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An Uncertain Sandwich Impulsive Control System with Impulsive Time Windows

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Abstract: In this paper, we formulate a new system, named the uncertain sandwich impulsive control system with impulsive time windows. The presented system shows that the linear entry matrix of the system is indeterminate. We first investigate the exponential stability of the considered system by linear matrix inequalities (LMIs) and inequalities techniques, then extend the considered system to a more general one and further study the stability of the general system. Finally, numerical simulations are delivered to demonstrate the effectiveness of the theoretical results.

Keywords: sandwich impulsive control system; LMIs; exponential stability criterion; impulsive time windows

MSC: 34D06; 34A37

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1. Introduction

In our real life, there are many actual artificial or natural systems whose states gradually change continuously over some time intervals, and for some reason, their states will be suddenly changed at some moments. Because the time of change is often quite short, the process of mutation or jump can be seen as occurring at a moment in time. We call this phenomenon the impulse phenomenon. This phenomenon can not be described by traditional continuous or discrete systems.

Research on impulse dynamic systems began in the 1960s. In their book *On the stability of motion in the presence of impulses*, Milman and Myshkis [1] first mentioned and made a preliminary study of their stability. Impulsive systems are a special kind of hybrid system which consist of three parts: continuous dynamics described by differential equations that control the motion of the system between impulses; discrete dynamics described by difference equations that control the jump or reset of the instantaneous state at the moment of the pulse; an impulsive law that determines the time when the impulse occurs. Some basic theories of impulsive systems can be found in [2–4] and the references therein. Generally speaking, impulsive systems can be divided into three classes: impulsive control systems, impulsive disturbance systems, and rigid collision systems induced by discontinuities. The study of the first class mainly focuses on unstable continuous dynamics and further makes them stable. For example, the related works can be found in [5,6]. On the other hand, impulsive disturbance systems can be analyzed as a class of robustness in [7,8]. In addition, rigid collision systems exist in many mechanical models. When the moving track reaches the collision surface, there will be an instantaneous impulse due to the presence of the collision recovery coefficient of the system; the relevant studies can be found in [9,10] and the references therein. In the following, we will focus on the introduction of impulsive control systems.

Impulsive control is a discontinuous control method that is based on an impulsive differential equation or differential equation with impulsive effect [3]. At present, impulsive control is applied widely to many fields of science and technology, such as communication networks [11], control technology [12], biology [13], artificial earth satellite [14], and pest control [15]. In the 21st century, besides the impulsive control system, other control systems have also been proposed, such as adaptive control [16] and feedback control [17]. Considering the present wide application and the strong robustness of digital communication systems, Stojanovski et al. in [18] proposed a new concept of impulsive synchronization. Since then, many studies have been devoted to investigating the synchronization and stability analysis of impulsive control systems; see examples [19–35] and the references therein. For example, He et al. in [19] investigated secure synchronization of multi-agent systems under deception attacks in the impulsive control framework and proposed a distributed impulsive controller to ensure the mean-square bounded synchronization. Lu et al. in [20] investigated the globally exponential synchronization of impulsive dynamical networks by considering two types of impulses: synchronizing impulses and desynchronizing impulses. Recently, Cui et al. in [21] investigated the synchronization of Kuramoto oscillator networks under event-triggered delayed impulsive control by Lyapunov stability theory. It is well known that the phenomenon of time delay widely exists in real physical systems. Therefore, like the common time-delay systems, the research of time-delay impulsive systems has never stopped. For example, stabilization of nonlinear time-delay systems: distributed-delay dependent impulsive control was investigated by using the Lyapunov–Razumikhin method in [23]. Li and Wu in [24] considered nonlinear differential systems with state-dependent delayed impulses and investigated the stability of nonlinear differential systems with state-dependent delayed impulses by using the impulsive control theory and some comparison arguments. In recent years, more and more researchers have paid attention to the stability analysis of memristor-based neural network systems with impulsive effects. Zhou et al. in [31] built inertial memristor-based neural networks with impulses and time-varying delays, and investigated the global exponential stability by an extended Halanay differential inequality and a new delay impulsive differential inequality. Rajchaki et al. in [32] analyzed the stability and passivity problems for a class of memristor-based fractional-order competitive neural networks by using Caputo's fractional derivation.

For some works on the stabilization and synchronization of impulsive control systems, the assumption of the occurrence of impulse is fixed, or the occurrence of impulse can be calculated. It is well known that any machine or computer has errors in the input of impulses, so the expected time is always different from the actual time. This time error is called the impulsive time window; some related works can be found in [36,37]. In recent years, many scholars have proposed various impulsive systems and made them stable by different methods. In [20], the authors have proposed an impulsive controller with an average impulsive interval. It is worth noting that some restrictions are defined on the average impulsive interval. In order to remove these restrictions and make the impulsive control system a more general one, Feng in [30] proposed a single-state jump impulsive system with periodic time windows and investigated the exponential stability of the new model. The corresponding impulsive control system can be written in the following form:

