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Article

On Impulsive Implicit ψ-Caputo Hybrid Fractional Differential Equations with Retardation and Anticipation

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Abstract: In this paper, we investigate the existence and Ulam–Hyers–Rassias stability results for a class of boundary value problems for implicit ψ-Caputo fractional differential equations with non-instantaneous impulses involving both retarded and advanced arguments. The results are based on the Banach contraction principle and Krasnoselskii's fixed point theorem. In addition, the Ulam–Hyers–Rassias stability result is proved using the nonlinear functional analysis technique. Finally, illustrative examples are given to validate our main results.

Keywords: fixed point theorem; ψ-Caputo fractional derivative; existence and uniqueness; Ulam–Hyers–Rassias stability; non-instantaneous impulses

MSC: 26A33; 34A08; 34B37; 34D20; 47H10

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1. Introduction

Because of its importance in the modeling and scientific understanding of natural processes, fractional calculus has long been an essential study topic in functional space theory. Several applications in viscoelasticity and electrochemistry have been studied. Non-integer derivatives of fractional order have been successfully utilized to generalize fundamental natural principles. We recommend the monograph [1] for some fundamental results on fractional calculus and its applications.

While determining the precise solution of differential equations is difficult or impossible in many contexts, such as nonlinear analysis and optimization, we explore approximate solutions. It should be noted that only steady approximations are allowed. For this reason, many techniques for stability analysis are used. Mathematician Ulam originally highlighted the stability problem in functional equations in a 1940 presentation at Wisconsin University. S. M. Ulam introduced the following challenge: "Under what conditions does an additive mapping exist near an approximately additive mapping?" [2]. The following year, in [3], Hyers provided an answer to Ulam's problem for additive functions defined on Banach spaces. In 1978, Rassias [4] demonstrated the existence of unique linear mappings near approximate additive mappings, generalizing Hyers' findings. In [5], Luo et al. established the new existence, uniqueness, and Hyers–Ulam stability results of Caputo fractional difference equations using some new criteria and by applying the Brouwer theorem and the contraction mapping principle. The authors of [6] addressed the Ulam stabilities of a k-generalized ψ-Hilfer fractional differential problem.

In [7–9], Shah et al. devoted their research work to the study of various kinds of Ulam stabilities for some classes of coupled systems of fractional differential equations. In the papers by Salim et al. [10,11], the authors addressed the existence, stability, and uniqueness of solutions to diverse hybrid problems with fractional differential equations using various fractional derivatives and different types of conditions. Wang et al. [12] studied the existence, uniqueness, and different kinds of stability results for a coupled system of a nonlinear implicit fractional anti-periodic boundary value problem.

Real-world processes and phenomena can exhibit rapid shifts in state. These modifications have a very brief duration in contrast to the entire longevity of the process and are thus irrelevant to the evolution of the examined process. In such instances, impulsive equations can be employed to construct appropriate mathematical models. Physics, biology, population dynamics, ecology, pharmacokinetics, and other fields all contain such operations. Non-instantaneous impulses are actions that begin at an arbitrary fixed moment and last for a specified time interval. Hernandez and O'Regan [13] studied the existence of solutions to a novel class of abstract differential equations with non-instantaneous impulses. In the papers by Alzabut et al. [14,15], Bai et al. [16], Salim et al. [17], and Wang et al. [18,19], the authors presented some fundamental results and recent developments on differential equations with instantaneous and non-instantaneous impulses.

The authors of [20] studied the nonlinear fractional differential hybrid system with periodic boundary conditions, given by

$$\begin{cases} {}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}(v(\vartheta)g_1(\vartheta,v(\vartheta))) = g_2(\vartheta,v(\vartheta)), \varrho \in (0,1), \\ v(a) = v(b), \end{cases}$$

where $\vartheta \in [a,b]$, ${}^{C}\mathcal{D}_{a^+}^{\varrho,\Psi}$ is the Ψ-Caputo fractional derivative; $g_1 : [a,b] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous with g_1 and g_2, which are identically zero at the origin; and $g_2(\vartheta,0) \equiv 0$. Their arguments are based on Dhage's fixed point theorem. The authors of [21] established existence and stability results, with relevant fixed point theorems, for the following boundary value problem:

$$\begin{cases} \left({}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}x\right)(\vartheta) = f\left(\vartheta, x(\vartheta), \left({}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}x\right)(\vartheta)\right); \ \vartheta \in \Omega_i, \ i = 0,\ldots,m, \\ x(\vartheta) = \Psi_i(\vartheta, x(\vartheta)); \ \vartheta \in \tilde{\Omega}_i, \ i = 1,\ldots,m, \\ \phi_1\left({}^{\zeta_1}\mathbb{J}_{a^+}^{1-\zeta_3}x\right)(a^+) + \phi_2\left({}^{\zeta_1}\mathbb{J}_{m^+}^{1-\zeta_3}x\right)(b) = \phi_3, \end{cases}$$

where ${}^{\zeta_1}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}$, ${}^{\zeta_1}\mathbb{J}_{a^+}^{1-\zeta_3}$ are the generalized Hilfer fractional derivative of order $\zeta_1 \in (0,1)$ and type $\zeta_2 \in [0,1]$ and the generalized fractional integral of order $1-\zeta_3$, respectively; $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$, $\Omega_i := (\varkappa_i, \vartheta_{i+1}]$; $i = 0,\ldots,m$, $\tilde{\Omega}_i := (\vartheta_i, \varkappa_i]$; $i = 1,\ldots,m$, $a = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \ldots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = b < \infty$, $x(\vartheta_i^+) = \lim_{\epsilon\to 0^+} x(\vartheta_i+\epsilon)$ and $x(\vartheta_i^-) = \lim_{\epsilon\to 0^-} x(\vartheta_i+\epsilon)$; $f : (a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a given function; and $\Psi_i : \tilde{\Omega}_i \times \mathbb{R} \to \mathbb{R}$; $i = 1,\ldots,m$ are given continuous functions.

Motivated by the above-mentioned papers, first, we present some existence, uniqueness, and Ulam stability results for the following fractional problem:

$$\begin{cases} {}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta)) = \varphi\left(\vartheta, y^{\vartheta}(\cdot), {}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta))\right); \ \vartheta \in \Omega_j, \ j = 0,\ldots,m, \\ y(\vartheta) = \Psi_j(\vartheta, y(\vartheta_j^-)); \ \vartheta \in \tilde{\Omega}_j, \ j = 1,\ldots,m, \\ \delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \\ y(\vartheta) = \hbar_1(\vartheta), \ \ \vartheta \in [-\varpi, 0], \ \varpi > 0, \\ y(\vartheta) = \hbar_2(\vartheta), \ \ \vartheta \in [\varkappa, \varkappa+\tilde{\varpi}], \ \tilde{\varpi} > 0, \end{cases} \tag{1}$$

where ${}^{C}D_{\varkappa_\jmath}^{\zeta;\psi}$ represents the ψ-Caputo derivative of order $0 < \zeta \leq 1$, $\Theta := [0, \varkappa]$, $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_3 \in \mathbb{R}$, where $\delta_1 \neq 0$, $\Omega_0 := [0, \vartheta_1]$, $\Omega_\jmath := (\varkappa_\jmath, \vartheta_{\jmath+1}]$; $\jmath = 1, \ldots, m$, $\tilde{\Omega}_\jmath := (\vartheta_\jmath, \varkappa_\jmath]$; $\jmath = 1, \ldots, m$, $0 = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \ldots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = \varkappa < \infty$, $y(\vartheta_\jmath^+) = \lim_{\epsilon \to 0^+} y(\vartheta_\jmath + \epsilon)$ and $y(\vartheta_\jmath^-) = \lim_{\epsilon \to 0^-} y(\vartheta_\jmath + \epsilon)$ represent the right and left hand limits of $y(\vartheta)$ at $\vartheta = \vartheta_\jmath$; $\varphi : \Theta \times PC([-\varpi, \tilde{\varpi}], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function; and $\Phi \in C(\Theta, \mathbb{R} \backslash \{0\})$, $\hbar_1 \in C([-\varpi, 0], \mathbb{R})$, $\hbar_2 \in C([\varkappa, \varkappa + \tilde{\varpi}], \mathbb{R})$, and $\Psi_\jmath : \tilde{\Omega}_\jmath \times \mathbb{R} \to \mathbb{R}$; $\jmath = 1, \ldots, m$ are given continuous functions. For y defined on $[-\varpi, \varkappa + \tilde{\varpi}]$ and for any $\vartheta \in [0, \varkappa]$, y^ϑ is given by

$$y^\vartheta(\varrho) = y(\vartheta + \varrho), \;\; \varrho \in [-\varpi, \tilde{\varpi}].$$

The following are the primary novelties of the current paper:

- Given the varied conditions we imposed on problem (1), our study may be viewed as a partial continuation of the ones in the aforementioned studies.
- The ψ-fractional derivative unifies a larger number of fractional derivatives in a single fractional operator and opens the door to new applications.
- If we take $\delta_1 = -\delta_2$, $\delta_3 = 0$, and remove the impulses, and the retarded and advanced arguments, we then obtain the problem studied in [20].
- We weaken the several conditions imposed in the study of [20], such as the requirement that functions g_1 and g_2 are identically zero at the origin and $g_2(\vartheta, 0) \equiv 0$.
- We study the Ulam–Hyers–Rassias stability of an implicit problem with non-instantaneous impulses, delay, and anticipation.

The following is how the current paper is arranged: In Section 2, we present certain notations and review some preliminary information on the ψ-Caputo fractional derivative and auxiliary results. Section 3 presents an existence result to problem (1) based on the Banach contraction principle and Krasnoselskii's fixed point theorem. The Ulam–Hyers–Rassias stability of our problem is discussed in Section 4. In the final part, we provide some examples to demonstrate the application of our study results.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts that are used throughout this paper.

