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About Some Monge–Kantovorich Type Norm and Their Applications to the Theory of Fractals

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Abstract: If X is a Hilbert space, one can consider the space $\mathrm{cabv}(X)$ of X valued measures defined on the Borel sets of a compact metric space, having a bounded variation. On this vector measures space was already introduced a Monge–Kantorovich type norm. Our first goal was to introduce a Monge–Kantorovich type norm on $\mathrm{cabv}(X)$, where X is a Banach space, but not necessarily a Hilbert space. Thus, we introduced here the Monge–Kantorovich type norm on $\mathrm{cabv}(L^q([0,1])), (1 < q < \infty)$. We obtained some properties of this norm and provided some examples. Then, we used the Monge–Kantorovich norm on cabv $(K^n)(K$ being $\mathbb{R}$ or $\mathbb{C})$ to obtain convergence properties for sequences of fractal sets and fractal vector measures associated to a sequence of iterated function systems.

Keywords: variation of a vector measure; Haar functions; attractor; fractal measure; Lipschitz functions; weak convergence of operators

MSC: 28C20; 46G12; 28B05; 28C15; 46C05

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1. Introduction

We consider two types of vector integrals, which were introduced in [1,2]. They involve vector functions and vector measures and the result of each of them is a scalar (real or complex). Using these integrals, one can introduce Monge–Kantorovich type norms on some spaces of vector measures (see [3]). In some particular cases, these norms have important applications in the theory of fractals (see [4,5]). Unlike [3], where the Monge–Kantorovich type norm was introduced on $\mathrm{cabv}(X)$ (X–Hilbert space), we introduce, in Sections 2–5 the Monge–Kantorovich type norm on the space of vector measures: $\mathrm{cabv}(L^q([0,1]))$ $(1 < q < \infty)$. To this aim, we use the Haar functions and the duality $(L^p)^* = L^q, \frac{1}{p} + \frac{1}{q} = 1$. We provide some properties of this norm. Some examples are, also provided. In the second part of the paper (Sections 6 and 7), we consider the Monge–Kantorovich type norm on $\mathrm{cabv}(X)$ (X—being a Hilbert space) and, more in particular, on cabv (K^n) ($K = \mathbb{R}$ or $\mathbb{C}$). We consider a sequence of iterated function systems (I.F.S.), built using a finite family of contractions and a sequence of linear and continuous operators. We take into account the convergence of the I.F.S. sequence, which is based on the topology of weak convergences of the operators. We study the problems of the convergence of attractos and fractal measures associated to the sequence of I.F.S. In the last part of the paper, we give an example of a sequence of operators which is convergent to an operator in the topology of weak convergence of operators, but is not convergent in the topology given by the operatorial norm. For more details regarding Monge–Kantorovich norm, one can consult [6–13]. About the fractals theory, you can read the following [14–17]. For more details regarding functional analysis, see [16,18,19].

2. Preliminaries

Let X be a Banach space over $\mathbb{R}$ and X' its conjugate. Let, also, (T,d) be a compact metric space. We denote by $\mathcal{B}$ the Borel subsets of T. If $\mu : \mathcal{B} \to X$ is a countable-additive measure and $A \in \mathcal{B}$, we define *the variation of μ on A*, by the formula:

$$|\mu|(A) = \sup\left\{ \sum_i \|\mu(A_i)\| \mid (A_i)_i \text{ is a finite partition of } A \text{ with Borel subsets} \right\}.$$

If $|\mu|(T) < \infty$ we say that μ has *bounded variation*. We denote by: $\text{cabv}\, X = \{\mu : \mathcal{B} \to \mathbb{R} | |\mu|(T) < \infty\}$, called *the vectorial norm*. One can prove that $\|\|$: $\text{cabv}\, X \to \mathbb{R}_+$ is a norm and $(\text{cabv}(X), \|\|)$ is a Banach space (see [18]). Now, we define the following function spaces:

$$S(X) = \{f : T \to X |\ f \text{ is a simple function}\};$$

$$TM(X) = \{f : T \to X |\ \exists (f_n)_n \subset S(X) \text{ such that } f_n \xrightarrow{u} f\}$$

(the space of totally measurable functions);

$$C(X) = \{f : T \to X | f \text{ is continuous}\}.$$

For any $A \in \mathcal{B}$, we denote by φ_A the characteristic function of A.

3. An Integral for Vector Function with Respect to Vector Measures

Definition 1 (see [2])**.** *Let $f = \sum_{i=1}^{m} \varphi_{A_i} x_i \in S(X)$, where $x_i \in X, A_i \in \mathcal{B}$. Let, also, $\mu \in$* $\text{cabv}(X')$*. We define the integral of f with respect to μ by the formula: $\int f d\mu = \sum_{i=1}^{m} \mu(A_i)(x_i)$.*

Obviously, we have: $\left| \int f d\mu \right| \leq \|\mu\| \cdot \|f\|_\infty$, hence, the linear application $f \mapsto \int f d\mu$ is continuous and can be extended to the closure of $S(X)$ with respect to $\|\|_\infty$, that is, to the space $TM(X)$: if $(f_n)_n \subset S(X)$ such that $f_n \xrightarrow{u} f \in TM(X)$, we define $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$ and the limit does not depend on the sequence of simple functions, which tends to f.

Example 1. *We will provide now an example of such sequence which will be called* the cannonical sequence *(see [1]), for the case when $f \in C(X)$.*

Let us denote: $\tilde{X} = f(T)$; f is continuous and T is compact, hence, $\tilde{X}$ is also compact. That means $\tilde{X}$ is precompact (totally bounded). Consequently, for any $m \in \mathbb{N}$, we will find the elements: $x_1^m = f(t_1^m), x_2^m = f(t_2^m), \ldots, x_{j(m)}^m = f\left(t_{j(m)}^m\right)$ such that $\tilde{X} \subset \bigcup_{i=1}^{j(m)} B\left(x_i^m, \frac{1}{m}\right)$.

We deduce that $t_i^m \in D_i^m \overset{def}{=} f^{-1}\left(B\left(x_i^m, \frac{1}{m}\right)\right)$ and $\bigcup_{i=1}^{j(m)} D_i^m = T$. We obtain the following partition of T:

$$C_1^m = D_1^m, C_2^m = D_2^m \setminus D_1^m, \ldots, C_p^m = D_p^m \setminus \bigcup_{i=1}^{p-1} D_i^m$$

(we consider only those sets C_i^m, which are not empty).

Let $y_i^m \in f\left(C_i^m\right)$, arbitrarily fixed. We define the simple function $f_m = \sum_{i=1}^{p} \varphi_{C_i^m} y_i^m$. If we take $t \in T$, arbitrarily, then there exists $i \in \{1, \ldots, p\}$ such that $t \in C_i^m$. Then, both $f(t)$ and y_i^m belong to $f\left(C_i^m\right)$. But, $f\left(C_i^m\right) \subset f\left(A_i^m\right) \subset B\left(x_i^m, \frac{1}{m}\right)$, hence $\|f(t) - f_m(t)\| = \|f(t) - y_i^m\| < \frac{2}{m}$, which means $f_m \xrightarrow{u} f$.

