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Comparing Compound Poisson Distributions by Deficiency: Continuous-Time Case

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Abstract: In the paper, we apply a new approach to the comparison of the distributions of sums of random variables to the case of Poisson random sums. This approach was proposed in our previous work (Bening, Korolev, 2022) and is based on the concept of statistical deficiency. Here, we introduce a continuous analog of deficiency. In the case under consideration, by continuous deficiency, we will mean the difference between the parameter of the Poisson distribution of the number of summands in a Poisson random sum and that of the compound Poisson distribution providing the desired accuracy of the normal approximation. This approach is used for the solution of the problem of determination of the distribution of a separate term in the Poisson sum that provides the least possible value of the parameter of the Poisson distribution of the number of summands guaranteeing the prescribed value of the $(1-\alpha)$-quantile of the normalized Poisson sum for a given $\alpha \in (0,1)$. This problem is solved under the condition that possible distributions of random summands possess coinciding first three moments. The approach under consideration is applied to the collective risk model in order to determine the distribution of insurance payments providing the least possible time that provides the prescribed Value-at-Risk. This approach is also used for the problem of comparison of the accuracy of approximation of the asymptotic $(1-\alpha)$-quantile of the sum of independent, identically distributed random variables with that of the accompanying infinitely divisible distribution.

Keywords: limit theorem; compound Poisson distribution; Poisson random sum; asymptotic expansion; asymptotic deficiency; kurtosis; accompanying infinitely divisible distribution

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1. Introduction

This paper is a complement to our previous work [1], where we considered a version of the problem of stochastic ordering and proposed an approach based on the concept of deficiency that is well-known in asymptotic statistics; see, e.g., [2] and later publications [3–6]. In the paper [1], we used the approach mentioned above in order to establish a kind of stochastic order for the distributions of sums of independent random variables (r.v.s) based on the comparison of the number of summands required for the distribution of the sum to have the desired asymptotic properties (for the problems and methods related to stochastic ordering, see, e.g., [7]). Here, we apply this approach to the comparison of the distributions of sums of random variables to the case of Poisson random sums.

In statistics, as well as in [1], the deficiency is measured in integer units and correspondingly has the meaning of either the number of additional observations required for a statistical procedure to attain the same quality as the 'optimal' procedure in statistics or the number of additional summands in the sum required to attain the desired accuracy of the normal approximation in [1]. Unlike these cases, in the present paper, we deal with the compound Poisson distributions and introduce a continuous analog of deficiency. The extension of the approach proposed in [1] for non-random sums of independent r.v.s to Poisson random sums is possible due to the asymptotic normality of the latter as the parameter of the Poisson distribution of the number of summands infinitely grows. In the case under consideration, by continuous deficiency, we mean the difference between the parameter of the Poisson distribution of the number of summands in a Poisson random sum and that of the compound Poisson distribution providing the desired accuracy of the normal approximation. This approach is used for the solution of the problem of determination of the distribution of a separate term in the Poisson sum that provides the least possible value of the parameter of the Poisson distribution of the number of summands guaranteeing the prescribed value of the $(1-\alpha)$-quantile of the normalized Poisson sum for a given $\alpha \in (0,1)$.

This problem is solved under the condition that possible distributions of random summands possess coinciding first three moments. Therefore, we can say that, in this problem, we deal with 'fine tuning' of the distribution of a separate summand since we assume that different possible distributions of random summands may differ only by their kurtosis. In the setting under consideration, the best distribution delivers the smallest value of the parameter of the compounding Poisson distribution. This problem is actually a particular case of the problem of quantification of the accuracy of approximations of the compound Poisson distributions provided by limit theorems of probability theory. The main mathematical tools used in the paper are asymptotic expansions for the compound Poisson distributions and their quantiles.

The formal setting mentioned above can be applied to solving some practical problems dealing with the collective risk insurance models where it is traditional to describe the cumulative insurance payments by the compound Poisson process. The approach under consideration makes it possible to determine the distribution of insurance payments providing the least possible time that provides the prescribed Value-at-Risk.

To make the above-mentioned more clear, consider an insurance company that starts its activity at time $t_0 = 0$. Within the classical collective risk model [8], the total insurance payments at some time t have the form of a sum of a random number (number of payments by the time t) of independent identically distributed r.v.s (insurance payments), that is, of a Poisson random sum. In this model, the number of insurance payments by time t follows the Poisson process $N_\lambda(t)$ with some intensity $\lambda > 0$. We assume that the parameter λ is uncontrollable and fixed. Since $N_\lambda(t)$ has the same distribution as $N_1(\lambda t)$ and the parameter λ is assumed fixed, the setting under consideration concerns the problem of determination of the distribution of an individual insurance payment providing the least possible t guaranteeing the prescribed Value-at-Risk for the average losses of the insurance company within the time interval $[0,t]$.

The approach considered in the paper can be used when the distributions of the summands (possible losses) are known only up to their first three moments, and the exact Value-at-Risk is not known for sure.

