mathematics

MDPI

Article

# On the Existence of Super Efficient Solutions and Optimality Conditions for Set-Valued Vector Optimization Problems

**Lu-Chuan Ceng [1], Ching-Feng Wen [2,3] and Yeong-Cheng Liou [3,4,*]**

1 Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; zenglc@hotmail.com
2 Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan; cfwen@kmu.edu.tw
3 Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 80708, Taiwan
4 Department of Healthcare Administration and Medical Informatics and Research Center of Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 807, Taiwan
* Correspondence: simplex_liou@hotmail.com

**Abstract:** In this paper, by using the normal subdifferential and equilibrium-like function we first obtain some properties for $K$-preinvex set-valued maps. Secondly, in terms of this equilibrium-like function, we establish some sufficient conditions for the existence of super minimal points of a $K$-preinvex set-valued map, that is, super efficient solutions of a set-valued vector optimization problem, and also attain necessity optimality terms for a general type of super efficiency.

**Keywords:** $K$-preinvex multi-valued map; normal subdifferential; equilibrium-like function; set-valued vector optimization problem; super efficient solutions

**MSC:** 26E25; 26B25; 90C29; 49J52



**Citation:** Ceng, L.-C.; Wen, C.-F.; Liou, Y.-C. On the Existence of Super Efficient Solutions and Optimality Conditions for Set-Valued Vector Optimization Problems. *Mathematics* **2022**, *10*, 316. https://doi.org/10.3390/math10030316

Academic Editor: Nikolai A. Kudryashov

Received: 27 December 2021
Accepted: 17 January 2022
Published: 20 January 2022



## 1. Introduction

During the past more than 20 years, extending and characterizing definitions/properties of generalized convexity from the real-valued to the multi-valued mappings had been investigated by many scholars; the readers are referred to Benoist and Popovici [1,2], Jabarootian and Zafarani [3], Oveisiha and Zafarani [4], Sach and Yen [5], Yang [6] and the references cited therein. In particular, Sach and Yen [5] provided certain necessity and sufficiency terms for a multi-valued $F$ to be $K$-convex by using a contingent derivative of the epigraphical multifunction of $F$ w.r.t. an ordering cone $K$. Subsequently, Yang [6] introduced Dini direction derivative for multifunctions and it was used to derive certain properties of $K$-convexity mappings.

Set-valued vector optimization problems received great attention from many authors, who considered and studied them via various kinds of methods and approaches; see, for example, Chen and Jahn [7], Chinaie and Zafarani [8,9], Chinchuluun and Pardalos [10], Durea and Strugariu [11], Floudas and Pardalos [12], Ha [13], Mohan and Neogy [14] and Mordukhovich [15,16]. Super efficiency was first put forth by Borwein and Zhuang [17] in linear normed space and then explored in a few papers: Bao and Mordukhovich [18], Huang [19], Rong and Wu [20], Zaffaroni [21], Zheng et al. [22]. Such a concept extracts and puries the notion of efficiency and other types of proper efficiency. So, Rong and Wu [20] gave certain characterizations of super efficiency by virtue of super duality with cone-convexlike conditions, Lagrange multipliers and scalarizing procedure. Zaffaroni [21] applied various scalarizing functions to present some characterizations of super minimizers and other different solutions of vector optimization issues. Bao and Mordukhovich [18] derived some necessity terms of super efficiency in constrained issues of multiobjective optimization by advanced techniques of variational analysis; also, see [23,24]. In 2013, Oveisiha and Zafarani [25] obtained a characterization of $K$-preinvexity mappings by virtue of the normal subdifferential notion and marginal functions. Meanwhile, two sufficiency conditions for which super minimal points exist were deduced using the $K$-preinvexity hypothesis. Certain necessity optimality conditions for a general type of super efficiency were also obtained. For the concepts of generalized invexity and invariant monotonicity w.r.t. a function, the reader is referred to: Jabarootian and Zafarani [26], Soleimani and Damaneh [27], Weir and Mond [28], Yang et al. [29,30], and the references cited therein. In addition, by virtue of the scalarizing technique, Oveisiha and Zafarani investigated Stampacchia variational-like inequalities by using a normal subdifferential for multifunctions and built their relations with set-valued vector optimization issues. Besides, they attained certain characterizations of the solution sets of pseudoinvexity extremum issues. Subsequently, motivated by Oveisiha and Zafarani [31], Ceng and Latif [32] considered Stampacchia equilibrium-like problems by virtue of a normal subdifferential for multifunctions, established their relations with set-valued vector optimization issues, and attained certain characterizations of a solution set of a set-valued generalized $K$-pseudoinvexity program. Very recently, Atarzadeh et al. [33] considered the nonsmooth composite minimization problem (NCMP) with inequality constraints and obtained some equivalent conditions for the Karush–Kuhn–Tucker (KKT) optimality condition of the NCMP. Atarzadeh et al. [34] studied Fritz John (FJ) and KKT multiplier rules of orders one and two for a set-valued vector optimization problem with inequality constraints and established sufficient conditions for the equivalence between the disjunction and multiplier rules in various cases.

In this paper, inspired by the above research works, we first deduce some properties for $K$-preinvex set-valued maps using their marginal functions, equilibrium-like function and normal subdifferential concept. Secondly, in terms of this equilibrium-like function we establish some sufficient conditions for the existence of super minimal points of a $K$-preinvex set-valued map, that is, super efficient solutions of a set-valued vector optimization problem, and also attain necessity optimality conditions for a general type of super efficiency. The structure of this paper is assigned below: Some concepts and basic tools are contained in Section 2. Certain properties of $K$-preinvexity mappings are obtained by virtue of a normal subdifferential and an equilibrium-like function in Section 3. Certain necessity and sufficiency optimality conditions are established for efficiency and a general type of super efficiency in Section 4.

## 2. Concepts and Basic Tools

Suppose that $X^*$ is the topological dual space of a Banach space $X$. We denote by the same notation $\|\cdot\|$ the norms in $X$ and $X^*$. Let the $\langle\cdot,\cdot\rangle$, $[v,\omega]$ and $(v,\omega)$ represent the duality pairing between $X^*$ and $X$, the line segment for $v,\omega \in X$ and the interior of $[v,\omega]$, respectively. Recall now certain notions of coderivatives and subdifferentials below.

Assume that $X$ is a normed linear space and $\emptyset \neq \Xi \subset X$. Let $\epsilon > 0$ and $v \in \Xi$. Define the set of $\epsilon$-normals to $\Xi$ at $v$ by

$$\widehat{N}_\varepsilon(v;\Xi) := \{v^* \in X^* : \limsup_{u \xrightarrow{\Xi} v} \frac{\langle v^*, u-v\rangle}{\|u-v\|} \leq \varepsilon\},$$

where $u \xrightarrow{\Xi} v$ indicates $u \to v$ along $u \in \Xi$. Whenever $\epsilon = 0$, the above set is written as $\widehat{N}(v;\Xi)$ and known as regular normal cone to $\Xi$ at $v$. In case $\bar{v} \in \Xi$, the basic normal cone to $\Xi$ at $\bar{v}$ is:

$$N(\bar{v};\Xi) := \limsup_{v\to\bar{v},\epsilon\downarrow 0} \widehat{N}_\epsilon(v;\Xi).$$

Suppose that $\phi : X \to \overline{\mathbf{R}}$ takes the finite-valued at $\bar{v} \in X$. The basic (limiting) subdifferential and regular (Fréchet) subdifferential (due to [15]) of $\phi$ at $\bar{v}$ are formulated below

$$\partial\phi(\bar{v}) := \{v^* \in X^* : (v^*, -1) \in N(\bar{v}, \phi(\bar{v}); \text{epi}\phi)\},$$

$$\widehat{\partial}\phi(\bar{v}) := \{v^* \in X^* : (v^*, -1) \in \widehat{N}((\bar{v}, \phi(\bar{v})); \text{epi}\phi)\},$$

respectively. In case $X$ is an Asplund space, that is, each continuous convex function on $X$ is of Fréchet differentiability on a dense set of points, the following holds,

$$\partial\phi(\bar{v}) = \limsup_{v \xrightarrow{\phi} \bar{v}} \widehat{\partial}\phi(v),$$

where $v \xrightarrow{\phi} \bar{v}$ means that $v \to \bar{v}$ with $\phi(v) \to \phi(\bar{v})$. According to [16], we know that the following conclusions hold:

(i) if $X$ is an Asplund space, $\phi_1$ is Lipschitz-continuous around $\bar{v}$ and $\phi_2$ is of l.s.c. property around this point, then $\partial(\phi_1 + \phi_2)(\bar{v}) \subset \partial\phi_1(\bar{v}) + \partial\phi_2(\bar{v})$ for the limiting subdifferentials;

(ii) if $X$ is a Banach space, $\phi_1, \phi_2$ are arbitrary extended-real-valued functions that are finite at $\bar{v}$ and $\widehat{\partial}^+\phi_2(\bar{v}) := -\widehat{\partial}(-\phi_2)(\bar{v})$ is nonempty, then $\widehat{\partial}(\phi_1 + \phi_2)(\bar{v}) \subset \bigcap_{v^* \in \widehat{\partial}^+\phi_2(\bar{v})}[v^* + \widehat{\partial}\phi_1(\bar{v})]$.