$$\begin{cases} \dot{u}(t) = Au(t) + \psi(u(t)) & mT \le t < mT + \theta_1, \\ u(t) = u(t^-) + Ju(t^-), & t = mT + \theta_1, \\ \dot{u}(t) = Au(t) + \psi(u(t)), & mT + \theta_1 < t < (m+1)T, \end{cases} \tag{1}$$

where $u(t) \in R^n$ is the state vector, $\psi : R^n \mapsto R^n$ is a continuous nonlinear function with $\psi(0) = 0$, and there exists a semi-definite diagonal matrix $L = \mathrm{diag}(l_1, l_2, \cdots, l_n)$ such that $\|\psi(u)\|^2 \le u^T Lu$. $Au(t)$ represents the linear part of the control system in which $A \in R^{n\times n}$ is a constant matrix with appropriate dimension. Meanwhile, $u(t^-)$ is defined as $u(t^-) = \lim_{b\to t^-} u(b)$. J is the coefficient of impulse intensity; $T > 0$ is the control period of the system; θ_1 represents the impulsive moment, which is defined by impulse time windows $[mT, (m+1)T]$. When $t = mT + \theta_1$, a random impulse $Ju(t^-)$ occurs. It is worth noting that the impulsive intensity J is always assumed as a constant matrix in [30,31]. Due to the need for engineering technology, we hope that the impulsive intensity J is flexible and unfixed. In [33], Feng et al. investigated a nonlinear impulsive control system with impulse time windows and an unfixed coefficient of impulsive intensity. They restricted the range of J as $J \leq \mu I$, in which μ is a constant and I is an unit matrix with proper dimension, and further investigated the stability of the considered system.

In real life, since the state of the system is quite complicated, one impulse J for system (1) is far from stabilizing the system. In [38], Feng formulated a new system that obtains two impulsive intensities, named the sandwich control system with impulsive time windows, and further investigated the stability of the considered system and obtained an exponential stability criterion. Different from most existing results for impulsive systems, they show that the impulse moments are unknown but limited to certain intervals. In [39], Liao et al. further studied the sandwich control system with dual stochastic impulses and obtained the exponential stability criterion. Particularly, a sandwich control system with dual stochastic impulses can be written as follows

$$\begin{cases} \dot{u}(t) = Au(t) + \psi(u(t)) + K_1 u(t), & mT \leq t < mT + \theta_1 \\ u(t) = u(t^-) + H_1 u(t^-), & t = mT + \theta_1 \\ \dot{u}(t) = Au(t) + \psi(u(t)) + K_2 u(t), & mT + \theta_1 < t < mT + \theta_2 \\ u(t) = u(t^-) + H_2 u(t^-), & t = mT + \theta_2 \\ \dot{u}(t) = Au(t) + \psi(u(t)) + K_3 u(t), & mT + \theta_2 < t < (m+1)T \end{cases} \tag{2}$$

where $K_1, K_2, K_3, H_1, H_2 \in R^{n\times n}$ are constant matrices with appropriate dimensions. θ_1 and θ_2 represent the impulsive moment, respectively. For system (2), the matrix A is a constant coefficient matrix. In fact, there exist many uncertain factors in various engineering, biological, and economic systems; see, for example, [40,41] and the references therein. To make the nonlinear impulse control system more reasonable, parameter uncertainty and bounded gain error are introduced into the corresponding impulsive differential equations. Hence, the research on the robustness of uncertain impulsive control systems is of great importance. In recent years, many scholars have focused on the system with uncertain parameters and obtained some results. In [42], Xie et al. investigated H_∞ control and quadratic stabilization of systems with parameter uncertainty via output feedback. Based on the notion of quadratic stability with disturbance attenuation, the problems of robust H_∞, control, and quadratic stabilization via linear dynamic output feedback have been solved. Ren in [43] studied a class of uncertain impulsive control systems and obtained a new sufficient condition for the considered system by the generalized Cauchy–Schwarz inequality method. Wen in [44] discussed fault-tolerant secure consensus tracking for multi-agent impulsive control systems with uncertain parameters. Using the impulsive control method, Lin in [45] investigated hyper-chaotic systems with uncertain parameters.

In the existing works on the study of the impulsive control system, most have focused on the constant coefficient matrix A. So far, the uncertain sandwich control system with impulsive time windows has not been studied. Based on the above discussion, in this paper we put forward an uncertain sandwich control system with impulse time windows which is can be written in the following form:

$$\begin{cases} \dot{u}(t) = (A + \Delta_1)u(t) + \psi(u(t)) + K_1 u(t), & mT < t < mT + \theta_1, \\ u(t) = u(t^-) + H_1 u(t^-), & t = mT + \theta_1, \\ \dot{u}(t) = (A + \Delta_2)u(t) + \psi(u(t)) + K_2 u(t), & mT + \theta_1 < t < mT + \theta_2, \\ u(t) = u(t^-) + H_2 u(t^-), & t = mT + \theta_2, \\ \dot{u}(t) = (A + \Delta_3)u(t) + \psi(u(t)) + K_3 u(t), & mT + \theta_2 < t \leq (m+1)T, \end{cases} \tag{3}$$

where $\Delta_i (i = 1, 2, 3)$ denote the parametric uncertainty in A and $\Delta_i^T \Delta_i \leq W$. According to [46], Δ_i can be defined as $\Delta_i = M_i Q_i N_i$, in which Q_i is an uncertain matrix and has the inequality $Q_i^T Q_i \leq I$. M_i and N_i are known constant matrices of proper dimensions. Although the impulsive control system with uncertain parameters is also studied in [43], the system we constructed is closer to the needs of real life because system (3) has multiple impulses which occur in an expected time but are limited to time windows. Therefore, it is of great significance to study the exponential stability of system (3). In this paper, our goal is to find proper constant matrix K_1, K_2, K_3 and impulsive intensity H_1, H_2 such that sandwich control system (3) reaches exponential stability.