Example 2. *Let $f \in C(X), a \in T, x \in X', \mu = \delta_a x$, δ_a being the Dirac measure concentrated at a. Let us compute $\int f d(\delta_a x)$. We consider the cannonical sequence $(f_m)_m$ associated to f. For any m,* *we denote by C_i^m the unique set from the partition of T such that $a \in C_i^m$. We have:* $\int f_m d(\delta_a x) = = [(\delta_a x)(C_i^m)](y_i^m) = x(y_i^m)$. *We conclude that:*

$$\int f d(\delta_a x) = \lim_{m\to\infty} \int f_m d(\delta_a x) = \lim_{m\to\infty} x(y_i^m) = x(f(a))$$

$(f(a), y_i^m \in f(C_i^m) \Longrightarrow \|y_i^m - f(a)\| < \frac{2}{m})$.

Proposition 1 (the properties of the integral)**.**

(a) *Let* $f \in TM(\mathbb{R}), \mu \in \text{cabv}(\mathbb{R})$, $x \in X, y \in X'$. *Then:*

$$(i)\ fx \in TM(X); \quad (ii)\ \mu y \in \text{cabv}(X'); \quad (iii)\ \int (fx) d(\mu y) = \left(\int f d\mu\right) y(x);$$

(b) *Let* $X = \mathbb{R}^n$ *and* $\{e_1, \dots e_n\}$ *its cannonical basis,* $f \in TM(X), \mu \in \text{cabv}(X'), f = \sum_{i=1}^{n} f_i e_i, \mu = \sum_{i=1}^{n} \mu_i e_i$. *Then,* $\int f d\mu = \sum_{i=1}^{n} (\int f_i d\mu_i)$.

4. The Haar Functions

Let $r \in (1, \infty)$. We consider the space $L^r([0,1])$ (with respect to the Lebesgue measure, denoted by λ) and the functions $\left(\gamma_n^k\right)_{k,n}$,

$\gamma_n^k : [0,1] \to \mathbb{R}$ defined as follows: $\gamma_0^0 = 1, \gamma_0^1(x) = \begin{cases} 1, x \in \left[0, \frac{1}{2}\right) \\ 0, x = \frac{1}{2} \\ -1, x \in \left(\frac{1}{2}, 1\right] \end{cases}$, ...,

$\gamma_n^k(x) = \begin{cases} \sqrt{2^n}, x \in \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right) \\ -\sqrt{2^n}, x \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right] \\ 0, \text{otherwise} \end{cases}$, ..., $k \in \{1, \dots, 2^n\}$. These functions may be written as a sequence $(g_j)_{j\geq 1}$ of functions, increasing k (for a fixed n) and then increasing n. One can prove the following results:

Lemma 1 (see [16])**.** $\int_{[0,1]} g_i g_j d\lambda = \delta_{i,j}, \forall i, j \in \mathbb{N}^*$.

Lemma 2 (see [16])**.** *For any $r \in (1, \infty)$, the functions $(g_j)_{j\geq 1}$ represents a Schauder basis for $L^r([0,1])$.*

Theorem 1 (see [2])**.** *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1, X = L^p([0,1]), X' = L^q([0,1])$, $f \in TM(X), \mu \in \text{cabv}(X'), f \in \sum_{n=1}^{\infty} f_n g_n, \mu = \sum_{m=1}^{\infty} \mu_m g_m$. Then, $\int f d\mu = \sum_{n=1}^{\infty} f_n d\mu_n$.*

5. The Monge–Kantorovich Type Norm on cabv(L^q), $1 < q < \infty$

We will denote: $L(X) = \{f : T \to X | f$ is a Lipschitz function$\}, BL(X) = \{f : T \to X | f$ is a bounded Lipschitz function$\}$, T being compact, $L(X) = BL(X) \subset C(X)$. For any $f \in L(X)$, we denote by $\|f\|_L$ the Lipschitz constant of f. It is easy to prove (see [3]):

Lemma 3. *The application $\|\|_{BL} : BL(X) \to \mathbb{R}_+, \|f\|_{BL} \stackrel{def}{=} \|f\|_\infty + \|f\|_L$ is a norm on $BL(X)$. Let: $BL_1(X) = \{f \in BL(X) | \|f\|_{BL} \leq 1\}$.*

We consider, now, $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $X = L^p([0,1]) \overset{not}{=} L^p$, $X' = L^q([0,1]) \overset{not}{=} L^q$. For any $\mu \in \text{cabv}(L^q)$, we define:

$$\|\mu\|_{MK} = \sup\left\{\left|\int f d\mu\right| \middle| f \in BL_1(L^p)\right\}.$$

Theorem 2. *The application* $\|\|_{MK} : \text{cabv}(L^q) \to \mathbb{R}_+$ *is a norm on* $\text{cabv}(L^q)$, *called* the Monge–Kantorovich type norm.

Proof.
(1°) For any $a \in \mathbb{R}$, $\|a\mu\|_{MK} = \sup\left\{\left|\int f d(a\mu)\right| \middle| f \in BL_1(L^p)\right\} =$
$= |a| \sup\left\{\left|\int f d\mu\right| \middle| f \in BL_1(L^p)\right\} = |a|\|\mu\|_{MK}$;
(2°) Let $\mu_1, \mu_2 \in \text{cabv}(L^q)$, $f \in BL_1(L^p)$. We have:

$$\left|\int f d(\mu_1 + \mu_2)\right| \le \left|\int f d\mu_1\right| + \left|\int f d\mu_2\right| \le \|\mu_1\|_{MK} + \|\mu_2\|_{MK} \Longrightarrow$$

$$\Longrightarrow \sup\left\{\left|\int f d(\mu_1 + \mu_2)\right| \middle| f \in BL_1(L^p)\right\} \le \|\mu_1\|_{MK} + \|\mu_2\|_{MK} \Longrightarrow$$

$$\Longrightarrow \|\mu_1 + \mu_2\|_{MK} \le \|\mu_1\|_{MK} + \|\mu_2\|_{MK}.$$

(3°) We consider $\mu \in \text{cabv}(L^q)$ such that $\|\mu\|_{MK} = 0$. We prove that $\mu = 0$. We will need the following result:

Lemma 4. *If* $\mu \in \text{cabv}(\mathbb{R})$ *and* $\|\mu\|_{MK} = 0$, *then* $\mu = 0$ *(for the proof one can see [3]).*