Within the framework of the collective risk model in the setting under consideration, the problem consists in the description of the best strategy of the insurance company. Here, the choice of the terms of a contract (e.g., the amount of insurance payment related to each possible insurance event) is meant as a strategy. That is, a strategy consists in the determination of the distribution of an insurance payment. Briefly, the problem is to choose an optimal distribution of a separate insurance payment among the distributions that have the same first three moments so that the desired goal is achieved within the least possible time interval.

We also consider the application of the proposed approach to the study of the asymptotic properties of non-random sums of independent identically distributed r.v.s as compared to those of the compound Poisson distributions with the same expectation. It is well-known that, in many respects, these properties coincide. This phenomenon manifests itself, for example, in the form of the method of accompanying infinitely divisible distributions (see, e.g., [9], Chapter 4, Section 24). Therefore, it is of certain interest to investigate the accuracy of the approximation of the characteristics of sums of independent r.v.s as compared to that of the accompanying infinitely divisible (that is, corresponding compound Poisson) laws. This problem was studied by many specialists; see, e.g., [10–13] and the references therein. Unlike most preceding works where the approximation of distribution functions was discussed, here we consider the application of accompanying laws to a somewhat inverse problem of approximation of quantiles.

The paper is organized as follows. Section 2 contains a short overview of the results concerning the asymptotic expansions for compound Poisson distributions. Here we also formulate basic lemmas to be used in the next sections. The main results are presented in Section 3. In Section 3.1, we introduce the notion of the α-reserve in the collective risk model and present some asymptotic expansions for this quantity. In Section 3.2, a continuous-time analog of the notion of deficiency is introduced. Here we also prove some general results concerning the continuous-time deficiency. In Section 3.3, we consider the problem of comparison of compound Poisson distributions by deficiency and present the asymptotic formula for the deficiency of one compound Poisson distribution with respect to the other. In Section 3.4, we deal with the problem of comparison of the distributions of Poisson random sums with those of non-random sums. Actually, this problem consists in the comparison of the accuracy of approximation of the asymptotic $(1-\alpha)$-quantile of the sum of independent identically distributed random variables with that of the accompanying infinitely divisible distribution.

2. Notation and Auxiliary Results

Throughout what follows, we will assume that all the random variables and processes are defined on the same probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. The expectation and variance with respect to the measure P will be, respectively, denoted E and D. The set of real numbers and natural numbers will be, respectively, denoted $\mathbb{R}$ and $\mathbb{N}$. The distribution function of the standard normal law will be denoted $\Phi(x)$,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \varphi(y)dy, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{x^2}{2}\Big\}, \quad x \in \mathbb{R}.$$

The distribution of a random variable X will be denoted $\mathcal{L}(X)$.

Let $X_1, X_2, \ldots$ be independent identically distributed random variables. Let N_λ be the random variable with the Poisson distribution with parameter λ. Assume that for each $\lambda > 0$, the random variables $N_\lambda, X_1, X_2, \ldots$ are independent. Let S_λ be the Poisson random sum, $S_\lambda = X_1 + \ldots + X_{N_\lambda}$. If $N_\lambda = 0$, then S_λ is assumed to equal to zero. Assume that $\mathsf{E}X_1 = a$ and $\mathsf{D}X_1 = \sigma^2 > 0$ exist. For integer $k \geq 0$, denote $\mathsf{E}X_1^k = \alpha_k$. Of course, $\alpha_0 = 1$, $\alpha_1 = a$ and $\alpha_2 = \sigma^2 + a^2$.

Recall some facts concerning the asymptotic expansions for the compound Poisson distributions (sf. [8,14,15]).

Denote the characteristic functions of the random variables X_1 and S_λ as $f(t)$ and $h_\lambda(t)$, respectively. It is well-known that if $f(t)$ has r continuous derivatives, then, as $t \to 0$, we have

$$f(t) = 1 + iat - \frac{\alpha_2 t^2}{2} + (it)^2 \sum\nolimits_{k=1}^{r-2} \frac{(it)^k \alpha_{k+2}}{(k+2)!} + o(t^r). \tag{1}$$

A random variable X_1 is said to satisfy the Cramér condition (C), if

$$\limsup_{|t|\to\infty} |f(t)| < 1. \tag{2}$$

For $k = 0, 1, 2, \ldots$ define the function $H_k(x) : \mathbb{R} \to \mathbb{R}$ as

$$H_k(x) \equiv (-1)^k \frac{\phi^{(k)}(x)}{\phi(x)}.$$

The function $H_k(x)$, $x \in \mathbb{R}$, so defined, is a polynomial of degree k and is called the *Hermite polynomial* of degree k.

It is easy to calculate that

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3,$$

$$H_5(x) = x^5 - 10x^3 + 15x, \quad H_6(x) = x^6 - 15x^4 + 45x^2 - 15.$$

Let m be a nonnegative integer and $q_k \in \mathbb{R}$, $k = 0, \ldots, m$. Consider the polynomial

$$q(x) = \sum_{k=0}^{m} q_k x^k.$$

Let $H_0(x), \ldots, H_m(x)$ be Hermite polynomials. Let

$$Q(x) = \sum_{k=0}^{m} q_k H_k(x).$$

Then it is easy to make sure that the function $v(t) = q(it)\exp\{-t^2/2\}$ is the Fourier transform of the function $V(x) = Q(x)\phi(x)$. Throughout what follows, we will assume that $r \geq 3$ is a fixed integer number.