This paper is organized as follows. In Section 2, we give some definitions and lemmas. In Section 3, we discuss the exponential stability of system (3) and give an exponential stability criterion. Then, numerical simulations are given to show the effectiveness of our results in Section 4. Finally, Section 5 gives some conclusions.

In this paper, the maximum eigenvalue, the minimum eigenvalue, and the transpose of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ are defined by $\lambda_m(P)$ and $\lambda_M(P)$ and P^T, respectively. We use $P > 0$ (<0, $\leq$0, $\geq$0) to denote a symmetrical positive (negative, semi-negative, semi-positive) definite matrix P. The Euclidean norm of vector $x \in R^n$ is defined as $\|x\|$.

2. Preliminaries

In order to investigate the stability of system (3), we give two lemmas and two definitions as follows.

Lemma 1 ([33]). *For any $x, y \in R^n$, and $\sigma > 0$, then*

$$2x^T y \leq \sigma x^T x + \sigma^{-1} y^T y. \tag{4}$$

Lemma 2 ([47]). *Let B, C, D, and G be real matrices of appropriate dimensions, and G satisfying $G = G^T$, then*

$$G + BCD + D^T C^T B^T < 0, \tag{5}$$

for all $C^T C \leq I$, if and only if there exists a scalar $\xi > 0$ such that

$$G + \xi^{-1} BB^T + \xi D^T D < 0. \tag{6}$$

Definition 1 ([3]). *The right-upper Dini's derivative of a function $V : [0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}_+$ is defined by*

$$D^+V(t, x(t)) = \lim_{h\to 0^+} \sup \frac{[V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))]}{h}. \tag{7}$$

Definition 2 ([48]). *System (3) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that any solution of system (3) satisfies the ineqality*

$$\|u(t)\| \leq \mu e^{-\alpha t} \|u(0)\|, \quad \forall t \geq 0.$$

3. Exponential Stability Analysis

In this section, rigorous mathematical proof about the exponential stability of system (3) is presented.

Theorem 1. *If there exists a symmetric and positive definite matrix $P \in R^{n\times n}$ and real constants $T > 0, \eta > 0, g_i > 0, \varepsilon_i > 0$ $(i = 1, 2, 3)$ as well as $\theta_2 > \theta_1 > 0, \lambda_1 > 0, \lambda_2 > 0$ such that the following inequalities hold*

$$g_1\theta_1 + g_2(\theta_2 - \theta_1) + g_3(T - \theta_2) - \ln \lambda_1 - \ln \lambda_2 > 0, \tag{8}$$

$$\mathcal{H}_i + \eta^{-1}\left(PM_iM_i^T P^T + \eta^2 N_i^T N_i\right) \leq 0, \tag{9}$$

then system (3) is exponentially stable, where

$$\begin{aligned}
\mathcal{H}_i &= \varepsilon_i^{-1}L + PA + A^TP + PK_i + K_i^TP + \varepsilon_iP^2 + g_iP, \\
\lambda_1 &= \lambda_M((I+H_1)^TP(I+H_1))/\lambda_m(P), \\
\lambda_2 &= \lambda_M((I+H_2)^TP(I+H_2))/\lambda_m(P).
\end{aligned}$$

Proof. The Lyapunov function is constructed as follows,

$$V(u(t)) = u^T(t)Pu(t). \tag{10}$$

From (10), we have that

$$\lambda_m(P)\|u(t)\|^2 \le V(u(t)) \le \lambda_M(P)\|u(t)\|^2 \tag{11}$$

If $mT \le t < mT + \theta_1$, it can be obtained from (3), (4), (7), and (10) that

$$\begin{aligned}
D^+(V(u(t))) =& 2u^TP\dot{u} \\
=& 2u^TP[(A+\Delta_1)u + \psi(u) + K_1u] \\
=& u^T[P((A+\Delta_1) + (A+\Delta_1)^TP + PK_1 + K_1^TP]u + 2u^TP\psi(u) \\
\le& u^T[P(A+\Delta_1) + (A+\Delta_1)^TP + PK_1 + K_1^TP]u + \varepsilon_1u^TP^2u \\
&+ \varepsilon_1^{-1}u^TLu \\
=& -g_1u^TPu + u^T[P(A+\Delta_1) + (A+\Delta_1)^TP + PK_1 + K_1^TP \\
&+ \varepsilon_1P^2 + \varepsilon_1^{-1}L + g_1P]u \\
=& -g_1u^TPu + u^T[\mathcal{H}_1 + P\Delta_1 + \Delta_1^TP]u.
\end{aligned} \tag{12}$$