The Banach space of all continuous functions from Θ to $\mathbb{R}$ is denoted by $C(\Theta, \mathbb{R})$ equipped with the norm

$$\|\xi\|_\infty = \sup\{|\xi(\vartheta)| : \vartheta \in \Theta\}.$$

Let $\mathcal{X} = C([-\varpi, 0], \mathbb{R})$ and $\tilde{\mathcal{X}} = C([\varkappa, \varkappa + \tilde{\varpi}], \mathbb{R})$ be the spaces endowed, respectively, with the norms

$$\|\xi\|_{\mathcal{X}} = \sup\{|\xi(\theta)| : \theta \in [-\varpi, 0]\} \quad \text{and} \quad \|\xi\|_{\tilde{\mathcal{X}}} = \sup\{|\xi(\theta)| : \theta \in [\varkappa, \varkappa + \tilde{\varpi}]\}.$$

Consider the Banach space

$$\begin{aligned} PC(\Theta, \mathbb{R}) \quad &= \Big\{ y : \Theta \to \mathbb{R} : y|_{\tilde{\Omega}_\jmath} = \Psi_\jmath; \jmath = 1, \ldots, m, \;\; y|_{\Omega_\jmath} \in C(\Omega_\jmath, \mathbb{R}); \jmath = 0, \ldots, m, \\ &\qquad \text{and there exist } u(\vartheta_\jmath^-),\ y(\vartheta_\jmath^+), y(\varkappa_\jmath^-), \text{ and } y(\varkappa_\jmath^+) \text{ with } y(\vartheta_\jmath^-) = y(\vartheta_\jmath) \Big\}, \end{aligned}$$

equipped with the norm

$$\|y\|_{PC} = \sup_{\vartheta \in \Theta} |y(\vartheta)|.$$

Consider the weighted Banach space

$$\begin{aligned} PC([-\varpi,\tilde{\varpi}],\mathbb{R}) \;=\; & \Big\{ y:[-\varpi,\tilde{\varpi}]\to\mathbb{R} : y|_{[\tau_j,\tilde{\tau}_{j+1}]}\in C([\tau_j,\tilde{\tau}_{j+1}],\mathbb{R}); j=0,\dots,m, \\ & \text{for each } \vartheta\in\Omega_j,\ y|_{[\tilde{\tau}_j,\tau_j]}\in C([\tilde{\tau}_j,\tau_j],\mathbb{R}); j=1,\dots,m, \\ & \text{for each } \vartheta\in\tilde{\Omega}_j,\ \tau_j=\varkappa_j-\vartheta,\ \tilde{\tau}_j=\vartheta_j-\vartheta, \\ & \text{there exist } y(\tau_j^-), y(\tilde{\tau}_j^+), y(\tilde{\tau}_j^-) \text{ and } y(\tau_j^+); j=1,\dots,m, \\ & \text{with } y(\tau_j^-)=y(\tau_j)\Big\}, \end{aligned}$$

equipped with the norm

$$\|y\|_{[-\varpi,\tilde{\varpi}]} = \sup_{\tau\in[-\varpi,\tilde{\varpi}]} |y^{\vartheta}(\tau)|.$$

Next, we consider the Banach space

$$\mathbb{F} = \Big\{ y:[-\varpi,\varkappa+\tilde{\varpi}]\to\mathbb{R} : y|_{[-\varpi,0]}\in\mathcal{X},\, y|_{[\varkappa,\varkappa+\tilde{\varpi}]}\in\tilde{\mathcal{X}} \text{ and } y|_{[0,\varkappa]}\in PC(\Theta,\mathbb{R})\Big\},$$

equipped with the norm

$$\|y\|_{\mathbb{F}} = \max\{\|y\|_{\mathcal{X}}, \|y\|_{\tilde{\mathcal{X}}}, \|y\|_{PC}\}.$$

Let $\psi\in C^1(\Theta,\mathbb{R})$ be an increasing differentiable function such that $\psi'(\vartheta)\neq 0$, for all $\vartheta\in\Theta$. Now, we start by defining ψ-fractional integral and derivative operators as follows.

Definition 1 ([22])**.** *The ψ–Riemann–Liouville fractional integral of order $\zeta>0$ for an integrable function $\xi\colon\Theta\longrightarrow\mathbb{R}$ is given by*

$$\mathbb{I}_{0^+}^{\zeta;\psi}\xi(\vartheta) = \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\zeta-1}\xi(s)ds,$$

where Γ is the Gamma function.

One can deduce that

$$D_{\vartheta}\left(\mathbb{I}_{0^+}^{\zeta;\psi}\xi(\vartheta)\right) = \psi'(\vartheta)\mathbb{I}_{0^+}^{\zeta-1;\psi}\xi(\vartheta),\ \zeta>1,$$

where $D_{\vartheta}=\frac{d}{dt}$.

Definition 2 ([23])**.** *For $n-1<\zeta<n$ $(n\in\mathbb{N})$ and $\xi,\psi\in C^n(\Theta,\mathbb{R})$, the ψ-Caputo fractional derivative of a function ξ of order ζ is given by*

$$^{C}D_{0^+}^{\zeta;\psi}\xi(\vartheta) = \mathbb{I}_{0^+}^{n-\zeta;\psi}\left(\frac{D_{\vartheta}}{\psi'(\vartheta)}\right)^n\xi(\vartheta),$$

where $n=[\zeta]+1$ for $\zeta\notin\mathbb{N}$ and $n=\zeta$ for $\zeta\in\mathbb{N}$.

From the above definition, we can express the ψ-Caputo fractional derivative with the following formula:

$$^{C}D_{0^+}^{\zeta;\psi}\xi(\vartheta) = \begin{cases} \int_0^{\vartheta}\frac{\psi'(s)(\psi(\vartheta)-\psi(s))^{n-\zeta-1}}{\Gamma(n-\zeta)}\left(\frac{D_{\vartheta}}{\psi'(s)}\right)^n\xi(s)ds, & \text{if } \zeta\notin\mathbb{N}, \\ \left(\frac{D_{\vartheta}}{\psi'(\tau)}\right)^n\xi(\vartheta), & \text{if } \zeta\in\mathbb{N}. \end{cases}$$

Lemma 1 ([22,23]). *For $\zeta, \beta > 0$, and $\xi \in C(\Theta, \mathbb{R})$, we have*

$$\mathbb{I}_{0^+}^{\zeta;\psi}\mathbb{I}_{0^+}^{\beta;\psi}\xi(\vartheta) = \mathbb{I}_{a^+}^{\zeta+\beta;\psi}\xi(\vartheta),\ \vartheta \in \Theta.$$

Lemma 2 ([22,23]). *Let $\zeta > 0$. If $\xi \in C(\Theta, \mathbb{R})$, then*

$${}^{C}D_{0^+}^{\zeta;\psi}\mathbb{I}_{0^+}^{\zeta;\psi}\xi(\vartheta) = \xi(\vartheta),\ \vartheta \in \Theta,$$

and if $\xi \in C^{n-1}(\Theta, \mathbb{R})$, then

$$\mathbb{I}_{0^+}^{\zeta;\psi}\ {}^{C}D_{0^+}^{\zeta;\psi}\xi(\vartheta) = \xi(\vartheta) - \sum_{k=0}^{n-1}\frac{\left(\frac{D_\vartheta}{\psi'(\vartheta)}\right)^k\xi(a)}{k!}[\psi(\vartheta) - \psi(a)]^k, \quad \vartheta \in \Theta.$$

Lemma 3 ([22,23]). *For $\vartheta > 0$, $\zeta \geq 0$ and $\beta > 0$. Then,*

- $\mathbb{I}_{0^+}^{\zeta;\psi}(\psi(\vartheta) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\zeta)}(\psi(\vartheta) - \psi(0))^{\beta+\zeta-1}$;
- ${}^{C}D_{0^+}^{\zeta;\psi}(\psi(\vartheta) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\zeta)}(\psi(\vartheta) - \psi(0))^{\beta-\zeta-1}$;
- ${}^{C}D_{0^+}^{\zeta;\psi}(\psi(\vartheta) - \psi(0))^{k} = 0$, *for all* $k \in \{0, \dots, n-1\}, n \in \mathbb{N}$.

Theorem 1 (Banach fixed point theorem [24]). *Let E be a Banach space and $\mathcal{H} : E \longrightarrow E$ a contraction, i.e., there exists $\jmath \in [0,1)$ such that*

$$\|\mathcal{H}(\xi_1) - \mathcal{H}(\xi_2)\| \leq \jmath\|\xi_1 - \xi_2\|, \qquad for\ all\ \xi_1, \xi_2 \in E.$$

Then, $\mathcal{H}$ has a unique fixed point.

Theorem 2 (Krasnoselskii's fixed point theorem [25]). *Let D be a closed, convex, and nonempty subset of a Banach space E, and A and B the operators such that (1) $Ax + By \in D$ for all $x, y \in D$; (2) A is compact and continuous; (3) B is a contraction mapping. Then, there exists $z \in D$ such that $z = Az + Bz$.*

3. Main Results

We study the fractional differential equation that follows:

$${}^{C}D_{\varkappa_\jmath}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta)) = \sigma(\vartheta);\ \vartheta \in \Omega_\jmath,\ \jmath = 0, \dots, m, \tag{2}$$

where $0 < \zeta \leq 1$, with the conditions

$$y(\vartheta) = \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-));\ \vartheta \in \tilde{\Omega}_\jmath,\ \jmath = 1, \dots, m, \tag{3}$$

$$\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \tag{4}$$

$$y(\vartheta) = \hbar_1(\vartheta),\ \ \vartheta \in [-\varpi, 0],\ \varpi > 0, \tag{5}$$

$$y(\vartheta) = \hbar_2(\vartheta),\ \ \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}],\ \tilde{\varpi} > 0, \tag{6}$$

where $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_3 \in \mathbb{R}$, $\delta_1 \neq 0$, $\sigma(\cdot) \in C(\Theta, \mathbb{R})$, $\Phi \in C(\Theta, \mathbb{R}\backslash\{0\})$, $\hbar_1 \in C([-\varpi, 0], \mathbb{R})$, $\hbar_2 \in C([\varkappa, \varkappa + \tilde{\varpi}], \mathbb{R})$, and $\Psi_\jmath : \tilde{\Omega}_\jmath \times \mathbb{R} \to \mathbb{R}; \jmath = 1, \dots, m$ are given continuous functions.

