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An Approximation Formula for Nielsen's Beta Function Involving the Trigamma Function

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Abstract: We prove that the function $\sigma(s)$ defined by $\beta(s) = \frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{\psi'(s)}{2} - \frac{\sigma(s)}{2s^5}, s > 0$, is strictly increasing with the sharp bounds $0 < \sigma(s) < \frac{49}{120}$, where $\beta(s)$ is Nielsen's beta function and $\psi'(s)$ is the trigamma function. Furthermore, we prove that the two functions $s \mapsto (-1)^{1+\mu}\left[\beta(s) - \frac{6s^2+12s+5}{3s^2(2s+3)} + \frac{\psi'(s)}{2} + \frac{49\mu}{240s^5}\right], \mu = 0, 1$ are completely monotonic for $s > 0$. As an application, double inequality for $\beta(s)$ involving $\psi'(s)$ is obtained, which improve some recent results.

Keywords: Nielsen's beta function; trigamma function; approximation formula; completely monotonic; sharp bound

MSC: 33B15; 26A48; 26D15; 41A30

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1. Introduction and Preliminaries

The diagamma function is defined by $\psi = \Gamma'/\Gamma$ and its derivative ψ' is called the trigamma function, where Γ is the Euler's gamma function [1]. The Nielsen's beta function [2,3] is defined by

$$\beta(s) = \frac{1}{2}\left[\psi\left(\frac{s}{2} + \frac{1}{2}\right) - \psi\left(\frac{s}{2}\right)\right], \quad s \neq 0, -1, -2, \dots \tag{1}$$

and satisfies

$$\beta(s+1) + \beta(s) = \frac{1}{s}, \tag{2}$$

$$\beta(s) = \int_0^\infty \frac{e^{-st}}{1+e^{-t}} dt, \quad s > 0 \tag{3}$$

and

$$\beta(s) = \frac{1}{2s}\, {}_2F_1\left(1, 1; 1+s; \frac{1}{2}\right), \tag{4}$$

where ${}_2F_1$ is Gauss hypergeometric function [4].

In some literature, one finds the function $G(s) = 2\beta(s)$, which is called the Bateman's $G-$function [5,6].

Qiu and Vuorinen [7] established the bounds

$$\frac{(3 - 2\ln 4)}{s^2} < \beta(s) - \frac{1}{2s} < \frac{1}{4s^2}, \qquad s > 1/2 \tag{5}$$

and Mortici [8] established the relation

$$0 < \psi(s+b) - \psi(s) \leq \psi(b) + \gamma - b + b^{-1}, \quad s \geq 1;\ b \in (0,1), \tag{6}$$

where γ is the Euler constant. Then, Mahmoud and Agarwal [6] established the asymptotic formula

$$2\beta(s) - s^{-1} \sim \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{k} s^{-2k}, \quad s \to \infty, \tag{7}$$

where $B_r s$ are the Bernoulli numbers [9]. In addition, they obtained the relation

$$\frac{1}{2s^2 + \frac{3}{2}} < 2\beta(s) - s^{-1} < \frac{1}{2s^2}, \quad s > 0 \tag{8}$$

which improves the lower bound of the inequality (5) for $s > \sqrt{\frac{9-12\ln 2}{16\ln 2 - 11}} \approx 2.74$. In [10], Mahmoud and Almuashi presented the inequality

$$\sum_{k=1}^{2m} \frac{(2^{2k}-1)}{k} B_{2k} s^{-2k} < 2\beta(s) - s^{-1} < \sum_{k=1}^{2m-1} \frac{(2^{2k}-1)}{k} B_{2k} s^{-2k}, \quad m \in \mathbb{N}, \tag{9}$$

where the constants $\frac{(2^{2k}-1)}{k} B_{2k}$ are the best possible. Mahmoud, Talat, and Moustafa [11] presented the family

$$\varrho(\xi, s) = \ln\left(1 + \frac{1}{s+\xi}\right) + \frac{2}{s(s+1)}, \quad \xi \in [1,2];\ s > 0$$

which is asymptotically equivalent to $2\beta(s)$ for $s \to \infty$.