Given a proper zero matrix O, we have that

$$\begin{bmatrix} \Delta_1 & O \\ O & O \end{bmatrix} = \begin{bmatrix} M_1Q_1N_1 & O \\ O & O \end{bmatrix} = \begin{bmatrix} M_1 \\ O \end{bmatrix} Q_1 \begin{bmatrix} N_1 & O \end{bmatrix},$$

and furthermore, obtain that

$$\begin{aligned}
\begin{bmatrix} \mathcal{H}_1 + P\Delta_1 + \Delta_1^TP - \varepsilon_1P^2 & -P \\ -P & -\varepsilon_1^{-1}I \end{bmatrix} =& \begin{bmatrix} \mathcal{H}_1 - \varepsilon_1P^2 & -P \\ -P & -\varepsilon_1^{-1}I \end{bmatrix} + \begin{bmatrix} PM_1 \\ O \end{bmatrix} Q_1 \begin{bmatrix} N_1 & O \end{bmatrix} \\
&+ \begin{bmatrix} N_1^T \\ O \end{bmatrix} Q_1^T \begin{bmatrix} M_1^TP^T & O \end{bmatrix}.
\end{aligned} \tag{13}$$

By Lemma 2 and the condition (9) for $(i = 1)$, we have that

$$\begin{bmatrix} \mathcal{H}_1 + P\Delta_1 + \Delta_1^TP - \varepsilon_1P^2 & -P \\ -P & -\varepsilon_1^{-1}I \end{bmatrix} \le 0, \tag{14}$$

which is equivalent to the following equation

$$\mathcal{H}_1 + P\Delta_1 + \Delta_1^TP \le 0, \tag{15}$$

so, it can be obtained from (12) and (15) that

$$D^+(V(u(t))) \le -g_1V(u(t)), \tag{16}$$

which implies that

$$V(u(t)) \le V(u(mT)^-)\exp(-g_1(t - mT)). \tag{17}$$

When $t = mT + \theta_1$, then

$$\begin{aligned} V(u(t))|_{t=mT+\theta_1} =&\big(u(t^-) + H_1u(t^-)\big)^T P\big(u(t^-) + H_1u(t^-)\big) \\ =&u(t^-)^T(I + H_1)^T P(I + H_1)u(t^-) \\ \leq&\lambda_1 V(u(t^-)), \end{aligned} \tag{18}$$

where $\lambda_1 = \lambda_M((I + H_1)^T P(I + H_1))/\lambda_m(P)$.

If $mT + \theta_1 < t < mT + \theta_2$, it can be obtained from (3), (4), (7), and (10) that

$$\begin{aligned} \dot{V}(u) =&2u^T P\dot{u} \\ \leq&u^T[P(A + \Delta_2) + (A + \Delta_2)^T P + PK_2 + K_2^T P]u \\ &+ \varepsilon_2 u^T P^2 u + \varepsilon_2^{-1} u^T L u \\ =&- g_2 u^T P u + u^T[\mathcal{H}_2 + P\Delta_2 + \Delta_2^T P]u. \end{aligned} \tag{19}$$

Similarly, by Lemma 2 and the condition (9) for $(i = 2)$, we have that

$$\dot{V}(u) \leq -g_2 V(u(t)), \tag{20}$$

which implies that

$$V(u(t)) \leq \lambda_1 V(u(mT + \theta_1)^-) \exp(-g_2(t - mT - \theta_1)). \tag{21}$$

When $t = mT + \theta_2$, then

$$\begin{aligned} V(u(t))|_{t=mT+\theta_2} =&(u(t^-) + H_2u(t^-))^T P(u(t^-) + H_2u(t^-)) \\ =&u(t^-)^T(I + H_2)^T P(I + H_2)u(t^-) \\ \leq&\lambda_2 V(u(t^-)). \end{aligned} \tag{22}$$

If $mT + \theta_2 < t < (m + 1)T$, it can be obtained from (3), (4), (7), and (10) that

$$\begin{aligned} \dot{V}(u) =&2u^T P\dot{u} \\ \leq&u^T[P(A + \Delta_3) + (A + \Delta_3)^T P + PK_3 + K_3^T P]u + \varepsilon_3 u^T P^2 u \\ &+ \varepsilon_3^{-1} u^T L u \\ \leq&- g_3 u^T P u + u^T[\mathcal{H}_3 + P\Delta_3 + \Delta_3^T P]u. \end{aligned} \tag{23}$$

Similarly, by Lemma 2 and the condition (9) for $(i = 3)$, we have that

$$\dot{V}(u) \leq -g_3 V(u), \tag{24}$$

which implies that

$$V(u(t)) \leq \lambda_2 V(u(mT + \theta_2)^-) \exp(-g_3(t - mT - \theta_2)). \tag{25}$$