Let now, $(g_j)_{j \ge 1}$ the Haar functions sequence and we denote by $\|\|_p$ the norm on L^p. Let $f : T \to \mathbb{R}$, $f \in BL_1(\mathbb{R})$ and, for, $j \in \mathbb{N}^*$, arbitraily, fixed, $f_j = \frac{f \cdot g_j}{\|g_j\|_p}$.
We can write: $\|f_j(t_1) - f_j(t_2)\|_p = \frac{1}{\|g_j\|_p}\left(\int |f(t_1) - f(t_2)|^p |g_j|^p d\lambda\right)^{\frac{1}{p}} \le$
$\le \|f\|_L d(t_1, t_2) \cdot \|g_j\|_p \cdot \frac{1}{\|g_j\|_p}, \forall t_1, t_2 \in T \Longrightarrow f_j \in BL(L^p)$ and $\|f_j\|_L = \|f\|_L$. Then $\|f_j\|_\infty = \max_{t \in T} \|f_j(t)\|_p = \max_{t \in T}\left(\int |f(t)|^p |g_j|^p d\lambda\right)^{\frac{1}{p}} \frac{1}{\|g_j\|_p} = \max_{t \in T} |f(t)| = \|f\|_\infty; \Longrightarrow \|f_j\|_{BL} =$
$= \|f\|_{BL} \le 1 \Longrightarrow f_j \in BL_1(L^p)$.
But $\|\mu\|_{MK} = 0 \Longrightarrow \int f_j d\mu = 0 \Longrightarrow 0 = \int \frac{f g_j}{\|g_j\|_p} d\left(\sum_{i=1}^{\infty} \mu_i g_i\right) \overset{\text{lemma 1}}{=} \frac{1}{\|g_j\|_p}\left(\int f d\mu_j\right) \Longrightarrow$
$\Longrightarrow \int f d\mu_j = 0, \forall f \in BL_1(\mathbb{R}) \Longrightarrow \|\mu_j\|_{MK} = 0 \overset{lemma4}{\Longrightarrow} \mu_j = 0$. So, $\mu_j = 0, \forall j \in \mathbb{N}^*$, hence $\mu = 0$. □

Definition 2. *The norm defined by Theorem 2 is called* the Monge–Kantorovich type norm.

Proposition 2. *We have the inequality:* $\|\mu\|_{MK} \le \|\mu\|, \forall \mu \in \text{cabv}(L^q)$.

Proof. For any $f \in BL_1(L^p)$ and $\mu \in \text{cabv}(L^q)$, we have:

$$\left|\int f d\mu\right| \le \|\mu\| \cdot \underbrace{\|f\|_\infty}_{\le 1} \le \|\mu\| \Longrightarrow \|\mu\|_{MK} \le \|\mu\|.$$

□

Example 3. *Let* $a \in T, x \in L^q$ *such that* $\|x\|_q = 1$. *We will compute* $\|\delta_a x\|_{MK}$.

(i) *For* $f \in BL_1(L^p)$ *we have:* $|\int f d(\delta_a x)| \overset{example2}{=} |x(f(a))| = |\int x f(a) d\lambda| \leq$ $\leq \|f(a)\|_p \underbrace{\|x\|_q}_{=1} \leq \|f\|_\infty \leq \|f\|_{BL} \leq 1$. *Taking the supremum for* $f \in BL_1(L^p)$, *we get:* $\|\delta_a x\|_{MK} \leq 1$.

(ii) *Consider the function* $f(t) = x^{q-1}\,\mathrm{sgn}(x), \forall t \in T$. *We have:*

$$\int |f(t)|^p d\lambda = \int |x|^{pq-p} d\lambda = \int |x|^q d\lambda < \infty \Longrightarrow f(t) \in L^p, \forall t \in T.$$

$$\|f\|_{BL} = \|f\|_\infty + \underbrace{\|f\|_L}_{=0\ (f \text{ is constant})} = \|f(t)\|_p = \|x^{q-1}\|_p = \left(\int |x|^q d\lambda\right)^{\frac{1}{p}} =$$

$$= \left[\left(\int |x|^q d\lambda\right)^{\frac{1}{q}}\right]^{\frac{q}{p}} = \|x\|_q^{\frac{q}{p}} = 1.$$

For $\mu = \delta_a x$, *we have:* $|\int f d\mu| = |x(f(a))| = |\int x f(a) d\lambda| = |\int x^q\,\mathrm{sgn}(x) d\lambda| =$ $= \int |x|^q d\lambda = \|x\|_q^q = 1 \Longrightarrow \|\delta_a x\|_{MK} \geq 1$. *From (i) and (ii) we deduce that:* $\|\delta_a x\|_{MK} = 1$.

Theorem 3. *We suppose that T is infinite. Let us denote by* τ_{MK} *and* τ *the topologies generated on* $\mathrm{cabv}(L^q)$ *by the norms* $\|\|_{MK}$, *respectively* $\|\|$. *Then* $\tau_{MK} \subset \tau$, *the inclusion being strictly.*

Proof. From the inequality $\|\mu\|_{MK} \leq \|\mu\|, \forall \mu \in \mathrm{cabv}(L^q)$, it results that $\tau_{MK} \subset \tau$. We will prove that $\tau \not\subset \tau_{MK}$. Let us suppose the contrary: $\tau \subset \tau_{MK}$. Then, it would result that the identity application $I : (\mathrm{cabv}(L^q), \|\|_{MK}) \to (\mathrm{cabv}(L^q), \|\|)$ is continuous. Then, for any sequence $(\mu_n)_n \subset (\mathrm{cabv}(L^q))$ and $\mu \in (\mathrm{cabv}(L^q))$ such that $\mu_n \overset{\|\|_{MK}}{\to} \mu$, we would have: $\mu_n \overset{\|\|}{\to} \mu \quad (*)$. Presently, we need the following result: □

Lemma 5. *For any* $a, b \in T, a \neq b$ *and for any* $x \in L^q$, *with* $\|x\|_q = 1$, *we have:*

(i) $\|\delta_a x - \delta_b x\| = 2$;

(ii) $\|\delta_a x - \delta_b x\|_{MK} \leq d(a, b)$.

Proof of Lemma 5.

(i) Let $(A_j)_{1 \leq j \leq n}$ a partition of T with Borel sets. We can have two cases:
1°) If $\exists j_0 \in \{1, \dots, n\}$ such that $a, b \in A_{j_0}$, then, denoting $\mu = \delta_a x - \delta_b x$, we have: $\sum_{j=1}^{n} \|\mu(A_j)\| = \|x - x\|_q = 0$.
2°) If $\exists j_1 \neq j_2$ such that $a \in A_{j_1}, b \in A_{j_2}$, then $\sum_{j=1}^{n} \|\mu(A_j)\| = \|x\|_q + \|x\|_q = 2$. Therefore,

$$\|\delta_a x - \delta_b x\| = \|\mu\| = \sup\left\{\sum_j \|\mu(A_j)\| \,\middle|\, (A_j)_j \text{ finite partition of } T \text{ with Borel subsets}\right\} = 2.$$

(ii) Let $f \in BL_1(L^p), \mu = \delta_a x - \delta_b x$. We have:

$$\left|\int f d\mu\right| = \left|\int f d(\delta_a x) - \int f d(\delta_b x)\right| = |x(f(a)) - x(f(b))| = \int x \cdot [f(a) - f(b)] d\lambda \leq$$

$$\leq \int |x||f(a) - f(b)| d\lambda \leq \|x\|_q \|f(a) - f(b)\|_p \leq \|f\|_L d(a, b) \leq d(a, b).$$

