 1$, then it immediately follows that

$$\frac{1}{2}|f(0)-Tf(0)| = \frac{2}{9} < \frac{1}{3} = |f(0)-g(0)|.$$

Turning toward the right side of (1), we find

$$\alpha|Tf(0)-g(0)| + \alpha|Tg(0)-f(0)| + (1-2\alpha)|f(0)-g(0)| = \frac{1}{9}\alpha + \frac{1}{3} < \frac{4}{9} = |Tf(0)-Tg(0)|,$$

so the implications fails to be satisfied, which leads to the conclusion that T is not generalized α-nonexpansive. □

We prove next that there exist mappings which satisfy condition (E) on a subset C but fail to be (α,β) hybrids. In this respect, we analyze the following two examples.

Example 3. *Consider $X = \mathbb{R}^3$ with the usual Euclidean norm and let*

$$C = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}.$$

We shall consider the mapping

$$T\colon C \to C, \quad T\colon \begin{pmatrix} (0,0,0), & (1,0,0), & (0,1,0), & (0,0,1) \\ (1,0,0), & (0,0,0), & (0,0,1), & (0,1,0) \end{pmatrix}.$$

Then T satisfies condition (E) of García-Falset but is not (α,β) hybrid.

Proof. Clearly, $\|x-Ty\| \leq \sqrt{2}$ and $\|x-Tx\| \neq 0$, for all $x,y \in C$, so T is a García-Falset mapping with $\mu = \sqrt{2}$. In order to prove that T is not (α,β) hybrid, we shall take $x = (0,0,1)$ and $y = (0,0,0)$. It follows that $\|Tx-Ty\|^2 = 2$, $\|Tx-y\|^2 = 1$, $\|Ty-x\|^2 = 2$ and $\|x-y\|^2 = 1$. Evaluating inequality (3) for these values obviously leads to a contradiction, thus T is not (α,β) hybrid. □

Example 4. *Let $C = [0,1]^2$ endowed with the 1-norm on $\mathbb{R}^2$ and consider the mapping*

$$T\colon C \to C, \quad Tf(x) = \begin{cases} \left(\frac{x_1}{2}, \frac{2}{3}x_2\right), & (x_1,x_2) \in [0,1) \times [0,1] \\ \left(\frac{4}{5}, \frac{2}{3}x_2\right), & (x_1,x_2) \in \{1\} \times [0,1] \end{cases}$$

We shall next verify that T satisfies condition (E) on C but it is not a (α,β)-generalized hybrid mapping.

Proof. We begin by showing that (α,β)-generalized hybrid condition (3) fails to be satisfied for the pair $x = \left(1, \frac{2}{3}\right)$ and $y = \left(\frac{4}{5}, \frac{9}{10}\right)$. Indeed, for this pair of points, the left side of (3) is equal to $\frac{9}{25}$, while the right side equals $\frac{1}{4}$ under the 1-norm, which yields a contradiction.

To prove that T is a García-Falset mapping, we shall analyze the following combinations.

Case 1: Let $x, y \in [0,1) \times [0,1]$. We have

$$\begin{aligned}
\|x - Ty\| &= \left\|(x_1, x_2) - \left(\frac{1}{2}y_1, \frac{2}{3}y_2\right)\right\| \\
&= \left|x_1 - \frac{1}{2}y_1\right| + \left|x_2 - \frac{2}{3}y_2\right| \\
&\leq \frac{1}{2}|x_1 - y_1| + \frac{2}{3}|x_2 - y_2| \\
&\leq \frac{1}{2}x_1 + \frac{1}{3}x_2 + |x_1 - y_1| + |x_2 - y_2| \\
&= \|x - Tx\| + \|x - y\|,
\end{aligned}$$

so T has the (E_1)-property.

Case 2: Let $x, y \in \{1\} \times [0,1]$. It follows

$$\begin{aligned}
\|x - Ty\| &= \left\|(x_1, x_2) - \left(\frac{4}{5}, \frac{2}{3}y_2\right)\right\| \\
&= \left|1 - \frac{4}{5}\right| + \left|x_2 - \frac{2}{3}x_2\right| \\
&\leq \frac{1}{5} + \frac{1}{3}x_2 + \frac{2}{3}|x_2 - y_2| \\
&\leq \frac{1}{5} + \frac{1}{3}x_2 + |x_2 - y_2| \\
&= \|x - Tx\| + |x - y|,
\end{aligned}$$

so again, T has the (E_1)-property.

