mathematics

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Non-Instantaneous Impulsive BVPs Involving Generalized Liouville–Caputo Derivative

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Abstract: This manuscript investigates the existence, uniqueness and Ulam–Hyers stability (UH) of solution to fractional differential equations with non-instantaneous impulses on an arbitrary domain. Using the modern tools of functional analysis, we achieve the required conditions. Finally, we provide an example of how our results can be applied.

Keywords: non-instantaneous impulses; Generalized Liouville–Caputo derivative; Leray–Schauder alternative theorem

MSC: 34A08; 34A12; 47H08; 47H10; 46B45

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Citation: Salem, A.; Abdullah, S. Non-Instantaneous Impulsive BVPs Involving Generalized Liouville–Caputo Derivative. *Mathematics* **2022**, *10*, 291. https://doi.org/10.3390/math10030291

Academic Editors: António M. Lopes, Alireza Alfi, Liping Chen and Sergio A. David

Received: 13 December 2021
Accepted: 13 January 2022
Published: 18 January 2022

1. Introduction

The study of differential equations with fractional order has become increasingly popular in recent decades. The reasons behind it are fractional order derivatives provide powerful tools for describing inherited or defined properties in a wide range of science and engineering fields [1–8].

There are several approaches of fractional derivatives, Riemann–Liouville, Caputo, Hadamard, Hilfer, etc. It is important to cite that the Caputo derivative is useful to affront problems where initial conditions are done in the function and in the respective derivatives of integer order. Due to the importance of the Caputo version, there are many versions established as generalization of it, such as Caputo–Katugampola, Caputo–Hadamard, Caputo–Fabrizio, etc. Furthermore, it is drown attention of huge number of contributors to study physical and mathematical modelings contain it and its related versions, see [9–13] and references cited therein.

Finding exact solutions to the differential equations, whether they are ordinary, partial, or fractional, is a extremely difficult and complex issue, and that is why mathematicians have resorted to studying the properties of solutions such as existence, uniqueness, stability, invariant, controllability and others. The most important of these properties are existence and uniqueness which attracted the attention of many contributors to their study [14–20]. Furthermore, Ulam–Hyers stability analysis that is necessary for nonlinear problems in terms of optimization and numerical solutions and plays a key role in numerical solutions where exact solutions are difficult to get.

The fractional differential equations (FDEs) with instantaneous impulses are increasingly being used to analyze abrupt shifts in the evolution pace of dynamical systems, such as those brought about by shocks, disturbances, and natural disasters [21,22]. The duration of instantaneous impulses is relatively short in comparison to the duration of the overall process. However, certain dynamics of evolution processes have been observed to be inexplicable by instantaneous impulsive dynamic systems. As an instance, the injection and absorption of drugs in the blood is a gradual and continuous process. Here, each spontaneous, the action begins in an arbitrary fixed position and lasts for a finite amount of time. This type of system is known as a non-instantaneous impulsive system, which are more suitable for investigating the dynamics of evolutionary processes [23–25] and the references cited therein. Hernandaz and O'Regan [26] discussed the evolution equations involving non-instantaneous impulses of the form:

$$\begin{cases} x' = Ax(t) + f(t, x(t)), & t \in (s_k, t_{k+1}], k = 0, 1, \cdots, m, \\ y(t) = g_k(t, x(t)), & t \in (t_k, s_k], k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases}$$

Liu et al. [27] explored generalized Ulam–Hyers–Rassias stability for the following fractional differential equation:

$$\begin{cases} {}^cD^v_{0,w} z(w) = f(w, z(w)), & w \in (w_k, s_k],\ k = 0, 1, \cdots, m,\ 0 < v < 1, \\ z(w) = g_k(w, z(w)), & w \in (s_{k-1}, w_k],\ k = 1, \cdots, m \end{cases}$$

where ${}^cD^v_{0,w}$ is a Caputo derivative of fractional order $0 < v < 1$ with the lower limit 0. Ho and Ngo [28] analyzed generalized Ulam–Hyers–Rassias stability for the following fractional differential equation:

$$\begin{cases} {}^cD^{\alpha,\rho}_{a^+} x(t) = f(t, x(t)), & t \in (t_k, s_k],\ k = 0, 1, \cdots, m,\ 0 < \alpha < 1, \\ x(t) = I_k(t, x(t)), & t \in (s_{k-1}, t_k],\ k = 1, \cdots, m, \\ x(a^+) = x_0 \end{cases}$$

where ${}^cD^{\alpha,\rho}_{a^+}$ is a Caputo–Katugampola derivative of fractional order $0 < \alpha < 1$. Recently, Abbas [29] has studied non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function by using the nonlinear alternative Leray–Schauder type and the Banach contraction mapping principle

$$\begin{cases} {}_aD^{\alpha,\rho,g} y(t) = f(t, y(t), {}_aI^{\beta,\rho,g} y(t)), & t \in (s_k, t_{k+1}], k = 0, 1, \cdots, m, \\ y(t) = \Psi_k(t, y(t_k^+)), & t \in (t_k, s_k], k = 1, 2, \cdots, m, \\ {}_aI^{\beta,\rho,g} y(a) = y_0, & y_0 \in \mathbb{R} \end{cases}$$

where $0 < \alpha \leq 1, \beta, \rho > 0, {}_aD^{\alpha,\rho,g}$ is the proportional fractional derivative of order α with respect to another function g.

It is remarkable that the most of contributions focus on the case when the order of fractional derivative lies in the unit interval $(0, 1)$. This observation encourages us to study these equations when the order of fractional derivative lies in the unit interval $(1, 2)$. Furthermore, although the Generalized Liouville–Caputo fractional derivative is considered a generalization of Caputo and Hadamard fractional derivatives, there is a rareness of the studies with this approach.

Inspire of the above, we investigate the existence of solutions for non-instantaneous impulsive fractional boundary value problems in this paper. Specifically, we consider the following problem:

$$\begin{cases} {}^cD^{\beta,\rho}_{0^+} y(\tau) = h(\tau, y(\tau), \tau^{1-\rho} y'(\tau)), & \tau \in (s_r, \tau_{r+1}], r = 0, 1, \cdots, k, \\ y(\tau) = \Phi_r(\tau, y(\tau), y(\tau_r - 0), & \tau \in (\tau_r, s_r], r = 1, 2, \cdots, k, \\ y'(\tau) = \tau^{\rho-1} \Psi_r(\tau, y(\tau), y(\tau_r - 0), & \tau \in (\tau_r, s_r], r = 1, 2, \cdots, k, \\ y(0) = y_0, \quad \lim_{\tau \to 0} \tau^{1-\rho} y'(\tau) = y_1, & y_0, y_1 \in \mathbb{R} \end{cases} \tag{1}$$

where all intervals are subset of $J = [0, T]$, ${}^cD^{\beta,\rho}$ is a generalized Caputo–Liouville (Katugampola) derivative of order $1 < \beta \leq 2$ and type $0 < \rho \leq 1$ and $h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Here, $0 = s_0 < \tau_1 < s_1 < \cdots < \tau_k < s_k < \tau_{k+1} = T, k \in \mathbb{N}$ are fixed real numbers and Φ_r and $\Psi_r : (\tau_r, s_r) \to \mathbb{R}, r = 1, \cdots, k$ are non-instantaneous impulses.

The main objectives of our work are to develope the existence theory and Ulam–Hyers stability of non-instantaneous impulsive BVPs involving Generalized Liouville–Caputo derivatives. This work is based on modern functional analysis techniques. Three basic results introduce: the first two deal with the existence and uniqueness of solutions by applying a nonlinear Leray–Schauder alternative theorem and the Banach fixed point theorem, respectively. While the third concerns the Ulam–Hyers stability analysis of solutions for the given problem by establishing a criterion for ensuring various types of Ulam–Hyers stability.

For the rest of the paper, it is arranged as follows: Section 2 provides some preliminary concepts about our work and a key lemma that deals with the linear variant of the given problem, along with giving a formula for converting the given problem into a fixed point problem.Using the Banach contraction mapping principle and the Leray–Schauder nonlinear alternative, the existence and uniqueness of problem (1) are presented in Section 3.