Theorem 3. *Function $y(\cdot)$ verifies (2)–(6) if and only if it verifies*

$$y(\vartheta)=\begin{cases}\frac{1}{\Phi(\vartheta)}\bigg[\frac{\delta_3\Phi(0)}{\delta_1}-\frac{\delta_2\Phi(0)\Phi(\varkappa_m)\Psi_m(\varkappa_m,y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)}\\ -\frac{\delta_2\Phi(0)}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m+}\sigma(\varkappa)+\mathbb{I}^{\zeta,\psi}_{0^+}\sigma(\vartheta)\bigg], \quad \vartheta\in\Omega_0,\\ \frac{1}{\Phi(\vartheta)}\Big[\Phi(\varkappa_\jmath)\Psi_\jmath(\varkappa_\jmath,y(\vartheta_\jmath^-))+\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath+}\sigma(\vartheta)\Big], \quad \vartheta\in\Omega_\jmath;\jmath=1,\dots,m,\\ \Psi_\jmath(\vartheta,y(\vartheta_\jmath^-)),\ \vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\dots,m,\\ \hbar_1(\vartheta),\ \ \vartheta\in[-\varpi,0],\\ \hbar_2(\vartheta),\ \ \vartheta\in[\varkappa,\varkappa+\tilde{\varpi}].\end{cases} \tag{7}$$

Proof. Let us assume that y satisfies (2)–(6). If $\vartheta\in\Omega_0$, then

$$^{C}D_0^{\zeta;\psi}\Phi(\vartheta)y(\vartheta)=\sigma(\vartheta).$$

By applying fractional operator $\mathbb{I}^{\zeta;\psi}_{0^+}$ on both sides of (2) and employing Lemma 2, we obtain

$$\Phi(\vartheta)y(\vartheta)=\mathbb{I}^{\zeta,\psi}_{0^+}\sigma(\vartheta)+c_0.$$

If $\vartheta\in\tilde{\Omega}_1$, then we have $y(\vartheta)=\Psi_1(\vartheta,y(\vartheta_1^-))$.
If $\vartheta\in\Omega_1$, then Lemma 2 implies that

$$\Phi(\vartheta)y(\vartheta)=\Phi(\varkappa_1)y(\varkappa_1)+\mathbb{I}^{\zeta,\psi}_{\varkappa_1+}\sigma(\vartheta)=\Phi(\varkappa_1)\Psi_1(\varkappa_1,y(\vartheta_1^-))+\mathbb{I}^{\zeta,\psi}_{\varkappa_1+}\sigma(\vartheta).$$

If $\vartheta\in\tilde{\Omega}_2$, then we have $y(\vartheta)=\Psi_2(\vartheta,y(\vartheta_2^-))$.
If $\vartheta\in\Omega_2$, then Lemma 2 implies that

$$\Phi(\vartheta)y(\vartheta)=\Phi(\varkappa_2)y(\varkappa_2)+\mathbb{I}^{\zeta,\psi}_{\varkappa_2+}\sigma(\vartheta)=\Phi(\varkappa_2)\Psi_2(\varkappa_2,y(\vartheta_2^-))+\mathbb{I}^{\zeta,\psi}_{\varkappa_2+}\sigma(\vartheta).$$

Repeating the process in this way, for $\vartheta\in\Theta$, we can obtain

$$y(\vartheta)=\begin{cases}\frac{1}{\Phi(\vartheta)}\Big[\mathbb{I}^{\zeta,\psi}_{0^+}\sigma(\vartheta)+c_0\Big], \quad \vartheta\in\Omega_0,\\ \frac{1}{\Phi(\vartheta)}\Big[\Phi(\varkappa_\jmath)\Psi_\jmath(\varkappa_\jmath,y(\vartheta_\jmath^-))+\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath+}\sigma(\vartheta)\Big], \quad \vartheta\in\Omega_\jmath;\jmath=1,\dots,m,\\ \Psi_\jmath(\vartheta,y(\vartheta_\jmath^-)),\ \vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\dots,m.\end{cases} \tag{8}$$

Taking $\vartheta=\varkappa$ in (8), we obtain

$$\Phi(\varkappa)y(\varkappa)=\Phi(\varkappa_m)\Psi_m(\varkappa_m,y(\vartheta_m^-))+\mathbb{I}^{\zeta,\psi}_{\varkappa_m+}\sigma(\varkappa).$$

Using condition (4), we obtain

$$\Phi(0)y(0)=\frac{\delta_3\Phi(0)}{\delta_1}-\frac{\delta_2\Phi(0)\Phi(\varkappa_m)\Psi_m(\varkappa_m,y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)}-\frac{\delta_2\Phi(0)}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m+}\sigma(\varkappa).$$

Substituting the value of $y(0)$ in (8), we obtain (7).

Reciprocally, for $\vartheta\in\Omega_0$, taking $\vartheta=0$, we obtain

$$y(0)=\frac{\delta_3}{\delta_1}-\frac{\delta_2\Phi(\varkappa_m)\Psi_m(\varkappa_m,y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)}-\frac{\delta_2}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m+}\sigma(\varkappa),$$

and for $\vartheta \in \Omega_m$, taking $\vartheta = \varkappa$, we obtain

$$y(\varkappa) = \frac{1}{\Phi(\varkappa)}\Big[\Phi(\varkappa_m)\Psi_m(\varkappa_m, y(\vartheta_m^-)) + \mathbb{I}^{\zeta,\psi}_{\varkappa_m{}^+}\sigma(\varkappa)\Big].$$

Thus, we can obtain $\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3$, which implies that (4) is verified. Next, we apply ${}^{RC}_{\varkappa_\jmath}D^{\zeta}_{\vartheta_{\jmath+1}}(\cdot)$ on both sides of (7), where $\jmath = 0, \ldots, m$. Then, using Lemma 2, we obtain Equation (2). In addition, it is clear that y verifies (3), (5), and (6). □

Lemma 4. *Let $0 < \zeta \le 1$, $\varphi : \Theta \times PC([-\varpi, \tilde{\varpi}], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be a given function, $\hbar_1(\cdot) \in \mathcal{X}$, and $\hbar_2(\cdot) \in \tilde{\mathcal{X}}$, $\Phi \in C(\Theta, \mathbb{R}\backslash\{0\})$; then, $y \in \mathbb{F}$ verifies (1) if and only if y is the fixed point of operator $\aleph : \mathbb{F} \to \mathbb{F}$, defined by*

$$\aleph y(\vartheta) = \begin{cases} \dfrac{1}{\Phi(\vartheta)}\Bigg[\dfrac{\delta_3\Phi(0)}{\delta_1} - \dfrac{\delta_2\Phi(0)\Phi(\varkappa_m)\Psi_m(\varkappa_m, y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)} \\ -\dfrac{\delta_2\Phi(0)}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m{}^+}\sigma(\varkappa) + \mathbb{I}^{\zeta,\psi}_{0^+}\sigma(\vartheta)\Bigg], \quad \vartheta \in \Omega_0, \\ \dfrac{1}{\Phi(\vartheta)}\Big[\Phi(\varkappa_\jmath)\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) + \mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath{}^+}\sigma(\vartheta)\Big], \quad \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)), \ \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m, \\ \hbar_1(\vartheta), \ \ \vartheta \in [-\varpi, 0], \\ \hbar_2(\vartheta), \ \ \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{9}$$

where σ is a function satisfying the following functional equation:

$$\sigma(\vartheta) = \varphi\Big(\vartheta, y^{\vartheta}(\cdot), \sigma(\vartheta)\Big).$$

Obviously, the fixed points of operator $\aleph$ are solutions of problem (1).

Proof. We can see that the proof follows the same processes as the proof of Theorem 3. In fact, it is a direct consequence of Theorem 3. □

Let us assume the following assumptions:

(A1) Function $\varphi : \Theta \times PC([-\varpi, \tilde{\varpi}], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A2) There exist constants $\psi_1, \wp_\jmath > 0$ and $0 < \psi_2 < 1$ such that

$$\begin{aligned} |\varphi(\vartheta, \xi, \gamma) - \varphi(\vartheta, \bar{\xi}, \bar{\gamma})| &\le \psi_1\|\xi - \bar{\xi}\|_{[-\varpi, \tilde{\varpi}]} + \psi_2|\gamma - \bar{\gamma}|, \\ |\Psi_\jmath(\vartheta, \gamma) - \Psi_\jmath(\vartheta, \bar{\gamma})| &\le \wp_\jmath|\gamma - \bar{\gamma}|, \end{aligned}$$

for any $\xi, \bar{\xi} \in PC([-\varpi, \tilde{\varpi}], \mathbb{R})$, $\gamma, \bar{\gamma} \in \mathbb{R}$, and $\vartheta \in \Omega_\jmath$; $\jmath = 0, \ldots, m$, where $\wp^* = \max_{\jmath=1,\ldots,m}\{\wp_\jmath\}$.

(A3) Function Φ is continuous on Θ, and there exists a positive real constant $\mathcal{M}$ such that

$$|\Phi(\vartheta)| \ge \mathcal{M}.$$

Set

$$\overline{\Phi} = \max_{\jmath=0,\ldots,m+1}|\Phi(\varkappa_\jmath)| \quad \text{and} \quad \tilde{\Phi} = \min_{\jmath=0,\ldots,m+1}|\Phi(\varkappa_\jmath)|.$$

We are now in a position to prove the existence result of problem (1) based on the Banach contraction principle.

Theorem 4. *Let us assume that assumptions (A1)–(A3) hold. If*

$$\beta := \frac{1}{\mathcal{M}}\left[\wp^*\overline{\Phi} + \frac{\wp^*|\delta_2|\overline{\Phi}^2}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\psi_2)\Gamma(\zeta+1)} + \frac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right] + \wp^* < 1, \tag{10}$$

then implicit fractional problem (1) *has a unique solution on* Θ.

Proof. We show that operator $\aleph$ defined in (9) is a contraction in $\mathbb{F}$.