It follows from (17)–(25) that
Case 1 when $m = 0$, then
(1) If $0 \leq t < \theta_1$, then we have that

$$V(u(t)) \leq V(u(0)) \exp(-g_1 t), \tag{26}$$

thus

$$V(u(\theta_1^-)) \leq V(u(0)) \exp(-g_1\theta_1). \tag{27}$$

(2) If $t = \theta_1$, then we have that

$$V(u(\theta_1)) \leq \lambda_1 V(u(0)) \exp(-g_1\theta_1). \tag{28}$$

(3) If $\theta_1 < t < \theta_2$, then we have that

$$V(u(t)) \leq \lambda_1 V(u(0)) \exp(-g_1\theta_1 - g_2(t - \theta_1)), \tag{29}$$

thus

$$V(u(\theta_2^-)) \leq \lambda_1 V(u(0)) \exp(-g_1\theta_1 - g_2(\theta_2 - \theta_1)). \tag{30}$$

(4) If $t = \theta_2$, then we have that

$$V(u(\theta_2)) \leq \lambda_1\lambda_2 V(u(0)) \exp(-g_1\theta_1 - g_2(\theta_2 - \theta_1)). \tag{31}$$

(5) If $\theta_2 < t < T$, then we have that

$$\begin{aligned} V(u(t)) \leq& \lambda_2 V(u(\theta_2^-)) \exp(-g_3(t - \theta_2)) \\ \leq& \lambda_1\lambda_2 V(u(0)) \exp(-g_1\theta_1 - g_2(\theta_2 - \theta_1) - g_3(t - \theta_2)). \end{aligned} \tag{32}$$

Hence,

$$V(u(T)) \leq \lambda_1\lambda_2 V(u(0)) \exp(-g_1\theta_1 - g_2(\theta_2 - \theta_1) - g_3(T - \theta_2)).$$

Case 2 when $m = 1$, then
(6) If $T \leq t < T + \theta_1$, then we have that

$$V(u(t)) \leq \lambda_1\lambda_2 V(u(0)) \exp(-g_1\theta_1 - g_2(\theta_2 - \theta_1) - g_3(T - \theta_2) - g_1(t - T)). \tag{33}$$

(7) If $t = T + \theta_1$, then we have that

$$V(u(t)) \leq \lambda_1^2\lambda_2 V(u(0)) \exp(-2g_1\theta_1 - g_2(\theta_2 - \theta_1) - g_3(T - \theta_2)). \tag{34}$$

(8) If $T + \theta_1 < t < T + \theta_2$, then we have that

$$\begin{aligned} V(u(t)) \leq& \lambda_1^2\lambda_1 V(u(0)) \exp(-2g_1\theta_1 - g_2(\theta_2 - \theta_1) - g_3(T - \theta_2) \\ &- g_2(t - T - \theta_1)). \end{aligned} \tag{35}$$

(9) If $t = T + \theta_2$, then we have that

$$V(u(t)) \leq \lambda_1^2\lambda_2^2 V(u(0)) \exp(-2g_1\theta_1 - 2g_2(\theta_2 - \theta_1) - g_3(T - \theta_2)). \tag{36}$$

(10) If $T + \theta_2 < t < 2T$, then we have that

$$\begin{aligned} V(u(t)) \leq& \lambda_1^2\lambda_2^2 V(u(0)) \exp(-2g_1\theta_1 - 2g_2(\theta_2 - \theta_1) \\ &- g_3(T - \theta_2) - g_3(t - T - \theta_2)). \end{aligned} \tag{37}$$

Case n when $m = n - 1$, then
(11) If $(n-1)T < t < (n-1)T + \theta_1$, then we have that

$$\begin{aligned} V(u(t)) \leq& \lambda_1^{n-1}\lambda_2^{n-1} V(u(0)) \exp(-(n-1)g_1\theta_1 - (n-1)g_2(\theta_2 - \theta_1) \\ &- (n-1)g_3(T - \theta_2) - g_1(t - (n-1)T)). \end{aligned} \tag{38}$$

(12) If $t = (n-1)T + \theta_1$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^n \lambda_2^{n-1} V(u(0)) \exp(-n g_1 \theta_1 - (n-1) g_2 (\theta_2 - \theta_1) \\ & - (n-1) g_3 (T - \theta_2)). \end{aligned} \tag{39}$$

(13) If $(n-1)T + \theta_1 < t < (n-1)T + \theta_2$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^n \lambda_2^{n-1} V(u(0)) \exp(-n g_1 \theta_1 - (n-1) g_2 (\theta_2 - \theta_1) \\ & - (n-1) g_3 (T - \theta_2) - g_2 (t - (n-1)T - \theta_1)). \end{aligned} \tag{40}$$

(14) If $t = (n-1)T + \theta_2$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^n \lambda_2^n V(u(0)) \exp(-n g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - (n-1) g_3 (T - \theta_2)). \end{aligned} \tag{41}$$

(15) If $(n-1)T + \theta_2 < t \leq nT$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^n \lambda_2^n V(u(0)) \exp(-n g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - (n-1) g_3 (T - \theta_2) - g_3 (t - (n-1)T - \theta_2)). \end{aligned} \tag{42}$$

Case $\mathbf{n+1}$ when $m = n$, then
(16) If $nT < t < nT + \theta_1$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1} \lambda_2^n V(u(0)) \exp(-n g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - n g_3 (T - \theta_2) - g_1 (t - nT)). \end{aligned} \tag{43}$$