Recently, Nantomah [12] established the inequality

$$\frac{1}{4}\psi'(s/2 + 1/4) < \beta(s) < \frac{1}{8}\left[\psi'(s/2) + \psi'((s+1)/2)\right], \quad s > 0. \tag{10}$$

For more inequalities and approximations of the Nielsen's beta $\beta(s)$ or Bateman's $G-$function $G(s)$, refer to [7,12–15] and the references therein.

Nielsen's $\beta-$function is very useful in evaluating and estimating several integrals [1], as well as some mathematical constants such as

$$G = \frac{-\beta'(1/2)}{4} \approx 0.9159655\ldots, \quad \pi = 2\beta(1/2), \quad \zeta(r+1) = \frac{(-2)^r \beta^{(r)}(1)}{r!(2^r - 1)};\ r \in \mathbb{N},$$

where G is the Catalan's constant and $\zeta(s)$ is the Riemann zeta function. The function $\beta(s)$ is also related to the Euler's beta function $B(s,v)$ by (see [16])

$$\frac{d}{ds}(\ln B(s/2, 1/2)) = -\beta(s) \quad \text{and} \quad B(s, 1-s) = \beta(s) + \beta(1-s).$$

Furthermore, the function $\beta(s)$ is related to the important alternating series [6]

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{s+r} = \beta(s), \qquad s \neq 0, -1, -2, \ldots .$$

Hence, our results enable us to estimate the errors of the numerical values of the Nielsen's $\beta-$function and its related constants, integrals, series, and mathematical constants.

In this paper, we prove that the function

$$\sigma(s) = s^5\left[\frac{2(6s^2 + 12s + 5)}{3s^2(2s+3)} - \psi'(s) - 2\beta(s)\right], \quad s > 0$$

is strictly increasing with the sharp bounds $0 < \sigma(s) < \frac{49}{120}$. Furthermore, we prove the completely monotonicity property of two functions related to $\sigma(s)$. As an application, we obtain the double inequality

$$\frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{1}{2}\psi'(s) - \frac{49}{240s^5} < \beta(s) < \frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{1}{2}\psi'(s), \quad s > 0$$

whose upper and lower bounds are better than the ones in (10) for $s > 0$ and $s \geq 2$, respectively. Furthermore, its upper (lower) bound is better than the counterpart in (8) for $s > 1/2$ and $s \geq 2.4$, respectively.

2. Main Results

Recall that an infinite differentiable function $M(s)$ on $s > 0$ is said to be completely monotonic [17] if, for $s > 0$ and $r \geq 0$, we have $(-1)^r M^{(r)}(s) \geq 0$. The necessary and sufficient condition for $M(s)$ to be completely monotonic function on $s > 0$ is the convergence of the integral [18]

$$M(s) = \int_0^\infty e^{-st} d\theta(t), \tag{11}$$

where $\theta(t)$ is bounded and non-decreasing for $t \geq 0$.

Theorem 1. *The function*

$$H(s) = 2\beta(s) - \frac{2\left(6s^2+12s+5\right)}{3s^2(2s+3)} + \frac{49}{120s^5} + \psi'(s) \tag{12}$$

is completely monotonic for $s > 0$.

Proof. Using the integral representation (3), we have

$$H(s) = \int_0^\infty \frac{e^{-st}h(t)}{8640(e^{2t}-1)} e^{-\frac{3t}{2}} dt,$$

where

$$\begin{aligned} h(t) &= -147e^{\frac{3t}{2}}t^4 + 147e^{\frac{7t}{2}}t^4 + 9600e^{\frac{3t}{2}}t + 8640e^{\frac{5t}{2}}t - 960e^{\frac{7t}{2}}t + 16640e^{\frac{3t}{2}} \\ &\quad -17280e^{\frac{5t}{2}} + 640e^{\frac{7t}{2}} - 640e^{2t} + 640 \\ &= \sum_{r=11}^{\infty} U(r)\frac{2^{4-r}t^r}{1323\, r!} \\ &\quad + \frac{290803t^{10}}{1120} + \frac{747299t^9}{1920} + \frac{25283t^8}{56} + \frac{10159t^7}{28} + 153t^6 \end{aligned}$$