Let $y, z \in C(\Theta, \mathbb{R})$. Then, for each $\vartheta \in [-\varpi, 0] \cup [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$|\aleph y(\vartheta) - \aleph z(\vartheta)| = 0.$$

Moreover, for $\vartheta \in \Omega_0$, we have

$$\begin{aligned}|\aleph y(\vartheta) - \aleph z(\vartheta)| \leq \frac{1}{|\Phi(\vartheta)|}\Bigg[&\frac{|\delta_2\Phi(0)\Phi(\varkappa_m)| \times |\Psi_m(\varkappa_m, y(\vartheta_m^-)) - \Psi_m(\varkappa_m, z(\vartheta_m^-))|}{|\delta_1\Phi(\varkappa)|} \\ &+ \frac{|\delta_2\Phi(0)|}{|\delta_1\Phi(\varkappa)|}\Big(\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}|\sigma(s) - \sigma_z(s)|\Big)(\varkappa) + \Big(\mathbb{I}^{\zeta,\psi}_{0^+}|\sigma(s) - \sigma_z(s)|\Big)(\vartheta)\Bigg],\end{aligned}$$

where σ and σ_z are functions satisfying the following functional equations:

$$\sigma(\vartheta) = \varphi(\vartheta, y^{\vartheta}(\cdot), \sigma(\vartheta)) \quad \text{and} \quad \sigma_z(\vartheta) = \varphi(\vartheta, z^{\vartheta}(\cdot), \sigma_z(\vartheta)).$$

Using hypothesis $(A2)$, for $\vartheta \in \Theta$, we have

$$\begin{aligned}|\sigma(\vartheta) - \sigma_z(\vartheta)| &= |\varphi\Big(\vartheta, y^{\vartheta}(\cdot), \sigma(\vartheta)\Big) - \varphi\Big(\vartheta, z^{\vartheta}(\cdot), \sigma_z(\vartheta)\Big)| \\ &\leq \psi_1\|y^{\vartheta} - z^{\vartheta}\|_{[-\varpi,\tilde{\varpi}]} + \psi_2|\sigma(\vartheta) - \sigma_z(\vartheta)|,\end{aligned}$$

which implies

$$|\sigma(\vartheta) - \sigma_z(\vartheta)| \leq \frac{\psi_1}{1-\psi_2}\|y^{\vartheta} - z^{\vartheta}\|_{[-\varpi,\tilde{\varpi}]} \leq \frac{\psi_1}{1-\psi_2}\|y - z\|_{\mathbb{F}}.$$

Then, using (A2), we find that

$$\begin{aligned}&|\aleph y(\vartheta) - \aleph z(\vartheta)| \\ &\leq \frac{\|y-z\|_{\mathbb{F}}}{|\Phi(\vartheta)|}\left[\frac{\wp_\jmath|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1}{|\delta_1\Phi(\varkappa)|(1-\psi_2)}\Big(\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}(1)\Big)(\varkappa) + \frac{\psi_1}{1-\psi_2}\Big(\mathbb{I}^{\zeta,\psi}_{0^+}(1)\Big)(\vartheta)\right] \\ &\leq \frac{\|y-z\|_{\mathbb{F}}}{|\Phi(\vartheta)|}\left[\frac{\wp_\jmath|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1(\psi(\varkappa) - \psi(\varkappa_m))^{\zeta}}{|\delta_1\Phi(\varkappa)|(1-\psi_2)\Gamma(\zeta+1)} + \frac{\psi_1(\psi(\vartheta) - \psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right] \\ &\leq \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\frac{\wp^*|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{|\delta_1\Phi(\varkappa)|(1-\psi_2)\Gamma(\zeta+1)} + \frac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right] \\ &\leq \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\frac{\wp^*|\delta_2|\overline{\Phi}^2}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\psi_2)\Gamma(\zeta+1)} + \frac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right].\end{aligned}$$

For $\vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}|\aleph y(\vartheta) - \aleph z(\vartheta)| \leq \frac{1}{|\Phi(\vartheta)|}\Big[&|\Phi(\varkappa_\jmath)| \times \Big|\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) - \Psi_\jmath(\varkappa_\jmath, z(\vartheta_\jmath^-))\Big| \\ &+ \Big(\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}|\sigma(s) - \sigma_z(s)|\Big)(\vartheta)\Big].\end{aligned}$$

Then, using $(A2)$ and $(A3)$, we find that

$$\begin{aligned}|\aleph y(\vartheta)-\aleph z(\vartheta)| &\le \frac{\|y-z\|_{\mathbb{F}}}{|\Phi(\vartheta)|}\left[\wp_{\jmath}|\Phi(\varkappa_{\jmath})|+\frac{\psi_1}{1-\psi_2}\left(\mathbb{I}^{\zeta,\psi}_{\varkappa_{\jmath}^{+}}(1)\right)(\vartheta)\right] \\ &\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\wp^{*}|\Phi(\varkappa_{\jmath})|+\frac{\psi_1(\psi(\vartheta)-\psi(\varkappa_{\jmath}))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right] \\ &\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\wp^{*}\overline{\Phi}+\frac{\psi_1(\psi(\varkappa)-\psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right].\end{aligned}$$

For $\vartheta \in \tilde{\Omega}_{\jmath}; \jmath = 1, \dots, m$, we have

$$|\aleph y(\vartheta)-\aleph z(\vartheta)| \le \left|\Psi_{\jmath}(\vartheta, y(\vartheta_{\jmath}^{-}))-\Psi_{\jmath}(\vartheta, z(\vartheta_{\jmath}^{-}))\right| \le \wp^{*}\|y-z\|_{\mathbb{F}}.$$

Thus, we can conclude that

$$\|\aleph y-\aleph z\|_{\mathbb{F}} \le \beta\|y-z\|_{\mathbb{F}}.$$

Consequently, using the Banach contraction principle, operator $\aleph$ has a unique fixed point, which is a solution to problem (1). □

Our second result is based on Krasnoselskii's fixed point theorem.

Remark 1. *Let us put*

$$\lambda_2(\vartheta)=|\varphi(\vartheta,0,0)|,\ \Lambda_2^{\jmath}(\vartheta)=|\Psi_{\jmath}(\vartheta,0)|,\ \psi_1=\tilde{\lambda}_1,\ \psi_2=\lambda_1,\ \wp^{*}=\Lambda_1,$$

then hypothesis $(A2)$ implies that

$$|\varphi(\vartheta,\xi,\gamma)| \le \tilde{\lambda}_1\|\xi\|_{[-\varpi,\tilde{\varpi}]}+\lambda_1|\gamma|+\lambda_2(\vartheta) \quad \text{and} \quad |\Psi_{\jmath}(\vartheta,\gamma)| \le \Lambda_1|\gamma|+\Lambda_2,$$

for $\vartheta \in \Theta$, $\xi \in PC([-\varpi,\tilde{\varpi}],\mathbb{R})$, $\gamma \in \mathbb{R}$, and $\lambda_2, \Lambda_2^{\jmath} \in C(\Theta,\mathbb{R}_{+})$, with

$$\tilde{\lambda}_2=\sup_{\vartheta\in\Theta}\lambda_2(\vartheta),\ \Lambda_2=\sup_{\vartheta\in\Theta}\Lambda_2^{\jmath}(\vartheta).$$

Set

$$\begin{aligned}\Xi_1 &= \frac{(\overline{\Phi}+\mathcal{M})\Lambda_1}{\mathcal{M}}+\frac{1}{\mathcal{M}}\left[\frac{|\delta_2|\overline{\Phi}^2\Lambda_1}{|\delta_1|\tilde{\Phi}}+\frac{(|\delta_2|\overline{\Phi}+|\delta_1|\tilde{\Phi})\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right], \\ \Xi_2 &= \frac{(\overline{\Phi}+\mathcal{M})\Lambda_2}{\mathcal{M}}+\frac{1}{\mathcal{M}}\left[\frac{|\delta_3|\overline{\Phi}}{|\delta_1|}+\frac{|\delta_2|\overline{\Phi}^2\Lambda_2}{|\delta_1|\tilde{\Phi}}+\frac{(|\delta_2|\overline{\Phi}+|\delta_1|\tilde{\Phi})\tilde{\lambda}_2(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right], \\ \Xi_3 &= \frac{1}{\mathcal{M}}\left[\frac{\Lambda_1\overline{\Phi}(|\delta_2|\overline{\Phi}+|\delta_1|\tilde{\Phi})}{|\delta_1|\tilde{\Phi}}+\frac{|\delta_2|\overline{\Phi}\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right].\end{aligned}$$

Theorem 5. *Let us assume that (A1)–(A3) hold. If*

$$\tilde{\beta} := \max\{\Xi_1, \Xi_3\} < 1, \tag{11}$$

then the problem (1) has at least one solution in $\mathbb{F}$.

Proof. Consider the set

$$\Upsilon_{\varepsilon}=\{\xi \in \mathbb{F} : \|\xi\|_{\mathbb{F}} \le \varepsilon\}, \quad \text{where} \quad \varepsilon \ge \frac{2\Xi_2}{1-\Xi_1}.$$

We define operators $\aleph_1$ and $\aleph_2$ on Υ_ε as

$$\aleph_1 y(\vartheta) = \begin{cases} \dfrac{1}{\Phi(\vartheta)}\left[\dfrac{\delta_3\Phi(0)}{\delta_1} - \dfrac{\delta_2\Phi(0)\Phi(\varkappa_m)\Psi_m(\varkappa_m, y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)}\right. \\ \left. -\dfrac{\delta_2\Phi(0)}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}\sigma(\varkappa)\right], \quad \vartheta \in \Omega_0, \\ \dfrac{1}{\Phi(\vartheta)}\left[\Phi(\varkappa_\jmath)\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-))\right], \quad \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ 0, \text{ if } \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m, \\ 0, \text{ if } \vartheta \in [-\varpi, 0], \\ 0, \text{ if } \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{12}$$

and

$$\aleph_2 y(\vartheta) = \begin{cases} \dfrac{1}{\Phi(\vartheta)}\left[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}\sigma(\vartheta)\right], \quad \vartheta \in \Omega_\jmath; \jmath = 0, \ldots, m, \\ \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)), \ \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m, \\ \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0], \\ \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{13}$$

Then, we can write the following operator equation:

$$\aleph y(\vartheta) = \aleph_1 y(\vartheta) + \aleph_2 y(\vartheta), \ \ y \in \mathbb{F}.$$

We shall use Krasnoselskii's fixed point theorem to prove in several steps that operator $\aleph$ defined in (9) has a fixed point.

Step 1: We prove that $\aleph_1 x + \aleph_2 y \in \Upsilon_\varepsilon$ for any $x, y \in \Upsilon_\varepsilon$.