Remark 1. *For fractional differential equation for non-instantaneous impulsive (1).The intervals $(\tau_r, s_r], r = 1, \cdots, k$ are known as non-instantaneous impulse intervals, and the functions $\Phi_r(\tau, y(\tau), y(\tau_r - 0)), r = 1, \cdots, k$ are known as non-instantaneous impulsive functions. The fractional differential equation with non-instantaneous impulses (1) is reduced to a fractional differential equation with instantaneous impulses if $\tau_r = s_{r-1}, r = 1, \cdots, k$.*

2. Preliminaries

Let the space of continuous real-valued functions on J be denoted by $\mathcal{C}(J, \mathbb{R})$. Consider the space

$$\mathcal{PC}(J, \mathbb{R}) = \{y : J \to \mathbb{R} : y \in C((\tau_k, \tau_{k+1}], \mathbb{R})\}$$

and there exist $y(\tau_k^-)$ and $y(\tau_k^+), k = 1, \cdots, r$ with $y(\tau_k^-) = y(\tau_k)$.

Furthermore, consider the space:

$$\mathcal{PC}^1_\delta(J, \mathbb{R}) = \{y : J \to \mathbb{R} : \delta y \in PC(J, \mathbb{R})\}$$

such that $\delta y(\tau_k^+)$ and $\delta y(\tau_k^-)$ exist and δy is left continuous at τ_k for $k = 1, \cdots, r$ and $\delta = \tau^{1-\rho} d/d\tau$. The space $\mathcal{PC}^1_\delta(J, \mathbb{R})$ equipped with the norm:

$$||y|| = \sup_{\tau \in J}\{|y(\tau)|_{\mathcal{PC}} + |\delta y(\tau)|_{\mathcal{PC}^1_\delta}\} = ||y(\tau)||_{\mathcal{PC}} + ||\delta y(\tau)||_{\mathcal{PC}^1_\delta}.$$

Furthermore, we recall that:

$$\mathcal{AC}^n(J, \mathbb{R}) = \{h : J \to \mathbb{R} : h, h', ..., h^{n-1} \in \mathcal{C}(J, R)\}$$

and $h^{(n-1)}$ is absolutely continuous.

For $0 \le \varepsilon < 1$, we define the space:

$$\mathcal{C}_{\varepsilon,\rho}(J, \mathbb{R}) = \{f : J \to \mathbb{R} : (\tau^\rho - a^\rho)^\varepsilon f(\tau) \in \mathcal{C}(J, \mathbb{R})\}$$

endowed with the norm

$$||f||_{\mathcal{C}_{\varepsilon,\rho}} = ||(\tau^\rho - a^\rho)^\varepsilon f(\tau)||_{\mathcal{C}}.$$

Furthermore, we define a class of functions f that is absolutely continuous $\delta^{n-1}, n \in \mathbb{N}$ derivative, denoted by $\mathcal{AC}^n_\delta(J, \mathbb{R})$ as follows:

$$\mathcal{AC}^n_\delta(J, \mathbb{R}) = \left\{f : J \to \mathbb{R} : \delta^k f \in \mathcal{AC}(J, \mathbb{R}), \delta = \tau^{1-\rho}\frac{d}{d\tau}, k = 0, 1, \cdots, n-1\right\}$$

Equipped with the norm

$$||f||_{\mathcal{C}^n_\delta} = \sum_{k=0}^{n-1} ||\delta^k f||_{\mathcal{C}}.$$

Generally, a space of functions that is endowed with the norm

$$||f||_{\mathcal{C}^n_{\delta,\varepsilon}} = \sum_{k=0}^{n-1} ||\delta^k f||_{\mathcal{C}} + ||\delta^n f||_{\mathcal{C}_{\varepsilon,\rho}}$$

is defined by

$$\mathcal{C}^n_{\delta,\varepsilon}(J,\mathbb{R}) = \{f : J \to \mathbb{R} : f \in \mathcal{AC}^n_\delta(J,\mathbb{R}), \delta^n f \in \mathcal{C}_{\varepsilon,\rho}(J,\mathbb{R})\}.$$

Note that $\mathcal{C}^n_{\delta,0} = \mathcal{C}^n_\delta$.

Definition 1 ([30])**.** *The left-sided and right-sided generalized fractional integrals of order $\alpha > 0$ and type $0 < \rho \le 1$ are defined, respectively, by:*

$$I^{\alpha,\rho}_{a^+} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} f(t)dt,$$
$$I^{\alpha,\rho}_{b^-} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} f(t)dt.$$

Definition 2 ([31])**.** *Let $n = [\alpha] + 1$, $n \in \mathbb{N}$, $0 \le a < b < \infty$ and $f \in \mathcal{AC}^n_\delta[a,b]$. The left-sided and right-sided Generalized Liouville–Caputo-type (Katugampola) fractional derivatives of order $\alpha > 0$ and type $0 < \rho \le 1$ are defined via the above generalized integrals, respectively, as*

$$({}^cD^{\alpha,\rho}_{a^+} f)(x) = \left(I^{n-\alpha,\rho}_{a^+}\left(x^{1-\rho}\frac{d}{dx}\right)^n f\right)(x) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n f(t)dt,$$
$$({}^cD^{\alpha,\rho}_{b^-} f)(x) = \left(I^{n-\alpha,\rho}_{b^-}\left(-x^{1-\rho}\frac{d}{dx}\right)^n f\right)(x) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_x^b \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\left(-t^{1-\rho}\frac{d}{dt}\right)^n f(t)dt.$$

Lemma 1 ([31])**.** *Let $n - 1 < \alpha \le n$; $n \in \mathbb{N}$ and $f \in \mathcal{AC}^n_\delta[a,b]$ or $f \in \mathcal{C}^n_\delta[a,b]$. Then,*

$$I^{\alpha,\rho}_{a^+}{}^cD^{\alpha,\rho}_{a^+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!}\left(\frac{t^\rho - a^\rho}{\rho}\right)^k,$$
$$I^{\alpha,\rho}_{b^-}{}^cD^{\alpha,\rho}_{b^-} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k\delta^k f(b)}{k!}\left(\frac{b^\rho - t^\rho}{\rho}\right)^k.$$

In particular, for $1 < \alpha \le 2$, we have:

$$I^{\alpha,\rho}_{a^+}{}^cD^{\alpha,\rho}_{a^+} f(x) = f(x) - f(a) - \frac{t^\rho - a^\rho}{\rho}\delta f(a),$$
$$I^{\alpha,\rho}_{b^-}{}^cD^{\alpha,\rho}_{b^-} f(x) = f(x) - f(b) + \frac{b^\rho - t^\rho}{\rho}\delta f(b).$$

Lemma 2. *Let $1 < \beta < 2$ and $v : J \to \mathbb{R}$ be an integrable function. Then, there is a solution to the linear problem:*

$$\begin{aligned}
{}^cD^{\beta,\rho}_{s_r} y(\tau) &= v(\tau) && \tau \in (s_r, \tau_{r+1}], r = 0,1,\cdots,k\\
y(\tau) &= \Phi_r(\tau, y(\tau), y(\tau_r - 0)), && \tau \in (\tau_r, s_r], r = 1,2,\cdots,k\\
\tau^{1-\rho}y'(\tau) &= \Psi_r(\tau, y(\tau), y(\tau_r - 0)), && \tau \in (\tau_r, s_r], r = 1,2,\cdots,k\\
y(0) &= y_0, \quad \lim_{\tau\to 0}\tau^{1-\rho}y'(\tau) = y_1, && y_0, y_1 \in \mathbb{R}
\end{aligned} \tag{2}$$

given by:

$$y(\tau) = \begin{cases} \dfrac{\rho^{1-\beta}}{\Gamma(\beta)} \displaystyle\int_0^{\tau} t^{\rho-1}(\tau^{\rho} - t^{\rho})^{\beta-1} v(t)dt + y_0 + \tfrac{y_1}{\rho}\tau^{\rho}, & \tau \in [0, \tau_1], \\ \Phi_r(\tau, y(\tau), y(\tau_r - 0)), & \tau \in (\tau_r, s_r], \\ \dfrac{\rho^{1-\beta}}{\Gamma(\beta)} \displaystyle\int_{s_r}^{\tau} t^{\rho-1}(\tau^{\rho} - t^{\rho})^{\beta-1} h(t)dt + \Phi_r(s_r, y(s_r), y(\tau_r - 0)) & \\ + \dfrac{\tau^{\rho} - s_r^{\rho}}{\rho} \Psi_r(s_r, y(s_r), y(\tau_r - 0)), & \tau \in (s_r, \tau_{r+1}]. \end{cases} \tag{3}$$

Proof. Applying the operator $I_{s_r}^{\beta,\rho}$ to fractional differential equation in (2) and using Lemma 1, we have:

$$y(\tau) = I_{s_r}^{\beta,\rho} v(\tau) + c_{1,r} + c_{2,r} \frac{\tau^{\rho} - s_r^{\rho}}{\rho} \qquad \text{and} \qquad \tau^{1-\rho} y'(\tau) = I_{s_r}^{\beta-1,\rho} v(\tau) + c_{2,r}$$

where $c_{1,r}, c_{2,r} \in \mathbb{R}$, $r = 0, 1, \cdots, k$ are constants to be determined.