Case 3: Let $x \in \{1\}$ and $y \in [0,1) \times [0,1]$. We obtain

$$\begin{aligned}
\|x - Ty\| &= \left\|(x_1, x_2) - \left(\frac{1}{2}y_1, \frac{2}{3}y_2\right)\right\| \\
&= \left|1 - \frac{1}{2}y_1\right| + \left|x_2 - \frac{2}{3}y_2\right| \\
&\leq \frac{1}{2} + \frac{1}{2}|1 - y_1| + \frac{1}{3}x_2 + \frac{2}{3}|x_2 - y_2| \\
&\leq \frac{5}{2}\left(\frac{1}{5} + \frac{1}{3}x_2\right) + |1 - y_1| + |x_2 - y_2| \\
&= \frac{5}{2}\|x - Tx\| + \|x - y\|,
\end{aligned}$$

and therefore, T has the $(E_{\frac{5}{2}})$-property.

Case 4: Let $x \in [0,1) \times [0,1]$ and $y \in \{1\} \times [0,1]$. Since $\|x - Tx\| = \dfrac{1}{2}x_1 + \dfrac{1}{3}x_2$ and $\|x - Ty\| = \left|x_1 - \dfrac{4}{5}\right| + \left|x_2 - \dfrac{2}{3}y_2\right|$, condition (2) can be written as

$$\left|x_1 - \frac{4}{5}\right| + \left|x_2 - \frac{2}{3}y_2\right| \le \mu\left(\frac{1}{2}x_1 + \frac{1}{3}x_2\right) + |x_1 - 1| + |x_2 - y_2|.$$

Actually, $\left|x_2 - \dfrac{2}{3}\right| \le \dfrac{1}{3}x_2 + \dfrac{2}{3}|x_2 - y_2| \le \dfrac{1}{3}x_2 + |x_2 - y_2|$, so it remains to show that

$$\left|x_1 - \frac{4}{5}\right| \le \mu\frac{1}{2}x_1 + |x_1 - 1|.$$

If $x_1 \ge \dfrac{4}{5}$, then the above inequality reduces to $x_1 \le \mu\dfrac{1}{4}x_1 + \dfrac{2}{5}$, so T satisfies (E_4)-property. Instead, if $x_1 \le \dfrac{4}{5}$, then it can be easily noticed that T has the (E_2)-property.

Considering all these combinations and taking the maximum value of the admissible parameter μ, we conclude that T has the (E_4) property. Consequently, it is a García-Falset mapping. □

Regarding the converse proof (i.e., there exists a (α, β)-generalized hybrid mapping which is not García-Falset), we can state this is true by looking at the patterns of conditions (2) and (3). Indeed, no subordination or equivalence relationship seems possible to be established between the two of them. Nevertheless, we leave it as an open issue to find an example of an operator which belongs to the class of (α, β)-hybrid mappings but does not have condition (E).

4. Convergence Theorems

Next, we provide an iterative algorithm as well as a convergence study regarding this algorithm with respect to the common fixed point of a pair of García-Falset and (α, β)-hybrid mappings. Throughout this section, we shall consider T as a García-Falset mapping and S as a (α, β)-generalized hybrid mapping. We denote by p the common fixed point of T and S, that is, $Tp = p = Sp$. For constructing the iteration procedure, we start from $\mathcal{U}_n$ iteration defined in [21] and studied only in connection with (α, β)-hybrid mappings on uniformly convex Banach spaces.

Algorithm 1 ([21])**.** *Let C be a nonempty convex set and $S\colon C \to C$ be a given operator. For an arbitrary initial point $x_0 \in C$, construct the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n &= (1-\xi_n)x_n + \xi_n Sx_n \\ y_n &= S((1-\zeta_n)Sx_n + \zeta_n Sz_n) \\ x_{n+1} &= (1-\eta_n-\delta_n)Sx_n + \eta_n Sy_n + \delta_n Sz_n, \end{cases} \tag{10}$$

where $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ and $\{\eta_n + \delta_n\}$ are sequences of real numbers in $(0,1)$.