with

$$\begin{aligned} U(r) &= 1512(189(r-5)5^r - (35)4^r - 5(3r-7)7^r + 70(5r+13)3^r) \\ &\quad -((2401)3^r - (81)7^r)(r-3)(r-2)(r-1)r \\ &> 3^r\left(81r^4 - 486r^3 + 891r^2 - 23166r + 52920\right) \\ &\quad +3^r\left(-2401r^4 + 14406r^3 - 26411r^2 + 543606r + 1375920\right) \\ &\quad +5^r(285768r - 1428840) - (52920)5^r \\ &> 3^r(1428840 + 520440r - 25520r^2 + 13920r^3 - 2320r^4) \\ &\quad +5^r(285768r - 1481760), \quad r \geq 11. \end{aligned}$$

Using induction, we obtain

$$\frac{2320r^4 - 13920r^3 + 25520r^2 - 520440r - 1428840}{285768r - 1481760} < \left(\frac{5}{3}\right)^r, \quad r \geq 11,$$

where

$$\begin{aligned} & \frac{5}{3}\left(\frac{2320r^4 - 13920r^3 + 25520r^2 - 520440r - 1428840}{285768r - 1481760}\right) \\ & - \frac{2320(r+1)^4 - 13920(r+1)^3 + 25520(r+1)^2 - 520440(r+1) - 1428840}{285768(r+1) - 1481760} \\ = \ & \frac{5Q(r)}{3969(27r-140)(27r-113)} > 0, \quad r \geq 11 \end{aligned}$$

with

$$Q(r+11) = 3132r^5 + 126266r^4 + 2004712r^3 + 15048292r^2 + 47492336r + 25456347 > 0, \quad r \geq 0.$$

Hence $U(r) > 0$ for $r \geq 11$, which completes the proof. □

From Theorem 1, the function $H(s)$ is completely monotonic for $s > 0$, and therefore it is positive, so we get the following result:

Corollary 1. *Nielsen's beta function satisfies the inequality*

$$\beta(s) > \frac{6s^2 + 12s + 5}{3s^2(2s+3)} - \frac{49}{240s^5} - \frac{1}{2}\psi'(s), \quad s > 0. \tag{13}$$

Theorem 2. *The function*

$$F(s) = \frac{2\left(6s^2 + 12s + 5\right)}{3s^2(2s+3)} - \psi'(s) - 2\beta(s) \tag{14}$$

is completely monotonic for $s > 0$.

Proof. Using the integral representation (3), we have

$$F(s) = \int_0^\infty \frac{e^{-st} f(t)}{27(e^{2t} - 1)} e^{-\frac{3t}{2}} dt,$$

where

$$\begin{aligned} f(t) &= -27e^{\frac{5t}{2}}(t-2) + 2e^{2t} + e^{\frac{7t}{2}}(3t-2) - 2e^{\frac{3t}{2}}(15t+26) - 2 \\ &= \frac{147t^5}{160} + \frac{291t^6}{160} + \frac{8469t^7}{4480} + \sum_{r=8}^{\infty} P(r) \frac{2^{1-r}t^r}{35\, r!} \end{aligned}$$

with

$$P(r) = -(350r)3^r - (189r)5^r + (15r)7^r - (910)3^r + (35)4^r + (189)5^{r+1} - (5)7^{r+1}, \quad r \geq 8.$$

Using

$$\begin{aligned} & 3\left((910)3^r - (35)4^r - (189)5^{r+1} + (5)7^{r+1}\right) - 7((-350)3^r - (189)5^r + (15)7^r) \\ & = 7((740)3^r - (15)4^r - (216)5^r) < 0, \quad r \geq 8. \end{aligned}$$