Now, we continue the Proof of Theorem 3:

T being compact and infinite, we will find $(t_n)_n \subset T, t \in T$ such that $t_n \to t$, $t_n \neq t, \forall n \geq 1$. Let $x \in L^q$ with $\|x\|_q = 1$. According to Lemma 5 (i), $\|\delta_{t_n}x - \delta_t x\| = 2$, hence $\delta_{t_n}x \overset{\|\|}{\not\to} \delta_t x$. On the other hand, from Lemma 5 (ii), $\|\delta_{t_n}x - \delta_t x\|_{MK} \leq d(t_n, t) \to 0$, so, $\delta_{t_n}x \xrightarrow{\|\|_{MK}} \delta_t x$. But, this is in contradiction with $(*)$.
We conclude that $\tau \not\subset \tau_{MK}$. $\square$

6. The Integral and Monge–Kantorovich Type Norm in the Particular Case Where the Functions and Measures Take Values in a Hilbert Space

Let $(X, \langle , \rangle)$ be a Hilbert space, (T, d) a compact metric space and we denote, as before, by $\mathcal{B}$ the Borel subsets of T.

Definition 3 (see [1])**.** *Let $f \in S(X)$, $f = \sum_{i=1}^{m} \varphi_{A_i} x_i$, where $(A_i)_{1 \leq i \leq m}$ is a partition of T with Borel sets and φ_{A_i} is the characteristic function of $A_i, x_i \in X$. The number $\sum_{i=1}^{m} \langle x_i, \mu(A_i) \rangle$ is called* the integral of f with respect to μ *and is denoted by $\int f d\mu$ (it is easy to prove that the value of the integral does not depend on the representation of f).*

Definition 4 (see [1])**.** *If $f \in TM(X)$, we define $\int f d\mu = \lim_{n\to\infty} (\int f_n d\mu)$, $(f_n)_{n \geq 1}$ being a sequence of simple functions which converges uniformly to f (one can prove that this integral does not depend on the sequence $(f_n)_{n \geq 1}$, uniformly convergent to f).*

For more details about Definitions 3 and 4, one can consult [1].

Lemma 6 (see [3])**.**

(a) *The application $\| \cdot \|_{MK} : \mathrm{cabv}(X) \to [0, \infty)$ defined by $\|\mu\|_{MK} = \sup\{|\int f d\mu|, f \in BL_1(X)\}$ is a norm on $\mathrm{cabv}(X)$, called* the Monge-Kantorovich type norm*;*

(b) *Let $a > 0$ and $K = \mathbb{R}$ or $\mathbb{C}$. We denote: $B_a(K^n) = \{\mu \in \mathrm{cabv}(K^n) | \|\mu\| \leq a\}$. Then the topology generated on $B_a(K^n)$ by $\| \cdot \|_{MK}$ is the same with the weak-$*$ topology;*

(c) *$B_a(K^n)$ equipped with the metric generated by $\| \cdot \|_{MK}$, which is a compact metric space.*

7. The Particular Case When the Functions and Measure Take Values in $\mathbb{R}^n$: Applications on Fractals Theory

We first provide some results that were already proved in previous papers, which we will use.

Let us denote $\mathcal{L}(X) = \{R : X \to X | R$ is linear and continuous$\}$. Let $N \in \mathbb{N}^*$; for any $i \in \{1, \ldots, N\}$, we consider the contraction $\omega_i : T \to T$, with its ratio r_i and $R_i \in \mathcal{L}(X)$. One can define the following operator, denoted by H, via:
$H(\mu) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}))$, (this means: $H(\mu)(A) = \sum_{i=1}^{N} R_i(\mu(\omega_i^{-1}(A)))$, for any $A \in \mathcal{B}$ and $\mu \in \mathrm{cabv}(X)$).

It can be proved that for any $\mu \in \mathrm{cabv}(X), H(\mu) \in \mathrm{cabv}(X)$ and $\|H\| \leq \sum_{i=1}^{N} \|R_i\|_o$ ($\| \cdot \|_o$ being the operatorial norm on $\mathcal{L}(X)$).

Lemma 7 (Change of variable formula (see [5]))**.** *For any $f \in C(X)$ and H as before, we have: $\int f dH(\mu) = \int g d\mu$, where $g = \sum_{i=1}^{N} R_i^* \circ f \circ \omega_i$ (R_i^* being the adjoint of R_i).*

Theorem 4 (see [5]). *Let us suppose that* $\sum_{i=1}^{N} \|R_i\|_o(1+r_i) < 1$. *Let* $a > 0$, $\mu^0 \in \mathrm{cabv}(K^n)$; *we define* $P : \mathrm{cabv}(K^n) \to \mathrm{cabv}(K^n), P(\mu) = H(\mu) + \mu^0$. *Let, also,* $A \subset B_a(K^n)$ *be non-empty, weak-*$*$ *close, such that* $P(A) \subset A$. *We denote by* P_0 *the restriction of* P *to* A. *Then, there is a unique measure* $\mu^* \in A$, *such that* $P_0(\mu^*) = \mu^*$. *If* $\mu^0 = 0$ *(the zero-measure) then* $\mu^* = 0$.

Definition 5 (see [5]). *The measure* μ^* *introduced by Theorem 4 is called* the Hutchinson vector measure *(or the fractal vector measure).*

Let X, Y Banach spaces and $\omega : Y \to X$ a contraction of ratio r.

Let, also, $(T_n)_{n\geq 1} \subset \mathcal{L}(X,Y)$ such that $\alpha \overset{not}{=} \sup_{n\geq 1} \|T_n\|_o < \frac{1}{r}$. For any $n \geq 1$ we consider the operators $U_n : Y \to Y, U_n \overset{def}{=} T_n \circ \omega$.

The following two lemmas were proved in [4].

Lemma 8. *For any* n, U_n *is a contraction of ratio less or equal to* $\alpha \cdot r$.

Remark 1 (see [4]). *In the Proof of Lemma 9, for an arbitrarily and fixed* $\varepsilon > 0$, *we find a rank* N_0 *such that for any* $n \geq N_0$, $\delta(U_n(K), U(K)) \leq \varepsilon$. *This rank depends not only on* ε, *but also on* K. *However, if we take* $Y_0 \subset Y$, *compact, such that* $U_n(Y_0) \subset Y_0$ *and* $U(Y_0) \subset Y_0$, *denoting again by* U_n *and* U *the restrictions of these functions on* Y_0, *it is easy to prove that* N_0 *depends only on* ε. *Hence, in this case,* $U_n(K) \xrightarrow{\delta} U(K)$, *uniformly with respect to* $K \subset Y_0$. *For example, if* Y *is the finite dimensional, we can take* $Y_0 = B[0,R] = \{x \in Y | \|x\| \leq R\}$, *with* $R \geq \frac{\alpha\|\omega(0)\|}{1-\alpha r}$:

$$\begin{aligned}\|U_n(x)\| &= \|T_n(\omega(x))\| \leq \|T_n(\omega(x)) - T_n(\omega(0))\| + \|T_n(\omega(0))\| \\ &\leq \|T_n\|_o \cdot \|\omega(x) - \omega(0)\| + \|T_n\|_o\|\omega(0)\| \leq \alpha(r\|x\| + \|\omega(0)\|) \\ &\leq \alpha(rR + \|\omega(0)\|) \leq R,\end{aligned}$$

according to the condition satisfied by R.