We adapt this algorithm for the problem of approximating common fixed points of a pair of mappings by properly including T into its definition:

Algorithm 2 (**modified-$\mathcal{U}_n$ iteration**)**.** *Let C be a nonempty convex set and $S, T\colon C \to C$ be two given operators. For an arbitrary initial point $x_0 \in C$, construct the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n &= (1-\xi_n)x_n + \xi_n Tx_n \\ y_n &= S((1-\zeta_n)Tx_n + \zeta_n Sz_n) \\ x_{n+1} &= (1-\eta_n-\delta_n)Sx_n + \eta_n Sy_n + \delta_n Sz_n, \end{cases} \tag{11}$$

where $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ and $\{\eta_n + \delta_n\}$ are sequences of real numbers in $(0,1)$. Additionally, we assume further that the parametric sequence $\{\xi_n\}$ satisfies $0 < p \le \xi_n \le q < 1$, and $\{\zeta_n\}$ is a convergent sequence to some $\zeta \in (0,1)$.

Henceforth, we shall call this procedure modified-$\mathcal{U}_n$ iteration. It is worth pointing out that, if $T = S$ (for example, both are the same nonexpansive mapping), then modified-$\mathcal{U}_n$ iteration reduces to $\mathcal{U}_n$ iteration.

We start by exposing a technical lemma concerning modified-$\mathcal{U}_n$ iteration, which will often accompany us in proving the rest of our results.

Lemma 4. *Let C be a nonempty, closed and convex subset of a Banach space X. Let $T\colon C \to C$ be a mapping satisfying condition (E) and $S\colon C \to C$ a (α, β)-generalized hybrid mapping such that $F(T) \cap F(S) \neq \emptyset$. Suppose the sequence $\{x_n\}$ is generated by iteration (11). Then, $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T) \cap F(S)$.*

Proof. Let $p \in F(T) \cap F(S)$. Since T and S have at least one fixed point, they are quasinonexpansive. Therefore, keeping in mind our iteration scheme and the properties of the norm, we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\xi_n)x_n + \xi_n T x_n - p\| \\ &\le (1-\xi_n)\|x_n - p\| + \xi_n \|Tx_n - p\| \\ &\le (1-\xi_n)\|x_n - p\| + \xi_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{12}$$

Proceeding in much the same way for $\{y_n\}$ and keeping in mind relation (12), we obtain

$$\begin{aligned} \|y_n - p\| &= \|S((1-\zeta_n)Tx_n + \zeta_n S z_n) - p\| \\ &\le \|(1-\zeta_n)Tx_n + \zeta_n S z_n - p\| \\ &\le (1-\zeta_n)\|Tx_n - p\| + \zeta_n \|Sz_n - p\| \\ &\le (1-\zeta_n)\|x_n - p\| + \zeta_n \|z_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{13}$$

Lastly, considering both inequality (12) and inequality (13), we find

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\eta_n-\delta_n)Sx_n + \eta_n S y_n + \delta S z_n - p\| \\ &\le (1-\eta_n-\delta_n)\|Sx_n - p\| + \eta_n\|Sy_n - p\| + \delta_n \|Sz_n - p\| \\ &\le (1-\eta_n-\delta_n)\|x_n - p\| + \eta_n\|y_n - p\| + \delta_n \|z_n - p\| \\ &\le (1-\eta_n-\delta_n)\|x_n - p\| + \eta_n\|x_n - p\| + \delta_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{14}$$

This shows that the sequence $\{\|x_n - p\|\}$ is bounded and nonincreasing for any $p \in F(T) \cap F(S)$; thus, $\lim_{n\to\infty}\|x_n - p\|$ exists. □

The following theorem provides a necessary and sufficient condition for T and S to have a common fixed point. For proving this outcome, we recall the following property of lim sup:

Lemma 5. *Let $\{a_n\}$ and $\{b_n\}$ be two bounded real sequences. Then, for $c_n = (1-\alpha_n)a_n + \alpha_n b_n$ with $\alpha_n \in [0,1]$ convergent to a real number $\alpha \in [0,1]$,*

$$\limsup_{n\to\infty} c_n \le (1-\alpha)\limsup_{n\to\infty} a_n + \alpha \limsup_{n\to\infty} b_n.$$

Theorem 1. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T\colon C \to C$ be a mapping satisfying condition (E) and $S\colon C \to C$ be a (α, β)-generalized hybrid mapping. Suppose the sequence $\{x_n\}$ is generated iteratively by the procedure (11). Then, $F(T) \cap F(S) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and, $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.*