(17) If $t = nT + \theta_1$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1} \lambda_2^n V(u(0)) \exp(-(n+1) g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - n g_3 (T - \theta_2)). \end{aligned} \tag{44}$$

(18) If $nT + \theta_1 < t < nT + \theta_2$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1} \lambda_2^n V(u(0)) \exp(-(n+1) g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - n g_3 (T - \theta_2) - g_2 (t - nT - \theta_1)). \end{aligned} \tag{45}$$

(19) If $t = nT + \theta_2$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1} \lambda_2^{n+1} V(u(0)) \exp(-(n+1) g_1 \theta_1 - (n+1) g_2 (\theta_2 \\ & - \theta_1) - n g_3 (T - \theta_2)). \end{aligned} \tag{46}$$

(20) If $nT + \theta_2 < t \leq (n+1)T$, then we have that

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1} \lambda_2^{n+1} V(u(0)) \exp(-(n+1) g_1 \theta_1 - n g_2 (\theta_2 - \theta_1) \\ & - n g_3 (T - \theta_2) - g_3 (t - nT - \theta_2)). \end{aligned} \tag{47}$$

From (45) and (46), we can obtain the two inequalities as fellows

$$\begin{aligned} V(u(t)) \leq & \lambda_1^n \lambda_2^n V(u(0)) \exp(-(n+1) g_1 \theta_1 \\ & - (n+1) g_2 (\theta_2 - \theta_1) - n g_3 (T - \theta_2)) \\ \leq & V(u(0)) \exp(-(g_1 \theta_1 + g_2 (\theta_2 - \theta_1) + g_3 (T - \theta_2) \\ & - \ln \lambda_1 - \ln \lambda_2) n + \ln \lambda_1 - g_1 \theta_1)), \end{aligned} \tag{48}$$

and

$$\begin{aligned} V(u(t)) \leq & \lambda_1^{n+1}\lambda_2^{n+1}V(u(0))\exp(-(n+1)g_1\theta_1 \\ & -(n+1)g_2(\theta_2-\theta_1)-ng_3(T-\theta_2)-g_3(t-nT-\theta_2)) \\ \leq & V(u(0))\exp(-(g_1\theta_1+g_2(\theta_2-\theta_1)+g_3(T-\theta_2) \\ & -\ln\lambda_1-\ln\lambda_2)(n+1)). \end{aligned} \quad (49)$$

By the inequalities (49) and (50) and the condition (8), we can obtain that

$$\lim_{t\to\infty} V(u(t)) = 0. \quad (50)$$

From (50) and Definition 2, it is easy to obtain that system (3) is exponentially stable. □

Remark 1. *The choice of parameters in Theorem 1 mainly depends on solving the LMIs equations and inequality (9). In other words, given parameters $K_1, K_2, K_3, T, \theta_1, \theta_2, \eta$, we can find a flexible solution $g_1, g_2, g_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, P, H_1, H_2$ through the LMIs method given in [49] and inequality (9).*

In system (3), we only consider two impulse intensities, H_1 and H_2. If n impulse intensities $H_1, \cdots, H_n$ are inserted in a period unit T, system (3) can be written in the following form:

$$\begin{cases} \dot{u}(t) = (A+\Delta_1)u(t)+\psi(u(t))+K_1u(t), & mT < t < mT+\theta_1, \\ u(t) = u(t^-)+H_1u(t^-), & t = mT+\theta_1, \\ \dot{u}(t) = (A+\Delta_i)u(t)+\psi(u(t))+K_iu(t), & mT+\theta_i < t < mT+\theta_{i+1}, \\ u(t) = u(t^-)+H_iu(t^-), & t = mT+\theta_i, \\ \dot{u}(t) = (A+\Delta_{i+1})u(t)+\psi(u(t))+K_{i+1}u(t), & mT+\theta_{i+1} < t \leq (m+1)T, \end{cases} \quad (51)$$

where $i = 2, \cdots, n$.

Corollary 1. *If there exists a symmetric and positive definite matrix $P \in R^{n\times n}$ and real constants $T > 0, \eta > 0, g_i > 0, \varepsilon_i > 0 (i = 1, \cdots, n)$ as well as $\theta_j > \theta_i > 0 (j > i), \lambda_i > 0$ such that the following inequalities hold*

$$g_1\theta_1 + \sum_{i=2}^{n} g_i(\theta_i-\theta_{i-1}) + g_{n+1}(T-\theta_n) - \ln\Pi_{i=1}^{n}\lambda_i > 0, \quad (52)$$

$$\mathcal{H}_i + \eta^{-1}\left(PM_iM_i^TP^T + \eta^2N_i^TN_i\right) \leq 0, \quad (53)$$

then system (51) is exponentially stable, where

$$\begin{aligned} \mathcal{H}_i &= \varepsilon_i^{-1}L + PA + A^TP + PK_i + K_i^TP + \varepsilon_iP^2 + g_iP, \\ \lambda_i &= \lambda_M((I+H_i)^TP(I+H_i))/\lambda_m(P). \end{aligned}$$

Remark 2. *Conditions (52) and (53) can be obtained by the same method from Theorem 1. In many engineering applications, since the complexity of the system, two impulses can not make the system stable. Corollary 1 gives a generalization for impulse intensities n. Compared with the results in [30,35,38,39,43], we not only consider the n impulses intensities in one period T, but also investigate the uncertainty of parameters for systems (3) and (51). Obviously, the results we get are more in line with the needs of the project.*

4. Simulation

In the following, we take Chua's system with uncertain parameters as an example to show the effectiveness of our theoretical results. Let $X = (x, y, z)^T$.