Let (T,d) be a metric space. We denote by $\mathcal{P}^*(T)$ the family of non-empty and bounded subsets of T. For any $x \in T$ and $A \in \mathcal{P}^*(T)$, we will denote: $d(x,A) = \inf_{y\in A} d(x,y)$.
If $A, B \in \mathcal{P}^*(T)$ we define $d(A,B) = \sup_{x\in A} d(x,B)$. In a similar way, we define:
$d(B,A) = \sup_{y\in B} d(y,A)$. Presently, we denote:
$\delta(A,B) = \max\{d(A,B), d(B,A)\}$. Let us define

$$\mathcal{K}^*(T) = \{K \subset T | K \text{ is compact and non-empty}\}.$$

Proposition 3.

(i) $\delta : \mathcal{K}^*(T) \times \mathcal{K}^*(T) \to [0,\infty)$ *is a metric on* $\mathcal{K}^*(T)$;
(ii) *If* $\omega : T \to T$ *is a Lipschitz function , then* $\delta(\omega(A), \omega(B)) \leq L \cdot \delta(A,B)$, L *being the Lipschitz constant of* ω;
(iii) *if* $(A_i)_{1\leq i\leq n} \subset \mathcal{K}^*(T)$, $(B_i)_{1\leq i\leq n} \subset \mathcal{K}^*(T)$, *then:*

$$\delta\left(\bigcup_{i=1}^{n} A_i, \bigcup_{i=1}^{n} B_i\right) \leq \max_{1\leq i\leq n} \delta(A_i, B_i).$$

Definition 6 (see [20]). *The metric* δ *introduced by Proposition 3 is called* the Hausdorff-Pompeiu metric.

Proposition 4.

(i) *If (T, d) is complete, then $(\mathcal{K}^*(T), \delta)$ is also complete;*

(ii) *If (T, d) is compact, $(\mathcal{K}^*(T), \delta)$ is also compact.*

Lemma 9. *Let us suppose that there exists $T \in \mathcal{L}(X, Y)$ such that $T_n \xrightarrow{\|\cdot\|_o} T$. Then, for any $K \in \mathcal{K}^*(Y), U_n(K) \xrightarrow{\delta} U(K)$, where we denoted: $U = T \circ \omega$.*

Definition 7 (see [20]). *Let (T, d) be a complete metric space and $(\omega_i)_{1\le i\le m}$, $\omega_i : T \to T$, $i = \overline{1, m}$ such that any ω_i is a contraction of ratio $r_i \in [0, 1)$. The family $(\omega_i)_{1\le i\le m}$ is called* the iterated function system *(I.F.S.).*

Definition 8 (see [20]). *If $(\omega_i)_{1\le i\le m}$ is an I.F.S. on the complete metric space (T, d), we define: $S : \mathcal{K}^*(T) \to \mathcal{K}^*(T), S(E) = \bigcup\limits_{i=1}^{m} \omega_i(E), \forall E \in \mathcal{K}^*(T)$.*

Proposition 5. *The function S above defined is a contraction of ratio $r \le \max\limits_{1\le i\le n} r_i$. Hence, using the contraction principle, we deduce that there is an unique set $K \in \mathcal{K}^*(T)$, such that $K = S(K)$.*

Definition 9 (see [20]). *The set K introduced by Proposition 5 is called* the attractor *(or:* the fractal*) associated to the I.F.S. $(\omega_i)_{1\le i\le n}$.*

Remark 2 ([4]). *Let now $(\omega_j)_{1\le j\le m}$, $\omega_j : Y_0 \to X$ be contractions of ratio r_j, Y_0 being a compact and non-empty subset of a Banach space Y. We denote $r = \max\limits_i r_i$. Let us consider $T_n, T \in \mathcal{L}(X, Y)$ such that $\alpha \overset{not}{=} \sup\limits_n \|T_n\|_o < \frac{1}{r}$ and $T_n \xrightarrow{\|\cdot\|_o} T$. We denote $U_j^n = T_n \circ \omega_j$, $U_j = T \circ \omega_j$ and we will suppose, as before, that $U_j^n(Y_0) \subset Y_0$, $U_j(Y_0) \subset Y_0$. Using Lemma 8, we have that the functions $U_j^n : Y_0 \to Y_0$ and $U_j : Y_0 \to Y_0$ are contractions of ratios less or equal by αr. Here, if Y is finite dimensional, we can take $Y_0 = B[0, R], R \ge \frac{\alpha\beta}{1-\alpha r}, \beta = \max\limits_j \|\omega_j(0)\|$]. We can deduce that $(U_j^n)_j$ is an I.F.S. on $\mathcal{K}^*(Y_0)$. Y_0 being compact in the Banach space Y, it results that Y_0 is a complete metric space (with respect to the metric given by the restriction on Y_0 of the norm on Y). Consequently, (Proposition 4), $\mathcal{K}^*(Y_0)$ is complete. Hence (Proposition 5) there exists an unique set $K_n \in \mathcal{K}^*(Y_0)$ such that $K_n = \bigcup\limits_{j=1}^{m} U_j^n(K_n)$ (the attractor associated to the I.F.S. $(U_j^n)_j$. Similar, $(U_j)_j$ is an I.F.S. with its attractor $K = \bigcup\limits_{j=1}^{m} U_j(K)$.*

Lemma 10. *For any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for any $n \ge N_0, x \in Y_0$ and $j \in \{1, \dots, m\}$ we have: $\|U_j^n(x) - U_j(x)\| < \varepsilon$.*

Proof. Y_0 being compact, it is precompact, that means: for a given ε, there exists $p \in \mathbb{N}$ and $\{x_1, x_2, \dots, x_p\} \subset Y_0$ such that for any $x \in Y_0$, we can find $i \in \overline{1, p}$ with $\|x - x_i\| < \frac{\varepsilon}{3}$ $(\star)$; let $x \in Y_0$, arbitrarily, fixed and x_i which satisfies $(\star)$. We can write:

$$\|U_j^n(x) - U_j(x)\| \le \underbrace{\|U_j^n(x) - U_j^n(x_i)\|}_{\overset{not}{=} a} + \underbrace{\|U_j^n(x_i) - U_j(x_i)\|}_{\overset{not}{=} b} + \underbrace{\|U_j(x_i) - U_j(x)\|}_{\overset{not}{=} c}.$$