- For $\tau \in [0, \tau_1]$, we obtain:

  $$y(\tau) = I_0^{\beta,\rho} v(\tau) + c_{1,0} + c_{2,0} \frac{\tau^{\rho}}{\rho} \qquad \text{and} \qquad \tau^{1-\rho} y'(\tau) = I_0^{\beta-1,\rho} v(\tau) + c_{2,0}.$$

  Applying the initial conditions $y(0) = y_0$ and $\lim_{\tau \to 0} \tau^{\rho-1} y'(\tau) = y_1$ give $c_{1,0} = y_0$ and $c_{2,0} = y_1$ which imply that:

  $$y(\tau) = I_0^{\beta,\rho} v(\tau) + y_0 + y_1 \frac{\tau^{\rho}}{\rho} \qquad \text{and} \qquad \tau^{1-\rho} y'(\tau) = I_0^{\beta-1,\rho} v(\tau) + y_1.$$

- For $\tau \in (\tau_1, s_1]$. Then,

  $$y(\tau) = \Phi_1(\tau, y(\tau), y(\tau_1 - 0)) \qquad \text{and} \qquad y'(\tau) = \tau^{\rho-1} \Psi_1(\tau, y(\tau), y(\tau_1 - 0)).$$

- For $\tau \in (s_1, \tau_2]$. Then,

  $$y(\tau) = I_{s_1}^{\beta,\rho} v(\tau) + c_{1,1} + c_{2,1} \frac{\tau^{\rho} - s_1^{\rho}}{\rho} \qquad \text{and} \qquad \tau^{1-\rho} y'(\tau) = I_{s_1}^{\beta-1,\rho} v(\tau) + c_{2,1}.$$

  Due to the previous impulsive conditions, we get

  $$c_{1,1} = \Phi_1(s_1, y(s_1), y(\tau_1 - 0)) \qquad \text{and} \qquad c_{2,1} = \Psi_1(s_1, y(s_1), y(\tau_1 - 0))$$

  which imply that

  $$y(\tau) = I_{s_1}^{\beta,\rho} v(\tau) + \Phi_1(s_1, y(s_1), y(\tau_1 - 0)) + \Psi_1(s_1, y(s_1), y(\tau_1 - 0)) \frac{\tau^{\rho} - s_1^{\rho}}{\rho},$$
  $$\tau^{1-\rho} y'(\tau) = I_{s_1}^{\beta-1,\rho} v(\tau) + \Psi_1(s_1, y(s_1), y(\tau_1 - 0)).$$

- By similar process. For $\tau \in (s_r, \tau_{r+1}]$. Then,

  $$y(\tau) = I_{s_r}^{\beta,\rho} v(\tau) + \Phi_r(s_r, y(s_r), y(\tau_r - 0)) + \frac{\tau^{\rho} - s_r^{\rho}}{\rho} \Psi_r(s_r, y(s_r), y(\tau_r - 0)).$$

Hence, from the previous, we obtain the solution (3). By direct computation, the converse follows. The proof is complete. □

Next, we present the concept of Ulam stability for problem (1). First, consider $\mathcal{E} = \mathcal{PC}^1_\delta(J,\mathbb{R}) \cap \mathcal{AC}^2_\delta(J,\mathbb{R})$ with $y \in \mathcal{E}$ and $\epsilon > 0$. Let us introduce the following inequality

$$\begin{cases} \|{}^cD^{\beta,\rho}_{s_r} y(\tau) - h(\tau)\| & \leq \epsilon, \qquad \tau \in (s_r, \tau_{r+1}], r = 0,1,\cdots,k \\ \|y(\tau) - \Phi_r\| & \leq \epsilon, \qquad \tau \in (\tau_r, s_r], r = 1,\cdots,k \\ \|\tau^{1-\rho} y'(\tau) - \Psi_r\| & \leq \epsilon, \qquad \tau \in (\tau_r, s_r], r = 1,\cdots,k \end{cases} \tag{4}$$

Definition 3 ([32]). *If there is a constant $\Lambda > 0$ and $\epsilon > 0$ such that for any solution $\widetilde{y} \in \mathcal{E}$ of the inequality (4), there is a unique solution $y \in \mathcal{E}$ to the problem (1) fulfilling*

$$\|\widetilde{y}(\tau) - y(\tau)\| \leq \Lambda\epsilon.$$

Then the problem (1) is said to be UH stable.

Definition 4 ([32]). *If there is a function $\mu \in (\mathbb{R}^+, \mathbb{R}^+), \mu(0) = 0$, for $\epsilon > 0$ such that for any solution $\widetilde{y} \in \mathcal{E}$ of the inequality (4), there is a unique solution $y \in \mathcal{E}$ to the problem (1) fulfilling*

$$\|\widetilde{y}(\tau) - y(\tau)\| \leq \mu(\epsilon).$$

Then the problem (1) is said to be GUH stable.

Remark 2. *If one has a function $\varrho \in \mathcal{E}$ together with a sequences $\varrho_r, r = 0,\cdots,r$ dependent on y. Then $y \in \mathcal{E}$ is called a solution of the inequality (4) such that:*

$$\begin{array}{lll} (a) & |\varrho(\tau)| \leq \epsilon, |\varrho_r| \leq \epsilon, & \tau \in J, r = 0,\cdots,k \\ (b) & {}^cD^{\beta,\rho}_{s_r}\widetilde{y}(\tau) = \widetilde{h}(\tau) + \varrho(\tau), & \tau \in (s_r, \tau_{r+1}], r = 0,1,\cdots,k \\ (c) & \widetilde{y}(\tau) = \Phi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0) + \varrho_r, & \tau \in (\tau_r, s_r], r = 1,2,\cdots,k \\ (d) & \tau^{1-\rho}\widetilde{y}'(\tau) = \Psi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0) + \varrho_r, & \tau \in (\tau_r, s_r], r = 1,2,\cdots,k. \end{array}$$

3. Existence and Uniqueness Results

Our results for uniqueness and existence for problem (1) are presented in this section. By using Lemma 2, we convert the non-instantaneous fractional differential Equation (1) into a fixed point problem. define the operator $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ by:

$$\mathcal{G}y(\tau) = \begin{cases} \dfrac{\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_0^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1} h(t)dt + y_0 + \frac{y_1}{\rho}\tau^\rho, & \tau \in [0, \tau_1], \\ \Phi_r(\tau, y(\tau), y(\tau_r - 0)), & \tau \in (\tau_r, s_r], \\ \dfrac{\rho^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_{s_r}^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1} h(t)dt + \Phi_r(s_r, y(s_r), y(\tau_r - 0)) & \\ +\dfrac{\tau^\rho - s_r^\rho}{\rho}\Psi_r(s_r, y(s_r), y(\tau_r - 0)), & \tau \in (s_r, \tau_{r+1}]. \end{cases} \tag{5}$$

where $h(\tau) = h(\tau, y(\tau), \tau^{1-\rho}y'(\tau))$.

To explain and prove our main results, we first introduce these hypotheses. Consider the following

($\mathfrak{H}_1$) The function $h : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\Phi_r, \Psi_r : [\tau_r, s_r] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions $\forall r = 1,\cdots,k$ and $k \in \mathbb{N}$.

($\mathfrak{H}_2$) $|\widehat{h}(\tau)| = |h(\tau, y, \tau^{1-\rho}y')| \leq q(\tau)v(|y|)$, where $q \in C([0,T], \mathbb{R}^+)$ and $v : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

($\mathfrak{H}_3$) There exist constants $\vartheta_r > 0, \vartheta^*_r > 0,\ r = 1,\cdots,k; k \in \mathbb{N}$ such that

$$|\Phi_r(\tau, y, v)| \leq \vartheta_r, \qquad \text{and} \qquad |\Psi_r(\tau, y, v)| \leq \vartheta^*_r$$

$\forall \tau \in [\tau_r, s_r] \quad , \quad y, v \in \mathbb{R}.$

($\mathfrak{H}_4$) There exist $\mathcal{A} > 0$ satisfies $||y||_{\mathcal{E}} \neq \mathcal{A}$ for some $y \in \mathcal{E}$.