It is well-known that mean-value theorems play an important role in nonsmooth analysis. We here recall a mean-value theorem for limiting subdifferentials.

**Theorem 1** ([16]). *Suppose that $X$ is an Asplund space and $\phi$ is Lipschitz continuous on an open set containing $[a, b]$ in $X$. Then, $\exists c \in [a, b)$ and $\exists v^* \in \partial\phi(c)$ s.t. $\langle v^*, b - a\rangle \geq \phi(b) - \phi(a)$.*

Let $\emptyset \neq \Xi \subset X$, and $\eta : \Xi \times \Xi \to X$ be a map. Then $\eta$ is referred to as being skew if $\eta(v, \omega) + \eta(\omega, v) = 0\ \forall v, \omega \in \Xi$. According to [28], the set $\Xi$ is referred to as being invex w.r.t. $\eta$ if $v + t\eta(\omega, v) \in \Xi\ \ \forall v, \omega \in \Xi, t \in [0, 1]$.

Next, we always suppose that $\Xi \subset X$ is an invex subset w.r.t. $\eta : \Xi \times \Xi \to X$. Motivated by Theorem 1, we define a mean-value condition for limiting subdifferential $\partial\phi$ w.r.t. $\psi$.

**Definition 1.** *Suppose that the space $X$ is Asplund one, and $\psi : X^* \times \Xi \times \Xi \to \mathbf{R}$. Given $v, \omega \in \Xi$ and Lipschitz-continuous $\phi : \Xi \to \mathbf{R}$ on some open subset containing $[v, \omega]$. Then $\phi$ is referred to as satisfying mean-value condition for limiting subdifferential $\partial\phi$ w.r.t. $\psi$ iff $\exists u \in [v, \omega)$ and $\exists \xi^* \in \partial\phi(u)$ s.t. $\psi(\xi^*, v, \omega) \geq \phi(\omega) - \phi(v)$.*

Given a multi-valued map $\Gamma : X \to 2^Y$ with $Y$ being partially ordered by a convex and closed cone $K \neq \emptyset$. We denote by "$\leq_K$" the ordering relation on $Y$, that is,

$$v_1 \leq_K v_2 \quad \Leftrightarrow \quad v_2 - v_1 \in K.$$

Define $\mathrm{dom}\Gamma := \{v \in X : \Gamma(v) \neq \emptyset\}$, $\mathrm{gr}\Gamma := \{(v, \omega) : v \in \mathrm{dom}\Gamma, \omega \in \Gamma(v)\}$ and $\mathrm{epi}\Gamma := \{(v, \omega) : v \in X, \omega \in \Gamma(v) + K\}$. In what follows, we recall some definitions and results involving coderivatives and subdifferentials of set-valued mappings.

Suppose that $\Gamma : X \to 2^Y$ is a set-valued mapping between Banach spaces and $(\bar{v}, \bar{\omega}) \in \mathrm{gr}\Gamma$. Then the Fréchet coderivative of $\Gamma$ at $(\bar{v}, \bar{\omega})$ (see [16]) is the set-valued mapping $\widehat{D}^*\Gamma(\bar{v}, \bar{\omega}) : Y^* \to 2^{X^*}$ formulated by

$$\widehat{D}^*\Gamma(\bar{v}, \bar{\omega})(\omega^*) := \{v^* \in X^* : (v^*, -\omega^*) \in \widehat{N}((\bar{v}, \bar{\omega}); \mathrm{gr}\Gamma)\},$$

and the normal coderivative of $\Gamma$ at $(\bar{v}, \bar{\omega})$ (see [16]) is the set-valued mapping $D^*_N\Gamma(\bar{v}, \bar{\omega}) : Y^* \to 2^{X^*}$ formulated by

$$D^*_N\Gamma(\bar{v}, \bar{\omega})(\omega^*) := \{v^* \in X^* : (v^*, -\omega^*) \in N((\bar{v}, \bar{\omega}); \mathrm{gr}\Gamma)\}.$$

If $\Gamma = \phi : X \to Y$ is single-pointed, then $D^*_N\phi(\bar{v})$ and $\widehat{D}^*\phi(\bar{v})$ stand for its normal and Fréchet coderivatives at $(\bar{v}, \phi(\bar{v}))$, respectively.

On basis of the coderivative of epigraphical multifunction, Bao and Mordukhovich [23] formulated the mild extensions of subdifferential concept from extended-real-valued functions to vector-valued and multi-valued mappings with values in partially ordered spaces, and provided certain applications to multiobjective optimization issues in [18,24].

Given a multi-valued map $\Gamma : X \to 2^Y$. We formulate epigraphical multifunction $\mathcal{E}_\Gamma : X \to 2^Y$ (see [1]) below

$$\mathcal{E}_\Gamma(v) := \{\omega \in Y : \omega \in \Gamma(v) + K\}.$$

The Fréchet subdifferential and normal subdifferential of $\Gamma$ at $(\bar{v}, \bar{\omega}) \in \mathrm{epi}\Gamma$ in the direction $\omega^* \in Y^*$ (see [1]) are formulated, respectively, by

$$\widehat{\partial}\Gamma(\bar{v}, \bar{\omega})(\omega^*) := \widehat{D}^*\mathcal{E}_\Gamma(\bar{v}, \bar{\omega})(\omega^*),$$

$$\partial\Gamma(\bar{v}, \bar{\omega})(\omega^*) := D_N^*\mathcal{E}_\Gamma(\bar{v}, \bar{\omega})(\omega^*).$$

The normal subdifferential and Fréchet subdifferential of $\Gamma$ at $(\bar{v}, \bar{\omega}) \in \mathrm{epi}\Gamma$ are written, successively, as

$$\partial\Gamma(\bar{v}, \bar{\omega}) := \{v^* \in X^* : v^* \in D_N^*\mathcal{E}_\Gamma(\bar{v}, \bar{\omega})(\omega^*), -\omega^* \in N(0; K), \|\omega^*\| = 1\},$$

$$\widehat{\partial}\Gamma(\bar{v}, \bar{\omega}) := \{v^* \in X^* : v^* \in \widehat{D}^*\mathcal{E}_\Gamma(\bar{v}, \bar{\omega})(\omega^*), -\omega^* \in \widehat{N}(0; K), \|\omega^*\| = 1\}.$$

It is worth noting (see [23]) that if $K = \mathbf{R}_+$ and $\Gamma = \phi : X \to \overline{\mathbf{R}}$, then normal subdifferential and Fréchet subdifferential for multifunctions, successively, revert to limiting subdifferential and Fréchet subdifferential formulated as above.

Suppose that $\Gamma : \Xi \subset X \to 2^Y$ with $\mathrm{dom}\Gamma \neq \emptyset$ and $B_Y$ is the closed unit ball of $Y$. Recall that $F$ is referred to as being (see [16]):

(i) Lipschitz around $\bar{v} \in \mathrm{dom}\Gamma$ if $\exists$ (neighborhood) $U$ at $\bar{v}$ and $\exists \ell \geq 0$ s.t.

$$\Gamma(v) \subset \Gamma(u) + \ell\|v - u\|B_Y, \quad \forall v, u \in \Xi \cap U;$$

(ii) epi-Lipschitz around $\bar{v} \in \mathrm{dom}\Gamma$ if $\mathcal{E}_\Gamma$ is Lipschitz around this point;

(iii) Lipschitz-like around $(\bar{v}, \bar{\omega}) \in \mathrm{gr}\Gamma$ if $\exists$ (neighborhood) $U$ at $\bar{v}$, $\exists$ (neighborhood) $V$ at $\bar{\omega}$ and $\exists \ell \geq 0$ s.t.

$$\Gamma(v) \cap V \subset \Gamma(u) + \ell\|v - u\|B_Y, \quad \forall v, u \in \Omega \cap U;$$

(iv) epi-Lipschitz-like around $(\bar{v}, \bar{\omega}) \in \mathrm{epi}\Gamma$ if $\mathcal{E}_\Gamma$ is Lipschitz-like around this point.

Suppose that $\Gamma : X \to 2^Y$ and $\emptyset \neq \Xi \subset X$. The set-valued map $\Gamma_\Xi : X \to 2^Y$ associated with $\Xi$ and $\Gamma$, is formulated below

$$\Gamma_\Xi(v) = \begin{cases} \Gamma(v) & \text{if } v \in \Xi, \\ \emptyset & \text{if } v \notin \Xi. \end{cases}$$

The relationship between $\Gamma$ and normal coderivatives of $\Gamma_\Xi$ is formulated in [16] (Proposition 3.12).