For a complex z, let

$$\widetilde{f}(z) = \sum_{k=1}^{r-2} \frac{\alpha_{k+2} z^k}{(k+2)!}.$$

Obviously, $\widetilde{f}(z)$ is a polynomial of degree $\leq r-2$ with real coefficients; moreover, $\widetilde{f}(0) = 0$. From (1), it follows that

$$f(t) - 1 - iat + \frac{\alpha_2 t^2}{2} = (it)^2 \widetilde{f}(it) + o(t^r)$$

as $t \to 0$. For $\lambda > 0$ and a complex z let

$$p_\lambda(z) = \sum_{k=1}^{r-2} \frac{1}{k!} \left[\frac{z^2}{\alpha_2} \widetilde{f}\left(\frac{z}{\sqrt{\lambda \alpha_2}} \right) \right]^k. \tag{3}$$

It can be easily made sure that there exist integer $m \geq 3$ and polynomials $q_k(z)$ with real coefficients, $k = 3, \ldots, m$, not depending on λ such that

$$p_\lambda(z) = \sum_{k=3}^{m} \lambda^{-k/2+1} q_k(z) \tag{4}$$

for all $\lambda > 0$ and complex z. Moreover, these polynomials $q_k(z)$ are uniquely determined by (3) and (4). Let

$$q_k(z) = \sum_{j=3}^{L_k} q_{k,j} z^j \tag{5}$$

be the corresponding representation of $q_k(z)$ with $q_{k,j} \in \mathbb{R}$ $(j = 3, \ldots, L_k)$, $L_k \geq 3$ $(k = 3, \ldots, m)$. Let $H_j(x)$ be the Hermite polynomials. For $x \in \mathbb{R}$ and $k = 3, \ldots, m$ let

$$R_k(x) = -\sum_{j=3}^{L_k} q_{k,j} H_{j-1}(x). \tag{6}$$

The function $R_k(x)$ is called the *Edgeworth polynomial* of degree k.

For $\lambda > 0$ and complex z from (3) and (4), we easily obtain

$$p_\lambda(z) = \sum_{k=3}^{(r-2)^2+2} \lambda^{-k/2+1} \sum_{\frac{k-2}{r-2} \leq j \leq k-2} \alpha_{k,j} z^{k+2(j-1)},$$

where

$$j!\alpha_{k,j} = \sum_{\substack{3\le n_1\le\ldots\le n_j\le r\\ n_1+\ldots+n_j=k+2(j-1)}} \frac{\alpha_{n_1}\cdot\ldots\cdot\alpha_{n_j}}{n_1!\cdot\ldots\cdot n_j!}\alpha_2^{-k/2-j+1}.$$

Therefore, in (4) and (5), we should set $m = (r-2)^2 + 2$ and $L_k = 3(k-2)$ ($k = 3,\ldots,m$).

For $x \in \mathbb{R}$, $\lambda > 0$ and $r \in \mathbb{N}$ define the functions $G_{\lambda,r}(x)$ as

$$G_{\lambda,r}(x) = \Phi(x) + \phi(x)\sum_{k=3}^{r}\lambda^{-k/2+1}R_k(x).$$

In particular, for $r = 3$, we have

$$R_3(x) = -\frac{\alpha_3}{6\alpha_2^{3/2}}H_2(x)$$

and

$$G_{\lambda,3}(x) = \Phi(x) - \frac{\alpha_3}{6\alpha_2^{3/2}\sqrt{\lambda}}(x^2-1)\phi(x). \tag{7}$$

For $r = 4$, we have

$$R_4(x) = -\frac{\alpha_4}{24\alpha_2^2}H_3(x) - \frac{\alpha_3^2}{72\alpha_2^3}H_5(x)$$

and

$$G_{\lambda,4}(x) = \Phi(x) - \frac{\alpha_3}{6\alpha_2^{3/2}\sqrt{\lambda}}(x^2-1)\phi(x) - \frac{\phi(x)}{\lambda}\left[\frac{\alpha_4}{24\alpha_2^2}(x^3-3x) - \frac{\alpha_3^2}{72\alpha_2^3}(x^5-10x^3+15x)\right]. \tag{8}$$

Moreover, if $\varkappa_3(S_\lambda)$ and $\varkappa_4(S_\lambda)$ are the skewness and kurtosis of the random variable S_λ,

$$\varkappa_3(S_\lambda) \equiv \mathsf{E}\left(\frac{S_\lambda - \mathsf{E}S_\lambda}{\sqrt{\mathsf{D}S_\lambda}}\right)^3 = \mathsf{E}\left(\frac{S_\lambda - \alpha_1\lambda}{\sqrt{\lambda\alpha_2}}\right)^3 = \frac{\alpha_3}{\sqrt{\lambda}\alpha_2^{3/2}},$$

$$\varkappa_4(S_\lambda) \equiv \mathsf{E}\left(\frac{S_\lambda - \mathsf{E}S_\lambda}{\sqrt{\mathsf{D}S_\lambda}}\right)^4 - 3 = \mathsf{E}\left(\frac{S_\lambda - \alpha_1\lambda}{\sqrt{\lambda\alpha_2}}\right)^4 - 3 = \frac{\alpha_4}{\lambda\alpha_2^2},$$

then (7) and (8) can be rewritten as

$$G_{\lambda,3}(x) = \Phi(x) - \frac{\varkappa_3(S_\lambda)}{6}\Phi^{(3)}(x)$$

and

$$G_{\lambda,4}(x) = \Phi(x) - \frac{\varkappa_3(S_\lambda)}{6}\Phi^{(3)}(x) + \frac{\varkappa_4(S_\lambda)}{24}\Phi^{(4)}(x) + \frac{\varkappa_3^2(S_\lambda)}{72}\Phi^{(6)}(x).$$