For $\vartheta \in \Omega_0$, using (12) and Remark 1, we obtain

$$|(\aleph_1 x)(\vartheta)| \leq \frac{1}{|\Phi(\vartheta)|}\left[\frac{|\delta_3\Phi(0)|}{|\delta_1|} + \frac{|\delta_2\Phi(0)\Phi(\varkappa_m)| \times |\Psi_m(\varkappa_m, x(\vartheta_m^-))|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|}{|\delta_1\Phi(\varkappa)|}\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}|\sigma_x(\varkappa)|\right],$$

where σ_x is a function satisfying the following functional equations:

$$\sigma_x(\vartheta) = \varphi(\vartheta, x^\vartheta(\cdot), \sigma_x(\vartheta)).$$

Using Remark 1, we have

$$\begin{aligned} |\sigma_x(\vartheta)| &= |\varphi(\vartheta, x^\vartheta(\cdot), \sigma_x(\vartheta))| \\ &\leq \tilde{\lambda}_1\|x^\vartheta\|_{[-\varpi,\tilde{\varpi}]} + \lambda_1|\sigma_x(\vartheta)| + \lambda_2(\vartheta) \\ &\leq \tilde{\lambda}_1\|x\|_{\mathbb{F}} + \lambda_1|\sigma_x(\vartheta)| + \tilde{\lambda}_2 \\ &\leq \tilde{\lambda}_1\varepsilon + \lambda_1|\sigma_x(\vartheta)| + \tilde{\lambda}_2, \end{aligned}$$

which implies that

$$|\sigma_x(\vartheta)| \leq \frac{\tilde{\lambda}_1\varepsilon + \tilde{\lambda}_2}{1 - \lambda_1}.$$

Thus,

$$|(\aleph_1 x)(\vartheta)| \le \frac{1}{|\Phi(\vartheta)|}\Bigg[\frac{|\delta_3\Phi(0)|}{|\delta_1|} + \frac{|\delta_2\Phi(0)\Phi(\varkappa_m)|(\Lambda_1\varepsilon+\Lambda_2)}{|\delta_1\Phi(\varkappa)|}$$
$$+\frac{|\delta_2\Phi(0)|(\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2)(\psi(\varkappa)-\psi(\varkappa_m))^{\zeta}}{|\delta_1\Phi(\varkappa)|(1-\lambda_1)\Gamma(\zeta+1)}\Bigg]$$
$$\le \frac{1}{\mathcal{M}}\left[\frac{|\delta_3|\overline{\Phi}}{|\delta_1|} + \frac{|\delta_2|\overline{\Phi}^2(\Lambda_1\varepsilon+\Lambda_2)}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}(\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2)(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right].$$

For $\vartheta \in \Omega_\jmath; \jmath = 1,\dots,m$, we have

$$|(\aleph_1 x)(\vartheta)| \le \frac{1}{|\Phi(\vartheta)|}\Big[|\Phi(\varkappa_\jmath)| \times |\Psi_\jmath(\varkappa_\jmath, x(\vartheta_\jmath^-))|\Big] \le \frac{\overline{\Phi}(\Lambda_1\varepsilon+\Lambda_2)}{\mathcal{M}}.$$

Then, we deduce that for each $\vartheta \in \Theta$, we obtain

$$\|\aleph_1 x\|_{\mathbb{F}} \le \max\left\{\frac{\overline{\Phi}\Lambda_1}{\mathcal{M}}\varepsilon + \frac{\overline{\Phi}\Lambda_2}{\mathcal{M}}, \frac{1}{\mathcal{M}}\left[\frac{|\delta_3|\overline{\Phi}}{|\delta_1|} + \frac{|\delta_2|\overline{\Phi}^2\Lambda_2}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\tilde{\lambda}_2(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right]\right.$$
$$\left. + \frac{\varepsilon}{\mathcal{M}}\left[\frac{|\delta_2|\overline{\Phi}^2\Lambda_1}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right]\right\}. \tag{14}$$

For $\vartheta \in \Omega_\jmath; \jmath = 0,\dots,m$, using (13) and Remark 1, we obtain

$$|(\aleph_2 y)(\vartheta)| \le \frac{1}{|\Phi(\vartheta)|}\left[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}|\sigma(\vartheta)|\right] \le \frac{\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2}{|\Phi(\vartheta)|(1-\lambda_1)}\left[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}(1)\right] \le \frac{(\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2)(\psi(\varkappa)-\psi(0))^{\zeta}}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)},$$

and for $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1,\dots,m$, we have

$$|(\aleph_2 y)(\vartheta)| \le \left|\Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\right| \le \Lambda_1\varepsilon + \Lambda_2,$$

then, for each $\vartheta \in \Theta$ we obtain

$$\|\aleph_2 y\|_{\mathbb{F}} \le \max\left\{\Lambda_1\varepsilon+\Lambda_2, \frac{\tilde{\lambda}_2(\psi(\varkappa)-\psi(0))^{\zeta}}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)} + \varepsilon\frac{\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^{\zeta}}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)}\right\}. \tag{15}$$

From (14) and (15), for each $\vartheta \in \Theta$, we have,

$$\|\aleph_1 x + \aleph_2 y\|_{\mathbb{F}} \le \|\aleph_1 x\|_{PC} + \|\aleph_2 y\|_{\mathbb{F}} \le \varepsilon,$$

thus, $\aleph_1 x + \aleph_2 y \in \Upsilon_\varepsilon$.

Step 2: $\aleph_1$ is a contraction.

Let $y, z \in C(\Theta, \mathbb{R})$. Then, for $\vartheta \in \Omega_0$, we have

$$|\aleph_1 y(\vartheta) - \aleph_1 z(\vartheta)| \le \frac{\|y-z\|_{\mathbb{F}}}{|\Phi(\vartheta)|}\left[\frac{\wp_\jmath|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1}{|\delta_1\Phi(\varkappa)|(1-\psi_2)}\left(\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}(1)\right)(\varkappa)\right]$$
$$\le \frac{\|y-z\|_{\mathbb{F}}}{|\Phi(\vartheta)|}\left[\frac{\wp_\jmath|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1(\psi(\varkappa)-\psi(\varkappa_m))^{\zeta}}{|\delta_1\Phi(\varkappa)|(1-\psi_2)\Gamma(\zeta+1)}\right]$$
$$\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\frac{\wp^*|\delta_2\Phi(0)\Phi(\varkappa_m)|}{|\delta_1\Phi(\varkappa)|} + \frac{|\delta_2\Phi(0)|\psi_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1\Phi(\varkappa)|(1-\psi_2)\Gamma(\zeta+1)}\right]$$
$$\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}\left[\frac{\wp^*|\delta_2|\overline{\Phi}^2}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\psi_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\psi_2)\Gamma(\zeta+1)}\right].$$

For $\vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}|\aleph y(\vartheta) - \aleph z(\vartheta)| &\le \frac{1}{|\Phi(\vartheta)|}\Big[|\Phi(\varkappa_\jmath)| \times \Big|\Psi_\jmath(\varkappa_\jmath, y(\vartheta_\jmath^-)) - \Psi_\jmath(\varkappa_\jmath, z(\vartheta_\jmath^-))\Big|\Big] \\ &\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}(\wp^*|\Phi(\varkappa_\jmath)|) \\ &\le \frac{\|y-z\|_{\mathbb{F}}}{\mathcal{M}}(\wp^*\overline{\Phi}).\end{aligned}$$

Thus, using Remark 1, we find that for each $\vartheta \in \Theta$, we have

$$\|\aleph_1 y - \aleph_1 z\|_{\mathbb{F}} \le \frac{1}{\mathcal{M}}\left[\frac{\Lambda_1\overline{\Phi}(|\delta_2|\overline{\Phi} + |\delta_1|\tilde{\Phi})}{|\delta_1|\tilde{\Phi}} + \frac{|\delta_2|\overline{\Phi}\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^\zeta}{|\delta_1|\tilde{\Phi}(1-\lambda_1)\Gamma(\zeta+1)}\right]\|y-z\|_{\mathbb{F}} \le \Xi_3\|y-z\|_{\mathbb{F}}.$$

Then, using (11), operator $\aleph_1$ is a contraction.

Step 3: $\aleph_2$ is continuous and compact. Let $\{y_n\}$ be a sequence where $y_n \to y$ in $\mathbb{F}$. Then, for each $\vartheta \in [-\varpi, 0] \cup [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$|\aleph y_n(\vartheta) - \aleph y(\vartheta)| = 0.$$

For $\vartheta \in \Omega_\jmath; \jmath = 0, \ldots, m$, we have

$$|(\aleph_2 y_n)(\vartheta) - (\aleph_2 y)(\vartheta)| \le \frac{1}{\Phi(\vartheta)}\Big[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}|\sigma_n(\vartheta) - \sigma(\vartheta)|\Big],$$

where σ and σ_n are functions satisfying the following functional equations:

$$\sigma(\vartheta) = \varphi(\vartheta, y^\vartheta(\cdot), \sigma(\vartheta)) \quad \text{and} \quad \sigma_n(\vartheta) = \varphi(\vartheta, y_n^\vartheta(\cdot), \sigma_n(\vartheta)).$$

For each $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$, we have,

$$|(\aleph_2 y_n)(\vartheta) - (\aleph_2 y)(\vartheta)| \le \Big|\Psi_\jmath(\vartheta, y_n(\vartheta_\jmath^-)) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\Big|.$$

Since $y_n \to y$ and since φ and $\Psi_\jmath$ are continuous, we may obtain

$$\|\aleph_2 y_n - \aleph_2 y\|_{PC} \to 0 \text{ as } n \to \infty.$$

Then, $\aleph_2$ is continuous. Now, we demonstrate that $\aleph_2$ is uniformly bounded on Υ_ε. Let $y \in \Upsilon_\varepsilon$. Thus, for $\vartheta \in \Theta$,

$$\|\aleph_2 y\|_{\mathbb{F}} \le \max\left\{\Lambda_1\varepsilon + \Lambda_2, \frac{\tilde{\lambda}_2(\psi(\varkappa)-\psi(0))^\zeta}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)} + \varepsilon\frac{\tilde{\lambda}_1(\psi(\varkappa)-\psi(0))^\zeta}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)}\right\}.$$

Consequently, $\aleph_2$ is uniformly bounded on Υ_ε. We take $y \in \Upsilon_\varepsilon$ and $0 < \gamma_1 < \gamma_2 \le \varkappa$. Then, for $\gamma_1, \gamma_2 \in \Omega_\jmath; \jmath = 0, \ldots, m$,

$$\begin{aligned}&|(\aleph_2 y)(\gamma_1) - (\aleph_2 y)(\gamma_2)| \\ &\le \left|\frac{1}{\Phi(\gamma_1)}\Big[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}\sigma(\gamma_1)\Big] - \frac{1}{\Phi(\gamma_2)}\Big[\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}\sigma(\gamma_2)\Big]\right| \\ &\le \left|\frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\gamma_1}\psi'(s)\frac{(\psi(\gamma_1)-\psi(s))^{\zeta-1}}{\Phi(\gamma_1)}\sigma(s)ds - \frac{1}{\Gamma(\zeta)}\int_{\varkappa_\jmath}^{\gamma_2}\psi'(s)\frac{(\psi(\gamma_2)-\psi(s))^{\zeta-1}}{\Phi(\gamma_2)}\sigma(s)ds\right|\end{aligned}$$

$$\leq \frac{1}{\Gamma(\zeta)}\int_{\varkappa_j}^{\gamma_1}\psi'(s)\left|\frac{(\psi(\gamma_1)-\psi(s))^{\zeta-1}}{\Phi(\gamma_1)}-\frac{(\psi(\gamma_2)-\psi(s))^{\zeta-1}}{\Phi(\gamma_2)}\right||\sigma(s)|ds$$
$$+\frac{\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2}{\Gamma(\zeta)}\int_{\gamma_1}^{\gamma_2}\psi'(s)\frac{(\psi(\gamma_2)-\psi(s))^{\zeta-1}}{\Phi(\gamma_2)}ds$$
$$\leq \frac{1}{\Gamma(\zeta)}\int_{\varkappa_j}^{\gamma_1}\psi'(s)\left|\frac{(\psi(\gamma_1)-\psi(s))^{\zeta-1}}{\Phi(\gamma_1)}-\frac{(\psi(\gamma_2)-\psi(s))^{\zeta-1}}{\Phi(\gamma_2)}\right||\sigma(s)|ds$$
$$+\frac{(\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2)(\psi(\gamma_2)-\psi(\gamma_1))^{\zeta}}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)}$$
$$\leq \frac{(\tilde{\lambda}_1\varepsilon+\tilde{\lambda}_2)\left[(\psi(\gamma_1)-\psi(0))^{\zeta}-(\psi(\gamma_2)-\psi(0))^{\zeta}\right]}{\mathcal{M}(1-\lambda_1)\Gamma(\zeta+1)}.$$