($\mathfrak{H}_5$) There exist positive constants $\kappa_{1r}, \kappa_{2r}, \kappa^*_{1r}$ and κ^*_{2r}, $r = 1, \cdots, k; k \in \mathbb{N}$ such that:

$$
\begin{aligned}
|\Phi_r(\tau, y_1, v_1) - \Phi_r(\tau, y_2, v_2)| &\leq \kappa_{1r}|y_1 - y_2| + \kappa_{2r}|v_1 - v_2|, \\
|\Psi_r(\tau, y_1, v_1) - \Psi_r(\tau, y_2, v_2)| &\leq \kappa^*_{1r}|y_1 - y_2| + \kappa^*_{2r}|v_1 - v_2|
\end{aligned}
$$

for each $\tau \in [\tau_r, s_r]$ and $y_1, y_2, v_1, v_2 \in \mathbb{R}$.

($\mathfrak{H}_6$) There exists $\mathcal{L} > 0$ satisfies

$$
|h(\tau, y, \delta y) - h(\tau, u, \delta u)| \leq \mathcal{L}(|y - u| + \delta|y - u|)
$$

$\forall \tau \in [0, T]$ and $y, u \in \mathbb{R}$.

Below are the short constants that we will use later to simplify handling:

$$
\Omega = \Omega(\beta) + \Omega(\beta - 1) \tag{6}
$$

$$
\Omega_r = \Omega_r(\beta) + \Omega_r(\beta - 1), \tag{7}
$$

$$
\mathcal{Q} = \frac{\mathcal{A}}{\Omega\|q\|v(\mathcal{A}) + |y_0| + \frac{|y_1|}{\rho}(\rho + \tau_1^\rho)}, \tag{8}
$$

$$
\mathcal{Q}_{1r} = \frac{\mathcal{A}}{\vartheta_r + \vartheta_r^*}, \tag{9}
$$

$$
\mathcal{Q}_{2r} = \frac{\mathcal{A}}{\Omega_r\|q\|v(\mathcal{A}) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)} \tag{10}
$$

where $r = 1, 2, \cdots, k; k \in \mathbb{N}$,

$$
\Omega(\beta) = \frac{\tau_1^{\rho\beta}}{\rho^\beta \Gamma(\beta + 1)} \quad \text{and} \quad \Omega_r(\beta) = \frac{(T^\rho - s_r^\rho)^\beta}{\rho^\beta \Gamma(\beta + 1)}.
$$

Lemma 3 ([33,34]). *(Leray–Schauder nonlinear alternative) Assume that $\mathbb{E}$ is a Banach space, B is a convex closed subset of $\mathbb{E}$, and $Y \subset B$ is an open subset and $0 \in Y$. If $\mathcal{F} : \overline{Y} \to B$ is continuous and compact, then either*

- *In $\overline{Y}$, $\mathcal{F}$ has a fixed point; or*
- *For some $\lambda \in (0, 1)$, there exists $y \in \partial Y$ and $y = \lambda \mathcal{F} y$.*

Theorem 1. *Consider Hypotheses ($\mathfrak{H}_1$)–($\mathfrak{H}_4$) satisfied. If*

$$
\max_r\{\mathcal{Q}, \mathcal{Q}_{1r}, \mathcal{Q}_{2r}\} > 1
$$

where $\mathcal{Q}$, $\mathcal{Q}_{1r}$ and $\mathcal{Q}_{2r}$ are given by Equations (8), (9) and (10), respectively. Then, the problem in Equation (1) has at least one solution in $[0, T]$.

Proof. Verifying the hypotheses of Leray–Schauder nonlinear alternative involves a number of steps. The first step is to demonstrate that the operator $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ defined by Equation (5) maps bounded sets into bounded sets in $\mathcal{E}$. In other word, we show that for a positive number ω, there exists a positive constant $\mathcal{I}$ such that $\|\mathcal{G}y\|_{\mathcal{E}} \leq \mathcal{I}$ for any $y \in B_\omega$ where B_ω is a closed bounded set defined as

$$
B_\omega = \left\{(y, \delta y) : y \in \mathcal{E} \wedge \|y\|_{\mathcal{E}} = \|y\|_{\mathcal{PC}} + \|\delta y\|_{\mathcal{PC}^1_\delta} \leqslant \omega\right\}
$$

with the radius:

$$
\omega \geqslant \max\left\{\Omega\|q\|v(\omega) + |y_0| + \frac{|y_1|}{\rho}(\rho + \tau_1^\rho), \vartheta_r + \vartheta_r^*, \Omega_r\|q\|v(\omega) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)\right\}.
$$

Then, in light of $(\mathfrak{H}_2)$ and $(\mathfrak{H}_3)$, we have

- **Case I.** For each $\tau \in [0, \tau_1]$ and $(y, \delta y) \in B_\omega$. Using (6), we have

$$\|\mathcal{G}y\|_{\mathcal{PC}} \leq \sup_{\tau \in [0,\tau_1]} I_{0^+}^{\beta,\rho} \left|\widehat{h}(t)\right| + |y_0| + \left|\frac{y_1}{\rho}\tau^\rho\right| \leq \Omega(\beta)||q||v(\omega) + |y_0| + \frac{|y_1|}{\rho}\tau_1^\rho.$$

Similarly, one can establish that

$$\|\delta\mathcal{G}y\|_{\mathcal{PC}_\delta^1} \leq \sup_{\tau \in [0,\tau_1]} I_{0^+}^{\beta-1,\rho} \left|\widehat{h}(t)\right| + |y_1| \leq \Omega(\beta - 1)\|q\|v(\omega) + |y_1|.$$

Consequently, we have

$$\|\mathcal{G}y\|_{\mathcal{E}} \leq \Omega\|q\|v(\omega) + |y_0| + \frac{|y_1|}{\rho}(\rho + \tau_1^\rho) := \mathcal{I}_1.$$

- **Case II.** For each $\tau \in (\tau_r, s_r],\ r = 1, 2, \cdots, k$ and $(y, \delta y) \in B_\omega$, we get

$$\|\mathcal{G}y\|_{\mathcal{E}} = \|\mathcal{G}y\|_{\mathcal{PC}} + \|\delta\mathcal{G}\|_{\mathcal{PC}_\delta^1} \leq \vartheta_r + \vartheta_r^* := \mathcal{I}_{2r}.$$

- **Case III.** For each $\tau \in (s_r, \tau_{r+1}],\ r = 1, 2, \cdots, k$ and $(y, \delta y) \in B_\omega$. Using (7), we have

$$\begin{aligned}\|\mathcal{G}y\|_{\mathcal{PC}} \leq& \sup_{\tau \in (s_r,\tau_{r+1}]} I_{s_r}^{\beta,\rho} \left|\widehat{h}(t)\right| + \left|\Phi_r(s_r, y(s_r), y(\tau_r - 0))\right| + \left|\frac{\tau^\rho - s_r^\rho}{\rho}\Psi_r(s_r, y(s_r), y(\tau_r - 0))\right| \\ \leq& \Omega_r(\beta)\|q\|v(\omega) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(T^\rho - s_r^\rho).\end{aligned}$$

In a similar manner, one can obtain:

$$\|\delta\mathcal{G}y\|_{\mathcal{PC}_\delta^1} \leq \Omega_r(\beta - 1)\|q\|v(\omega) + \vartheta_r^*.$$

Hence, we deduce that:

$$\|\mathcal{G}y\|_{\mathcal{E}} \leq \Omega_r\|q\|v(\omega) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho) := \mathcal{I}_{3r}.$$

From the above three inequalities, we can conclude that $\|\mathcal{G}y\|_{\mathcal{E}} \leq \mathcal{I}$ where $\mathcal{I} = \max_r\{\mathcal{I}_1, \mathcal{I}_{2r}, \mathcal{I}_{3r}\}$. Thus, the operator $\mathcal{G}$ maps bounded sets into bounded sets of the space $\mathcal{E}$.

In the next step, we check that the operator $\mathcal{G}$ maps bounded sets into equicontinuous sets in $\mathcal{E}$.Considering the condition $(\mathfrak{H}_1)$, $\mathcal{G}$ is continuous.