**Proposition 1** ([16])**.** *Suppose that the spaces $Y, X$ are Asplund ones, the closed set $\Xi \subset X$ and the local closedness map $\Gamma : X \to 2^Y$ is Lipschitz-like around $(\bar{v}, \bar{\omega}) \in \mathrm{gr}\Gamma$. Then for each $\omega^* \in Y^*$ the inclusion holds below:*

$$D_N^*\Gamma_\Xi(\bar{v}, \bar{\omega})(\omega^*) \subset D_N^*\Gamma(\bar{v}, \bar{\omega})(\omega^*) + N(\bar{v}; \Xi).$$

Suppose that $K$ is a pointed convex closed cone in $Y$ enjoying $K^+ := \{y^* \in Y^* : y^*(u) \geq 0\ \forall u \in K\}$. Next, we aim to marginal functions associated with $\Gamma$. Associated with $\Gamma$ and $\omega^* \in Y^*$, the marginal function and minimum set are formulated, successively, as

$$\phi_{\omega^*}(v) := \inf\{\omega^*(\omega) : \omega \in \Gamma(v)\},$$

$$M_{\omega^*}(v) := \{\omega \in \Gamma(v) : \phi_{\omega^*}(v) = \omega^*(\omega)\}.$$

In what follows, we give the proposition below for certain properties of $\phi_{\omega^*}$ and $M_{\omega^*}$. Because the demonstration is simple, we omit it.

**Proposition 2** ([25]). *Assume $\bar{v} \in \mathrm{dom}\Gamma$. In case $\Gamma(\bar{v})$ is of weak compactness, one has $M_{\omega^*}(\bar{v}) \neq \emptyset$ for any $\omega^* \in Y^*$. Moreover, in case $M_{\omega^*}(v)$ is nonempty around $\bar{v}$ and multi-valued $\Gamma$ is u.s.c., the real-valued $\phi_{\omega^*}$ is l.s.c. at this point.*

Next, we always assume that $\mathrm{gr}\Gamma$ is of closedness and $M_{\omega^*}(v) \neq \emptyset \ \forall v \in \mathrm{dom}\Gamma, \omega^* \in K^+$.

**Theorem 2** ([13]). *Assume that $\Gamma : X \to 2^Y$ is u.s.c. and its graph is of both convexity and closedness. Then for each $v \in X$, $\omega^* \in Y^*$ and $\omega \in M_{\omega^*}(v)$, the following holds:*

$$\partial\phi_{\omega^*}(v) = D_N^*\Gamma(v,\omega)(\omega^*).$$

Oveisiha and Zafarani [25] obtained the property of basic normal cone for invexity set. Its demonstration is analogous to the ones of Propositions 1.5 and 1.3 in [16].

**Lemma 1** ([25]). *Suppose that $\emptyset \neq \Xi \subset X$, $\Xi$ is of both closedness and invexity w.r.t. $\eta$ s.t. $\eta$ is of continuity in the 2nd variable and $\bar{v} \in \Xi$. Then for each $v^* \in N(\bar{v};\Xi)$, one has*

$$\langle v^*, \eta(v,\bar{v})\rangle \leq 0 \quad \forall v \in \Xi.$$

Let $\psi : X^* \times \Xi \times \Xi \to \mathbf{R}$ be a function. Then $\psi$ is referred to as an equilibrium function if the following holds:

$$\psi(\xi, v, \omega) + \psi(\xi, \omega, v) = 0 \quad \forall(\xi, v, \omega) \in X^* \times \Xi \times \Xi.$$

Inspired by Lemma 1, we present the following definition concerning the basic normal cone w.r.t. $\eta$ and $\psi$.

**Condition B.** Suppose that $X$ is a Banach space and $\Xi \subset X$ is a closed and invex set w.r.t. $\eta$ s.t. $\eta$ is continuous in the second variable. Let $\bar{v} \in \Xi$ and $\psi : X^* \times \Xi \times \Xi \to \mathbf{R}$. Then for any $v^* \in N(\bar{v};\Xi)$, one has

$$\psi(v^*, \bar{v}, v) \leq 0 \quad \forall v \in \Xi.$$

**Definition 2.** *Suppose that $\Xi$ is an invex set w.r.t. $\eta$. Let $\psi : X^* \times \Xi \times \Xi \to \mathbf{R}$ and $\phi : \Xi \to \mathbf{R}$. Then $\phi$ is referred to as being*

*(i) preinvex w.r.t. $\eta$ on $\Xi$ if for any $\forall v_1, v_2 \in \Xi$ and $\lambda \in [0,1]$, one has*

$$\phi(v_2 + \lambda\eta(v_1, v_2)) \leq \lambda\phi(v_1) + (1-\lambda)\phi(v_2);$$

*(ii) invex w.r.t. $\psi$ on $\Xi$ if for any $v_1, v_2 \in \Xi$ and $\xi \in \partial\phi(v_2)$, one has*

$$\psi(\xi, v_2, v_1) \leq \phi(v_1) - \phi(v_2);$$

*(iii) weakly invex w.r.t. $\psi$ on $\Xi$ if $\forall v_1, v_2 \in \Xi$, $\exists\xi \in \partial\phi(v_2)$, s.t.*

$$\psi(\xi, v_2, v_1) \leq \phi(v_1) - \phi(v_2).$$

In addition, $\partial_M\phi$ is referred to as being invariant monotone on $\Xi$ w.r.t. $\psi$ if for any $v_i \in \Xi$ and $\xi_i \in \partial\phi(v_i)$, $(i = 1, 2)$, one has

$$\psi(\xi_1, v_1, v_2) + \psi(\xi_2, v_2, v_1) \leq 0.$$

**Remark 1.** *If we set* $\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle \ \ \forall (\xi, v, \omega) \in X^* \times \Xi \times \Xi$, *then Condition B and Definition 2 reduce to Lemma 2.12 and Definition 2.13 in [25], respectively. Moreover, for the concepts of generalized invexity and invariant monotonicity w.r.t.* $\eta$, *the reader is referred to [26,27,29] and the references cited therein.*

Suppose that $\Xi \subset X$ is an invex set w.r.t. $\eta$ and $\Gamma : \Xi \subset X \to 2^Y$ is a multi-valued function. $\Gamma$ is referred to as being $K$-preinvex w.r.t. $\eta$ on $\Omega$ (see [3] if

$$\lambda\Gamma(v_1) + (1-\lambda)\Gamma(v_2) \subset \Gamma(v_2 + \lambda\eta(v_1, v_2)) + K \quad \forall v_1, v_2 \in \Xi,\ \lambda \in [0,1].$$

In addition, a single-valued $\Gamma = \phi : \Xi \to Y$ is known as being of $K$-preinvexity w.r.t. $\eta$ on $\Xi$ iff

$$\phi(v_2 + \lambda\eta(v_1, v_2)) \leq_K \lambda\phi(v_1) + (1-\lambda)\phi(v_2) \quad \forall v_1, v_2 \in \Xi,\ \lambda \in [0,1].$$

**Remark 2.** *In the case when* $\eta(v_1, v_2) = v_1 - v_2$, *we obtain the concept of K-convexity in [6].*

The following conditions will be used in the proof of our main results later on.

**Condition A** (see [3]). A mapping $\Gamma : \Xi \to 2^Y$ from an invex set $\Xi$ w.r.t. $\eta$ to an ordered Banach space is referred to as enjoying Condition A if

$$\Gamma(v_1) \subset \Gamma(v_2 + \eta(v_1, v_2)) + K \quad \forall v_1, v_2 \in \Xi.$$

It should be noted that in case $K = \mathbf{R}_+$ and $\Gamma = \phi : \Xi \to \mathbf{R}$, we attain Condition A for real-valued functions of [29]:

$$\phi(v_2 + \eta(v_1, v_2)) \leq \phi(v_1) \quad \forall v_1, v_2 \in \Xi.$$

It is clear that in $\Gamma : \Xi \to 2^Y$ suits to Condition A, for each $\omega^* \in K^+$, $\phi_{\omega^*}$ suits to Condition A for real-valued functions.

Motivated by the Condition C of [14], we put forth the novel one below.

