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Editorial

Polynomial Sequences and Their Applications †

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† This book is dedicated to Prof. Francesco Aldo Costabile on the occasion of his 75th birthday.

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The purpose of this Special Issue is to present, albeit partially, the state of the art on the theory and application of polynomial sequences.

Polynomials are incredibly useful mathematical tools, as they are simply defined and can be calculated quickly on computer systems. They can be differentiated and integrated easily and can be pieced together to form spline curves.

Stemming from Weierstrass's well-known approximation theorem (1885) [1], sequences of polynomials perform an important role in several branches of science: mathematics, physics, engineering, etc. For example, polynomial sequences arise in physics and approximation theory as the solutions of certain ordinary differential equations. Among these sequences of polynomials, we highlight orthogonal polynomials. In statistics, Hermite polynomials are very important, and they are also orthogonal polynomials and Sheffer A-type zero polynomials [2]. In algebra and combinatorics, umbral polynomials are used, such as rising factorials, falling factorials and Abel, Bell, Bernoulli, Euler, Boile, ciclotomic, Dickson, Fibonacci, Lucas and Touchard polynomials. Finally, in computational and numerical mathematics, polynomial sequences are particularly important and frequently used.

This volume contains both theoretical works and practical applications in the field of polynomial sequences and their applications. In the following, a brief overview of the published papers is presented.

Contributions

In [3], Extended Dynamic Mode Decomposition (EDMD) is used for the approximation of the Koopman operator in the form of a truncated (finite dimensional) linear operator in a lifted space of nonlinear observable functions. Orthogonal polynomials are used for the expression of the observable functions, in conjunction with an order-reduction procedure called p-q quasi-norm reduction. The authors present a Matlab library to automate the computation of the EDMD. The performance of this library is illustrated with a few representative examples.

Motivated by the improvements of Bernstein polynomials in computational disciplines, Özger et al. [4] propose a new generalization of Bernstein–Kantorovich operators involving shape parameters λ, α and a positive integer as an original extension of Bernstein–Kantorovich operators. Some approximation and convergence results are presented. Finally, illustrative graphics that demonstrate the approximation behavior and consistency of the proposed operators are provided by a computer program.

In [5], odd and even polynomial sequences associated with the $\delta^2(\cdot)$ operator are determined, being $\delta(\cdot)$ the known central difference operator. Many aspects, including matrix and determinant forms, recurrence formulas, generating functions and an algorithm for effective calculation, are covered. New examples of odd and even central polynomial sequences are presented.

In [6], the authors address the problem associated with the construction of polynomial complexity computer programs. One of the new approaches to the problems is representing a class of polynomial algorithms as a certain class of special logical programs. One of the main contributions of this paper is the construction of a new logical programming language describing the class of polynomial algorithms. Particularly, the authors find $p-$iterative terms that simulate the work of the Turing machine. This language allows one to create fast and reliable programs and describes any algorithms of polynomial complexity. Its main limitation is that the implementation of algorithms of complexity is not higher than polynomial.

Niu et al. [7] give some important approximation results of Chebyshev polynomials in the Legendre norm. Particularly, interpolation operators at the Chebyshev–Gauss–Lobatto points are studied. The single-domain and multidomain cases for both one dimension and multi-dimensions are analyzed.

In [8], the authors focus on the coefficients of the block tridiagonal matrices in linear systems obtained from block iterative cyclic reductions. They examine the roots of characteristic polynomials by regarding each block cyclic reduction as a composition of two types of matrix transformations and then examining changes in the existence range of roots. The fact that the roots are not very scattered allows one to accurately solve linear systems in floating-point arithmetic.

A general method for proving whether a certain set is $p-$computable or not is proposed in [9]. The method is based on a polynomial analogue of the classical Gandy fixed-point theorem. Gandy's theorem deals with the extension of a predicate through a special operator and states that the smallest fixed point of this operator is a Σ-set. In this paper, a new type of operator is used which extends predicates so that the smallest fixed point remains a $p-$computable set. Polynomial algorithms for checking if a certain element belongs to a given data type or not are used.

Paper [10] deals with the general linearization problem of Jacobi polynomials. The authors provide two approaches for finding closed analytical forms of the linearization coefficients of these polynomials. An application of some of the derived linearization formulas to the solution of the non-linear Riccati differential equation based on the application of the spectral tau method is presented.

In paper [11], the authors analyze some manuscript documents of Peter Winn that came to their knowledge after his death. They concern continued fractions, rational (Padé) approximation, Thiele interpolation, orthogonal polynomials, moment problems, series, and abstract algebra. The authors think that these works are valuable additions to the literature on these topics and that they can lead to new research and results.

A matrix calculus-based approach to general bivariate Appell polynomials is proposed in [12]. This approach, which is new in the literature, generates a systematic, simple theory which is in perfect analogy with the theory in the univariate case. Known and new basic results are given, such as recurrence relations, the generating function, determinant forms and differential equations.

Paper [13] deals with monic orthogonal polynomials with respect to the perturbed Meixner–Pollaczek measure. By introducing a time variable to the Meixner–Pollaczek measure, the authors find some interesting properties such as some recursive relations, moments of finite order, concise hypergeometric formulae and orthogonality relations. Moreover, certain analytic properties of the zeros of the corresponding monic perturbed Meixner–Pollaczek polynomials are studied. Finally, some practical applications are considered.