Lemma 1. *Let $r > 3$. Assume that the distribution of the random variable X_1 satisfies the Cramér condition (C) (see (2)). Then*

$$\sup_x\left|\mathsf{P}\left(\frac{S_\lambda - a\lambda}{\sqrt{\lambda(a^2+\sigma^2)}} < x\right) - G_{\lambda,r}(x)\right| = o\big(\lambda^{-r/2+2}\big),$$

that is,

$$\lim_{\lambda\to\infty}\lambda^{r/2-1}\sup_x\left|\mathsf{P}\left(\frac{S_\lambda - a\lambda}{\sqrt{\lambda(a^2+\sigma^2)}} < x\right) - G_{\lambda,r}(x)\right| = 0.$$

This statement is a particular case of Theorem 4.4.1 in [15].

Our further reasoning is based on the following general statement dealing with the asymptotic behavior of the quantiles of univariate distributions of a random process.

Let $Z(t)$, $t \geq 0$, be a random process. Assume that for each $t \geq 0$ the distribution of the random variable $Z(t)$ is continuous. For $\beta \in (0,1)$ and $t \geq 0$, the left β-quantile of the random variable $Z(t)$ will be denoted $u_\beta(t)$:

$$u_\beta(t) = \inf\{u : \mathsf{P}(Z(t) < u) \geq \beta\}.$$

Lemma 2. *Assume that, as $t \to \infty$, the distribution function of the random process $Z(t)$ admits the asymptotic expansion of the form*

$$\mathsf{P}(Z(t) < x) = \Psi_0(x) + t^{-1/2}\Psi_1(x) + t^{-1}\Psi_2(x) + o(t^{-1}).$$

Moreover, let the functions $\Psi_0''(x)$, $\Psi_1'(x)$ and $\Psi_2(x)$ be continuous and $\Psi_0'(x) > 0$. Then for any $\beta \in (0,1)$, we have

$$u_\beta(t) = u_\beta - \frac{\Psi_1(u_\beta)}{\Psi_0'(u_\beta)\sqrt{t}} + \frac{\Psi_0'(u_\beta)\Psi_1(u_\beta)\Psi_1'(u_\beta) - \left(\Psi_0'(u_\beta)\right)^2\Psi_2(u_\beta) - \frac{1}{2}\Psi_1^2(u_\beta)\Psi_0''(u_\beta)}{\left(\Psi_0'(u_\beta)\right)^3 t} + o(t^{-1}),$$

where u_β is the left β-quantile of the distribution function $\Psi_0(x)$: $\Psi_0(u_\beta) = \beta$.

For the proof of this statement, see [15], Section 4.5.

Remark 1. *If we set*

$$\overline{u}_\beta(t) = u_\beta - \frac{\Psi_1(u_\beta)}{\Psi_0'(u_\beta)\sqrt{t}} + \frac{\Psi_0'(u_\beta)\Psi_1(u_\beta)\Psi_1'(u_\beta) - \left(\Psi_0'(u_\beta)\right)^2\Psi_2(u_\beta) - \frac{1}{2}\Psi_1^2(u_\beta)\Psi_0''(u_\beta)}{\left(\Psi_0'(u_\beta)\right)^3 t},$$

then it is not difficult to make sure that under the conditions of Lemma 2, we have

$$\mathsf{P}(Z(t) < \overline{u}_\beta(t)) = \beta + o(t^{-1}).$$

From Lemmas 1 and 2, it follows that if $\alpha_4 = \mathsf{E}X_1^4 < \infty$ and the random variable X_1 satisfies the Cramér (C) condition (2), then

$$\mathsf{P}\left(\frac{S_\lambda - a\lambda}{\sqrt{\lambda(a^2+\sigma^2)}} < x\right) = \Phi(x) + \frac{\Psi_1(x)}{\sqrt{\lambda}} + \frac{\Psi_2(x)}{\lambda} + o(\lambda^{-1}) \tag{9}$$

where

$$\Psi_1(x) = -\frac{\alpha_3}{6\alpha_2^{3/2}}\phi(x)H_2(x), \quad \Psi_2(x) = -\phi(x)\left[\frac{\alpha_4}{24\alpha_2^2}H_3(x) + \frac{\alpha_3^2}{72\alpha_2^3}H_5(x)\right].$$

Therefore, setting $t = \lambda$, $Z(t) = S_\lambda$, $\Psi_0(x) = \Phi(x)$, from Lemma 2, we obtain the following result. For $\beta \in (0,1)$, let $w_\beta(\lambda)$ and u_β be the β-quantiles of the random variable S_λ and of the standard normal distribution, respectively.