**Condition C.** Suppose that $\emptyset \neq \Xi \subset X$ and $\Xi$ is of invexity w.r.t. $\eta$. Then $\eta$ is referred to as satisfying Condition C w.r.t. $\psi$ iff for any $v, \omega \in \Xi$ and $t \in [0,1]$,

(a) $\eta(v, v + t\eta(\omega, v)) = -t\eta(\omega, v)$ and

$$\psi(\xi, v + t\eta(\omega, v), v) = -t\psi(\xi, v, \omega) \quad \forall \xi \in X^*;$$

(b) $\eta(\omega, v + t\eta(\omega, v)) = (1-t)\eta(\omega, v)$ and

$$\psi(\xi, v + t\eta(\omega, v), \omega) = (1-t)\psi(\xi, v, \omega) \quad \forall \xi \in X^*.$$

Let us note that if we set $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle$ for all $(\xi, v, \omega) \in X^* \times \Xi \times \Xi$, then $\eta$ suits to condition C w.r.t. $\psi$, with $\langle \cdot, \cdot \rangle$ indicating the duality pairing between $X^*$ and $X$. Yang et al. [30] had shown that

$$\eta(v + t\eta(\omega, v), v) = t\eta(\omega, v) \quad \forall v, \omega \in \Xi,\ t \in [0,1].$$

Suppose that $\mathcal{H}$ is a Hausdorff metric on the family $CB(X)$ of all bounded, closed and nonempty sets in $X$, derived by the $d(u, v) = \|u - v\|$, that is written as

$$\mathcal{H}(U, V) = \max\{\sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\|\} \quad \forall U, V \in CB(X).$$

According to [35], in case $U$ and $V$ are of compactness in $X$, we know that $\forall u \in U$, $\exists v \in V$ s.t. $\|u - v\| \leq \mathcal{H}(U, V)$.

**Definition 3.** *Suppose that* $\emptyset \neq \Xi \subset X$ *and* $\Xi$ *is of invexity w.r.t.* $\eta$. *A compact-valued* $T : \Xi \to 2^{\mathcal{L}(X,Y)}$ *is referred to as being of* $\mathcal{H}$*-hemicontinuity iff the map* $t \mapsto T(v + t\eta(\omega, v))$ *is continuous at* $0^+$, *where* $\mathcal{L}(X,Y)$ *is the family of all linear bounded mappings of* $X$ *into* $Y$ *and* $CB(\mathcal{L}(X,Y))$ *is endowed with the metric topology derived by* $\mathcal{H}$.

For unspecified terms, we are referred to [16].

## 3. $K$-Preinvexity Mappings

Using a normal subdifferential, we first put forth the notions of $K$-invexity w.r.t. $\phi$, weak $K$-invexity w.r.t. $\phi$ and invariant $K$-monotonicity w.r.t. $\phi$ for set-valued maps, and then establish the relations between them and $K$-preinvex maps.

**Definition 4.** *Let* $\emptyset \neq \Xi \subset X$, $\Gamma : \Xi \subset X \to 2^Y$ *and* $\psi : X^* \times \Xi \times \Xi \to \mathbf{R}$.

*(i)* $\Gamma$ *is referred to as being* $K$*-invex w.r.t.* $\psi$ *on* $\Xi$ *if for any* $\omega^* \in K^+$, $v_i \in \Xi$, $\omega_i \in M_{\omega^*}(v_i)$, $(i = 1, 2)$ *and* $\xi \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, *one has*

$$\psi(\xi, v_1, v_2) \leq \omega^*(\omega_2) - \omega^*(\omega_1).$$

*(ii)* $\Gamma$ *is referred to as being weakly* $K$*-invex w.r.t.* $\psi$ *on* $\Xi$ *if for any* $\omega^* \in K^+$, $v_i \in \Xi$, $\omega_i \in M_{v^*}(v_i)$, $(i = 1, 2)$, *there exists* $\xi \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, *such that*

$$\psi(\xi, v_1, v_2) \leq \omega^*(\omega_2) - \omega^*(\omega_1).$$

*(iii) The set-valued map* $\partial\Gamma : X \times Y \times Y^* \to 2^{X^*}$ *is referred to as being invariant* $K$*-monotone w.r.t.* $\psi$ *on* $\Xi$ *if for any* $\omega^* \in K^+$, $v_i \in \Xi$, $\omega_i \in M_{\omega^*}(v_i)$ *and* $\xi_i \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, $(i = 1, 2)$, *one has*

$$\psi(\xi_1, v_1, v_2) + \psi(\xi_2, v_2, v_1) \leq 0.$$

Note that in case $\Xi \subset X$ is an invex set w.r.t. $\eta$ and

$$\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle \quad \forall (\xi, v, \omega) \in X^* \times \Xi \times \Xi,$$

Definition 4 reduces to [25] (Definition 3.1). In addition, in case the function $\Gamma = \phi : X \to \mathbf{R}$ is real-valued, the last concept reverts to [25] (Definition 2.13) involving the invexity, weak invexity and invariant monotonicity for real-valued ones.

**Lemma 2** ([25])**.** *Assume that* $\emptyset \neq \Xi \subset X$ *and* $\Gamma : \Xi \to 2^Y$ *is multi-valued and* $\bar{v} \in \mathrm{dom}\Gamma$. *In case* $\Gamma$ *is epi-Lipschitz around* $\bar{v}$ *and* $\omega^* \in K^+$, *the real-valued* $\phi_{\omega^*}$ *is locally Lipschitz at* $\bar{v}$.

**Lemma 3** ([25])**.** *Let* $\Gamma : X \to 2^Y$ *and* $0 \neq \omega^* \in K^+$. *Assume that* $\bar{v} \in \mathrm{dom}\Gamma$ *and* $\bar{\omega} \in M_{\omega^*}(\bar{v})$. *Then*

$$\widehat{\partial}\phi_{\omega^*}(\bar{v}) \subset \widehat{\partial}\Gamma(\bar{v}, \bar{\omega})(\omega^*) \subset \widehat{D}^*\Gamma(\bar{v}, \bar{\omega})(\omega^*).$$

Next, let us recall the relationship between limiting subdifferential of marginal functions for $\Gamma$ and its normal coderivative.

**Theorem 3** ([25])**.** *Let* $\Gamma : X \to 2^Y$ *and* $\omega^* \in K^+$ *where* $X$ *and* $Y$ *both are Asplund spaces. Assume that* $\bar{v} \in \mathrm{dom}\Gamma$, $\bar{\omega} \in M_{\omega^*}(\bar{v})$ *and* $\Gamma$ *is Lipschitz around* $\bar{v}$. *Then*

$$\partial\phi_{\omega^*}(\bar{v}) \subset D_N^*\Gamma(\bar{v}, \bar{\omega})(\omega^*).$$

**Proof.** It is enough to use Theorem 3.38 in [16]. □

Applying the last theorem for $\mathcal{E}_\Gamma$, we obtain the conclusion below.

**Corollary 1.** *Let $\Gamma : X \to 2^Y$ and $\omega^* \in K^+$ where X and Y both are Asplund spaces. Assume that $\bar{v} \in \mathrm{dom}\Gamma, \bar{\omega} \in M_{\omega^*}(\bar{v})$ and $\Gamma$ is epi-Lipschitz around $\bar{v}$. Then*

$$\partial\phi_{\omega^*}(\bar{v}) \subset \partial\Gamma(\bar{v}, \bar{\omega})(\omega^*).$$

**Lemma 4** ([25])**.** *Let $\Gamma : X \to 2^Y$ be K-preinvex w.r.t. $\eta$. Then, for each $\omega^* \in K^+$, $\phi_{\omega^*}$ is of preinvexity w.r.t. $\eta$.*

**Proof.** Via certain mild corrections in the demonstration of [1] (Lemma 1.1 and Proposition 2.1), we can obtain the desired conclusion. □

**Lemma 5.** *Let $\Gamma : X \to 2^Y$ be K-invex w.r.t. $\psi$. Then $\partial\Gamma$ is invariant K-monotone w.r.t. $\psi$.*

**Proof.** The conclusion follows directly from Definition 4. □

**Theorem 4.** *Let $\Gamma : X \to 2^Y$ be a locally epi-Lipschitz map satisfying Condition A, where X and Y both are Asplund spaces. Assume that for any $\omega^* \in K^+ \setminus \{0\}$, $\phi_{\omega^*}$ suits to mean-value condition for limiting subdifferential $\partial\phi_{\omega^*}$ w.r.t. $\psi$. If $\eta$ satisfies Condition C w.r.t. $\psi$ and $\partial\Gamma$ is invariant K-monotone w.r.t. $\psi$, then $\Gamma$ is K-invex w.r.t. $\psi$.*

**Proof.** Let $\partial\Gamma$ be invariant $K$-monotone w.r.t. $\psi$ and $v_1, v_2 \in X$. Let $\varrho = v_2 + \frac{1}{2}\eta(v_1, v_2)$ and fix $\omega^* \in K^+$ arbitrarily. By Lemma 2, $\phi_{\omega^*}$ is locally Lipschitz. Now, note that $\eta$ satisfies Condition C w.r.t. $\psi$ and $\phi_{\omega^*}$ suits to mean-value condition for $\partial\phi_{\omega^*}$ w.r.t. $\psi$. So, $\exists\lambda_1, \lambda_2$ s.t. $0 < \lambda_2 \le \frac{1}{2} < \lambda_1 \le 1$, $\exists\xi_1 \in \partial\phi_{\omega^*}(u_1)$ and $\exists\xi_2 \in \partial\phi_{\omega^*}(u_2)$ s.t.