Proof. Consider first proving the direct implication. Suppose thus that $F(T) \cap F(S) \neq \emptyset$ and let $p \in F(T) \cap F(S)$. Let us denote

$$r = \lim_{n\to\infty}\|x_n - p\|.$$

We begin with inequality (12), which, by applying $\limsup_{n\to\infty}$ on its both members, implies

$$\limsup_{n\to\infty}\|z_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| = r. \tag{15}$$

Again, since T is quasinonexpansive, we also have

$$\limsup_{n\to\infty}\|Tx_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| = r.$$

As it can be seen in (14),

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \eta_n - \delta_n)\|x_n - p\| + \eta_n\|x_n - p\| + \delta_n\|z_n - p\| \\ &= (1 - \delta_n)\|x_n - p\| + \delta_n\|z_n - p\| \\ &= \|x_n - p\| - \delta_n\|x_n - p\| + \delta_n\|z_n - p\|, \end{aligned} \tag{16}$$

which further gives that

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\delta_n} \leq \|z_n - p\| - \|x_n - p\|$$

or, even more,

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\delta_n} \leq \|z_n - p\| - \|x_n - p\|$$

leading to

$$\|x_{n+1} - p\| \leq \|z_n - p\|.$$

Further, by taking $\liminf_{n\to\infty}$, this yields

$$r \leq \liminf_{n\to\infty}\|z_n - p\| \leq \limsup_{n\to\infty}\|z_n - p\| \leq r$$

so

$$\limsup_{n\to\infty}\|z_n - p\| = r.$$

This last result can also be written as

$$\limsup_{n\to\infty}\|z_n - p\| = \limsup_{n\to\infty}\|(1 - \xi_n)(x_n - p) + \xi_n(Tx_n - p)\| = r.$$

All conditions of Lemma 3 are fulfilled now; thus,

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{17}$$

In order to show that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$, we shall start with the following inequality

$$\|Sz_n - x_n\| \leq \|Sz_n - Tx_n\| + \|Tx_n - x_n\|. \tag{18}$$

Using again the fact that S is quasinonexpansive, we have

$$\|Sz_n - p\| \le \|z_n - p\| \le \|x_n - p\|$$

that implies

$$\limsup_{n\to\infty}\|Sz_n - p\| \le r.$$

Inequality (13) also gives

$$\limsup_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le r.$$

Due to the recurrence of $\{y_n\}$ in our procedure, we can write

$$\begin{aligned}
\limsup_{n\to\infty}\|y_n - p\| &= \limsup_{n\to\infty}\|S((1-\zeta_n)Tx_n + \zeta_n Sz_n) - p\| \\
&\le \limsup_{n\to\infty}\|(1-\zeta_n)Tx_n + \zeta_n Sz_n - p\| \\
&= \limsup_{n\to\infty}\|(1-\zeta_n)(Tx_n - p) + \zeta_n(Sz_n - p)\| \\
&\le (1-\zeta)\limsup_{n\to\infty}\|Tx_n - p\| + \zeta\limsup_{n\to\infty}\|Sz_n - p\| \\
&\le (1-\zeta)\limsup_{n\to\infty}\|x_n - p\| + \zeta\limsup_{n\to\infty}\|z_n - p\| \\
&= r.
\end{aligned} \tag{19}$$

On the other hand, using (14), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1-\eta_n-\delta_n)\|x_n - p\| + \eta_n\|y_n - p\| + \delta_n\|x_n - p\| \\
&= (1-\eta_n)\|x_n - p\| + \eta_n\|y_n - p\| \\
&= \|x_n - p\| - \eta_n\|x_n - p\| + \eta_n\|y_n - p\|,
\end{aligned} \tag{20}$$

which implies

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\eta_n} \le \|y_n - p\| - \|x_n - p\|$$

or, even more,

$$\|x_{n+1} - p\| - \|x_n - p\| \le \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\eta_n} \le \|y_n - p\| - \|x_n - p\|$$

that is

$$\|x_{n+1} - p\| \le \|y_n - p\|.$$

By taking $\liminf_{n\to\infty}$, this implies

$$r \le \liminf_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}\|y_n - p\| \le r$$

so

$$\limsup_{n\to\infty}\|y_n - p\| = r. \tag{21}$$

From relations (19) and (21), we obtain

$$r \le \limsup_{n\to\infty}\|(1-\zeta_n)(Tx_n - p) + \zeta_n(Sz_n - p)\| \le r$$