Then,
$$\frac{\big((910)3^r-(35)4^r-(189)5^{r+1}+(5)7^{r+1}\big)}{-(350)3^r-(189)5^r+(15)7^r}<\frac{7}{3}<r,\quad r\geq 8$$
and hence $P(r)>0$ for $r\geq 8$, which completes the proof. □

From Theorem 2, the function $F(s)$ is completely monotonic for $s>0$, and therefore it is positive, so we obtain the following result:

Corollary 2. *Nielsen's beta function satisfies the inequality*
$$\beta(s)<\frac{6s^2+12s+5}{3s^2(2s+3)}-\frac{1}{2}\psi'(s),\quad s>0. \tag{15}$$

Theorem 3. *The function*
$$\sigma(s)=s^5\left[\frac{2\big(6s^2+12s+5\big)}{3s^2(2s+3)}-\psi'(s)-2\beta(s)\right],\quad s>0 \tag{16}$$
is strictly increasing with the sharp bounds $0<\sigma(s)<\frac{49}{120}$.

Proof. Using the relation
$$\sigma(s)=s^5F(s),$$
we obtain
$$\frac{d}{ds}\sigma(s)=s^4\int_0^\infty\frac{e^{-st}m(t)}{27(e^{2t}-1)^2}e^{-\frac{3t}{2}}\,dt,$$
where
$$\begin{aligned}
m(t)\;=\;&-27e^{\frac{5t}{2}}t^2-54e^{\frac{7t}{2}}t^2-27e^{\frac{9t}{2}}t^2\\
&+90e^{\frac{3t}{2}}t+135e^{\frac{5t}{2}}t-207e^{\frac{7t}{2}}t-27e^{\frac{9t}{2}}t+9e^{\frac{11t}{2}}t-6e^{2t}t+3e^{4t}t+3t\\
&+208e^{\frac{3t}{2}}-216e^{\frac{5t}{2}}-200e^{\frac{7t}{2}}+216e^{\frac{9t}{2}}-8e^{\frac{11t}{2}}-16e^{2t}+8e^{4t}+8\\
=\;&\sum_{r=27}^{\infty}V(r)\frac{2^{-2-r}t^r}{40425\,r!}\\
&+\frac{6236986350913269689905490t^{26}}{4698686079358795899312537600000}+\frac{16256634668429586992570t^{25}}{27888687555548408709120000}\\
&+\frac{1226896997083781839990t^{24}}{5019963759998713567641600}+\frac{354941924939380361890t^{23}}{36376548985497924403200}\\
&+\frac{428947480570000501t^{22}}{115996648550694912000}+\frac{2568895942714902679t^{21}}{1937671288290017280000}\\
&+\frac{123968816386459403 3t^{20}}{2768101840414310400 00}+\frac{27562668794800823t^{19}}{1942527607308288000}\\
&+\frac{10183224174059897t^{18}}{242815950913536000}+\frac{11397477274003t^{17}}{99189522432000}+\frac{7182473003897t^{16}}{24797380608000}\\
&+\frac{16503914196013t^{15}}{24797380608000}+\frac{38836308383t^{14}}{28178841600}+\frac{719523341t^{13}}{283852800}+\frac{72073021t^{12}}{17740800}\\
&+\frac{23771543t^{11}}{4300800}+\frac{3304163t^{10}}{537600}+\frac{23563t^{9}}{4480}+\frac{3473t^{8}}{1120}+\frac{153t^{7}}{160}
\end{aligned}$$