Example 1. *Chua's oscillator [39] can be expressed as*

$$\begin{cases} \dot{x} = \alpha y - \alpha x - \alpha g(x), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y, \end{cases} \tag{54}$$

where

$$g(x) = bx + 0.5(a-b)(|x+1| - |x-1|),$$

α, β are parameters and constant a and b satisfy $a < b < 0$.

Take parameters $\alpha = 9.2156, \beta = 15.9946, a = -1.24905, b = -0.75735$, and the initial condition $X(0) = [4, 0, -3]^T$, the chaotic trajectory of system (54) is shown in Figure 1.

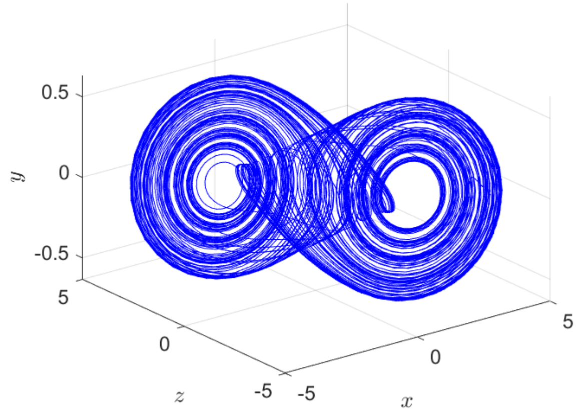

Figure 1. The chaotic phenomenon of system (54) with the initial condition $X(0) = [4, 0, -3]^T$.

Furthermore, system (54) can be written in the following form

$$\dot{X} = AX + \psi(X), \tag{55}$$

where

$$A = \begin{bmatrix} -\alpha(1+b) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

and

$$\psi(X) = \begin{bmatrix} -0.5\alpha(a-b)(|x+1| - |x-1|) \\ 0 \\ 0 \end{bmatrix}. \tag{56}$$

From (56), we have that

$$\|\psi(X)\|^2 = 0.5\alpha^2(a-b)^2(x^2 + 1 - |x^2 - 1|) \leq \alpha^2(a-b)^2 x^2,$$

so, we can select $L = \mathrm{diag}(\alpha^2(a-b)^2, 0, 0)$.

In order to better illustrate the importance of parameter perturbation, for matrix A of system (55), we give two kinds of uncertain parameter perturbation Δ^1 and Δ^2.

For the fist kind of parameter perturbation $\Delta_i^1 = M_i^1 Q_i^1 N_i^1$ $(i = 1, 2, 3)$, we select $M_1^1 = M_2^1 = M_3^1 = N_1^1 = N_2^1 = N_3^1 = I$ and $Q_1^1 = Q_2^1 = Q_3^1 = \mathrm{diag}(0.2\cos t, 0.2\cos t, 0.2\cos t)$, so $\Delta^1 := \Delta_1^1 = \Delta_2^1 = \Delta_3^1$. Then system (55) under uncertain factors can be written to the following form

$$\dot{X} = (A + \Delta^1)X + \varphi(X). \tag{57}$$

Choosing the same parameters and initial values as in system (55), we plot the phase diagram of system (57), and the result is shown in Figure 2. Then choosing $K_1 = \mathrm{diag}(-90, -80, -50), K_2 = \mathrm{diag}(-80, -80, -50), K_3 = \mathrm{diag}(-90, -80, -60)$ *with* $\eta = 0.2, \theta_1 = 0.005, \theta_2 = 0.01$*, and* $T = 0.02$*. By solving LMIs and inequality (9), we can obtain a feasible solution:* $g_1 = 10, g_2 = 80, g_3 = 60, \varepsilon_1 = 0.2, \varepsilon_2 = 8, \varepsilon_3 = 9, H_1 = \mathrm{diag}(-0.07, -0.26, 0.03), H_2 = \mathrm{diag}(-0.45 - 0.4 - 0.4)$ *and*

$$P = \begin{bmatrix} 0.9120 & -0.1267 & 0.1338 \\ -0.1267 & 0.9317 & -0.0032 \\ 0.1338 & -0.0032 & 0.0793 \end{bmatrix}.$$

Obviously, these parameters we select satisfy Equations (8) and (9). According to Theorem 1, system (57) is exponentially stable under the action of pulses H_1 *and* H_2*. The time response curves of Chua's oscillator are shown in Figure 3.*

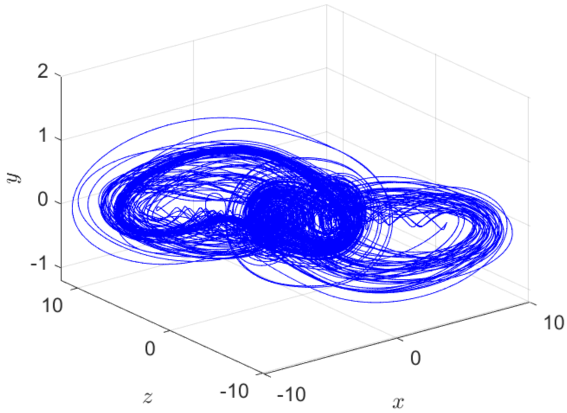

Figure 2. The chaotic phenomenon of system (57) with the initial condition $X(0) = [4, 0, -3]^T$.