so

$$\limsup_{n\to\infty}\|(1-\zeta_n)(Tx_n - p) + \zeta_n(Sz_n - p)\| = r.$$

One more time, the conditions of Lemma 3 are accomplished, so

$$\lim_{n\to\infty}\|Tx_n - Sz_n\| = 0. \tag{22}$$

Obviously, from the same arguments we can extract that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0 \tag{23}$$

holds too. Now we can handle inequality (18). Having (17) and (22), and letting $n \to \infty$, it leads to

$$\lim_{n\to\infty}\|Sz_n - x_n\| = 0. \tag{24}$$

Furthermore, by the properties of the norm, we have

$$\|Sz_n - z_n\| \le \|Sz_n - x_n\| + \|z_n - x_n\|,$$

from where, by taking the limit and using (23) and (24), it follows

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0. \tag{25}$$

On the other side, by the properties of the norm, we may clearly obtain

$$-\|Sz_n - z_n\| \le \|Sz_n - Sx_n\| - \|Sx_n - z_n\| \le \|Sz_n - z_n\|.$$

Letting $n \to \infty$ in this last inequality and using relation (25), it follows

$$\lim_{n\to\infty}[\|Sz_n - Sx_n\| - \|Sx_n - z_n\|] = 0. \tag{26}$$

Now, having in mind the assumption that S is a (α, β)-generalized hybrid mapping and the iterative formula of our algorithm, we have

$$\alpha\|Sz_n - Sx_n\|^2 + (1-\alpha)\|z_n - Sx_n\|^2 \le \beta\|Sz_n - x_n\|^2 + (1-\beta)\|z_n - x_n\|^2$$

which can be written more conveniently as

$$\begin{aligned}\alpha(\|Sz_n - Sx_n\| - \|z_n - Sx_n\|)(\|Sz_n - Sx_n\| + \|z_n - Sx_n\|) + \|z_n - Sx_n\|^2 \\ \le \beta\|Sz_n - x_n\|^2 + (1-\beta)\|z_n - x_n\|^2\end{aligned}$$

Taking the limit on both sides of this last inequality and using relation (26) along with (23) and (24), we can eventually conclude that

$$\lim_{n\to\infty}\|z_n - Sx_n\| = 0. \tag{27}$$

At this point, we are able to show that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. Since

$$\|Sx_n - x_n\| \le \|Sx_n - z_n\| + \|x_n - z_n\| \tag{28}$$

by letting $n \to \infty$ and considering (27) together with (23), it follows $\lim\limits_{n\to\infty}\|Sx_n - x_n\| = 0$, and this finishes this part of the proof.

Conversely, suppose $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. A similar proof as for Lemma 2 can clearly lead to

$$\varlimsup_{n\to\infty}\|x_n - Sp\| \le \varlimsup_{n\to\infty}\|x_n - p\|, \tag{29}$$

for all $p \in C$.

On the other hand, from the García-Falset condition (2), letting $x = p \in C$, $y = x_n$ and $n \to \infty$, we find also

$$\lim_{n\to\infty}\|x_n - Tp\| \leq \lim_{n\to\infty}\|x_n - p\|. \tag{30}$$

Assume $p \in A(C, \{x_n\})$. Both from (29) and (30), we find

$$r(Tp, \{x_n\}) = \limsup_{n\to\infty}\|x_n - Tp\| \leq \limsup_{n\to\infty}\|x_n - p\| = r(p, \{x_n\})$$

and

$$r(Sp, \{x_n\}) = \limsup_{n\to\infty}\|x_n - Sp\| \leq \limsup_{n\to\infty}\|x_n - p\| = r(p, \{x_n\}),$$

so, $Tp \in A(C, \{x_n\})$ and $Sp \in A(C, \{x_n\})$. However, from Edelstein's conclusion, the set $A(C, \{x_n\})$ must be a singleton for each bounded sequence $\{x_n\}$, and thus, we obtain $p = Tp = Sp$. □

Considering the previously two results, we are now ready to phrase our weak and strong convergence theorems.