We have: $a = \|T_n(\omega_j(x) - \omega_j(x_i))\| \le \|T_n\|_o \|\omega_i(x) - \omega_i(x_i)\| \le \alpha r\|x - x_i\| < \|x - x_i\| < \frac{\varepsilon}{3}$; similar $c \le \|T\|_o \|\omega_j(x) - \omega_j(x_i)\| < \|x - x_i\| < \frac{\varepsilon}{3}$; using the fact that $T_n \xrightarrow{\|\cdot\|_o} T$, for any $i \in \overline{1, p}$ there exists $N_i \in \mathbb{N}$ such that for any $n \ge N_i$, $\|U_j^n(x_i) - U_j(x_i)\| < \frac{\varepsilon}{3}$. Let

$N_0 = \max_i N_i$. Then, for $n \geq N_0$, we obtain $b < \frac{\varepsilon}{3}$. Hence, for any $n \geq N_0$, $j \in \overline{1,m}$, $x \in Y_0$ we have:$\|U_j^n(x) - U_j(x)\| < \varepsilon$. $\square$

Consequence 1. *Let N_0 given by lemma 10. We have:* $\max_j \sup_{x \in Y_0} \|U_j^n(x) - U_j(x)\| \leq \varepsilon, \forall n \geq N_0$.

Now, we will suppose that there exists $T \in \mathcal{L}(X,Y)$, with $\|T\|_o \leq \alpha$, such that $T_n \to T$ in the topology of the weak convergences of the operators, that is: for any $y' \in Y^*$ and for any $x \in Y, y'(T_n(x)) \to y'(T(x))$.
(Y' is the conjugate of $Y : Y' = \{y' : Y \to K | y'$ is linear and continuous$\}$). We consider again the operators $(U_n)_{n \geq 1}$ as before.

Lemma 11. *Let $Y_0 \subset Y$, compact, such that $U_n(Y_0) \subset Y_0$ and $U(Y_0) \subset Y_0$ (for example, as in the remark after lemma 9). Then, for any $a \in Y_0$, we can find a subquence $(U_{j_n}(a)) \subset (U_n(a))_{n \geq 1}$ with the property:* $\lim_{n \to \infty} \|U_{j_n}(a) - U(a)\| = 0$.

Proof. From the hypothesis, the sequence $((U_n - U)(a))_{n \geq 1}$ is included in $Y_0 - Y_0$, which is compact, hence, we can find $j_1 < j_2 < \ldots < j_n < \ldots$ and $z \in Y_0$ such that $U_{j_n}(a) - U(a) \xrightarrow{\|\cdot\|} z$. Let $y' \in Y^*$, arbitrarily. We have: $\lim_{n \to \infty} y'(T_{j_n}(\omega(a))) = y'(T(\omega(a)))$, hence, $0 = \lim_{n \to \infty} y'(T_{j_n}(\omega(a)) - T(\omega(a))) = y'(z)$. Using a consequence of Hahn-Banach theorem, we find $y' \in Y^*$ such that $y'(z) = \|z\|$ and $\|y'\| = 1$. We deduce that: $0 = y'(z) = \|z\| \Longrightarrow z = 0 \Longrightarrow \lim_{n \to \infty} \|U_{j_n}(a) - U(a)\| = 0$. $\square$

Consequence 2. *There exists a subsequence $(U_{j_n})_{n \geq 1} \subset (U_n)_{n \geq 1}$ such that $\lim_{n \to \infty} \|U_{j_n}(a) - U(a))\| = 0$ uniformly with respect to $a \in Y_0$.*

Proof. Y_0 being compact, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and $z_1, z_2, \ldots, z_N \in Y_0$ such that $\forall a \in Y_0$, $\exists p \in \{1, 2, \ldots, N\}$ with $\|a - z_p\| < \frac{\varepsilon}{3}$. We will find the subsequences:

(1) $(U_{j_{n_1}})_{n_1} \subset (U_{j_n})_n$ such that $\|U_{j_{n_1}}(z_1) - U(z_1)\| \to 0$

(2) $(U_{j_{n_2}})_{n_2} \subset (U_{j_{n_1}})_{n_1}$ such that $\|U_{j_{n_2}}(z_2) - U(z_2)\| \to 0$

$\vdots$

(N) $(U_{j_{n_N}})_{n_N} \subset (U_{j_{n_{N-1}}})_{n_{N-1}}$ such that $\|U_{j_{n_N}}(z_N) - U(z_N)\| \to 0$.

Hence, for any $p \in \{1, 2, \ldots, N\}$ we will find $N^{(p)} \in \mathbb{N}$ such that for $n_p \geq N^{(p)}$, we have: $\|U_{j_{n_p}}(z_p) - U(z_p)\| < \frac{\varepsilon}{3}$. We denote: $N_0 = \max_{1 \leq p \leq N} N^{(p)}$. Let now $a \in Y_0$, arbitrarily; we find $p \in \{1, 2, \ldots, N\}$ such that $\|a - z_p\| < \frac{\varepsilon}{3}$. We deduce successively:

$$\|U_{j_{n_N}}(a) - U(a)\| \leq \underbrace{\|U_{j_{n_N}}(a) - U_{j_{n_N}}(z_p)\|}_{\leq \|a - z_p\| < \frac{\varepsilon}{3}} + \underbrace{\|U_{j_{n_N}}(z_p) - U(z_p)\|}_{< \frac{\varepsilon}{3} \text{ for } n_N \geq N_0} + \underbrace{\|U(z_p) - U(a)\|}_{\leq \|z_p - a\| < \frac{\varepsilon}{3}} < \varepsilon,$$

for any $n_N > N_0$. Hence, denoting again by $(U_{j_n})_n$ the subsequence $(U_{j_{n_N}})_{n_N}$, we can write that $\lim_{n \to \infty} \|U_{j_n}(a) - U(a)\| = 0$, uniformly, with respect to $a \in Y_0$. $\square$

Lemma 12. *For any $K \in \mathcal{K}^*(Y_0)$ there exists $(U_{j_n})_{n \geq 1} \subset (U_n)_{n \geq 1}$ such that:* $\lim_{n \to \infty} \delta(U_{j_n}(K), U(K)) = 0$, *uniformly with respect to $K \in \mathcal{K}^*(Y_0)$.*

Proof. Let $\varepsilon > 0$, arbitrarily, fixed. For any $t \in U_n(K)$, we find $a \in K$ such that $t = T_n(\omega(a))$. We have:

$$d(t, U(K)) = \inf\{\|T_n(\omega(a)) - T(\omega(b))\| | b \in K\} \leq$$
$$\inf\{\|T_n(\omega(a)) - T(\omega(a))\| + \|T(\omega(a)) - T(\omega(b))\| | b \in K\} \stackrel{b=a}{=}$$
$$\|T_n(\omega(a)) - T(\omega(a))\| = \|U_n(a) - U(a)\|.$$

Using Lemma 11 and its Consequence 2, we find the subsequence $(U_{j_n})_n \subset (U_n)_n$ and $n_\varepsilon \in \mathbb{N}$ such that $\|U_{j_n}(a) - U(a)\| < \varepsilon, \forall n > n_\varepsilon$.