- **Case I.** For each $0 \leqslant \zeta_1 < \zeta_2 \leqslant \tau_1$ and $(y, \delta y) \in B_\omega$, we obtain that

$$\begin{aligned}|(\mathcal{G}y)(\zeta_2) - (\mathcal{G}y)(\zeta_1)| \leq& \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_1} t^{\rho-1}\left[(\zeta_2^\rho - t^\rho)^{\beta-1} - (\zeta_1^\rho - t^\rho)^{\beta-1}\right]|\widehat{h}(t)|dt \\ &+ \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{\zeta_1}^{\zeta_2} t^{\rho-1}(\zeta_2^\rho - t^\rho)^{\beta-1}|\widehat{h}(t)|dt + \frac{|y_1|}{\rho}\left(\zeta_2^\rho - \zeta_1^\rho\right) \\ \leq& ||q||v(|y|)\frac{1}{\rho^\beta\Gamma(\beta+1)}\left(\zeta_2^{\rho\beta} - \zeta_1^{\rho\beta}\right) + \frac{|y_1|}{\rho}\left(\zeta_2^\rho - \zeta_1^\rho\right) \\ \Rightarrow& 0 \quad \text{as} \quad \zeta_2 \to \zeta_1.\end{aligned}$$

Similarly, one can establish that:

$$
\begin{aligned}
&|(\delta\mathcal{G}y)(\zeta_2)-(\delta\mathcal{G}y)(\zeta_1)| \\
&\leq ||q||v(|y|)\frac{\rho^{2-\beta}}{\Gamma(\beta-1)}\left(\int_0^{\zeta_1} t^{\rho-1}\left[(\zeta_1^\rho-t^\rho)^{\beta-2}-(\zeta_2^\rho-t^\rho)^{\beta-2}\right]dt+\int_{\zeta_1}^{\zeta_2} t^{\rho-1}(\zeta_2^\rho-t^\rho)^{\beta-2}dt\right) \\
&\leq 2||q||v(|y|)\frac{1}{\rho^{\beta-1}\Gamma(\beta)}\left(\zeta_2^\rho-\zeta_1^\rho\right)^{\beta-1} \\
&\Rightarrow 0 \quad \text{as} \quad \zeta_2\to\zeta_1.
\end{aligned}
$$

- **Case II.** For each$\tau_r \leqslant \zeta_1 < \zeta_2 < s_r, r = 1, 2, \cdots, k$ and $(y, \delta y) \in B_\omega$, we have

$$
\begin{aligned}
|(\mathcal{G}y)(\zeta_2)-(\mathcal{G}y)(\zeta_1)| &\leq |\Phi_r(\zeta_2, y(\zeta_2), y(\tau_r-0))|-|\Phi_r(\zeta_1, y(\zeta_1), y(\tau_r-0))| \\
|(\delta\mathcal{G}y)(\zeta_2)-(\delta\mathcal{G}y)(\zeta_1)| &\leq |\Psi_r(\zeta_2, y(\zeta_2), y(\tau_r-0))|-|\Psi_r(\zeta_1, y(\zeta_1), y(\tau_r-0))|.
\end{aligned}
$$

Due to the continuity of both functions. It is clear that the above inequality approaches zero when letting $\zeta_2 \to \zeta_1$.

- **Case III.** For each $s_r \leqslant \zeta_1 < \zeta_2 < \tau_{r+1}, r = 1, 2, \cdots, k$, and $(y, \delta y) \in B_\omega$, we get

$$
\begin{aligned}
|(\mathcal{G}y)(\zeta_2)-(\mathcal{G}y)(\zeta_1)| &\leq \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{s_r}^{\zeta_1} t^{\rho-1}\left[(\zeta_2^\rho-t^\rho)^{\beta-1}-(\zeta_1^\rho-t^\rho)^{\beta-1}\right]\left|\widehat{h}(t)\right|dt \\
&+\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{\zeta_1}^{\zeta_2} t^{\rho-1}(\zeta_2^\rho-t^\rho)^{\beta-1}\left|\widehat{h}(t)\right|dt+\frac{\zeta_2^\rho-\zeta_1^\rho}{\rho}|\Psi_r(s_r, y(s_r), y(\tau_r-0))| \\
&\leq ||q||v(|y|)\frac{1}{\rho^\beta\Gamma(\beta+1)}\left[\left(\zeta_2^\rho-s_r^\rho\right)^\beta-\left(\zeta_1^\rho-s_r^\rho\right)^\beta\right]+\frac{\zeta_2^\rho-\zeta_1^\rho}{\rho}|\Psi_r(s_r, y(s_r), y(\tau_r-0))| \\
&\Rightarrow 0 \quad \text{as} \quad \zeta_2\to\zeta_1.
\end{aligned}
$$

Moreover, we have:

$$
\begin{aligned}
&|(\delta\mathcal{G}y)(\zeta_2)-(\delta\mathcal{G}y)(\zeta_1)| \\
&\leq ||q||v(|y|)\frac{\rho^{2-\beta}}{\Gamma(\beta-1)}\left(\int_{s_r}^{\zeta_1} t^{\rho-1}\left[(\zeta_1^\rho-t^\rho)^{\beta-2}-(\zeta_2^\rho-t^\rho)^{\beta-2}\right]dt+\int_{\zeta_1}^{\zeta_2} t^{\rho-1}(\zeta_2^\rho-t^\rho)^{\beta-2}dt\right) \\
&\leq ||q||v(|y|)\frac{1}{\rho^{\beta-1}\Gamma(\beta)}\left[2(\zeta_2^\rho-\zeta_1^\rho)^{\beta-1}+(\zeta_1^\rho-s_r^\rho)^{\beta-1}-(\zeta_2^\rho-s_r^\rho)^{\beta-1}\right] \\
&\Rightarrow 0 \quad \text{as} \quad \zeta_2\to\zeta_1.
\end{aligned}
$$

As a result of the three inequalities above, we conclude that $\|(\mathcal{G}y)(\zeta_2)-(\mathcal{G}y)(\zeta_1)\|_\mathcal{E} \to 0$ independently of $(y, \delta y) \in B_\omega$ as $\zeta_2 \to \zeta_1$. Using the preceding arguments and the Arzela-Ascoli theorem, the operator $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ is completely continuous.

Finally, we show that there exist an open set $Y \subset \mathcal{E}$ with $y \neq \lambda\mathcal{G}y$ for $\lambda \in (0, 1)$ and $y \in \partial Y$. Consider the equation $y = \lambda\mathcal{G}y$ for $\lambda \in (0, 1)$. Then based on **Step 1** , we have the following cases:

- **Case I.** For each $\tau \in [0, \tau_1]$, one has

$$
\|y(\tau)\| = \|\lambda(\mathcal{G}y)(\tau)\| \leq \Omega\|q\|v(\|y\|)+|y_0|+\frac{|y_1|}{\rho}(\rho+\tau_1^\rho)
$$

which implies that:

$$
\frac{\|y\|_\mathcal{E}}{\Omega\|q\|v(\|y\|_\mathcal{E})+|y_0|+\frac{|y_1|}{\rho}(\rho+\tau_1^\rho)} \leq 1. \tag{11}
$$

- **Case II.** For each $\tau \in (\tau_r, s_r], r = 1, 2, \cdots, k$, one has

$$\|y(\tau)\| = \|\lambda(\mathcal{G}y)(\tau)\| \leq \vartheta_r + \vartheta_r^*$$

which implies that:

$$\frac{\|y\|_{\mathcal{E}}}{\vartheta_r + \vartheta_r^*} \leq 1. \tag{12}$$

- **Case III.** For each $\tau \in (s_r, \tau_{r+1}], r = 1, \cdots, k$, we obtain:

$$\|y(\tau)\| = \|\lambda(\mathcal{G}y)(\tau)\| \leq \Omega_r\|q\|v(\|y\|) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)$$

which implies that:

$$\frac{\|y\|_{\mathcal{E}}}{\Omega_r\|q\|v(\|y\|_{\mathcal{E}}) + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)} \leq 1. \tag{13}$$

If (11)–(13) are combined with $(\mathfrak{H}_4)$ and given condition $\max_r\{\mathcal{Q}, \mathcal{Q}_{1r}, \mathcal{Q}_{2r}\} > 1$. A positive number $\mathcal{A}$ such that $\|y\|_{\mathcal{E}} \neq \mathcal{A}$ can be found. Create a set $Y = \{y \in \mathcal{E} : \|y\|_{\mathcal{E}} < \mathcal{A}\}$ with the operator $\mathcal{G} : \overline{Y} \to \mathcal{E}$ being continuous and completely continuous. In light of the choice of Y, there is no $y \in \partial Y$ satisfying $y = \lambda\mathcal{G}y$ for $\lambda \in (0, 1)$. Thus, it follows from the nonlinear alternative of Leray–Schauder, the operator $\mathcal{G}$ has a fixed point $y \in \overline{Y}$ that corresponds to a solution to Equation (1). □

Using the contraction mapping principle, we ensure the uniqueness of solution to problem (1).