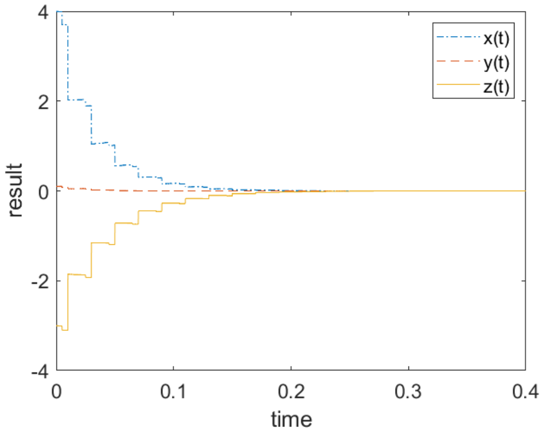

Figure 3. The time responding curves of system (57) with uncertain sandwich control and impulse time windows.

For the second kind of parameter perturbation $\Delta_i^2 = M_i^2 Q_i^2 N_i^2, (i = 1, 2, 3)$*, we select* $M_1^2 = M_2^2 = M_3^2 = N_1^2 = N_2^2 = N_3^2 = I$ *and* $Q_1^2 = Q_2^2 = Q_3^2 = \mathrm{diag}(0.02x^2(1) + 0.05x(1)x(3) + 0.09x(1)x(3))$*, so* $\Delta^2 := \Delta_1^2 = \Delta_2^2 = \Delta_3^2$*. Then system (55) under uncertain factors can be written to the following form*

$$\dot{X} = (A + \Delta^2)X + \varphi(X). \tag{58}$$

Choosing the same parameters and initial values as in system (55), we plot the phase diagram of system (58), and the result is shown in Figure 4. According to Theorem 1, system (58) is exponentially stable, and the time response curve of Chua's oscillator is shown in Figure 5.

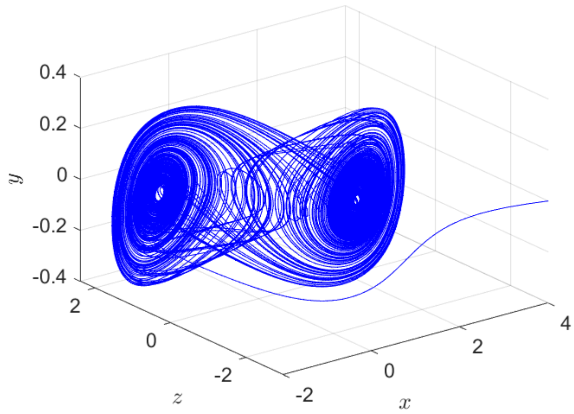

Figure 4. The chaotic phenomenon of system (58) with the initial condition $X(0) = [4, 0, -3]^T$.

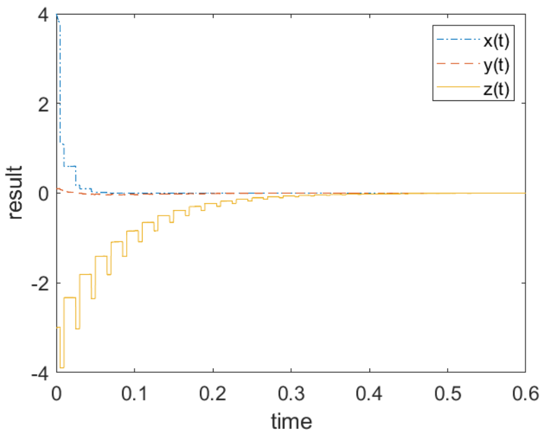

Figure 5. The time responding curves of system (58) with uncertain sandwich control and impulse time windows.

5. Conclusions

In this paper, a new model of a nonlinear impulsive control system with uncertain parameters and impulse time windows is proposed. A stability criterion is given in terms of linear matrix inequalities. By the LMIs method and some inequality techniques, a sufficient condition for exponential stability of system (3) is derived by Lyapunov stability. To make the conclusion more general, we insert a countable number of impulses in a period of time T and investigate its exponential stability. At last, numerical simulation verifies that the main theoretical results are effective. In this paper we only considered countable impulsive intensities n. In many engineering applications, countable impulses are far from making the system stable. To make the system a more general one, we can extend the results of this paper to a class of impulsive time sequences in one periodic unit with eventually uniformly bounded impulsive frequency and investigate the stability of the considered system in the future. We hope that this paper will provide a direction for the future study of uncertain parameter nonlinear impulsive control systems.

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