It results $d(t, U(K)) < \varepsilon, \forall t \in U_{j_n}(K), \forall n > n_\varepsilon$. Hence, $\sup_{t \in U_{j_n}} (t, U(K)) \leq \varepsilon$, that is: $d(U_{j_n}(K), U(K)) \leq \varepsilon, \forall n > n_\varepsilon$. Similar, $d(U(K), U_{j_n}(K)) = \sup_{t \in U(K)} \inf_{y \in U_{j_n}(K)} d(t, y)$. For $y \in U_{j_n}(K)$, there exists $b \in K$ such that $y = U_{j_n}(b)$. We can write as above in this proof: $\inf\{\|T(\omega(a)) - T_{j_n}(\omega(b))\| | b \in K\} \leq \|T(\omega(a)) - T_{j_n}(\omega(a))\| < \varepsilon, \forall n > n_\varepsilon$. We obtain $d(U(K), U_{j_n}(K)) \leq \varepsilon$, hence, $\delta(U_{j_n}(K), U(K)) \leq \varepsilon$,
$\forall n > n_\varepsilon, \forall K \in \mathcal{K}^*(Y_0)$. □

Theorem 5. *Let K_n, respectively K, the attractors associated to the I.F.S. $(U_j^n)_{1 \leq j \leq m}$ respectively $(U_j)_{1 \leq j \leq m}$. Then, there exists $(K_{i_n})_n \subset (K_n)_n$ such that $\lim_{n \to \infty} \delta(K_{i_n}, K) = 0$.*

Proof. Let $(U_{i_n})_n \subset (U_n)_n$ such that $\lim_{n \to \infty} \|U_{i_n}(a) - U(a)\| = 0$, uniformly with respect to $a \in Y_0$. We have:

$$\delta(K_{i_n}, K) = \delta\left(\bigcup_{j=1}^m U_j^{i_n}(K_{i_n}), \bigcup_{j=1}^m U_j(K)\right) \leq$$
$$\delta\left(\bigcup_{j=1}^m U_j(K), \bigcup_{j=1}^m U_j^{i_n}(K)\right) + \delta\left(\bigcup_{j=1}^m U_j^{i_n}(K), \bigcup_{j=1}^m U_j^{i_n}(K_{i_n})\right).$$

Let $\varepsilon > 0$, arbitrarily fixed. Using Proposition 3 (iii), we have:

$$\delta\left(\bigcup_{j=1}^m U_j(K), \bigcup_{j=1}^m U_j^{i_n}(K)\right) \leq \max_{1 \leq j \leq m} \delta(U_j(K), U_j^{i_n}(K)) \leq \varepsilon,$$

for n large enough (see Lemma 12);

$$\delta\left(\bigcup_{j=1}^m U_j^{i_n}(K), \bigcup_{j=1}^m U_j^{i_n}(K)\right) \leq \max_{1 \leq j \leq m} \delta(U_j^{i_n}(K), U_j^{i_n}(K)) \underset{2.1.1,\,(ii)}{\leq} r\alpha\delta(K_{i_n}, K).$$

We deduce that: $(\exists) n_\varepsilon \in \mathbb{N}$ such that $(1 - r\alpha)\delta(K_{i_n}, K) \leq \varepsilon, \forall n > n_\varepsilon$. Hence, $\lim_{n \to \infty} \delta(K_{i_n}, K) = 0$.
□

Remark 3. *With the same type of convergence of $(T_n)_n$ to T as in this whole section, we consider now the framework, regarding fractal vector measures. Let us suppose that all the conditions regarding the operators H^n and H are fulfilled and denote by μ_n^*, respectively μ; the fractal vector measures associated to P^n, respectively P.*

Theorem 6. *There exists a subsequence $(\mu_{i_n}^*)_n \subset (\mu_n^*)_n$ such that:*

$$\lim_{n \to \infty} \|\mu_{i_n}^* - \mu^*\|_{MK} = 0.$$

Proof. (see, also [4]) To make this proof easier to be read, in any place, we will use "n" instead of "i_n". For example, $\mu^*_{i_n}$ becomes μ^*_n, P^{i_n} becomes P^n and so on.

$$\|\mu^*_n - \mu^*\|_{MK} = \|P^n(\mu^*_n) - P(\mu^*)\|_{MK} \le \|P^n(\mu^*_n) - P^n(\mu^*)\|_{MK} +$$

$$+ \|P^n(\mu^*) - P^n(\mu^*)\|_{MK} \le q_n \|\mu^*_n - \mu^*\|_{MK} + \|P^n(\mu^*) - P(\mu^*)\|_{MK} \tag{1}$$

We have, obviously: $q_n \le \|H^n\|_o \le \left(\sum_{j=1}^{m} \|R_j\|_o\right)(1 + \alpha r) < 1$.

Hence $q \stackrel{def}{=} \sup_n q_n < 1$. According to (1), we can write:

$$\|\mu^*_n - \mu^*\|_{MK} \le q\|\mu^*_n - \mu^*\|_{MK} + \|P^n(\mu^*) - P(\mu^*)\|_{MK} \Longrightarrow$$

$$(1-q)\|\mu^*_n - \mu^*\|_{MK} \le \|P^n(\mu^*) - P(\mu^*)\|. \tag{2}$$

Let now $\varepsilon > 0$ arbitrarily fixed and $f \in BL_1(K^N)$. We can write:

$$\underbrace{\left|\sum_{j=1}^{m}\left(\int R^*_j \circ f \circ U^n_j d\mu^* - \int R^*_j \circ f \circ U_j d\mu^*\right)\right|}_{\stackrel{\text{lemma 7}}{=}\left|\int f dH^n(\mu^*) - \int f dH(\mu^*)\right|} =$$

$$\left|\sum_{j=1}^{m}\int R^*_j \circ (f \circ U^n_j - f \circ U_j) d\mu^*\right| \le \sum_{j=1}^{m}\left|\int R^*_j \circ (f \circ U^n_j - f \circ U_j) d\mu^*\right| \le$$

$$\sum_{j=1}^{m}\int \|R^*_j \circ (f \circ U^n_j - f \circ U_j)\| d|\mu^*| \le \sum_{j=1}^{m}\int \|R^*_j\|_o \|f \circ U^n_j - f \circ U_j\| d|\mu^*| \le$$

$$\sum_{j=1}^{m}\|R^*_j\|_o \int \|U^n_j(x) - U_j(x)\| d|\mu^*|(x) \le \max_j \sup_{x \in Y_0} \|U^n_j(x) - U_j(x)\| \cdot |\mu^*|(Y_0) < \varepsilon,$$

for n big enough, according the consequence of lemma 10 and using the fact that $\sum_{j=1}^{m} \|R^*_j\|_o < 1$.