$$\phi_{\omega^*}(v_2 + \eta(v_1, v_2)) - \phi_{\omega^*}(\varrho) \ge \frac{1}{2}\psi(\xi_1, v_2, v_1), \tag{1}$$

and

$$\phi_{\omega^*}(\varrho) - \phi_{\omega^*}(v_2) \ge \frac{1}{2}\psi(\xi_2, v_2, v_1), \tag{2}$$

where $u_1 = v_2 + \lambda_1\eta(v_1, v_2)$ and $u_2 = v_2 + \lambda_2\eta(v_1, v_2)$. By using Corollary 1, $\xi_i \in \partial\phi_{\omega^*}(u_i) \subset \partial\Gamma(u_i, \varrho_i)(\omega^*)$ that $\varrho_i \in M_{\omega^*}(u_i)$, $(i = 1, 2)$. Since $\partial\Gamma$ is invariant $K$-monotone w.r.t. $\psi$, we have

$$\psi(\xi_1, u_1, v_2) + \psi(w, v_2, u_1) \le 0, \tag{3}$$

for all $\omega_2 \in M_{\omega^*}(v_2)$ and $w \in \partial\Gamma(v_2, \omega_2)(\omega^*)$. Now, by Condition C, we get

$$\psi(\xi_1, u_1, x_2) = -\lambda_1\psi(\xi_1, v_2, v_1) \quad \text{and} \quad \psi(w, v_2, u_1) = \psi(w, v_2, v_1).$$

If we replace these relations in (3), we obtain:

$$\psi(\xi_1, v_2, v_1) \ge \psi(w, v_2, v_1).$$

Now, from (1) we have:

$$\phi_{\omega^*}(v_2 + \eta(v_1, v_2)) - \phi_{\omega^*}(\varrho) \ge \frac{1}{2}\psi(w, v_2, v_1).$$

In a similar way, we can derive:

$$\phi_{\omega^*}(\varrho) - \phi_{\omega^*}(v_2) \ge \frac{1}{2}\psi(w, v_2, v_1).$$

By adding these two inequalities, we obtain:

$$\phi_{\omega^*}(v_2 + \eta(v_1, v_2)) - \phi_{\omega^*}(v_2) \ge \psi(w, v_2, v_1).$$

Since it is clear that $\phi_{\omega^*}$ is the real-valued which suits to Condition A, one gets

$$\phi_{\omega^*}(v_1) - \phi_{\omega^*}(v_2) \geq \psi(w, v_2, v_1),$$

for all $\omega_i \in M_{\omega^*}(v_i),\ (i = 1, 2)$ and $w \in \partial\Gamma(v_2, \omega_2)(\omega^*)$. Consequently,

$$\omega^*(\omega_1) - \omega^*(\omega_2) \geq \psi(w, v_2, v_1).$$

This completes the proof. □

Observe that if $\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle$, then it is easy to see that Theorem 4 reduces to [25] ( Theorem 3.8).

**Theorem 5.** *Suppose that $\Gamma : X \to 2^Y$ is a locally epi-Lipschitz set-valued map satisfying Condition A. Let $\mathcal{E}_\Gamma$ be closed convex-valued for every $v$, and $\eta$ satisfy Condition C w.r.t. $\psi$. If $\Gamma$ is $K$-invex w.r.t. $\psi$, then $\Gamma$ is $K$-preinvex w.r.t. $\eta$.*

**Proof.** Suppose that $\Gamma$ is $K$-invex w.r.t. $\psi$. Then we can easily see that $\phi_{\omega^*}$ is invex w.r.t. $\psi$ for all $\omega^* \in K^+$. We claim that $\phi_{\omega^*}$ is preinvex w.r.t. $\eta$ for all $\omega^* \in K^+$. As a matter of fact, for any $v, \omega \in X$ and $\lambda \in (0, 1)$, we set $\bar{v} = \omega + \lambda\eta(v, \omega)$. By the invexity of $\phi_{\omega^*}$ w.r.t. $\psi$, we have

$$\phi_{\omega^*}(v) - \phi_{\omega^*}(\bar{v}) \geq \psi(\zeta, \bar{v}, v) \quad \forall\zeta \in \partial\phi_{\omega^*}(\bar{v}). \tag{4}$$

Similarly, the invexity condition of $\phi_{\omega^*}$ w.r.t. $\psi$ applied to the pair $\omega, \bar{v}$ yields

$$\phi_{\omega^*}(\omega) - \phi_{\omega^*}(\bar{v}) \geq \psi(\zeta, \bar{v}, \omega) \quad \forall\zeta \in \partial\phi_{\omega^*}(\bar{v}). \tag{5}$$

We note that by Condition C,

$$\psi(\zeta, \bar{v}, v) = (1 - \lambda)\psi(\zeta, \omega, v) \quad \text{and} \quad \psi(\zeta, \bar{v}, \omega) = -\lambda\psi(\zeta, \omega, v).$$

Now, multiplying (4) with $\lambda$ and (5) with $(1 - \lambda)$ and making the sum of them, one obtains that

$$\begin{aligned} &\lambda\phi_{\omega^*}(v) + (1 - \lambda)\phi_{\omega^*}(\omega) - \phi_{\omega^*}(\bar{v}) \\ &\geq \lambda\psi(\zeta, \bar{v}, v) + (1 - \lambda)\psi(\zeta, \bar{v}, \omega) \\ &= \lambda(1 - \lambda)\psi(\zeta, \omega, v) - (1 - \lambda)\lambda\psi(\zeta, \omega, v) = 0. \end{aligned}$$

This means that $\phi_{\omega^*}$ is preinvex w.r.t. $\eta$ for each $\omega^* \in K^+$. Now, from Theorem 3.1 in [3], we can deduce that $\Gamma$ is $K$-preinvex w.r.t. $\eta$. □

**Theorem 6.** *Suppose that $\Gamma : X \to 2^Y$ is a locally epi-Lipschitz map, where $X$ and $Y$ both are Asplund spaces. Let $\eta : X \times X \to X$ be of continuity in the 2nd variable s.t. the Condition C w.r.t. $\psi$ is valid. For any $\omega^* \in K^+$ and $\omega \in X$ we assume that*

*(i) $\psi(\cdot, \cdot, \omega) : X^* \times X \to \mathbf{R}$ is continuous;*

*(ii) $\partial\phi_{\omega^*}(\cdot) : X \to 2^{X^*}$ is $\mathcal{H}$-hemicontinuous with compact values;*

*(iii) $\phi_{\omega^*}$ satisfies the mean-value condition for limiting subdifferential $\partial\phi_{\omega^*}$ w.r.t. $\psi$.*

*If $\Gamma$ is $K$-preinvex w.r.t. $\eta$, then $\Gamma$ is weakly $K$-invex w.r.t. $\psi$.*

**Proof.** Using Lemmas 2 and 4, for each $\omega^* \in K^+$, $\phi_{\omega^*}$ is a function with locallLipschitz continuity and preinvexity. Take two points $v, \omega \in X$ arbitrarily. Now we choose sequences $\{v_n\} \subset X$ and $\{t_n\} \subset (0, 1)$ such that $v_n \to v$ and $t_n \to 0^+$. By the mean-value condition of $\phi_{\omega^*}$ for $\partial\phi_{\omega^*}$ w.r.t. $\psi$, from Condition C one obtains that, $\forall t_n \in (0, 1)$, $\exists t'_n \in (0, t_n]$ and $\exists\xi^*_n \in \partial\phi_{\omega^*}(v_n + t'_n\eta(\omega, v_n))$ s.t.

$$t_n\psi(\xi^*_n, v_n, \omega) = \psi(\xi^*_n, v_n, v_n + t_n\eta(\omega, v_n)) \leq \phi_{\omega^*}(v_n + t_n\eta(\omega, v_n)) - \phi_{\omega^*}(v_n). \tag{6}$$

By Nadler's result [35], there also exists $\zeta_n \in \partial\phi_{\omega^*}(v)$ such that

$$\|\xi_n^* - \zeta_n\| \le \mathcal{H}(\partial\phi_{\omega^*}(v_n + t_n'\eta(\omega, v_n)), \partial\phi_{\omega^*}(v)).$$

Since $\eta : X \times X \to X$ is continuous in the second variable and $\partial\phi_{\omega^*}(\cdot) : X \to 2^{X^*}$ is $\mathcal{H}$-hemicontinuous with nonempty compact values, we know that

$$\|v_n + t_n'\eta(\omega, v_n) - v\| \le \|v_n - v\| + t_n'\|\eta(\omega, v_n)\| \to 0 \quad (n \to \infty),$$

and hence

$$\|\xi_n^* - \zeta_n\| \le \mathcal{H}(\partial\phi_{\omega^*}(v_n + t_n'\eta(\omega, v_n)), \partial\phi_{\omega^*}(v)) \to 0 \quad (n \to \infty). \tag{7}$$