Lemma 3. *Let $\mathsf{E}X_1^4 < \infty$, and let the random variable X_1 satisfy the Cramér (C) condition (2). Then, as $\lambda \to \infty$, we have*

$$w_\beta(\lambda) = a\lambda + u_\beta\sqrt{\lambda\alpha_2} + \frac{\alpha_3 H_2(u_\beta)}{6\alpha_2} +$$

$$+\frac{1}{\sqrt{\lambda\alpha_2^{5/2}}}\left[\frac{\alpha_3^2}{72}\left(H_5(u_\beta) - 2H_2(u_\beta)H_3(u_\beta) + 4u_\beta H_2^2(u_\beta)\right) + \frac{\alpha_4\alpha_2}{24}H_3(u_\beta)\right] + o(\lambda^{-1/2})$$

where $H_k(x)$ are the Hermite polynomials.

3. Main Results

3.1. The Asymptotic Expansions for the α-Reserve in the Collective Risk Model

Let $X_1, X_2, \ldots$ be independent identically distributed r.v.s such that

$$X_1^2 > 0, \quad |X_1|^{4+\delta} < \infty, \quad \delta > 0. \tag{10}$$

Assume that the r.v. X_1 satisfies the Cramér (C) condition (2). For $t > 0$, let the r.v. N_t have the Poisson distribution with parameter λt, where $\lambda > 0$ is a fixed parameter. Assume that for each $t > 0$ the r.v.s $N_t, X_1, X_2, \ldots$ are independent. Consider the *Poisson random sum*

$$S_t = X_1 \ldots + X_{N_t}.$$

In terms of the collective risk model, the r.v.s X_j can be interpreted as individual insurance claims, and the r.v. S_t can be interpreted as the total insurance payment of an insurance company by the time t.

Let $\alpha \in (0,1)$. Define the *standardized α-reserve* $C_\alpha^*(t)$ by the formula

$$\mathsf{P}\left(\frac{S_t - \lambda t \mathsf{E} X_1}{\sqrt{\lambda t \mathsf{E} X_1^2}} \geq C_\alpha^*(t)\right) = \alpha + o(t^{-1}), \quad t \to \infty. \tag{11}$$

Along with the set $X_1, X_2, \ldots$ consider another set $Y_1, Y_2, \ldots$ of independent identically distributed r.v.s such that

$$Y_1^2 > 0, \quad |Y_1|^{4+\delta} < \infty, \quad \delta > 0. \tag{12}$$

Assume that the r.v. Y_1 satisfies the Cramér (C) condition (2). Also assume that for each $t > 0$, the r.v. N_t having the Poisson distribution with parameter λt is independent of the set $Y_1, Y_2, \ldots$ Denote

$$T_t = Y_1 + \ldots + Y_{N_t}.$$

In the same way as (11), define the *standardized α-reserve* $C_\alpha^{**}(t)$ for the sequence $Y_1, Y_2, \ldots$ as

$$\mathsf{P}\left(\frac{T_t - \lambda t \mathsf{E} Y_1}{\sqrt{\lambda t \mathsf{E} Y_1^2}} \geq C_\alpha^{**}(t)\right) = \alpha + o(t^{-1}), \quad t \to \infty.$$

Lemmas 2 and 3 directly imply the following statement. For $\alpha \in (0,1)$ let u_α be the $1-\alpha$-quantile of the standard normal distribution, that is, $\Phi(u_\alpha) = 1 - \alpha$.

Theorem 1. *Let $\alpha \in (0,1)$ and the r.v.s $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ satisfy conditions (10), (12) and (2). Then, as $t \to \infty$,*

$$C_\alpha^*(t) = u_\alpha + \frac{\mathsf{E}X_1^3(u_\alpha^2 - 1)}{6\sqrt{\lambda t}(\mathsf{E}X_1^2)^{3/2}} + \frac{1}{12\lambda t \mathsf{E}X_1^2}\left[\frac{(\mathsf{E}X_1^3)^2}{\mathsf{E}X_1^2}(5u_\alpha - 2u_\alpha^3) + \frac{\mathsf{E}X_1^4}{2(\mathsf{E}X_1^2)^2}(u_\alpha^3 - 3u_\alpha)\right] + o(t^{-1}),$$

$$C_\alpha^{**}(t) = u_\alpha + \frac{\mathsf{E}Y_1^3(u_\alpha^2 - 1)}{6\sqrt{\lambda t}(\mathsf{E}Y_1^2)^{3/2}} + \frac{1}{12\lambda t \mathsf{E}Y_1^2}\left[\frac{(\mathsf{E}Y_1^3)^2}{\mathsf{E}Y_1^2}(5u_\alpha - 2u_\alpha^3) + \frac{\mathsf{E}Y_1^4}{2(\mathsf{E}Y_1^2)^2}(u_\alpha^3 - 3u_\alpha)\right] + o(t^{-1}).$$

We see that if the first three moments of X_1 and Y_1 coincide, then $C_\alpha^*(t)$ and $C_\alpha^{**}(t)$ differ only by the terms of order $O(t^{-1})$.