with

$$\begin{aligned}
V(r) &= -(698544r^2)5^r - (712800r^2)7^r - (215600r^2)9^r + (9430344r)5^r - (8850600r)7^r \\
&\quad +(121275r)8^r - (754600r)9^r + (264600r)11^r - (121275r)4^{r+1} \\
&\quad +(1078000r)3^{r+2} + (11211200)3^{r+1} - (117600)11^{r+1} - (1397088)5^{r+2} \\
&\quad -(660000)7^{r+2} + (431200)9^{r+2} - (40425)4^{r+3} + (40425)2^{3r+5} \\
&> 11^r(264600r - 1293600) - 4^r(485100r + 2587200) \\
&\quad -9^r\Big(215600r^2 + 754600r - 34927200\Big) - 5^r\Big(698544r^2 - 9430344r + 34927200\Big) \\
&\quad -7^r\Big(712800r^2 + 8850600r + 32340000\Big), \quad r \geq 27 \\
&> -9^r\Big(215600r^2 + 754600r - 34927200\Big) - 9^r\Big(698544r^2 - 9430344r + 34927200\Big) \\
&\quad -9^r\Big(712800r^2 + 8850600r + 32340000\Big) + 11^r(264600r - 1293600) \\
&\quad -9^r(485100r + 2587200), \quad r \geq 27 \\
&> 11^r(264600r - 1293600) - 44\Big(36976r^2 + 14999r + 793800\Big)9^r, \quad r \geq 27.
\end{aligned}$$

Using induction, we obtain

$$\left(\frac{11}{9}\right)^r > \frac{44(36976r^2 + 14999r + 793800)}{264600r - 1293600}, \quad r \geq 27,$$

where

$$\begin{aligned}
&\frac{11}{9}\frac{44(36976r^2 + 14999r + 793800)}{264600r - 1293600} - \frac{44(36976(r+1)^2 + 14999(r+1) + 793800)}{264600(r+1) - 1293600} \\
&= \frac{11\epsilon(s)}{33075(9r-44)(9r-35)} > 0, \quad r \geq 27
\end{aligned}$$

with $\epsilon(s+27) = 5161745790 + 604101173s + 24298807s^2 + 332784s^3$ and hence $V(r) > 0$ for $r \geq 27$. Using the asymptotic expansions

$$\psi'(s) = \frac{\pi^2}{6} + \frac{1}{s^2} - 2\zeta(3)s + \frac{\pi^4 s^2}{30} + O(s^3), \quad \textit{as } s \to 0,$$

$$\psi'(s) = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{6s^3} - \frac{1}{30s^5} + O(s^{-7}), \quad \textit{as } s \to \infty,$$

$$\beta(s) = -\log(2) + \frac{1}{s} + \frac{\pi^2 s}{12} - \frac{3}{4}\zeta(3)s^2 + O(s^3), \quad \textit{as } s \to 0$$

and

$$\beta(s) = \frac{1}{2s} + \frac{1}{4s^2} - \frac{1}{8s^4} + O(s^{-6}), \quad \textit{as } s \to \infty,$$

where the Riemann zeta function [4] is defined by $\zeta(s) = \sum_{r=1}^{\infty} \frac{1}{r^s}$, $s > 1$, then we have

$$\lim_{s \to 0^+} \sigma(s) = 0 \quad \textit{and} \quad \lim_{s \to \infty} \sigma(s) = \frac{49}{120},$$

which completes the proof. □

Remark 1. *The function $\sigma(s)$ satisfies*

$$\sigma(s) = \frac{s^3}{9} + O(s^4), \ \ \textit{as } s \to 0^+ \quad \textit{and} \quad \sigma(s) = \frac{49}{120} + O(s^{-1}), \ \ \textit{as } s \to \infty.$$

Furthermore, the function $\beta(s)$ *satisfies*

$$\beta(s) - \frac{6s^2+12s+5}{3s^2(2s+3)} + \frac{1}{2}\psi'(s) = \left(\frac{\pi^2}{12} - \frac{2}{81} - \log(2)\right) - \frac{1}{18s^2} + \frac{1}{27s} + O(s), \quad \textit{as } s \to 0$$

and

$$\beta(s) - \frac{6s^2+12s+5}{3s^2(2s+3)} + \frac{49}{240s^5} + \frac{1}{2}\psi'(s) = \frac{17}{32s^6} + O(s^{-7}), \quad \textit{as } s \to \infty.$$