Thanks to the fact that $\partial\phi_{\omega^*}(v)$ is compact, we might suppose that $\zeta_n \to \zeta^* \in \partial\phi_{\omega^*}(v)$. So, from (7) it follows that $\xi_n^* \to \zeta^*$ as $n \to \infty$. Also, since $\Gamma : X \to 2^Y$ is $K$-preinvex w.r.t. $\eta$, by Lemma 4 we deduce that for every $\omega^* \in K^+$, $\phi_{\omega^*}$ is preinvex w.r.t. $\eta$. Consequently, from (6) and the preinvexity of $\phi_{\omega^*}$ w.r.t. $\eta$, we get

$$\psi(\xi_n^*, v_n, \omega) \le \phi_{\omega^*}(\omega) - \phi_{\omega^*}(v_n). \tag{8}$$

We note that for each $\omega \in X$, $\psi(\cdot, \cdot, \omega) : X^* \times X \to \mathbf{R}$ is continuous. Since $v_n \to v$ and $\xi_n^* \to \zeta^*$ as $n \to \infty$, from (8) we have

$$\psi(\zeta^*, v, \omega) \le \phi_{\omega^*}(\omega) - \phi_{\omega^*}(v).$$

This means that $\phi_{\omega^*}$ is weakly invex w.r.t. $\psi$. Hence, Corollary 1 implies that $\Gamma$ is weakly $K$-invex w.r.t. $\psi$. $\square$

**Theorem 7.** *Suppose that the set-valued map $\Gamma : X \to 2^Y$ is K-preinvex w.r.t. $\eta$. Let $\mathcal{Z} := X \times Y$, $\bar{\Xi} := \mathrm{epi}\Gamma$, $\bar{\eta} : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ be defined as $\bar{\eta}((v_1, \omega_1), (v_2, \omega_2)) := (\eta(v_1, v_2), \omega_1 - \omega_2)$ and $\bar{\psi} : \mathcal{Z}^* \times \bar{\Xi} \times \bar{\Xi} \to \mathbf{R}$ be defined as $\bar{\psi}((v^*, \omega^*), (v_1, \omega_1), (v_2, \omega_2)) = \psi(v^*, v_1, v_2) + \langle \omega^*, \omega_2 - \omega_1 \rangle$. Assume that Condition B holds for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}(= \mathrm{epi}\Gamma)$. If $\eta$ is continuous in the second variable, then for any $v_i \in X, \omega_i \in \Gamma(v_i)$, $(i = 1, 2), \omega^* \in Y^*$, and $v^* \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, one has*

$$\psi(v^*, v_1, v_2) \le \omega^*(\omega_2) - \omega^*(\omega_1).$$

*(Hence, $\partial\Gamma$ is invariant K-monotone w.r.t. $\psi$.)*

**Proof.** Since $\Gamma$ is $K$-preinvex w.r.t. $\eta$, we can deduce that $\mathrm{epi}\Gamma$ is an invex set w.r.t. $\bar{\eta}$. Since $v^* \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, we get $(v^*, -\omega^*) \in N((v_1, \omega_1); \mathrm{epi}\Gamma)$. Hence, by Condition B for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}(= \mathrm{epi}\Gamma)$, we obtain

$$\bar{\psi}((v^*, -\omega^*), (v_1, \omega_1), (v_2, \omega_2)) = \psi(v^*, v_1, v_2) - \langle \omega^*, \omega_2 - \omega_1 \rangle \le 0.$$

Therefore, for any $v_i \in X, \omega_i \in \Gamma(v_i), (i = 1, 2), \omega^* \in Y^*$, and $v^* \in \partial\Gamma(v_1, \omega_1)(\omega^*)$, we have

$$\psi(v^*, v_1, v_2) \le \omega^*(\omega_2) - \omega^*(\omega_1).$$

$\square$

**Remark 3.** *The above theorems extend the previous earlier results from real-valued cases to set-valued ones associated with $\psi$. From Theorems 4 and 5, we obtain that for a K-preinvexity mapping w.r.t. $\eta$, invariant K-monotonicity of its normal subdifferential w.r.t. $\psi$ is a sufficient condition for K-preinvexity w.r.t. $\eta$. Moreover, from Theorem 7, we can deduce that invariant K-monotonicity of a normal subdifferential w.r.t. $\psi$ is a necessary condition for K-preinvexity w.r.t. $\eta$. In the case of K-convex set-valued maps and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$, the Conditions A, B and C are trivially valid.*

## 4. Super Efficient Solutions

In this section, we aim to establish sufficiency terms for the existence of super minimizers to set-valued vector optimization issues. Moreover, we put forth an extension of super minimizers and obtain certain necessity optimality terms for it.

For a set-valued map $\Gamma : X \to 2^Y$, we consider the set-valued vector optimization issue below:

$$\text{minimize } \Gamma(v), \quad \text{subject to } v \in \Xi \subset X. \tag{9}$$

For recent research on set-valued vector optimization problems, the reader is referred to [25,31,32] and the references cited therein.

Borwein and Zhuang [17] put forward the concept of super minimal points to any set of partially ordered space. Let the Banach space $Y$ be ordered by a convex and closed cone $K \subset Y$. Given a set $A \subset Y$, $\bar{a} \in A$ is said to be a super minimal point of $A$ ($\bar{a} \in SE(A,K)$) iff $\exists M > 0$ s.t.

$$\text{cl}[\text{cone}(A - \bar{a})] \cap (\mathrm{B}_Y - K) \subset M\mathrm{B}_Y,$$

with the closed unit ball $\mathrm{B}_Y \subset Y$. Given $(\bar{v}, \bar{\omega}) \in \text{gr}\Gamma$ with $\bar{v} \in \Xi$. Then $(\bar{v}, \bar{\omega})$ is said to be a local super minimizer to problem (9) iff, $\exists$ (neighborhood) $U$ at $\bar{v}$ s.t. $\bar{\omega} \in SE(\Gamma(\Xi \cap U); K)$.

Recall a necessity term for super minimizers that had been demonstrated in [18] (Theorem 3.8).

**Theorem 8** ([18])**.** *Suppose that the spaces $X, Y$ are Asplund ones and $\Gamma$ be Lipschitz-like around $(\bar{v}, \bar{\omega})$. Let $\Xi$ and $\text{gr}\Gamma$ be locally closed around $\bar{v}$ and $(\bar{v}, \bar{\omega})$, respectively, $K \neq \{0\}$ and $\text{int}K^+ \neq \emptyset$. Assume that $(\bar{v}, \bar{\omega})$ is a local super minimizer to problem (9). Then $\exists \omega^* \in \text{int}K^+$ with $\|\omega^*\| = 1$ s.t.*

$$0 \in D_N^*\Gamma(\bar{v}, \bar{\omega})(\omega^*) + N(\bar{v}; \Xi).$$

*We also recall that $K$ has a bounded base if and only if $\text{int}K^+ \neq \emptyset$. By this relation, we prove a lemma that we need in the sequel.*

**Lemma 6** ([25])**.** *Let $K$ be an ordering pointed convex cone in $Y$, $\Xi$ be invex and $\Gamma$ be $K$-preinvexity mapping w.r.t. $\eta$. Assume that $\exists \omega^* \in \text{int}K^+$ and $\exists \bar{\omega} \in \Gamma(\bar{v})$ s.t.*

$$\omega^*(\bar{\omega}) \leq \omega^*(\omega) \quad \forall v \in \Xi, \omega \in \Gamma(v),$$

*Then problem (9) has the local super minimal point $(\bar{v}, \bar{\omega})$.*

**Proof.** Via mild corrections in the demonstration of [3] (Theorem 6.1 and Remark 6.1), we can derive the desired conclusion. □

**Remark 4.** *Rong and Wu [20] had shown that, in case there is a bounded closed base in a pointed convex cone $K$, one has $SE(A;K) = SE(A + K;K)$ for all nonempty sets $A \subset Y$. By this hypothesis and conditions of Theorem 8, we can obtain that*

$$0 \in \partial\Gamma(\bar{v}, \bar{\omega})(\omega^*) + N(\bar{v}; \Xi) \tag{10}$$

*is a necessity optimality term for super minimal points. Recently, it was proven in [18] that the relationship (10) is true under the normality property of the ordering cone $K$.*

Next, under the $K$-preinvexity of $\Gamma$ w.r.t. $\eta$, we demonstrate that the converse of Theorem 8 in the presence of (10) is true.