Now if we define the *α-reserves* $\widetilde{C}_\alpha^*(t)$ and $\widetilde{C}_\alpha^{**}(t)$ as

$$\mathsf{P}\big(S_t \geq \widetilde{C}_\alpha^*(t)\big) = \alpha + o(t^{-1}), \;\text{ and }\; \mathsf{P}\big(T_t \geq \widetilde{C}_\alpha^{**}(t)\big) = \alpha + o(t^{-1}), \;\; t \to \infty,$$

then

$$\widetilde{C}_\alpha^*(t) = \sqrt{\lambda t \mathsf{E}X_1^2} \cdot C_\alpha^*(t) + \lambda t \mathsf{E}X_1 \;\text{ and }\; \widetilde{C}_\alpha^{**}(t) = \sqrt{\lambda t \mathsf{E}Y_1^2} \cdot C_\alpha^*(t) + \lambda t \mathsf{E}Y_1.$$

3.2. A Continuous-Time Analog of Deficiency

In this section, we will propose an approach to the comparison of the two compound Poisson distributions in terms of the 'continuous' analog of deficiency. For the traditional definition of deficiency as the number of additional observations required for a statistical procedure to attain the desired quality, we refer the reader to the papers [1–3,5,6]. Here, we will introduce its continuous-time analog.

Consider two stochastic processes $X(t)$ and $Y(t)$, $t \geq 0$. We will be interested in the asymptotic behavior of the probabilities of $X(t)$ and $Y(t)$ to exceed a given threshold.

For $\alpha \in (0,1)$ let $c_\alpha(t)$ be the asymptotic $(1-\alpha)$-quantile of $X(t)$:

$$\mathsf{P}\big(X(t) \geq c_\alpha(t)\big) = \alpha + o(t^{-1}), \quad t \to \infty.$$

Lemma 2 directly implies the following statement.

Proposition 1. *Assume that there exist distribution function $G(x)$ and the functions $g_1(x)$ and $g_2(x)$ such that*

$$\sup_{x\in\mathbb{R}} \Big|\mathsf{P}\big(X(t) < x\big) - G(x) - \frac{1}{\sqrt{t}} g_1(x) - \frac{1}{t} g_2(x)\Big| = o(t^{-1}), \tag{13}$$

where the functions $G(x)$, $g_1(x)$ and $g_2(x)$ are smooth enough. Then the the asymptotic $(1-\alpha)$-quantile of $X(t)$ admits the asymptotic expansion

$$c_\alpha(t) = c_\alpha - \frac{g_1(c_\alpha)}{G'(c_\alpha)\sqrt{t}} - \frac{1}{t}\left[\frac{G''(c_\alpha)g_1^2(c_\alpha)}{2(G'(c_\alpha))^3} + \frac{G'(c_\alpha)g_2(c_\alpha) - g_1(c_\alpha){g_1}'(c_\alpha)}{(G'(c_\alpha))^2}\right] + o(t^{-1}),$$

where c_α is the $(1-\alpha)$-quantile of the distribution function $G(x)$, that is, $G(c_\alpha) = 1-\alpha$.

Assume that the asymptotic expansion for the distribution function of $Y(t)$ has the form

$$\mathsf{P}\big(Y(t) < x\big) = G(x) + \frac{1}{\sqrt{t}} g_1(x) + \frac{1}{t}\overline{g}_2(x) + o(t^{-1}), \tag{14}$$

where the functions $G(x)$, $g_1(x)$ and $\overline{g}_2(x)$ are smooth enough. The asymptotic expansion (14) differs from that for the distribution function of $X(t)$ in Proposition 1 only by the term of order t^{-1}, that is, the two distributions are close enough.

Define the positive function $m(t)$, $t > 0$, by the equality

$$\mathsf{P}\big(\sqrt{t}\,Y(m(t)) \geq c_\alpha(m(t))\big) = \alpha + o(t^{-1}), \quad t \to \infty. \tag{15}$$

If $m(t) - t = d + o(1)$, $d \in \mathbb{R}$, $t \to \infty$, then the number d is called *the asymptotic deficiency* of the distribution $\mathcal{L}(Y(t))$ with respect to the distribution $\mathcal{L}(X(t))$. In other words, d is the asymptotic 'additional' time required for the process $Y(t)$ to attain the quantile of the same order as that of $X(t)$.

Theorem 2. *Assume that conditions (13) and (14) hold. Then the asymptotic deficiency d of the distribution $\mathcal{L}(Y(t))$ with respect to the distribution $\mathcal{L}(X(t))$ has the form*

$$d = \frac{2\big[g_2(c_\alpha) - \overline{g}_2(c_\alpha)\big]}{G'(c_\alpha)c_\alpha} + o(1).$$

The proof of this statement repeats that of Theorem 3.1 in [1] up to notation (furthermore, unfortunately, in formula (16) of [1], the coefficient $\sqrt{n}$ analogous to $\sqrt{t}$ in (15) of the present paper was erroneously omitted).

3.3. The Comparison of Compound Poisson Distributions by Deficiency

In this section, we will discuss the asymptotic deficiency of the compound Poisson distributions providing a given $(1-\alpha)$-quantile of the normalized Poisson random sums. For this purpose, we will use Theorem 2.