Remark 2. *The upper bound in (15) is better than the counterpart in (10) for* $s > 0$, *where*

$$\begin{aligned}
&\left(\frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{\psi'(s)}{2}\right) - \frac{1}{8}\left(\psi'\left(\frac{s}{2}\right) + \psi'\left(\frac{s+1}{2}\right)\right) \\
&= \int_0^\infty -\frac{e^{-\frac{3t}{2}}\left(15e^{\frac{3t}{2}}t + 12e^{\frac{5t}{2}}t + 26e^{\frac{3t}{2}} - 26e^{\frac{5t}{2}} - e^t + 1\right)}{27(e^t-1)} e^{-st}dt \\
&= \int_0^\infty -\frac{e^{-(3/2+s)t}}{27(e^t-1)} \sum_{r=3}^{\infty} [10(5r+13)3^r - (5)2^r + 2(12r-65)5^r] \frac{2^{-r}t^r}{5\, r!} dt \\
&< 0, \quad \textit{for } s > 0.
\end{aligned}$$

Remark 3. *The lower bound in (13) is better than the counterpart in (10) for* $s \ge 2$. *To verify that, consider the function*

$$T(s) = \left(-\frac{49}{240s^5} + \frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{\psi'(x)}{2}\right) - \frac{1}{4}\psi'\left(\frac{s}{2} + \frac{1}{4}\right),$$

then $\lim_{s\to\infty} T(s+2) = 0$ *and*

$$T(s+2) - T(s+4) = \frac{2q(s)}{(s+2)^5(s+3)^2(s+4)^5(2s+5)^2(2s+7)(2s+11)} > 0,$$

where

$$\begin{aligned}
q(s) \;=\;& 480s^{12} + 19200s^{11} + 348600s^{10} + 3785880s^9 + 27307420s^8 + 137389520s^7 \\
&+ 492611791s^6 + 1262260470s^5 + 2278167311s^4 + 2792735404s^3 \\
&+ 2162321672s^2 + 908271720s + 137545680.
\end{aligned}$$

However, if $\pi(s)$ *is a real-valued function on* $s > 0$ *with* $\lim_{s\to\infty} \pi(s) = 0$ *and* $\pi(s) > \pi(s+r)$ *for all* $s > 0$, $r \in \mathbb{N}$, *then* $\pi(s) > 0$ *on* $s > 0$ *(see [19]). Hence* $T(s+2) > 0$ *for* $s > 0$ *or* $T(s) > 0$ *for* $s \ge 2$.

Remark 4. *The upper (lower) bound in inequality (15) (in inequality (13)) is better than the upper (lower) bound in inequality (8) for* $s > 0.5$ ($s \ge 2.4$), *respectively, since the two functions*

$$B(s) = \frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{\psi'(s)}{2} - \frac{1}{4s^2} - \frac{1}{2s}$$

and

$$A(s) = \frac{6s^2+12s+5}{3s^2(2s+3)} - \frac{49}{240s^5} - \frac{\psi'(s)}{2} - \frac{1}{4s^2+3} - \frac{1}{2s}$$

satisfy

$$\lim_{s\to\infty} B(s) = A(s) = 0$$

and

$$B(s) - B(s+1) = \frac{-24s-35}{12s^2(s+1)^2(4s^2+16s+15)} < 0, \quad s > 0.5,$$

$$A(s) - A(s+1) = \frac{C(s)}{240s^5(s+1)^5(2s+3)(2s+5)(4s^2+3)(4s^2+8s+7)} > 0, \quad s > 2.4,$$

where

$$\begin{aligned} C(s+2.4) \quad = \quad & 3840s^{11} + 134016s^{10} + 2082912s^9 + \frac{95187392s^8}{5} + \frac{2839058008s^7}{25} \\ & + \frac{289284768212s^6}{625} + \frac{4087044207104s^5}{3125} + \frac{7923834916229s^4}{3125} \\ & + \frac{50631576895694s^3}{15625} + \frac{194357236141326s^2}{78125} + \frac{1771594294687077s}{1953125} \\ & + \frac{431366838881769}{9765625} > 0, \quad s > 0. \end{aligned}$$

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