**Theorem 9.** *Suppose that the closed set $\Xi \subset X$ which is invex w.r.t. $\eta$ and the ordering cone $K \subset Y$, which is pointed, convex and closed. Let the map $\Gamma : \Xi \to 2^Y$ be $K$-preinvex w.r.t. $\eta$ which is of continuity in the 2nd variable. Let $\mathcal{Z} := X \times Y$, $\bar{\Xi} := \text{epi}\Gamma$, $\bar{\eta} : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ be defined as $\bar{\eta}((v_1, \omega_1), (v_2, \omega_2)) := (\eta(v_1, v_2), \omega_1 - \omega_2)$ and $\bar{\psi} : \mathcal{Z}^* \times \bar{\Xi} \times \bar{\Xi} \to \mathbf{R}$ be defined as $\bar{\psi}((v^*, \omega^*), (v_1, \omega_1), (v_2, \omega_2)) = \psi(v^*, v_1, v_2) + \langle \omega^*, \omega_2 - \omega_1 \rangle$. Assume that Condition B* *holds for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}(= \mathrm{epi}\Gamma)$ and for $X, \eta, \psi$ and $\Xi$, respectively. Suppose that $(\bar{v}, \bar{\omega}) \in \mathrm{gr}\Gamma$ and there is a $\omega^* \in \mathrm{int}K^+$ such that*

$$0 \in \partial\Gamma(\bar{v}, \bar{\omega})(\omega^*) + N(\bar{v}; \Xi). \tag{11}$$

*Then problem (9) has the local super minimizer $(\bar{v}, \bar{\omega})$.*

**Proof.** Utilizing the relationship (11), we know that, $\exists v_1^* \in \partial F(\bar{x}, \bar{y})(y^*)$ and $\exists x_2^* \in N(\bar{x}; \Omega)$ s.t. $x_1^* + x_2^* = 0$. Since $\mathrm{gr}\mathcal{E}_F = \mathrm{epi}\Gamma$, we obtain

$$(v_1^*, -\omega^*) \in N((\bar{v}, \bar{\omega}); \mathrm{gr}\mathcal{E}_\Gamma) = N((\bar{v}, \bar{\omega}); \mathrm{epi}\Gamma).$$

From the $K$-preinvexity of $\Gamma$ w.r.t. $\eta$, we can conclude that $\mathrm{epi}\Gamma$ is an invex set w.r.t. $\bar{\eta}$, where $\bar{\eta}((v_1, \omega_1), (v_2, \omega_2)) := (\eta(v_1, \omega_2), \omega_1 - \omega_2)$. Now, by Condition B for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}$, we get

$$\bar{\psi}((v_1^*, -\omega^*), (\bar{v}, \bar{\omega}), (v, \omega)) = \phi(v_1^*, \bar{v}, v) - \langle \omega^*, \omega - \bar{\omega} \rangle \le 0 \tag{12}$$

for any $v \in \Xi$ and $\omega \in \Gamma(v)$. Since $\Xi$ is invex w.r.t. $\eta$, and $-v_1^* = v_2^* \in N(\bar{v}; \Xi)$, by Condition B for $X, \eta, \psi$ and $\Xi$, we deduce that

$$\psi(v_1^*, \bar{v}, v) \ge 0 \quad \forall v \in \Xi. \tag{13}$$

By (12) and (13), we obtain:

$$\langle \omega^*, \omega - \bar{\omega} \rangle \ge 0 \quad \forall v \in \Xi, y \in \Gamma(v). \tag{14}$$

From (14) and Lemma 6, we conclude that problem (9) has the local super minimal point $(\bar{v}, \bar{\omega})$. □

**Remark 5.** *The following hypotheses exhibit crucial roles in Theorem 9:*
*(i) there exists an $\eta$ such that $\Gamma$ is K-preinvex w.r.t. $\eta$;*
*(ii) the Condition B holds for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}(= \mathrm{epi}\Gamma)$ and for $X, \eta, \psi$ and $\Xi$, respectively.*
*In particular, if $\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle$, then Theorem 9 reduces to [16] (Theorem 4.3). Moreover, whenever $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$, Theorem 9 is an existence theorem for K-convexity.*

**Example 1.** *Assume that $X = \Xi = \mathbf{R}^2$, $Y = \mathbf{R}$, $K = [0, +\infty)$, $(\bar{v}, \bar{\omega}) = ((0,0), 0)$ and $\Gamma : X \to 2^Y$ given by $\Gamma(v_1, v_2) = [\sqrt{v_1^2 + v_2^2}, +\infty)$. Let $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$. By some computation one deduces that $\Gamma$ is a K-preinvex map w.r.t. $\eta$ and $\partial\Gamma((0,0),0)(1) = \{(v_1, v_2) : v_1^2 + v_2^2 \le 1\}$. Hence, $0 \in \partial\Gamma(\bar{v}, \bar{\omega})(1) + N(\bar{v}; \Xi)$ and all the assumptions of Theorem 9 are examined. In this case, $(\bar{v}, \bar{\omega})$ is a local super minimizer to problem (9).*

**Example 2.** *Let $X = \Xi = \mathbf{R}^2$, $Y = \mathbf{R}$, $K = [0, +\infty)$ and $(\bar{v}, \bar{\omega}) = ((0,0), 0)$. Define $\Gamma : X \to 2^Y$ by $\Gamma(v_1, v_2) := [\sqrt{(\frac{2}{3}v_1 + \frac{1}{3}v_2)^2 + (\frac{1}{3}v_1 + \frac{2}{3}v_2)^2}, +\infty)$. Let $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$. Via certain calculation one obtains that $\Gamma$ is a K-preinvex map w.r.t. $\eta$ and $\partial\Gamma((0,0),0)(1) = \{(v_1, v_2) : (\frac{2}{3}v_1 + \frac{1}{3}v_2)^2 + (\frac{1}{3}v_1 + \frac{2}{3}v_2)^2 \le 1\}$. Thus, $0 \in \partial\Gamma(\bar{v}, \bar{\omega})(1) + N(\bar{v}; \Xi)$ and all the conditions of Theorem 9 are fulfilled. In this case, $(\bar{v}, \bar{\omega})$ is a local super minimizer to problem (9).*

**Example 3.** *Support that $X = \Xi = \mathbf{R}^2$, $Y = \mathbf{R}$, $K = [0, +\infty)$ and $(\bar{v}, \bar{\omega}) = ((0,0), 0)$. Define $\Gamma : X \to 2^Y$ by $\Gamma(v_1, v_2) := [|v_1| + |v_2|, +\infty)$. Let $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$. By certain computation one has that $\Gamma$ is a K-preinvex map w.r.t. $\eta$ and $\partial\Gamma((0,0),0)(1) = \{(v_1, v_2) : |v_1| + |v_2| \le 1\}$. Thus, $0 \in \partial\Gamma(\bar{v}, \bar{\omega})(1) + N(\bar{v}; \Xi)$ and all the conditions of Theorem 9 are satisfied. In this case, $(\bar{v}, \bar{\omega})$ is a local super minimizer to problem (9).*

**Example 4.** *Assume that* $X = \Xi = \mathbf{R}^2$, $Y = \mathbf{R}$, $K = [0, +\infty)$ *and* $(\bar{v}, \bar{\omega}) = ((0,0),0)$. *Define* $\Gamma : X \to 2^Y$ *by* $\Gamma(v_1, v_2) := [\max\{|v_1|, |v_2|\}, +\infty)$. *Let* $\eta(\omega, v) = \omega - v$ *and* $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$. *Using certain calculation one infers that* $\Gamma$ *is a K-preinvex map w.r.t.* $\eta$ *and* $\partial\Gamma((0,0),0)(1) = \{(v_1, v_2) : \max\{|v_1|, |v_2|\} \leq 1\}$. *Hence,* $0 \in \partial\Gamma(\bar{v}, \bar{\omega})(1) + N(\bar{v}; \Xi)$ *and all the hypotheses of Theorem 9 are verified. In this case, problem (9) has the local super minimizer* $(\bar{v}, \bar{\omega})$.

**Theorem 10.** *Suppose that the closed set* $\Xi \subset X$, *which is invex w.r.t.* $\eta$ *and the u.s.c. map* $\Gamma : \Xi \subset X \to 2^Y$ *is K-preinvex w.r.t.* $\eta$ *where* $\eta$ *is of continuity in the 2nd variable. Let* $\bar{v} \in \Xi$ *and assume that* $\exists \bar{\omega}^* \in \mathrm{int}K^+$ *s.t.*

$$0 \in \partial\phi_{\bar{\omega}^*}(\bar{v}) + N(\bar{v}; \Xi). \tag{15}$$

*Let* $\mathcal{Z} := X \times Y$, $\bar{\Xi} := \mathrm{epi}\phi_{\omega^*}$, $\bar{\eta} : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ *be defined as* $\bar{\eta}((v_1, \omega_1), (v_2, \omega_2)) := (\eta(v_1, v_2), \omega_1 - \omega_2)$ *and* $\bar{\psi} : \mathcal{Z}^* \times \bar{\Xi} \times \bar{\Xi} \to \mathbf{R}$ *be defined as* $\bar{\psi}((v^*, \omega^*), (v_1, \omega_1), (v_2, \omega_2)) = \psi(v^*, v_1, v_2) + \langle \omega^*, \omega_2 - \omega_1 \rangle$. *Assume that Condition B holds for* $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ *and* $\bar{\Xi}(= \mathrm{epi}\phi_{\omega^*})$ *and for* $X, \eta, \psi$ *and* $\Xi$, *respectively. Then, for all* $\bar{\omega} \in M_{\omega^*}(\bar{v})$, $(\bar{v}, \bar{\omega})$ *is a local super minimizer to problem (9).*

**Proof.** Note that $\bar{v} \in \Xi$ and there is a $\omega^* \in \mathrm{int}K^+$ such that (15) holds. By Lemma 4 we know that $\phi_{\omega^*}$ is preinvex w.r.t. $\eta$. Utilizing a similar inference to that of Theorem 9, we can derive the desired conclusion. □

It is worth emphasizing that, if $\psi(\xi, v, \omega) = \langle \xi, \eta(\omega, v) \rangle$, then it is clear that Condition B holds for $\mathcal{Z}, \bar{\eta}, \bar{\psi}$ and $\bar{\Xi}(= \mathrm{epi}\phi_{\omega^*})$ and for $X, \eta, \psi$ and $\Xi$, respectively. So, Theorem 10 reduces to [25] ( Theorem 4.5). In addition, if $\eta(\omega, v) = \omega - v$ and $\psi(\xi, v, \omega) = \langle \xi, \omega - v \rangle$, then, under the $K$-convexity condition, the similar theorem for Gâteaux differentiable vector-valued functions was proven in [36]. Hence, this theorem generalizes [36] (Theorem 5.14) to u.s.c. $K$-preinvex set-valued maps.