Theorem 2. *Suppose that Hypotheses* ($\mathfrak{H}_1, \mathfrak{H}_3, \mathfrak{H}_5$ *and* $\mathfrak{H}_6$) *are satisfied. If*

$$\Delta = \max_r \left\{\mathcal{L}\Omega, \mathcal{K}_r + \mathcal{K}_r^*, \mathcal{L}\Omega_r + \mathcal{K}_r + \frac{\mathcal{K}_r^*}{\rho}(\rho + T^\rho - s_r^\rho)\right\} < 1 \tag{14}$$

where $\mathcal{K}_r = \kappa_{1r} + \kappa_{2r}$ *and* $\mathcal{K}_r^* = \kappa_{1r}^* + \kappa_{2r}^*$. *Thus, the non-instantaneous impulsive fractional differential Equation (1) has a unique solution on* J.

Proof. Let us consider a set:

$$B_r = \left\{(y, \delta y) : y \in \mathcal{E} \wedge \|y\|_{\mathcal{E}} = \|y(\tau)\|_{\mathcal{PC}} + \|\delta y(\tau)\|_{\mathcal{PC}_\delta^1} \leqslant r\right\}$$

with radius

$$r \geq \max_r \left\{\frac{\Omega \mathrm{N} + |y_0| + \frac{|y_1|}{\rho}(\rho + \tau_1^\rho)}{1 - \mathcal{L}\Omega}, \vartheta_r + \vartheta_r^*, \frac{\Omega_r \mathrm{N} + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)}{1 - \mathcal{L}\Omega_r}\right\}$$

where $\sup_{\tau \in [0,T]} |h(\tau, 0, 0)| = \mathrm{N}$. Clearly, $\mathcal{G}$ is well defined and $\mathcal{G}y \in \mathcal{E}$ for all $y \in \mathcal{E}$. All that remains is to demonstrate that $\mathcal{G}$ is a contraction mapping. Thus, three cases are considered:

- **Case I.** For each $\tau \in [0, \tau_1]$ and $(y, \delta y), (v, \delta v) \in \mathcal{E}$. Using (6), we get

$$\begin{aligned}\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{PC}} &\leq \sup_{\tau \in [0, \tau_1]} \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}\left|h(t, y, t^{1-\rho}y') - h(t, v, t^{1-\rho}v')\right| dt \\ &\leq \mathcal{L}\Omega(\beta)\|y - v\|.\end{aligned}$$

Similarly, we can obtain:

$$\|\delta\mathcal{G}y - \delta\mathcal{G}v\|_{\mathcal{PC}^1_\delta} \leq \mathcal{L}\Omega(\beta-1)\|y-v\|$$

which implies that:

$$\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{E}} \leq \mathcal{L}\Omega\|y-v\|.$$

- **Case II.** For each $\tau \in (\tau_r, s_r], r = 1,2,\cdots,k$ and $(y,\delta y),(v,\delta v) \in \mathcal{E}$, we have:

$$\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{PC}} \leq (\kappa_{1r}+\kappa_{2r})\|y-v\|.$$

In addition:

$$\|\delta\mathcal{G}y - \delta\mathcal{G}v\|_{\mathcal{PC}^1_\delta} \leq (\kappa^*_{1r}+\kappa^*_{2r})\|y-v\|.$$

Consequently, we have:

$$\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{E}} \leq (\mathcal{K}_r+\mathcal{K}^*_r)\|y-v\|.$$

- **Case III.** For each $\tau \in (s_r, \tau_{r+1}], r = 1,2,\cdots,k$ and $(y,\delta y),(v,\delta v) \in \mathcal{E}$. Using (7), we obtain:

$$\begin{aligned}
&\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{PC}} \\
&\leq \sup_{\tau\in(s_r,\tau_{r+1}]} \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{s_r}^{\tau} t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}\left|h(t,y,t^{1-\rho}y') - h(t,v,t^{1-\rho}v')\right|dt \\
&+ |\Phi_r(s_r,y(s_r),y(\tau_r-0)) - \Phi_r(s_r,v(s_r),v(\tau_r-0))| \\
&+ \left|\frac{\tau^\rho - s_r^\rho}{\rho}(\Psi_r(s_r,y(s_r),y(\tau_r-0)) - \Psi_r(s_r,v(s_r),v(\tau_r-0)))\right| \\
&\leq \left[\mathcal{L}\Omega_r(\beta) + \mathcal{K}_r + \frac{\mathcal{K}^*_r}{\rho}(T^\rho - s_r^\rho)\right]\|y-v\|.
\end{aligned}$$

In a similar manner, it can be shown that:

$$\|\delta\mathcal{G}y - \delta\mathcal{G}v\|_{\mathcal{PC}^1_\delta} \leq [\mathcal{L}\Omega_r(\beta-1)+\mathcal{K}^*_r]\|y-v\|$$

which leads to:

$$\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{E}} \leq \left[\mathcal{L}\Omega_r + \mathcal{K}_r + \frac{\mathcal{K}^*_r}{\rho}(\rho + T^\rho - s_r^\rho)\right]\|y-v\|.$$

From the above, we obtain: $\|\mathcal{G}y - \mathcal{G}v\|_{\mathcal{E}} \leq \Delta\|y-v\|$ which, in view of the given condition $\Delta < 1$, shows that the operator $\mathcal{G}$ is a contraction. This implies that the problem in Equation (1) has a unique solution on$[0,T]$, according to the Banach contraction mapping principle. □

4. Stability Analysis

We present results regarding the Ulam–Hyers stability of our problem (1) in this section.

Theorem 3. *Suppose that Hypotheses ($\mathfrak{H}_1$), ($\mathfrak{H}_5$ and ($\mathfrak{H}_6$) are satisfied. Then, the non-instantaneous impulsive fractional differential Equation (1) is Ulam–Hyers stable and Generalized Ulam–Hyers stable if $\Delta < 1$ where Δ is defined as (14).*

Proof. Assuming a unique solution $y \in \mathcal{E}$ to the problem (1) corresponds to any solution $\widetilde{y} \in \mathcal{E}$ of the inequality (4). Then, in light of Lemma 2, we have:

$$y(\tau) = \begin{cases} \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}v(t)dt + y_0 + \frac{y_1}{\rho}\tau^\rho, & \tau \in [0, \tau_1], \\ \Phi_r(\tau, y(\tau), y(\tau_r - 0)), & \tau \in (\tau_r, s_r], \\ \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{s_r}^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}h(t)dt + \Phi_r(s_r, y(s_r), y(\tau_r - 0)) \\ +\frac{\tau^\rho - s_r^\rho}{\rho}\Psi_r(s_r, y(s_r), y(\tau_r - 0)), & \tau \in (s_r, \tau_{r+1}]. \end{cases}$$

Further, if $\widetilde{y}$ is the solution of inequality (4) and using Remark 2, we get:

$$\begin{aligned} {}^cD_{s_r}^{\beta,\rho}\widetilde{y}(\tau) &= \widetilde{h}(\tau) + \varrho(\tau) && \tau \in (s_r, \tau_{r+1}], r = 0, 1, \cdots, k \\ \widetilde{y}(\tau) &= \Phi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0) + \varrho_r, && r = 1, 2, \cdots, k \\ \tau^{1-\rho}\widetilde{y}'(\tau) &= \Psi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0) + \varrho_r, && r = 1, 2, \cdots, k \end{aligned}$$

where $\widetilde{h}(\tau) = h(\tau, \widetilde{y}(\tau), \tau^{1-\rho}\widetilde{y}'(\tau))$ and

$$\widetilde{y}(\tau) = \begin{cases} I_0^{\beta,\rho}\widetilde{h}(\tau) + I_0^{\beta,\rho}\varrho(\tau) + y_0 + \frac{y_1}{\rho}\tau^\rho, & \tau \in [0, \tau_1], \\ \Phi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0)) + \varrho_r, & \tau \in (\tau_r, s_r], \\ I_{s_r}^{\beta,\rho}\widetilde{h}(\tau) + I_{s_r}^{\beta,\rho}\varrho(\tau) + \Phi_r(s_r, \widetilde{y}(s_r), \widetilde{y}(\tau_r - 0)) \\ +\frac{\tau^\rho - s_r^\rho}{\rho}\Psi_r(s_r, \widetilde{y}(s_r), \widetilde{y}(\tau_r - 0)) + \frac{\varrho_r}{\rho}(\rho + \tau^\rho - s_r^\rho), & \tau \in (s_r, \tau_{r+1}]. \end{cases}$$

For each $\tau \in [0, \tau_1]$, we consider:

$$\begin{aligned} \|\widetilde{y}(\tau) - y(\tau)\|_{\mathcal{PC}} &\leq \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}\left|\widetilde{h}(t) - h(t)\right|dt + \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^\tau t^{\rho-1}(\tau^\rho - t^\rho)^{\beta-1}|\varrho(t)|dt \\ &\leq \mathcal{L}\Omega(\beta)\|\widetilde{y} - y\|_{\mathcal{E}} + \epsilon\Omega(\beta). \end{aligned}$$