Theorem 2. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X endowed with Opial's property, and let T, S and $\{x_n\}$ be as in Theorem 1. If $F(T) \cap F(S) \neq \emptyset$, then $\{x_n\}$ converges weakly to some $p \in F(T) \cap F(S)$.*

Proof. An obvious consequence of Lemma 4 is that the sequence $\{x_n\}$ is bounded. Therefore, it has a subsequence $\left\{x_{n_j}\right\}$ which converges weakly to an element $p \in X$. Since the set C is convex and closed, it is also weakly closed. Thus, it contains the weak limits of all of its weakly convergent sequences, so $p \in C$. Moreover, Theorem 1 provides $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$, so, according to Lemmas 1 and 2, we actually have $p \in F(T) \cap F(S)$. Further, we will prove that $\{x_n\}$ itself converges weakly to p. In order to do so, let us assume the contrary. Suppose there is another arbitrary subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$, which converges weakly to some element $p' \in X$ such that $p \neq p'$. In the same manner as for p, it follows that $p' \in F(T) \cap F(S)$ also. Now, keeping in mind that the sequence $\{\|x_n - p\|\}$ is convergent and using Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - p\| &= \liminf_{i\to\infty}\|x_{n_i} - p\| < \liminf_{i\to\infty}\left\|x_{n_i} - p'\right\| = \lim_{n\to\infty}\|x_n - p'\| \\
&= \liminf_{j\to\infty}\left\|x_{n_j} - p'\right\| < \liminf_{j\to\infty}\left\|x_{n_j} - p\right\| = \lim_{n\to\infty}\|x_n - p\|,
\end{aligned}$$

which, as expected, leads to a contradiction. Therefore, $p = p'$ and $\{x_n\}$ converges weakly to a common fixed point of T and S considered. □

An interesting corollary of the above theorem can be provided by limiting our setting X to a Hilbert space. Before displaying this result, let us first recall some important properties of the metric projection that will be involved in establishing our outcome.

Let C be a nonempty, closed and convex subset of a Hilbert space X. Then, for each $x \in X$, there is a unique closest point $x^* \in C$ such that

$$\|x - x^*\| = \inf_{y\in C}\|x - y\|.$$

Using this correspondence, we can define a mapping $P_C\colon X \to C$ by $P_C x = x^*$, known as the metric projection of X onto C. The following are characteristics of the projection mapping.

Lemma 6 ([33])**.** *Let C be a nonempty closed convex subset of a Hilbert space X. Then, given $x \in X$ and $q \in C$, $q = P_C x$ if and only if $\langle x - q, q - y\rangle \geq 0$, for all $y \in C$.*

Lemma 7 ([33])**.** *Let X be a Hilbert space and let C be a nonempty closed convex subset of X. Let $\{x_n\}$ be a sequence in X. Suppose that, for all $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \textit{for all } n \in \mathbb{N}.$$

Then, $\{P_C x_n\}$ converges strongly to some $z \in C$.

Corollary 1. *Let C be a nonempty closed convex subset of a Hilbert space X, and let T, S and $\{x_n\}$ be as in Theorem 1, $F(T) \cap F(S) \neq \emptyset$. Suppose $\{x_n\}$ converges weakly to a common fixed point p of T and S. Then, $p = \lim_{n\to\infty} P_{F(T)\cap F(S)} x_n$.*

Proof. Turning back to inequality (14) above, we conclude that $\|x_{n+1} - p\| \leq \|x_n - p\|$, for all $p \in F(T) \cap F(S)$. According to Lemma 7, this implies that the sequence $\left\{P_{F(T)\cap F(S)} x_n\right\}$ converges strongly to an element $z \in F(T) \cap F(S)$.

Further, from Lemma 6, letting $q = P_{F(T)\cap F(S)} x_n$, we have

$$\left\langle x_n - P_{F(T)\cap F(S)} x_n, P_{F(T)\cap F(S)} - y \right\rangle \geq 0,$$

for all $y \in F(T) \cap F(S)$. Now, keeping in mind that $x_n \rightharpoonup p \in F(T) \cap F(S)$ and that $P_{F(T)\cap F(S)} x_n \to z \in F(T) \cap F(S)$, we get

$$\langle p - z, z - y \rangle \geq 0,$$

for all $y \in F(T) \cap F(S)$. Substituting y with p above, it follows that $z = p$, thus, $\lim_{n\to\infty} P_{F(T)\cap F(S)} x_n = p$, completing the proof. □

The next theorem is a strong convergence outcome with respect to a subset C, which is additionally compact.