Kruchinin et al. in [14] study methods for obtaining explicit formulas for the coefficients of generating functions. These methods are based on using the powers of generating functions. The concept of compositae is generalized to the case of generating functions in two variables. Basic operations on such compositae are defined, such as composition, addition, multiplication, reciprocation and compositional inversion. These operations allow obtaining explicit formulas for compositae and coefficients of bivariate generating functions. Some applications are presented.

In paper [15], an extension of the two-variable Fubini polynomials is introduced by means of the polyexponential function. Some new relations are derived, including the

Stirling numbers of the first and second kinds, the usual Fubini polynomials, and the higher-order Bernoulli polynomials. Two-variable unipoly-Fubini polynomials are introduced and some relationships between the two-variable unipoly-Fubini polynomials, the Stirling numbers and the Daehee polynomials are derived.

In [16], as the centenary of the publication of I.M. Sheffer's famous paper approaches [2], the authors want to honor his memory by recalling some old and recent results. In particular, the classification of polynomials by means of suitable linear differential operators and Sheffer's method for the study of A-type zero polynomials. Moreover, the theory of Rota and his collaborators, the isomorphism between the group of Sheffer polynomial sequences and the so-called Riordan matrices group are considered. The interesting problem of orthogonality in the context of Sheffer sequences is also reported, recalling the results of Sheffer, Meixner, Shohat, and the very recent ones of Galiffa et al. and Costabile et al.

Some classes of multivalue methods for the numerical solution of ordinary and fractional differential equations are considered in [17]. Particularly, the authors focus on two-step and mixed collocation methods, Nordsieck GLM collocation methods for ordinary differential equations, and on two-step spline collocation methods for fractional differential equations. The convergence and stability of the proposed methods are reported and some numerical experiments are carried out to show the efficiency of the methods.

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References

1. Weierstrass, K. Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin* **1885**, *2*, 633–639.
2. Sheffer, I. Some properties of polynomial sets of type zero. *Duke Math. J.* **1939**, *5*, 590–622. [CrossRef]
3. Garcia-Tenorio, C.; Vande Wouwer, A. A Matlab Toolbox for Extended Dynamic Mode Decomposition Based on Orthogonal Polynomials and p-q Quasi-Norm Order Reduction. *Mathematics* **2022**, *10*, 3859. [CrossRef]
4. Özger, F.; Aljimi, E.; Temizer Ersoy, M. Rate of Weighted Statistical Convergence for Generalized Blending-Type Bernstein-Kantorovich Operators. *Mathematics* **2022**, *10*, 2027. [CrossRef]
5. Costabile, F.A.; Gualtieri, M.I.; Napoli, A. General Odd and Even Central Factorial Polynomial Sequences. *Mathematics* **2022**, *10*, 978. [CrossRef]
6. Goncharov, S.; Nechesov, A. Solution of the Problem P = L. *Mathematics* **2022**, *10*, 113. [CrossRef]
7. Niu, C.; Liao, H.; Ma, H.; Wu, H. Approximation Properties of Chebyshev Polynomials in the Legendre Norm. *Mathematics* **2021**, *9*, 3271. [CrossRef]
8. Shinjo, M.; Wang, T.; Iwasaki, M.; Nakamura, Y. Roots of Characteristic Polynomial Sequences in Iterative Block Cyclic Reductions. *Mathematics* **2021**, *9*, 3213. [CrossRef]
9. Goncharov, S.; Nechesov, A. Polynomial Analogue of Gandy's Fixed Point Theorem. *Mathematics* **2021**, *9*, 2102. [CrossRef]
10. Abd-Elhameed, W.M.; Badah, B.M. New Approaches to the General Linearization Problem of Jacobi Polynomials Based on Moments and Connection Formulas. *Mathematics* **2021**, *9*, 1573. [CrossRef]
11. Brezinski, C.; Norman, F.A.; Redivo-Zaglia, M. The Legacy of Peter Wynn. *Mathematics* **2021**, *9*, 1240. [CrossRef]
12. Costabile, F.A.; Gualtieri, M.I.; Napoli, A. General Bivariate Appell Polynomials via Matrix Calculus and Related Interpolation Hints. *Mathematics* **2021**, *9*, 964. [CrossRef]
13. Kelil, A.S.; Jooste, A.S.; Appadu, A.R. On Certain Properties and Applications of the Perturbed Meixner–Pollaczek Weight. *Mathematics* **2021**, *9*, 955. [CrossRef]
14. Kruchinin, D.; Kruchinin, V.; Shablya, Y. Method for Obtaining Coefficients of Powers of Bivariate Generating Functions. *Mathematics* **2021**, *9*, 428. [CrossRef]
15. Muhiuddin, G.; Khan, W.A.; Duran, U. Two-Variable Type 2 Poly-Fubini Polynomials. *Mathematics* **2021**, *9*, 281. [CrossRef]
16. Costabile, F.A.; Gualtieri, M.I.; Napoli, A. Towards the Centenary of Sheffer Polynomial Sequences: Old and Recent Results. *Mathematics* **2022**, *10*, 4435. [CrossRef]
17. Cardone, A.; Conte, D.; D'Ambrosio, R.; Paternoster, B. Multivalue Collocation Methods for Ordinary and Fractional Differential Equations. *Mathematics* **2022**, *10*, 185. [CrossRef]