Similarly, we can obtain:

$$\|\delta\widetilde{y}(\tau) - \delta y(\tau)\|_{\mathcal{PC}_\delta^1} \leq \mathcal{L}\Omega(\beta - 1)\|\widetilde{y} - y\|_{\mathcal{E}} + \epsilon\Omega(\beta - 1)$$

which implies that:

$$\|\widetilde{y}(\tau) - y(\tau)\|_{\mathcal{E}} \leq \mathcal{L}\Omega\|\widetilde{y} - y\|_{\mathcal{E}} + \epsilon\Omega.$$

Or, equivalently,

$$\|\widetilde{y} - y\|_{\mathcal{E}} \leq \frac{\epsilon\Omega}{1 - \mathcal{L}\Omega}, \qquad \mathcal{L}\Omega < 1.$$

For each $\tau \in (\tau_r, s_r], r = 1, 2, \cdots, k$, we consider:

$$\begin{aligned} \|\widetilde{y}(\tau) - y(\tau)\|_{\mathcal{PC}} &\leq |\Phi_r(\tau, \widetilde{y}(\tau), \widetilde{y}(\tau_r - 0)) - \Phi_r(\tau, y(\tau), y(\tau_r - 0))| + |\varrho_r|, \\ &\leq (\kappa_{1r} + \kappa_{2r})\|\widetilde{y} - y\| + \epsilon. \end{aligned}$$

In addition:

$$\|\delta\widetilde{y}(\tau) - \delta y(\tau)\|_{\mathcal{PC}_\delta^1} \leq (\kappa_{1r}^* + \kappa_{2r}^*)\|\widetilde{y} - y\|_{\mathcal{E}} + \epsilon.$$

Consequently, we have:

$$\|\widetilde{y}-y\|_{\mathcal{E}} \leq (\mathcal{K}_r+\mathcal{K}_r^*)\|\widetilde{y}-y\|_{\mathcal{E}}+2\epsilon.$$

Or, equivalently:

$$\|\widetilde{y}-y\|_{\mathcal{E}} \leq \frac{2\epsilon}{1-(\mathcal{K}_r+\mathcal{K}_r^*)}, \qquad \mathcal{K}_r+\mathcal{K}_r^*<1.$$

For each $\tau\in(s_r,\tau_{r+1}], r=1,2,\cdots,k$, we consider:

$$\begin{aligned}\|\widetilde{y}(\tau)-y(\tau)\|_{\mathcal{PC}} \leq & \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{s_r}^{\tau} t^{\rho-1}(\tau^\rho-t^\rho)^{\beta-1}\left|\widetilde{h}(t)-h(t)\right|dt+\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{s_r}^{\tau} t^{\rho-1}(\tau^\rho-t^\rho)^{\beta-1}|\varrho(t)|dt \\ &+|\Phi_r(\tau,\widetilde{y}(\tau),\widetilde{y}(\tau_r-0))-\Phi_r(\tau,y(\tau),y(\tau_r-0))|+|\varrho_r| \\ &+\left|\frac{\tau^\rho-s_r^\rho}{\rho}\right||\Psi_r(\tau,\widetilde{y}(\tau),\widetilde{y}(\tau_r-0))-\Psi_r(\tau,y(\tau),y(\tau_r-0))|+\left|\frac{\tau^\rho-s_r^\rho}{\rho}\varrho_r\right| \\ &\leq \left[\mathcal{L}\Omega_r(\beta)+\mathcal{K}_r+\frac{\mathcal{K}_r^*}{\rho}(T^\rho-s_r^\rho)\right]\|\widetilde{y}-y\|_{\mathcal{E}}+\epsilon(1+\frac{T^\rho-s_r^\rho}{\rho}).\end{aligned}$$

In a similar manner, it can be shown that:

$$\|\delta\widetilde{y}(\tau)-\delta y(\tau)\|_{\mathcal{PC}_\delta^1} \leq [\mathcal{L}\Omega_r(\beta-1)+\mathcal{K}_r^*]\|\widetilde{y}-y\|_{\mathcal{E}}+\epsilon$$

which leads to:

$$\|\widetilde{y}(\tau)-y(\tau)\|_{\mathcal{E}} \leq \frac{(2\rho+T^\rho-s_r^\rho)\varepsilon}{\rho\Big(1-\mathcal{L}\Omega_r-\mathcal{K}_r-\frac{\mathcal{K}_r^*}{\rho}(\rho+T^\rho-s_r^\rho)\Big)}, \quad \mathcal{L}\Omega_r+\mathcal{K}_r+\frac{\mathcal{K}_r^*}{\rho}(\rho+T^\rho-s_r^\rho)<1.$$

Then, for each $\tau\in J$, we obtain:

$$\|\widetilde{y}(\tau)-y(\tau)\|_{\mathcal{E}} \leq \Lambda\epsilon.$$

where $\Lambda=\max_r\left\{\frac{\Omega}{1-\mathcal{L}\Omega},\frac{2}{1-(\mathcal{K}_r+\mathcal{K}_r^*)},\frac{2\rho+T^\rho-s_r^\rho}{\rho\Big(1-\mathcal{L}\Omega_r-\mathcal{K}_r-\frac{\mathcal{K}_r^*}{\rho}(\rho+T^\rho-s_r^\rho)\Big)}\right\}$. $\square$

Thus, the solution of (1) is UH stable if $\Delta<1$. Additionally, by setting $\mu(\epsilon)=\Lambda$ and $\mu(0)=0$. Then, the solution of (1) becomes GUH stable.

5. Applications

In this section, we describe an application of our main results to demonstrate how they can be applied.

Example 1. *Consider the following non-instantaneous impulsive fractional differential equations:*

$$\begin{aligned} {}^cD_{s_r}^{\beta,\rho}y(\tau) &= h(\tau,y(\tau),\delta y(\tau)) && \tau\in(0,\tfrac{1}{3}]\cup(\tfrac{2}{3},1], \\ y(\tau) &= \frac{3}{4}\tau^2+\frac{1}{12}\sin y(\tau)+\frac{1}{8}\cos y(\tau_r-0), && \tau\in(\tfrac{1}{3},\tfrac{2}{3}], \\ \delta y(\tau) &= \frac{3}{2}\tau+\frac{1}{14}\cos y(\tau)+\frac{1}{10}\sin y(\tau_r-0), && \tau\in(\tfrac{1}{3},\tfrac{2}{3}], \\ y(0) &= 0, \qquad \lim_{\tau\to 0}\delta y(\tau)=1 \end{aligned} \tag{15}$$

where $J = [0,1], 0 = s_0 < \tau_1 = \dfrac{1}{3} < s_1 = \dfrac{2}{3} < \tau_2 = 1, \rho = \dfrac{1}{2}, \beta = \dfrac{5}{4}$ *and* $h(\tau, y(\tau), \delta y(\tau))$ *will be determined later. Using the given data, we can find that*

$$\begin{array}{lll} \Omega(\beta) \approx 1.05646621, & \Omega(\beta-1) \approx 1.14365822, & \Omega \approx 2.20012444, \\ \Omega_r(\beta) \approx 0.25212249, & \Omega_r(\beta-1) \approx 0.85871184, & \Omega_r \approx 1.11083434. \end{array}$$

In our example, we take

$$\Phi_1(\tau, y, v) = \frac{3}{4}\tau^2 + \frac{1}{12}\sin y + \frac{1}{8}\cos v,$$
$$\Psi_1(\tau, y, v) = \frac{3}{2}\tau + \frac{1}{14}\cos y + \frac{1}{10}\sin v.$$

It is clear that they are continuous on the interval $(\frac{1}{3}, \frac{2}{3}]$ *which meets the first assumption and satisfy*

$$|\Phi_1(\tau, y, v)| \leq \left|\frac{3}{4}\tau^2\right| + \left|\frac{1}{12}\sin y\right| + \left|\frac{1}{8}\cos v\right| \leq \frac{3}{4}\left(\frac{2}{3}\right)^2 + \frac{1}{12} + \frac{1}{8} = \frac{13}{24},$$
$$|\Psi_1(\tau, y, v)| \leq \left|\frac{3}{2}\tau\right| + \left|\frac{1}{14}\cos y\right| + \left|\frac{1}{10}\sin v\right| \leq 1 + \frac{1}{14} + \frac{1}{10} = \frac{82}{70}$$

for all $\tau \in (\frac{1}{3}, \frac{2}{3}]$ *and* $y, v \in \mathbb{R}$*. These lead to the third assumption is verified with* $\vartheta_1 = 13/24$ *and* $\vartheta_1^* = 82/70$*.*