Theorem 3. *Let C be a nonempty, compact and convex subset of a uniformly convex Banach space X and let T, S and $\{x_n\}$ be as in Theorem 1. If $F(T) \cap F(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Assume that $F(T) \cap F(S) \neq \emptyset$. Then, Theorem 1 provides $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. Keeping in mind that the subset C is compact, the sequence $\{x_n\}$ must have a subsequence $\{x_{n_j}\}$ that converges to a point $p \in C$. However, form (29) and (30),

$$\lim_{n\to\infty}\left\|Tx_{n_j} - x_n\right\| \leq \lim_{n\to\infty}\left\|x_{n_j} - p\right\|$$

and

$$\lim_{n\to\infty}\left\|Sx_{n_j} - x_n\right\| \leq \lim_{n\to\infty}\left\|x_{n_j} - p\right\|,$$

and therefore, $\{x_{n_j}\}$ converges to Tp. By the uniqueness of the limit, we have $p = Tp$ and $p = Sp$, so $p = Tp = Sp$. By Lemma 4, $\lim_{n\to\infty}\|x_n - p\|$ exists, so p is actually the strong limit of $\{x_n\}$. □

Not least, we shall give below our second strong convergence result regarding the modified-$\mathcal{U}_n$ iteration. Essential here is condition (A') of Fukhar-ud-din and Kahn [29].

Theorem 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T\colon C \to C$ be a mapping satisfying condition (E) and $S\colon C \to C$ a (α, β)-generalized hybrid mapping such that $F(T) \cap F(S) \neq \emptyset$. Suppose that T and S satisfy condition (A'). Then the sequence $\{x_n\}$ generated by (11) converges strongly to an element $p \in F(T) \cap F(S)$.*

Proof. Let us first denote $F = F(T) \cap F(S)$. By Lemma 4 and Theorem 1, we have already obtained that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$ and also that

$\lim_{n\to\infty}\|x_n - p\| = r$ exists for any $p \in F$. Therefore, $\lim_{n\to\infty} d(x_n, F)$ exists too. If $r = 0$, then the desired result follows. Suppose it is the case when $r > 0$. According to the assumptions on T and S, we have either

$$f(d(x_n, F)) \le \|Tx_n - x_n\|$$

or

$$f(d(x_n, F)) \le \|Sx_n - x_n\|.$$

Taking the limit in both cases, we find $\lim_{n\to\infty} f(d(x_n, F))$=0. By considering the properties of function f provided by Definition 7, we can deduce that $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence on C. Knowing that $\lim_{n\to\infty} d(x_n, F) = 0$ allows us to assert that there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \ge n_\varepsilon$, we have $d(x_n, F) \le \frac{\varepsilon}{2}$, for any $\varepsilon > 0$.

For $m, n \ge n_\varepsilon$ and $p \in F$, we have

$$\|x_n - x_m\| \le \|x_n - p\| + \|x_m - p\|.$$

Keeping in mind that the sequence $\{\|x_n - p\|\}$ is nonincreasing, this leads to

$$\|x_n - x_m\| \le 2\inf_{p\in F}\|x_{n_\varepsilon} - y\| = 2d(x_{n_\varepsilon}, F) \le \varepsilon,$$

which yields that, indeed, $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of X, it follows that $\{x_n\}$ converges to a point p in C. However, from $\lim_{n\to\infty} d(x_n, F) = 0$, we have $d(p, F) = 0$. Since T and S have at least one fixed point, they are quasinonexpansive, and thus, $F \in C$ is closed. Finally, this implies that $\{x_n\}$ converges strongly to $p \in F$, and the proof is complete. $\square$

5. Conclusions

In this paper, our purpose was to extend the classic approach of fixed-point searches from [21] by taking the $\mathcal{U}_n$-iteration and modifying the process by properly mixing a pair of two distinct types of operators into its structure. In other words, we brought together both García-Falset mappings and (α, β)-generalized hybrid mappings under the same iteration procedure. Under the resulted iteration process, we proved the existence of a common fixed point for a pair of García-Falset and (α, β)-generalized hybrid mappings. In the end, we proved several weak and strong convergence results to the common fixed point for the sequence of approximations generated by modified-$\mathcal{U}_n$ iteration. We underline that the present subject opens new research perspectives, such as the development of an associated ergodic theory.

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