Note that

$$|(\aleph_2 y)(\gamma_1)-(\aleph_2 y)(\gamma_2)|\to 0 \quad \text{as} \quad \gamma_1\to\gamma_2.$$

Moreover, for $\gamma_1,\gamma_2\in\tilde{\Omega}_j; j=1,\ldots,m$,

$$|(\aleph_2 y)(\gamma_1)-(\aleph_2 y)(\gamma_2)|\leq\left|\Psi_j(\gamma_1,y(\vartheta_j^-))-\Psi_j(\gamma_2,y(\vartheta_j^-))\right|,$$

Note, since Ψ_j are continuous, that

$$|(\aleph_2 y)(\gamma_1)-(\aleph_2 y)(\gamma_2)|\to 0 \quad \text{as} \quad \gamma_1\to\gamma_2.$$

Thus, $\aleph_2 Y_\varepsilon$ is equicontinuous on Θ, which implies that $\aleph_2 Y_\varepsilon$ is relatively compact. By the Arzelà–Ascoli theorem, $\aleph_2$ is compact. The equicontinuity for the other cases is obvious. Using Theorem 2, we conclude that $\aleph$ admits at least a fixed point, which is a solution to problem (1). □

4. Ulam–Hyers–Rassias Stability

Now, we consider the Ulam stability for problem (1). For this, we take inspiration from the papers [6,26–28] and the references therein. Let $y\in\mathbb{F}$, $\epsilon>0$, $\Delta_1,\Delta_2>0$, $\lambda>0$, and $\Im:\Theta\longrightarrow[0,\infty)$ be a continuous function. We consider the following inequalities:

$$\begin{cases}\left|{}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta))-\varphi\left(\vartheta,y^{\vartheta}(\cdot),{}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta))\right)\right|\leq\epsilon,\vartheta\in\Omega_j, j=0,\ldots,m,\\ \left|y(\vartheta)-\Psi_j(\vartheta,y(\vartheta_j^-))\right|\leq\epsilon,\quad\vartheta\in\tilde{\Omega}_j, j=1,\ldots,m,\\ |y(\vartheta)-\hbar_1(\vartheta)|\leq\epsilon,\quad\vartheta\in[-\varpi,0],\\ |y(\vartheta)-\hbar_2(\vartheta)|\leq\epsilon,\quad\vartheta\in[\varkappa,\varkappa+\tilde{\varpi}],\end{cases}\tag{16}$$

$$\begin{cases}\left|{}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta))-\varphi\left(\vartheta,y^{\vartheta}(\cdot),{}^{C}D_{\varkappa_j}^{\zeta;\psi}(\Phi(\vartheta)y(\vartheta))\right)\right|\leq\Im(\vartheta),\vartheta\in\Omega_j, j=0,\ldots,m,\\ \left|y(\vartheta)-\Psi_j(\vartheta,y(\vartheta_j^-))\right|\leq\lambda,\quad\vartheta\in\tilde{\Omega}_j, j=1,\ldots,m,\\ |y(\vartheta)-\hbar_1(\vartheta)|\leq\Delta_1,\quad\vartheta\in[-\varpi,0],\\ |y(\vartheta)-\hbar_2(\vartheta)|\leq\Delta_2,\quad\vartheta\in[\varkappa,\varkappa+\tilde{\varpi}],\end{cases}\tag{17}$$

and

$$\begin{cases} \left|{}^{C}D^{\zeta;\psi}_{\varkappa_\jmath}(\Phi(\vartheta)y(\vartheta)) - \varphi\Big(\vartheta, y^{\vartheta}(\cdot), {}^{C}D^{\zeta;\psi}_{\varkappa_\jmath}(\Phi(\vartheta)y(\vartheta))\Big)\right| \leq \epsilon\Im(\vartheta), \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left|y(\vartheta) - \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-))\right| \leq \epsilon\lambda, \quad \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ |y(\vartheta) - \hbar_1(\vartheta)| \leq \epsilon\Delta_1, \quad \vartheta \in [-\varpi, 0], \\ |y(\vartheta) - \hbar_2(\vartheta)| \leq \epsilon\Delta_2, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\omega}]. \end{cases} \tag{18}$$

Definition 3 ([6,27]). *Problem* (1) *is Ulam–Hyers (U-H) stable if there exists a real number $a_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (16), there exists a solution $y \in \mathbb{F}$ of* (1) *with*

$$|x(\vartheta) - y(\vartheta)| \leq \epsilon a_\varphi, \qquad \vartheta \in \Theta.$$

Definition 4 ([6,27]). *Problem* (1) *is generalized Ulam–Hyers (G.U-H) stable if there exists $K_\varphi : C([0,\infty), [0,\infty))$ with $K_\varphi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (16), there exists a solution $y \in \mathbb{F}$ of* (1) *with*

$$|x(\vartheta) - y(\vartheta)| \leq K_\varphi(\epsilon), \qquad \vartheta \in \Theta.$$

Definition 5 ([6,27]). *Problem* (1) *is Ulam–Hyers–Rassias (U-H-R) stable with respect to $(\Im, \lambda, \Delta_1, \Delta_2)$ if there exists a real number $a_{\varphi,\Im} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathbb{F}$ of inequality (18), there exists a solution $y \in \mathbb{F}$ of* (1) *with*

$$|x(\vartheta) - y(\vartheta)| \leq \epsilon a_{\varphi,\Im}(\Im(\vartheta) + \lambda + \Delta_1 + \Delta_2), \qquad \vartheta \in \Theta.$$

Definition 6 ([6,27]). *Problem* (1) *is generalized Ulam–Hyers–Rassias (G.U-H-R) stable with respect to $(\Im, \lambda, \Delta_1, \Delta_2)$ if there exists a real number $a_{\varphi,\Im} > 0$ such that for each solution $x \in \mathbb{F}$ of inequality (18), there exists a solution $y \in \mathbb{F}$ of* (1) *with*

$$|x(\vartheta) - y(\vartheta)| \leq a_{\varphi,\Im}(\Im(\vartheta) + \lambda + \Delta_1 + \Delta_2), \qquad \vartheta \in \Theta.$$

Remark 2. *It is clear that*

1. *Definition 3 $\Longrightarrow$ Definition 4.*
2. *Definition 5 $\Longrightarrow$ Definition 6.*
3. *Definition 5 for $\Im(.) = \lambda = \Delta_1 = \Delta_2 = 1 \Longrightarrow$ Definition 3.*

Remark 3. *A function $y \in \mathbb{F}$ is a solution of inequality (18) if and only if there exist $v \in \mathbb{F}$ and a sequence $v_\jmath, \jmath = 0, \ldots, m+2$ such that*

- $|v(\vartheta)| \leq \epsilon\Im(\vartheta)$, $\vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m$; $|v_\jmath| \leq \epsilon\lambda$, $\vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m$, $|v_{m+1}| \leq \epsilon\Delta_1$ *and* $|v_{m+2}| \leq \epsilon\Delta_2$.
- ${}^{C}D^{\zeta;\psi}_{\varkappa_\jmath}(\Phi(\vartheta)y(\vartheta)) = \varphi\Big(\vartheta, y^{\vartheta}(\cdot), {}^{C}D^{\zeta;\psi}_{\varkappa_\jmath}(\Phi(\vartheta)y(\vartheta))\Big) + v(\vartheta)$, $\vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m$.
- $y(\vartheta) = \Psi_\jmath(\vartheta, y(\vartheta_\jmath^-)) + v_\jmath$, $\vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m$.
- $y(\vartheta) = \hbar_1(\vartheta) + v_{m+1}, \quad \vartheta \in [-\varpi, 0]$.
- $y(\vartheta) = \hbar_2(\vartheta) + v_{m+2}, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\omega}]$.

Theorem 6. *Let us assume that in addition to* (A1)–(A3) *and (10), the following hypothesis holds:*

(A4) There exists a nondecreasing function $\Im : \Theta \longrightarrow [0, \infty)$ and $\ell_\Im > 0$ such that for each $\vartheta \in \Omega_\jmath; \jmath = 0, \ldots, m$, we have

$$(\mathbb{I}^{\zeta,\psi}_{\varkappa_\jmath^+}\Im)(\vartheta) \leq \ell_\Im \Im(\vartheta).$$

Then, problem (1) *is U-H-R stable with respect to $(\Im, \lambda)$.*

Proof. Let $x \in \mathbb{F}$ be a solution if inequality (18), and let us assume that y is the unique solution of problem

$$\begin{cases} {}^C D^{\zeta;\psi}_{\varkappa_J}(\Phi(\vartheta)y(\vartheta)) = \varphi\Big(\vartheta, y^{\vartheta}(\cdot), {}^C D^{\zeta;\psi}_{\varkappa_J}(\Phi(\vartheta)y(\vartheta))\Big);\ \vartheta \in \Omega_J,\ J = 0, \dots, m, \\ y(\vartheta) = \Psi_J(\vartheta, y(\vartheta_J^-));\ \vartheta \in \tilde{\Omega}_J,\ J = 1, \dots, m, \\ y(\vartheta) = \hbar_1(\vartheta), \quad \vartheta \in [-\varpi, 0], \\ y(\vartheta) = \hbar_2(\vartheta), \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \\ \delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \\ y(\varkappa_J) = x(\varkappa_J); J = 0, \dots, m, \\ y(\vartheta_J) = x(\vartheta_J); J = 1, \dots, m+1. \end{cases}$$