Theorem 4 (Application to Theorem 1)**.** *The Leray–Schauder nonlinear alternative theorem has been applied in Theorem 1 with the assumptions* $(\mathfrak{H}_1)$*–*$(\mathfrak{H}_3)$*. To illustrate our investigation, let us take*

$$h(\tau, y(\tau), \delta y(\tau)) = \frac{1}{2\sqrt{5-\tau}}\left[\frac{1}{15\pi}\sin(5\pi y) + \frac{3|\delta y(\tau)|}{4(|\delta y(\tau)|+1)}\right].$$

It is obvious that the function h is continuous which meets the first assumption and satisfies

$$|\widehat{h}(\tau)| = |h(\tau, y, \delta y)| \leq \frac{1}{2\sqrt{5-\tau}}\left(\frac{1}{3}\|y\| + \frac{3}{4}\right) := q(\tau)v(\|y\|)$$

where

$$q(\tau) = \frac{1}{2\sqrt{5-\tau}} \qquad \textit{and} \qquad v(\|y\|) = \frac{1}{3}\|y\| + \frac{3}{4}.$$

for all $\tau \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1]$*. It is obvious that the function* $q(\tau)$ *is nondecreasing function which admits the hypothesis* $(\mathfrak{H}_2)$ *with* $||q|| \leq q(1) = 1/4$*. The condition* $(\mathfrak{H}_4)$ *and (11)–(13) imply that*

$$\mathcal{A} > \max_r \left\{ \frac{\frac{3\|q\|}{4}\Omega + \frac{1}{\rho}(\rho + \tau_1^\rho)}{1 - \frac{\|q\|}{3}\Omega}, \vartheta_r + \vartheta_r^*, \frac{\frac{3\|q\|}{4}\Omega_r + \vartheta_r + \frac{\vartheta_r^*}{\rho}(\rho + T^\rho - s_r^\rho)}{1 - \frac{\|q\|}{3}\Omega_r} \right\}$$

$$\mathcal{A} > \max\{3.018702359, 1.713095238, 2.539578874\}$$

$$\mathcal{A} > 3.018702359.$$

Therefore, the conditions of Theorem (1) are satisfied, and consequently,on $[0,1]$ *there exists at least one solution to the boundary value problem (15).*

Theorem 5 (Application to Theorem 2)**.** *To demonstrate Theorem 2, which is based on the Banach fixed point theorem, we take*

$$h(\tau, y(\tau), \delta y(\tau)) = \frac{e^{-2\tau}(|y(\tau)| + |\delta y(\tau)|)}{(1+9e^{\tau})(1 + |y(\tau)| + |\delta y(\tau)|)}$$

It is clear that the function $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and that it fulfills the hypothesis $(\mathfrak{H}_2)$

$$\begin{aligned} |h(\tau, y, \delta y) - h(\tau, u, \delta u)| &\leq \frac{e^{-2\tau}|(|y| + |\delta y|) - (|u| + |\delta u|)|}{(1+9e^{\tau})|(1+|y|+|\delta y|)(1+|u|+|\delta u|)|} \\ &\leq \frac{1}{10}\Big||y| - |u|\Big| + \Big||\delta y| - |\delta u|\Big| \\ &\leq \frac{1}{10}(|y-u| + |\delta y - \delta u|). \end{aligned}$$

with $\mathcal{L} = 1/10$. For all $\tau \in (\frac{1}{3}, \frac{2}{3}]$ and $y_1, y_2, v_1, v_2 \in \mathbb{R}$, we get

$$\begin{aligned} |\Phi_1(\tau, y_1, v_1) - \Phi_1(\tau, y_2, v_2)| &\leq \frac{1}{12}|y_1 - y_2| + \frac{1}{8}|v_1 - v_2|, \\ |\Psi_1(\tau, y_1, v_1) - \Psi_1(\tau, y_2, v_2)| &\leq \frac{1}{14}|y_1 - y_2| + \frac{1}{10}|v_1 - v_2|. \end{aligned}$$

Thus, the condition $(\mathfrak{H}_5)$ of Theorem 2 is satisfied with

$$\begin{array}{lll} \kappa_{11} = \frac{1}{12}, & \kappa_{21} \frac{1}{8}, & \mathcal{K}_1 \approx 0.20833333, \\ \kappa^*_{11} = \frac{1}{14}, & \kappa^*_{22} = \frac{1}{10}, & \mathcal{K}^*_1 \approx 0.17142857. \end{array}$$

In conclusion, we have

$$\begin{aligned} \Delta &= \max_r \left\{ \mathcal{L}\Omega, \mathcal{K}_r + \mathcal{K}^*_r, \mathcal{L}\Omega_r + \mathcal{K}_r + \tfrac{\mathcal{K}^*_r}{\rho}(\rho + T^{\rho} - s_r^{\rho}) \right\} \\ &= \max\{0.22001244, 0.37976190, 0.55376046\} = 0.55376046 < 1. \end{aligned}$$

Hence, the problem in Equations (15) has a unique solution on $[0,1]$ by Theorem 2.

Theorem 6 (Application to Theorem 3)**.** *To demonstrate Theorem 3, we take*

$$h(\tau, y(\tau), \delta y(\tau)) = \frac{|y(\tau)|}{2(\tau+8)(1+|y(\tau)|)} + \frac{|\delta y(\tau)|}{(\tau+16)}$$

It is clear that the function $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and that it fulfills the hypothesis $(\mathfrak{H}_6)$

$$\begin{aligned} |h(\tau, y, \delta y) - h(\tau, u, \delta u)| &\leq \frac{|(|y| - |u|)|}{2(\tau+8)|(1+|y|)(1+|u|)|} + \frac{|(|\delta y| - |\delta u|)|}{(\tau+16)} \\ &\leq \frac{1}{16}\Big||y| - |u|\Big| + \Big||\delta y| - |\delta u|\Big| \\ &\leq \frac{1}{16}(|y-u| + |\delta y - \delta u|). \end{aligned}$$

Clearly the assumptions of Theorem 3 are fulfilled with

$$\mathcal{L} = \frac{1}{16}, \qquad \mathcal{K}_1 \approx 0.20833333, \qquad \mathcal{K}^*_1 \approx 0.17142857.$$

$$\Delta = \max\{0.137507777, 0.37976190, 0.55376046\} = 0.51210450 < 1.$$

In conclusion, we have:

$$\|\widetilde{y} - y\| \leq \Lambda\epsilon, \qquad \tau \in J,$$

where ϵ is any positive real constant, and

$$\Lambda = \max\left\{\frac{\Omega}{1 - \mathcal{L}\Omega}, \frac{2}{1 - (\mathcal{K}_r + \mathcal{K}_r^*)}, \frac{2\rho + T^\rho - s_r^\rho}{\rho\left(1 - \mathcal{L}\Omega_r - \mathcal{K}_r - \frac{\mathcal{K}_r^*}{\rho}(\rho + T^\rho - s_r^\rho)\right)}\right\},$$

$$\Lambda = \max\{2.55089191, 3.22456811, 0.24394774\},$$

$$\Lambda = 3.22456811 > 0.$$

Consequently,

$$\|\widetilde{y} - y\| \leq (3.22456811)\epsilon,$$

Thus, problem (15) is UH stable.

Moreover, by putting $\mu(\epsilon) = (3.22456811)\epsilon$ with $\mu(0) = 0$, problem (15) becomes GUH stable.

6. Conclusions

Our work involved the development of the existence theory and Ulam–Hyers stability of non-instantaneous impulsive BVPs involving Generalized Liouville–Caputo derivatives. This work is based on modern functional analysis techniques. Three conclusions have been obtained: the first two deal with the existence and uniqueness of solutions, while the third concerns the stability analysis of solutions for the given problem. The first existence result is based on a nonlinear Leray–Schauder alternative, while the second is based on the Banach fixed point theorem. The third conclusion establishes a criterion for ensuring various types of Ulam–Hyers stability, that is necessary for nonlinear problems in terms of optimization and numerical solutions and plays a key role in numerical solutions where exact solutions are difficult to get.

Author Contributions: Formal analysis, A.S. and S.A.; Investigation, A.S. and S.A.; Methodology, S.A.; Resources, A.S.; Software, S.A.; Supervision, A.S.; Writing—original draft, S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-310-130-1443). The authors, therefore, gratefully acknowledge DSR technical and financial support.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Our deep appreciation goes out to anonymous referees for their helpful suggestions and valuable advice.

Conflicts of Interest: The authors declare that they do not have any competing interests.

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