Define the *average Poisson random sums* $\overline{S}_t$ and $\overline{T}_t$ by the formulas

$$\overline{S}_t = \frac{S_t - \lambda t\,\mathsf{E}X_1}{t\sqrt{\lambda\mathsf{E}X_1^2}}, \quad \overline{T}_t = \frac{T_t - \lambda t\,\mathsf{E}Y_1}{t\sqrt{\lambda\mathsf{E}Y_1^2}}.$$

Define the *asymptotic deficiency* $d \in \mathbb{R}$ of $\overline{T}_t$ with respect to $\overline{S}_t$ by the formula

$$\mathsf{P}\big(\sqrt{t}\cdot\overline{T}_{\bar{t}} \geq C_\alpha^*(t)\big) = \alpha + o(t), \quad t \to \infty,$$

where $\bar{t} = t + d + o(1)$, that is, d is the 'additional time' required for the normalized average Poisson random sum $\sqrt{t}\cdot\overline{T}_t$ to exceed the asymptotic α-reserve $C_\alpha^*(t)$ of the normalized average Poisson random sum $\sqrt{t}\cdot\overline{S}_t$.

To apply Theorem 2, assume that

$$\frac{\mathsf{E}X_1^3}{(\mathsf{E}X_1^2)^{3/2}} = \frac{\mathsf{E}Y_1^3}{(\mathsf{E}Y_1^2)^{3/2}}. \tag{16}$$

Condition (16) holds, e.g., if the first three moments of X_1 and Y_1 coincide.

Theorem 2 directly implies the following statement.

Theorem 3. *Assume that the r.v.s $N_t, X_1, X_2, \ldots; Y_1, Y_2, \ldots$ satisfy conditions (2), (10) and (16). Then, as $t \to \infty$, the 'additional time' d has the form*

$$d = \frac{(3-u_\alpha^2)}{12}\left[\frac{\mathsf{E}X_1^4}{(\mathsf{E}X_1^2)^2} - \frac{\mathsf{E}Y_1^4}{(\mathsf{E}Y_1^2)^2}\right] + o(1). \tag{17}$$

Remark 2. *If $\mathsf{E}X_1 = \mathsf{E}Y_1 = 0$, then (17) can be rewritten as*

$$d = \tfrac{1}{12}(3-u_\alpha^2)\big(\varkappa_4(X_1) - \varkappa_4(Y_1)\big) + o(1),$$

That is, in this case, the continuous-time analog of asymptotic deficiency is determined by the difference of kurtoses.

3.4. Comparing the Distributions of Poisson Random Sums with Those of Non-Random Sums

It is well-known that the asymptotic properties of non-random sums of independent identically distributed r.v.s coincide with those of the compound Poisson distributions with the same expectation. This phenomenon manifests itself, for example, in the form of the method of accompanying infinitely divisible distributions (see, e.g., [9], Chapter 4, Section 24). Therefore, it is of certain interest to investigate the accuracy of the approximation of the characteristics of sums of independent r.v.s as compared to that of the accompanying infinitely divisible (that is, corresponding compound Poisson) laws. This problem was studied by many specialists, see, e.g., [10–13]. Unlike most preceding works where the approximation of distribution functions was discussed, here we consider the application of accompanying laws to a somewhat inverse problem of approximation of quantiles.

Here, we will not assume the possibility of the interpretation of the presented results in terms of a collective risk model where at least the expectations of X_j should be positive. Assume that the independent identically distributed r.v.s $X_1, X_2, \ldots$ are standardized:

$$\mathsf{E}X_1 = 0, \quad \mathsf{E}X_1^2 = 1. \tag{18}$$

Again, let N_t be an r.v. with the Poisson distribution with parameter λt, where $\lambda > 0$ is fixed. Assume that for each $t > 0$ the random variables $N_t, X_1, X_2, \ldots$ are independent. Consider the problem of comparison of the distribution of a normalized Poisson random sum

$$S_t^* = \frac{X_1 + \ldots + X_{N_t}}{\sqrt{\lambda t}}$$

with the distribution of the corresponding non-random sum

$$U_t^* = \frac{X_1 + \ldots + X_{[\lambda t]}}{\sqrt{[\lambda t]}}$$

as $t \to \infty$, where the symbol $[a]$ denotes the integer part of a real number a. For definiteness, if $N_t = 0$, then S_t^* is assumed to be equal to zero.