Using Theorem 3, we obtain, for each $\vartheta \in \Theta$,

$$y(\vartheta) = \begin{cases} \dfrac{1}{\Phi(\vartheta)}\Bigg[\dfrac{\delta_3\Phi(0)}{\delta_1} - \dfrac{\delta_2\Phi(0)\Phi(\varkappa_m)\Psi_m(\varkappa_m, y(\vartheta_m^-))}{\delta_1\Phi(\varkappa)} \\ -\dfrac{\delta_2\Phi(0)}{\delta_1\Phi(\varkappa)}\mathbb{I}^{\zeta,\psi}_{\varkappa_m^+}\sigma(\varkappa) + \mathbb{I}^{\zeta,\psi}_{0^+}\sigma(\vartheta)\Bigg], \quad \vartheta \in \Omega_0, \\ \dfrac{1}{\Phi(\vartheta)}\Big[\Phi(\varkappa_J)\Psi_J(\varkappa_J, y(\vartheta_J^-)) + \mathbb{I}^{\zeta,\psi}_{\varkappa_J^+}\sigma(\vartheta)\Big], \quad \vartheta \in \Omega_J; J = 1, \dots, m, \\ \Psi_J(\vartheta, y(\vartheta_J^-)),\ \vartheta \in \tilde{\Omega}_J; J = 1, \dots, m, \\ \hbar_1(\vartheta), \ \ \vartheta \in [-\varpi, 0], \\ \hbar_2(\vartheta), \ \ \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases}$$

where σ is a function satisfying the following functional equations:

$$\sigma(\vartheta) = \varphi(\vartheta, y^{\vartheta}(\cdot), \sigma(\vartheta)).$$

Since x is a solution of inequality (18), using Remark 3, we have

$$\begin{cases} {}^C D^{\zeta;\psi}_{\varkappa_J}(\Phi(\vartheta)x(\vartheta)) = \varphi\Big(\vartheta, x^{\vartheta}(\cdot), {}^C D^{\zeta;\psi}_{\varkappa_J}(\Phi(\vartheta)x(\vartheta))\Big) + v(\vartheta), \vartheta \in \Omega_J, J = 0, \dots, m, \\ x(\vartheta) = \Psi_J(\vartheta, x(\vartheta_J^-)) + v_J, \vartheta \in \tilde{\Omega}_J, J = 1, \dots, m, \\ x(\vartheta) = \hbar_1(\vartheta) + v_{m+1}, \quad \vartheta \in [-\varpi, 0], \\ x(\vartheta) = \hbar_2(\vartheta) + v_{m+2}, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]. \end{cases} \tag{19}$$

Clearly, the solution of (19) is given by

$$x(\vartheta) = \begin{cases} \dfrac{1}{\Phi(\vartheta)}\Big[\Phi(\varkappa_J)x(\varkappa_J) + \mathbb{I}^{\zeta,\psi}_{\varkappa_J^+}\sigma_x(\vartheta)\Big], \quad if\ \vartheta \in \Omega_J, J = 0, \dots, m, \\ \Psi_J(\vartheta, x(\vartheta_J^-)) + v_J, \qquad if\ \vartheta \in \tilde{\Omega}_J, J = 1, \dots, m, \\ \hbar_1(\vartheta) + v_{m+1}, \quad \vartheta \in [-\varpi, 0], \\ \hbar_2(\vartheta) + v_{m+2}, \quad \vartheta \in [\varkappa, \varkappa + \tilde{\varpi}], \end{cases}$$

where σ_x is a function satisfying the following functional equations:

$$\sigma_x(\vartheta) = \varphi(\vartheta, x_n^{\vartheta}(\cdot), \sigma_x(\vartheta)).$$

Hence, for each $\vartheta \in \Omega_J, J = 0, \dots, m$, we have

$$\begin{aligned} |x(\vartheta) - y(\vartheta)| &\le \frac{1}{\Phi(\vartheta)}\left[\frac{1}{\Gamma(\zeta)}\int_{\varkappa_J}^{\vartheta}\psi'(\varrho)(\psi(\vartheta) - \psi(\varrho))^{\zeta-1}|\sigma_x(\varrho) - \sigma(\varrho)|d\varrho + \left(\mathbb{I}^{\zeta,\psi}_{\varkappa_J^+}|v(\tau)|\right)\right] \\ &\le \frac{1}{\mathcal{M}}\left[\epsilon\ell_{\Im}\Im(\vartheta) + \frac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{(1 - \psi_2)\Gamma(\zeta + 1)}\|x - y\|_{\mathbb{F}}\right]. \end{aligned}$$

Moreover, for each $\vartheta \in \tilde{\Omega}_J, J = 1, \ldots, m$, we have

$$\begin{aligned}|x(\vartheta) - y(\vartheta)| &\leq |\Psi_J(\vartheta, x(\vartheta_J^-)) - \Psi_J(\vartheta, y(\vartheta_J^-))| + |v_J| \\ &\leq \wp^*|x(\vartheta) - y(\vartheta)| + \epsilon\lambda \\ &\leq \wp^*\|x - y\|_{\mathbb{F}} + \epsilon\lambda.\end{aligned}$$

For each $\vartheta \in [-\varpi, 0]$, we have

$$|x(\vartheta) - y(\vartheta)| \leq |v_{m+1}| \leq \epsilon\Delta_1.$$

Moreover, for each $\vartheta \in [\varkappa, \varkappa + \tilde{\varpi}]$, we have

$$|x(\vartheta) - y(\vartheta)| \leq |v_{m+2}| \leq \epsilon\Delta_2.$$

Thus,

$$\|x - y\|_{\mathbb{F}} \leq \left[\epsilon\ell_{\Im}\frac{\Im(\vartheta)}{\mathcal{M}} + \epsilon\lambda + \epsilon\Delta_1 + \epsilon\Delta_2\right] + \left[\wp^* + \frac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{\mathcal{M}(1 - \psi_2)\Gamma(\zeta + 1)}\right]\|x - y\|_{\mathbb{F}}.$$

Then, we have

$$\|x - y\|_{\mathbb{F}} \leq a_{\varphi,\Im}\epsilon(\lambda + \Im(\vartheta) + \Delta_1 + \Delta_2),$$

where

$$a_{\varphi,\Im} = \frac{1 + \dfrac{\ell_{\Im}}{\mathcal{M}}}{1 - \left[\wp^* + \dfrac{\psi_1(\psi(\varkappa) - \psi(0))^{\zeta}}{\mathcal{M}(1 - \psi_2)\Gamma(\zeta + 1)}\right]}.$$

Hence, problem (1) is U-H-R stable with respect to $(\Im, \lambda, \Delta_1, \Delta_2)$. □

Remark 4. *If conditions (A1)–(A3) and (10) are satisfied, then using Theorem 6 and Remark 2, it is clear that problem (1) is U-H-R stable and G.U-H-R stable. Moreover, if $\Im(.) = \lambda = \Delta_1 = \Delta_2 = 1$, then problem (1) is also G.U-H stable and U-H stable.*

5. Examples

Example 1. *Consider the following boundary value impulsive problem, which is an example of our problem (1):*

$$^{C}D_{\varkappa_J}^{\frac{1}{2};\psi}(\Phi(\vartheta)y(\vartheta)) = \varphi\left(\vartheta, y^{\vartheta}(\cdot),\ ^{C}D_{\varkappa_J}^{\frac{1}{2};\psi}(\Phi(\vartheta)y(\vartheta))\right);\ \vartheta \in \Omega_0 \cup \Omega_1, \tag{20}$$

$$y(\vartheta) = \Psi_1(\vartheta, y(\vartheta_1^-)) \in \tilde{\Omega}_1, \tag{21}$$

$$y(0) + y(\varkappa) = 0, \tag{22}$$

$$y(\vartheta) = \hbar_1(\vartheta),\ \ \vartheta \in [-\pi, 0],\ \varpi > 0, \tag{23}$$

$$y(\vartheta) = \hbar_2(\vartheta),\ \ \vartheta \in [\pi, 2\pi],\ \tilde{\varpi} > 0, \tag{24}$$

where $\Omega_0 = (0, 2]$, $\Omega_1 = (3, \pi]$, $\tilde{\Omega}_1 = (2, 3]$, $\varkappa_0 = 0$, $\vartheta_1 = 2$, and $\varkappa_1 = 3$, with $\zeta = \frac{1}{2}$, $\psi(\vartheta) = \vartheta$, $J \in \{0, 1\}$, $\delta_1 = \delta_2 = 1$, $\delta_3 = 0$, and $\varpi = \tilde{\varpi} = \pi$.

Set

$$\begin{aligned}\varphi(\vartheta, y_1, y_2) &= \frac{3 + 3|\sin(\vartheta)| + \|y_1\|_{[-\varpi,\tilde{\varpi}]} + |y_2(\vartheta)|}{2450 + 6230e^{\vartheta}},\ \vartheta \in \Omega_0 \cup \Omega_1, \\ \Phi(\vartheta) &= \frac{\sqrt{3}}{233}\left(\vartheta^2 + 3|\sin(\vartheta)| + 1\right), \\ \Psi_1(\vartheta, y_2(\vartheta_1^-)) &= \frac{|\cos(\vartheta)| + |y_2(\vartheta)|}{412e^{\vartheta}},\end{aligned}$$

where $y_1 \in PC([-\pi,\pi],\mathbb{R})$, $y_2 \in \mathbb{R}$.

Clearly, function φ *is continuous. Hence, condition* (A1) *is satisfied. For each* $x_1, y_1 \in PC([-\pi,\pi],\mathbb{R})$, $x_2, y_2 \in \mathbb{R}$, *and* $\vartheta \in \Theta$, *we have*

$$\begin{aligned}
|\varphi(\vartheta,x_1,x_2)-\varphi(\vartheta,y_1,y_2)| &\leq \frac{1}{2450+6230e^{\vartheta}}\Big(\|x_1-y_1\|_{[-\varpi,\tilde{\varpi}]}+|x_2-y_2|\Big)\\
&\leq \frac{1}{8680}\Big(\|x_1-y_1\|_{[-\varpi,\tilde{\varpi}]}+\|x_2-y_2\|\Big),\\
|\Psi_1(\vartheta,x_2(\vartheta_1^-))-\Psi_1(\vartheta,y_2(\vartheta_1^-))| &\leq \frac{|x_2(\vartheta)-y_2(\vartheta)|}{412e^{\vartheta}}\\
&\leq \frac{1}{412}|x_2(\vartheta)-y_2(\vartheta)|.
\end{aligned}$$

Hence, condition (A2) *is satisfied with* $\psi_1=\psi_2=\frac{1}{8680}$ *and* $\wp^*=\frac{1}{412}$.