Hence, $\sup_{f \in BL_1(K^N)} \left|\int f dH^n(\mu^*) - \int f dH(\mu^*)\right| \le \varepsilon;$

$$\Longrightarrow \|H^n(\mu^*) - H(\mu^*)\|_{MK} \le \varepsilon \Longrightarrow \|P^n(\mu^*) - P(\mu^*)\|_{MK} \le \varepsilon$$

$$\stackrel{(2)}{\Longrightarrow} (1-q)\|\mu^*_n - \mu^*\|_{MK} \le \varepsilon, \text{ for } n \text{ large enough.}$$

We deduce that $\mu^*_n \xrightarrow{\|\cdot\|_{MK}} \mu^*$. □

Example 4. *For any $f \in L^2$, we define :*

$$T_n(f)(x) = \begin{cases} f(x), & \text{if } x \in \left(\frac{1}{n}, 1\right] \\ f(x)\sqrt{n}, & \text{if } x \in \left[0, \frac{1}{n}\right]. \end{cases}$$

Let $T(f) = f$.

(a) *We prove that $T_n(f) \in L^2$:*

$$\int_{[0,1]} |T_n(f)(x)|^2 d\lambda = \int_{[0,\frac{1}{n}]} n f^2(x) d\lambda + \int_{(\frac{1}{n},1]} f^2(x) d\lambda.$$

Using the density of $C([0,1])$ *in* $L^1([0,1])$ *we find a sequence* $(F_k)_k \subset C([0,1])$ *such that* $F_k \xrightarrow{L^1([0,1])} f^2$ *and consequently, a subsequence* $(F_{k_j})_j \subset (F_k)_k$ *and the Borel set* A *with the properties:*

(i) $A \subset [0,1]$ *and* $\lambda(A) = 0$;

(ii) $\lim_{j\to\infty} F_{k_j}(x) = f^2(x), \forall x \in [0,1]\backslash A$.

For $\varepsilon > 0$, *arbitrarily, fixed, we will find* $k_j \in \mathbb{N}$ *such that* $|F_{k_j}(x) - f^2(x)| < \varepsilon, \forall x \in [0,1]\backslash A$. *We denote:* $M = \max_{x\in[0,\frac{1}{n}]} |F_{k_j}(x)|$. *We can write:*

$$\int_{[0,\frac{1}{n}]} nf^2(x)d\lambda \le \int_{[0,\frac{1}{n}]} n|f^2(x) - F_{k_j}(x)|d\lambda + n\int_{[0,\frac{1}{n}]} |F_{k_j}(x)|d\lambda \le$$

$$M \cdot n \cdot \frac{1}{n} + \int_{[0,\frac{1}{n}]\cap A} n|f^2(x) - F_{k_j}(x)|d\lambda + \int_{[0,\frac{1}{n}]\backslash A} n|f^2(x) - F_{k_j}(x)|d\lambda \le$$

$$M + n\cdot\varepsilon\cdot\frac{1}{n} = \varepsilon + M, \text{ and that implies } \int_{[0,1]} |T_n(f)(x)|^2 d\lambda \le \varepsilon + M + \int_{[\frac{1}{n},1]} f^2(x)d\lambda;$$

this inequality shows that $T_n(f) \in L^2$.

(b) *We prove that* $T_n \overset{\|\cdot\|_o}{\not\longrightarrow} T$. *We consider* $f : [0,1] \to \mathbb{R}, f(x) = 1, \forall x \in [0,1]$. *We have:*

$$\|T_n(f) - T(f)\|_2^2 = \int_{[0,\frac{1}{n}]} [f(x)\sqrt{n} - f(x)]^2 d\lambda = \int_{[0,\frac{1}{n}]} (\sqrt{n}-1)^2 d\lambda =$$

$$\frac{n - 2\sqrt{n} + 1}{n} \to 1, \text{ when } n \to \infty.$$

Hence, we found $f \in L^2$, *with* $\|f\| \le 1$ *such that* $\lim_{n\to\infty} \|(T_n - T)(f)\| \ne 0$, *that proves* $T_n \overset{\|\cdot\|_o}{\not\longrightarrow} T$.

(c) *Let* $g \in (L^2)^* = L^2, f \in L^2$. *We deduce:*

$$|g(T_n(f)) - g(T(f))| = |g(T_n(f)) - g(f)| = \left|\int_{[0,\frac{1}{n}]} g(x)f(x)(\sqrt{n}-1)d\lambda\right|.$$

Now, we use again the density of $C([0,1])$ *in* $L^1([0,1])$: *we find a sequence* $(h_k)_k \subset C([0,1])$ *such that* $h_k \xrightarrow{L^1([0,1])} fg$ *and, consequently, a subsequence* $(h_{k_j})_j \subset (h_k)_k$ *and the Borel set* B *with the properties:*

(i) $B \subset [0,1]$ *and* $\lambda(B) = 0$;

(ii) $\lim_{j\to\infty} h_{k_j}(x) = f(x)g(x), \forall x \in [0,1]\backslash B$.

Let $\varepsilon > 0$ *arbitrarily be fixed. We will find* $k_j \in \mathbb{N}$ *such that* $|h_{k_j}(x) - f(x)g(x)| < \varepsilon, \forall x \in [0,1]\backslash B$. *We have:*

$$\int_{[0,\frac{1}{n}]} |f(x)g(x) - h_{k_j}(x)|(\sqrt{n}-1)d\lambda = \int_{[0,\frac{1}{n}]\backslash B} |f(x)g(x) - h_{k_j}(x)|(\sqrt{n}-1)d\lambda \le$$

$\frac{\varepsilon(\sqrt{n}-1)}{n}$.

Hence,

$$\int_{[0,\frac{1}{n}]} |f(x)g(x)|(\sqrt{n}-1)d\lambda \leq \underbrace{\int_{[0,\frac{1}{n}]} |f(x)g(x)-h_{k_j}(x)|(\sqrt{n}-1)d\lambda}_{\leq \varepsilon \frac{\sqrt{n}-1}{n}} +$$

$$\int_{[0,\frac{1}{n}]} |h_{k_j}(x)|(\sqrt{n}-1)d\lambda \leq$$

$$\left(\max_{x\in[0,1]} |h_{k_j}(x)| + \varepsilon\right) \cdot \frac{\sqrt{n}-1}{n} \to 0.$$

We have proved that for any $g \in (L^2)^*, g(T_n(f)) \to g(T(f))$, *for any* $f \in L^2$, *that is* $T_n \to T$ *in the topology of weak convergence of operators.*

8. Conclusions

This paper shows, especially in its second part, the important role played by the Monge–Kantorovich norms in the vector measure theory and fractals theory.

In the future, we intend to concentrate our research work in two directions:

(a) To introduce Monge–Kantorovich type norms on more general measure space;
(b) To give convergence properties for families of fractal sets and fractal vector measures in a more general framework (for example, in the case of an uncountable family of iterated function systems).

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