**Author Contributions:** Conceptualization, L.-C.C., C.-F.W. and Y.-C.L.; methodology, L.-C.C., C.-F.W. and Y.-C.L.; software, C.-F.W.; validation, L.-C.C., C.-F.W. and Y.-C.L.; formal analysis, L.-C.C., C.-F.W. and Y.-C.L.; investigation, L.-C.C., C.-F.W. and Y.-C.L.; writing (original draft preparation), L.-C.C.; writing (review and editing), L.-C.C. and Y.-C.L.; supervision, C.L.C.; project administration, C.L.C.; funding acquisition, L.-C.C. and Y.-C.L. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau grant number 20LJ2006100, the Innovation Program of Shanghai Municipal Education Commission grant number 15ZZ068 and the Program for Outstanding Academic Leaders in Shanghai City grant number 15XD1503100. This research was supported by the Grant MOST grant number 106-2923-E-039-001-MY3. Y.-C.L. was partially supported by MOST project in Taiwan with grand no. MOST 110 2410 H 037 001 and NKUST and KMU joint R and D Project (NKUST KMU 110KK002).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Benoist, J.; Popovici, N. Characterizations of convex and quasiconvex set-valued maps. *Math. Methods Oper. Res.* **2003**, *57*, 427–435. [CrossRef]
2. Benoist, J.; Popovici, N. Generalized convex set-valued maps. *J. Math. Anal. Appl.* **2003**, *288*, 161–166. [CrossRef]
3. Jabarootian, T.; Zafarani, J. Characterizations of preinvex and prequasiinvex set-valued maps. *Taiwan. J. Math.* **2009**, *13*, 871–898. [CrossRef]
4. Oveisiha, M.; Zafarani, J. Vector optimization problem and generalized convexity. *J. Global Optim.* **2012**, *52*, 29–43. [CrossRef]
5. Sach, P.H.; Yen, N.D. Convexity criteria for set-valued maps. *Set-Valued Anal.* **1997**, *5*, 37–45. [CrossRef]

6. Yang, X.Q. Directional derivatives for set-valued mappings and applications. *Math. Methods Oper. Res.* **1998**, *48*, 273–285. [CrossRef]
7. Chen, G.Y.; Jahn, J. Optimality conditions for set-valued optimization problems. *Math. Methods Oper. Res*. **1998**, *48*, 1187–1200. [CrossRef]
8. Chinaie, M.; Zafarani, J. Image space analysis and scalarization of multivalued optimization. *J. Optim. Theory Appl.* **2009**, *142*, 451–467. [CrossRef]
9. Chinaie, M.; Zafarani, J. Image space analysis and scalarization for ε-optimization of multifunctions. *J. Optim. Theory Appl.* **2013**, *157*, 685–695. [CrossRef]
10. Chinchuluun, A.; Pardalos, P.M. A survey of recent developments in multiobjective optimization. *Ann. Oper. Res.* **2007**, *154*, 29–50. [CrossRef]
11. Durea, M.; Strugariu, R. Optimality conditions in terms of Bouligand derivatives for Pareto efficiency in set-valued optimization. *Optim. Lett.* **2011**, *5*, 141–151. [CrossRef]
12. Floudas, C.A.; Pardalos, P.M. *Encyclopedia of Optimization*; Springer: New York, NY, USA, 2009.
13. Ha, T.X.D. Lagrange multipliers for set-valued optimization problems associated with coderivatives. *J. Math. Anal. Appl.* **2005**, *311*, 647–663. [CrossRef]
14. Mohan, S.R.; Neogy, S.K. On invex sets and preinvex functions. *J. Math. Anal. Appl.* **1995**, *189*, 901–908. [CrossRef]
15. Mordukhovich, B.S. Maximum principle in the problem of time optimal response with nonsmooth constraints. *J. Appl. Math. Mech.* **1976**, *40*, 960–969. [CrossRef]
16. Mordukhovich, B.S. *Variational Analysis and Generalized Differentiation, I: Basic Theory*; Fundamental Principles of Mathematical Sciences Series; Springer: New York, NY, USA, 2006; Volume 330.
17. Borwein, J.M.; Zhuang, D. Super efficiency in vector optimization. *Trans. Am. Math. Soc.* **1993**, *338*, 105–122. [CrossRef]
18. Bao, T.Q.; Mordukhovich, B.S. Necessary conditions for super minimizers in constrained multiobjective optimization. *J. Glob. Optim.* **2009**, *43*, 533–552. [CrossRef]
19. Huang, H. The Lagrange multiplier rule for super efficiency in vector optimization. *J. Math. Anal. Appl.* **2007**, *342*, 503–513. [CrossRef]
20. Rong, W.D.; Wu, Y.N. Characterization of super efficiency in cone-convexlike vector optimization with set-valued maps. *Math. Methods Oper. Res.* **1998**, *48*, 247–258. [CrossRef]
21. Zaffaroni, A. Degrees of efficiency and degrees of minimality. *SIAM J. Control Optim.* **2003**, *42*, 1071–1086. [CrossRef]
22. Zheng, X.Y.; Yang, X.M.; Teo, K.L. Super efficiency of vector optimization in Banach spaces. *J. Math. Anal. Appl.* **2007**, *327*, 453–460. [CrossRef]
23. Bao, T.Q.; Mordukhovich, B.S. Variational principles for set-valued mappings with applications to multiobjective optimization. *Control Cybern.* **2007**, *36*, 531–562.
24. Bao, T.Q.; Mordukhovich, B.S. Relative Pareto minimizers in multiobjective optimization: Existence and optimality conditions. Math. Program. **2010**, *122*, 301–347. [CrossRef]
25. Oveisiha, M.; Zafarani, J. Super efficiency solutions for set-valued maps. *Optimization* **2013**, *62*, 817–834. [CrossRef]
26. Jabarootian, T.; Zafarani, J. Generalized invariant monotonicity and invexity of nondifferentiable functions. *J. Global Optim.* **2006**, *36*, 537–564. [CrossRef]
27. Soleimani-Damaneh, M. Characterizations and applications of generalized invexity and monotonicity in Asplund spaces. *Top* **2010**, *18*, 1–22. [CrossRef]
28. Weir, T.; Mond, B. Preinvex functions in multiple-objective optimization. *J. Math. Anal. Appl.* **1988**, *136*, 29–38. [CrossRef]
29. Yang, X.M.; Yang, X.Q.; Teo, K.L. Generalized invexity and generalized invariant monotonicity. *J. Optim. Theory Appl.* **2003**, *117*, 607–625. [CrossRef]
30. Yang, X.M.; Yang, X.Q.; Teo, K.L. Criteria for generalized invex monotonicity. *Eur. J. Oper. Res.* **2005**, *164*, 115–119. [CrossRef]
31. Oveisiha, M.; Zafarani, J. On characterization of solution sets of set-valued pseudoinvex optimization problems. *J. Optim. Theory Appl.* **2014**, *163*, 387–398. [CrossRef]
32. Ceng, L.C.; Latif, A. Characterizations of solution sets of set-valued generalized pseudoinvex optimization problems. *J. Nonlinear Sci. Appl.***2016**, *9*, 6382–6395. [CrossRef]
33. Atarzadeh, S.; Fakhar, M.; Zafarani, J. Optimality conditions of a nonsmooth composite minimization problem. *J. Nonlinear Convex Anal.* **2019**, *20*, 1623–1636.
34. Atarzadeh, S.; Fakhar, M.; Zafarani, J. Characterizing the Lagrange multiplier rule in nonconvex set-valued optimization. *Carpathian J. Math.* **2019**, *35*, 407–416. [CrossRef]
35. Nadler, S.B., Jr. Multivalued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
36. Gong, X.H. Scalarization and optimality conditions for vector equilibrium problems. *Nonlinear Anal.* **2010**, *73*, 3598–3612. [CrossRef]