Hypothesis (A3) *is verified with* $\mathcal{M}=\frac{\sqrt{3}}{233}$, *indeed we have* $|\Phi(\vartheta)| \geq \frac{\sqrt{3}}{233}$. *Condition* (10) *of Theorem 4 is verified, for*

$$\tilde{\Phi}=\frac{\sqrt{3}}{233} \quad \text{and} \quad \overline{\Phi}=\frac{\sqrt{3}(\pi^2+1)}{233}.$$

Then,

$$\begin{aligned}
\beta &= \frac{1}{\mathcal{M}}\left[\wp^*\overline{\Phi}+\frac{\wp^*|\delta_2|\overline{\Phi}^2}{|\delta_1|\tilde{\Phi}}+\frac{|\delta_2|\overline{\Phi}\psi_1(\psi(\varkappa)-\psi(0))^{\zeta}}{|\delta_1|\tilde{\Phi}(1-\psi_2)\Gamma(\zeta+1)}+\frac{\psi_1(\psi(\varkappa)-\psi(0))^{\zeta}}{(1-\psi_2)\Gamma(\zeta+1)}\right]+\wp^*\\
&= \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}(\pi^2+1)+\sqrt{3}(\pi^2+1)^2}{412}+\frac{466(\pi^2+1)+2}{8679}\right]+\frac{1}{412}\\
&\approx 0.652663491979853\\
&< 1.
\end{aligned}$$

Then, problem (20)–(24) *has a unique solution in* $\mathbb{F}$.

Now, if we want to check the result obtained in Theorem 5, using Remark 1, we deduce that all the requirements of Theorem 5 are verified. Indeed, we have

$$\tilde{\beta} \approx 0.6526635 < 1.$$

Consequently, problem (20)–(24) *has at least one solution in* $\mathbb{F}$.

Hypothesis (A4) *is satisfied with* $\lambda=\Delta_1=\Delta_2=1$, $\Im(\vartheta)=5\sqrt{\pi}$, *and* $\ell_{\Im}=2$. *Indeed, for each* $\vartheta \in \Omega_0 \cup \Omega_1$, *we obtain*

$$\mathbb{I}_{\varkappa_j+}^{\frac{1}{2},\psi}5\sqrt{\pi}=\frac{1}{\Gamma(\zeta)}\int_{\varkappa_j}^{\vartheta}(\vartheta-\varrho)^{-\frac{1}{2}}5\sqrt{\pi}d\varrho \leq 5\int_0^{\vartheta}(\vartheta-\varrho)^{-\frac{1}{2}}d\varrho \leq 10\sqrt{\pi}.$$

Consequently, Theorem 6 implies that problem (20)–(24) *is U-H-R stable.*

Example 2. *Consider the following initial value impulsive problem:*

$$^{C}D_{\varkappa_j}^{\frac{1}{4};\psi}(\Phi(\vartheta)y(\vartheta))=\varphi\Big(\vartheta,y^{\vartheta}(\cdot),\,^{C}D_{\varkappa_j}^{\frac{1}{4};\psi}(\Phi(\vartheta)y(\vartheta))\Big);\ \vartheta\in\Omega_0\cup\Omega_1, \tag{25}$$

$$y(\vartheta)=\Psi_1(\vartheta,y(\vartheta_1^-))\in\tilde{\Omega}_1, \tag{26}$$

$$y(0)=0, \tag{27}$$

$$y(\vartheta)=1+\vartheta^2,\ \ \vartheta\in[-e,0],\ \varpi>0, \tag{28}$$

$$y(\vartheta)=1-\vartheta^2,\ \ \vartheta\in[6,8],\ \tilde{\varpi}>0, \tag{29}$$

where $\Omega_0 = (0, e]$, $\Omega_1 = (2e, 6]$, $\tilde{\Omega}_1 = (e, 2e]$, $\varkappa_0 = 0$, $\vartheta_1 = e$, *and* $\varkappa_1 = 2e$, *with* $\zeta = \frac{1}{4}$, $\psi(\vartheta) = \vartheta^2$, $\jmath \in \{0, 1\}$, $\delta_1 = 1$, $\delta_3 = \delta_2 = 0$, *and* $\varpi = \tilde{\varpi} = 2$.

Set

$$\begin{aligned} \varphi(\vartheta, y_1, y_2) &= \frac{1 + \|y_1\|_{[-\varpi,\tilde{\varpi}]} + |y_2(\vartheta)|}{2022 + 2022e^{12\vartheta}}, \ \vartheta \in \Omega_0 \cup \Omega_1, \\ \Phi(\vartheta) &= \frac{\vartheta^2 + 1}{22}, \\ \Psi_1(\vartheta, y_2(\vartheta_1^-)) &= \frac{|\cos(\vartheta)| + |y_2(\vartheta)|}{312e^{12\vartheta}}, \end{aligned}$$

where $y_1 \in PC([-2, 2], \mathbb{R})$, $y_2 \in \mathbb{R}$.

Clearly, function φ *is continuous. Hence, condition* (A1) *is satisfied. For each* $x_1, y_1 \in PC([-2, 2], \mathbb{R})$, $x_2, y_2 \in \mathbb{R}$, *and* $\vartheta \in \Theta$, *we have*

$$\begin{aligned} |\varphi(\vartheta, x_1, x_2) - \varphi(\vartheta, y_1, y_2)| &\leq \frac{1}{2022 + 2022e^{12\vartheta}} \Big(\|x_1 - y_1\|_{[-\varpi,\tilde{\varpi}]} + |x_2 - y_2| \Big) \\ &\leq \frac{1}{4044} \Big(\|x_1 - y_1\|_{[-\varpi,\tilde{\varpi}]} + \|x_2 - y_2\| \Big), \\ |\Psi_1(\vartheta, x_2(\vartheta_1^-)) - \Psi_1(\vartheta, y_2(\vartheta_1^-))| &\leq \frac{|x_2(\vartheta) - y_2(\vartheta)|}{312e^{12\vartheta}} \\ &\leq \frac{1}{312} |x_2(\vartheta) - y_2(\vartheta)|. \end{aligned}$$

Hence, condition (A2) *is satisfied with* $\psi_1 = \psi_2 = \frac{1}{4044}$ *and* $\wp^* = \frac{1}{312}$.

Hypothesis (A3) *is verified with* $\mathcal{M} = \frac{1}{22}$, *and condition* (10) *of Theorem 4 is verified, for*

$$\tilde{\Phi} = \frac{1}{22} \quad \text{and} \quad \overline{\Phi} = \frac{4e^2 + 1}{22}.$$

Indeed, we have

$$\begin{aligned} \beta &= \frac{1}{\mathcal{M}} \left[\wp^* \overline{\Phi} + \frac{\wp^* |\delta_2| \overline{\Phi}^2}{|\delta_1| \tilde{\Phi}} + \frac{|\delta_2| \overline{\Phi} \psi_1 (\psi(\varkappa) - \psi(0))^{\zeta}}{|\delta_1| \tilde{\Phi} (1 - \psi_2) \Gamma(\zeta + 1)} + \frac{\psi_1 (\psi(\varkappa) - \psi(0))^{\zeta}}{(1 - \psi_2) \Gamma(\zeta + 1)} \right] + \wp^* \\ &= 22 \left[\frac{4e^2 + 1}{412 \times 2022} + \frac{36^{\frac{1}{4}}}{4043 \Gamma(\frac{5}{4})} \right] + \frac{1}{312} \\ &\approx 0.018717359521359 \\ &< 1. \end{aligned}$$

Then, problem (25)–(29) *has a unique solution in* $\mathbb{F}$. *Moreover, since* $\tilde{\beta} \approx 0.0187 < 1$, *using Theorem 5, problem* (25)–(29) *at least one solution in* $\mathbb{F}$. *As in the above example, we can easily verify that the requirements of Theorem 6 are verified, which implies that problem* (25)–(29) *is U-H-R stable.*

Example 3. *Let us consider problem* (25)–(29) *with the following modifications:* $\zeta = \frac{1}{3}$ *and* $\psi(\vartheta) = \dfrac{1}{1+e^{-\vartheta}}$.
By following the same steps as the above example, we obtain

$$\begin{aligned}
\beta &= \tilde{\beta} \\
&= 22\left[\frac{4e^2+1}{412\times 2022} + \frac{\left(\frac{1}{1+e^{-6}} - \frac{1}{2}\right)^{\frac{1}{3}}}{4043\Gamma(\frac{4}{3})}\right] + \frac{1}{312} \\
&\approx 0.00884062006887214 \\
&< 1.
\end{aligned}$$

Consequently, we obtain the existence result of our problem using Theorem 5 and the uniqueness result using Theorem 4.

Hypothesis (A4) is satisfied with $\lambda = \Delta_1 = \Delta_2 = 1$, $\Im(\vartheta) = 3e$, *and*

$$\ell_{\Im} = \frac{1}{\Gamma(\frac{4}{3})}\left(\frac{1}{1+e^{-6}} - \frac{1}{2}\right)^{\frac{1}{3}}.$$

Indeed, for each $\vartheta \in \Omega_0 \cup \Omega_1$, *we obtain*

$$\mathbb{I}_{\varkappa_j^+}^{\frac{1}{3},\psi} 3e = \frac{3e}{\Gamma(\frac{1}{3})}\int_{\varkappa_j}^{\vartheta} (\vartheta - \varrho)^{-\frac{2}{3}} d\varrho \leq \frac{3e}{\Gamma(\frac{1}{3})}\int_{0}^{\vartheta} (\vartheta - \varrho)^{-\frac{2}{3}} d\varrho \leq \frac{3e}{\Gamma(\frac{4}{3})}\left(\frac{1}{1+e^{-6}} - \frac{1}{2}\right)^{\frac{1}{3}}.$$

Consequently, Theorem 6 implies the U-H-R stability of our problem.

6. Conclusions

In the present research, we investigated existence and uniqueness criteria for the solutions of a boundary value problem for implicit ψ-Caputo fractional differential equations with non-instantaneous impulses involving both retarded and advanced arguments. To achieve the desired results for the given problem, the fixed-point approach was used, namely, the Banach contraction principle and Krasnoselskii's fixed point theorem. In addition, we dedicated a section to the investigation of various types of Ulam stability for problem (2). Examples are provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and substantially contribute to the literature on this new field of study. We feel that there are multiple potential study avenues, such as coupled systems, problems with infinite delays, and many more, due to the limited number of publications on implicit hybrid differential equations, particularly with non-instantaneous impulses. We hope that this article will serve as a starting point for such an undertaking.

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