If conditions (18), (10) and (2), then Lemmas 1 and 2 imply (see (9)) that, as $t \to \infty$,

$$\mathsf{P}(S_t^* < x) = \Phi(x) - \frac{\mathsf{E}X_1^3}{6\sqrt{\lambda t}}\varphi(x)(x^2-1) -$$

$$-\frac{\varphi(x)}{24\lambda t}\left[\mathsf{E}X_1^4(x^3-3x) + \frac{(\mathsf{E}X_1^3)^2}{3}(x^5-10x+15x)\right] + o(t^{-1}), \tag{19}$$

whereas the classical theory of asymptotic expansions in the central limit theorem (e.g., see [16]) yields that

$$\mathsf{P}(U_t^* < x) = \Phi(x) - \frac{\mathsf{E}X_1^3}{6\sqrt{[\lambda t]}}\varphi(x)(x^2-1) -$$

$$-\frac{\varphi(x)}{24[\lambda t]}\left[(\mathsf{E}X_1^4 - 3)(x^3-3x) + \frac{(\mathsf{E}X_1^3)^2}{3}(x^5-10x+15x)\right] + o(t^{-1}). \tag{20}$$

Note that (19) and (20) differ in that, in (19), the kurtosis of X_1 is present in the non-normalized form $\varkappa_4^*(X_1) = \mathsf{E}X_1^4$, whereas in (20), there stands the normalized kurtosis $\varkappa_4(X_1) = \mathsf{E}X_1^4 - 3$.

From the obvious inequalities

$$\lambda t - 1 \le [\lambda t] \le \lambda t$$

it follows that, as $t \to \infty$,

$$\frac{1}{\lambda t} \le \frac{1}{[\lambda t]} \le \frac{1}{\lambda t - 1} = \frac{1}{\lambda t}\Big(1 + \frac{1}{\lambda t} + O(t^{-2})\Big)$$

and

$$\frac{1}{\sqrt{[\lambda t]}} = \frac{1}{\sqrt{\lambda t}} + O(t^{-3/2}).$$

Therefore, relation (20) can be rewritten as

$$\mathsf{P}(U_t^* < x) = \Phi(x) - \frac{\mathsf{E}X_1^3}{6\sqrt{\lambda t}}\varphi(x)(x^2-1) -$$

$$-\frac{\varphi(x)}{24\lambda t}\left[\mathsf{E}(X_1^4 - 3)(x^3-3x) + \frac{(\mathsf{E}X_1^3)^2}{3}(x^5-10x+15x)\right] + o(t^{-1}). \tag{21}$$

Denote $\overline{U}_t^* = U_t^*/\sqrt{t}$. Let $\alpha \in (0,1)$. Define the asymptotic $(1-\alpha)$-quantile $C_\alpha(t)$ of S_t^* by the relation

$$\mathsf{P}\big(S_t^* \ge C_\alpha(t)\big) = \alpha + o(t^{-1}), \quad t \to \infty.$$

Define the number $d \in \mathbb{R}$ by the formula

$$\mathsf{P}\big(\sqrt{t}\,\overline{U}_{\bar{t}}^* \geq C_\alpha(\bar{t})\big) = \alpha + o(t^{-1}), \quad t \to \infty,$$

where $\bar{t} = t + d + o(1)$. Now relations (19), (21) and Theorem 2 directly imply the following statement.

Theorem 4. *Let $\alpha \in (0,1)$. Assume that the r.v.s $N_t, X_1, X_2, \ldots$ satisfy conditions (18), (10) and (2). Then*

$$d = \frac{3 - u_\alpha^2}{4} + o(1)$$

as $t \to \infty$, where $\Phi(u_\alpha) = 1 - \alpha$.

Remark 3. *The quantity d can be interpreted as the asymptotic deficiency of the distribution of a non-random sum with respect to the corresponding accompanying compound Poisson distribution. Note that under the conditions of Theorem 4, d does not depend on the distribution of X_1. If $\alpha > 0.0417$..., then d is asymptotically positive, that is, the (accompanying) compound Poisson distribution of the r.v. S_t^* provides better accuracy for the approximation of the asymptotic $(1-\alpha)$-quantile of U_t^*.*

4. Conclusions

This paper is a continuation of our previous work [1] and deals with a version of the problem of stochastic ordering. We follow an approach based on the concept of deficiency, which is well-known in asymptotic statistics. In the present paper, we considered compound Poisson distributions and introduced a continuous analog of deficiency. It was suggested to understand the continuous deficiency as the difference between the parameter of the compounding distribution of a Poisson random sum and that of the compound Poisson distribution providing the desired accuracy of the normal approximation. The asymptotic representations for the continuous deficiency were obtained under the condition that possible distributions of random summands possess coinciding first three moments. Therefore, we can say that, in this problem, we deal with 'fine tuning' of the distribution of a separate summand since we assume that different possible distributions of random summands can differ only by their kurtosis. In the setting under consideration, the best distribution delivers the smallest value of the parameter of the compounding Poisson distribution. The main mathematical tools used in the paper are asymptotic expansions for the compound Poisson distributions and their quantiles. The formal setting mentioned above was applied to solving some practical problems dealing with the collective risk insurance models where it is traditional to describe the cumulative insurance payments by the compound Poisson process. The approach under consideration makes it possible to determine the distribution of insurance payments providing the least possible time that provides the prescribed Value-at-Risk. We also considered the application of the proposed approach to the study of the asymptotic properties of non-random sums of independent identically distributed r.v.s as compared to those of the compound Poisson distributions with the same expectation. We investigate the accuracy of the approximation of the characteristics of sums of independent r.v.s as compared to that of the accompanying infinitely divisible (that is, corresponding compound Poisson) laws. Unlike most preceding works where the approximation of distribution functions was discussed, here we considered the application of accompanying laws to a somewhat inverse problem of approximation of quantiles.

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