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Computational Traveling Wave Solutions of the Nonlinear Rangwala–Rao Model Arising in Electric Field

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Abstract: The direct influence of the integrability requirement on mixed derivative nonlinear Schrödinger equations is investigated in this paper. A. Rangwala mathematically formalized these effects in 1990 and dubbed this form the Rangwala–Rao ($\mathcal{RR}$) equation. Our research focuses on innovative soliton wave solutions and their interactions in order to provide a clear picture of the slowly evolving envelope of the electric field and pulse propagation in optical fibers in terms of the dispersion effect. For creating unique solitary wave solutions to the investigated model, three contemporary computational strategies (extended direct (ExD) method, improved F–expansion (ImFE) method, and modified Kudryashov (MKud) method) are employed. These solutions are numerically computed to demonstrate the dynamical behavior of optical fiber pulse propagation. The originality of the paper's findings is proved by comparing our results to previously published results.

Keywords: nonlinear Schrödinger equations; Rangwala–Rao equation; optical fiber; soliton waves

MSC: 35Q60; 35E05; 35C08; 35Q51

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1. Introduction

Recently, the dynamical and physical characterizations of a system with one degree of freedom have attracted the attention of many mathematicians and physicists [1,2]. This study was given in the presence of a linear restoring force and nonlinear damping of the investigated model [3]. The mathematical model that describes this phenomenon is the Lienard equation formulated by the French physicist Alfred–Marie Liénard [4]. The Lienard equation is given by [5,6]

$$\frac{d^2x}{dt^2} + \Gamma(x)\,\frac{d\,x}{dt} + \Xi(x) = 0, \tag{1}$$

where Γ, Ξ are two continuously differentiable functions on $\mathbf{R}$, and, respectively, odd and even functions. A class of classical anharmonic oscillators of Equation (1) can be given by [7]

$$\mathscr{W}'' + r_1\,\mathscr{W} + r_2\,\mathscr{W}^3 + r_3\,\mathscr{W}^5 = 0, \tag{2}$$

where r_1, r_2, r_3 are arbitrary constants; $\mathscr{W} = \mathscr{W}(\chi)$ describes the waves' propagation of the the Alfvén wave with a small non-vanishing wave number. Furthermore, the general form of Equation (2) is given by [8]

$$\mathscr{W}'' + r_1\,\mathscr{W} + r_2\,\mathscr{W}^{p+1} + r_3\,\mathscr{W}^{2\,p+1} = 0, \tag{3}$$

where $p = 1, 2, 3, \cdots$. These models (2), (3) are beneficial for studying solitary wave solutions of some icons nonlinear evolution equations such as the generalized Ablowitz equation [9], the generalized Gerdjikov–Ivanov equation [10], the generalized one-dimensional Klein–Gordon equation [11], $\mathcal{RR}$ equation [12], Kundu–Eckhaus equation [13], the generalized Zakharov equations [14], the Chen–Lee–Lin equation [15], and the well-known nonlinear Schrödinger equation [16].

In mathematics and physics, a nonlinear partial differential equation is a partial differential equation with nonlinear components [17]. They have been used to solve mathematical problems such as the Poincaré conjecture and the Calabi–Yau conjecture, as well as to explain a broad range of physical phenomena such as gravity and fluid dynamics [18,19]. These are difficult to study since there are no general procedures that apply to all of these equations; instead, each one must be examined as a standalone issue [20]. Nonlinear and linear partial differential equations are distinguished by the features of the operator that defines the PDE [21].

All solutions to a PDE should ideally be wholly specified, which is possible for sure uncommon PDEs [22]. If the equation has a large symmetry group [23], a finite-dimensional compact manifold, perhaps with singularities, may be created. In this situation, the moduli space of solutions modulo the symmetry group [24] is all that matters. The moduli space may be explicitly compactified in the self-dual Yang–Mills equations, which are somewhat more complicated since the moduli space is finite-dimensional but not necessarily compact [25]. In totally integrable models, where the solutions are often a superposition of solitons, it is occasionally possible to describe all solutions [26]. For example, consider the Korteweg–De Vries equation. Many outstanding solutions may be stated as fundamental functions [27]. Ordinary differential equations, typically solved correctly, are a helpful place to start when looking for straightforward solutions to complicated problems [28–30].

Mathematical Analysis of Model

This paper studies the $\mathcal{RR}$ equation which is given by [31,32]

$$\mathcal{R}_{xt} - r_1\,\mathcal{R}_{xx} + \mathcal{R} + i\,r_2\,|\mathcal{R}|^2\,\mathcal{R}_x = 0, \tag{4}$$

where $\mathcal{R} = \mathcal{R}(x,t)$ represents a complex smooth envelope function of a spatial variable x and a temporal variable t; r_1, r_2 are real constants. Using the next transformation $\mathcal{R}(x,t) = e^{i\,\psi(\mathfrak{z})}\,e^{-i\,t\,\omega}\,\varphi(\mathfrak{z})$, $\mathfrak{z} = x - \lambda\,t$ to Equation (4), we obtain

$$\begin{cases} \text{Re:} \quad (-\lambda - r_1)\varphi'' + \varphi(\lambda + r_1)(\psi')^2 - r_2\varphi^3\psi' + \varphi\omega\psi' + \varphi = 0, \\ \text{Im:} \quad -\varphi'(2(\lambda + r_1)\psi' - r_2\varphi^2 + \omega) - \varphi(\lambda + r_1)\psi'' = 0. \end{cases} \tag{5}$$

Substituting $\psi = \int \frac{\left(r_2\varphi(x)^2 - \omega\right)^2}{4r_2(\lambda + r_1)\varphi(x)^2}\,dx$ into the first equation of (5), leads to

$$\left(-16\lambda^2r_2^2 - 32\lambda r_1r_2^2 - 16r_1^2r_2^2\right)\varphi^3\varphi'' + \varphi^4\left(16\lambda r_2^2 - 6r_2^2\omega^2 + 16r_1r_2^2\right) + 8r_2^3\omega\varphi^6 - 3r_2^4\varphi^8 + \omega^4 = 0. \tag{6}$$

Balancing the terms of Equation (6) by using the homogeneous balance rule and the suggested computational schemes' auxiliary equations $\Big[\mathcal{F}'(\mathfrak{z}) = \mathcal{J}_3\,\mathcal{F}(\mathfrak{z})^2 + \mathcal{J}_2\,\mathcal{F}(\mathfrak{z}) + \mathcal{J}_1 \mapsto$ (ExD method [33]) & $\mathscr{G}'(\mathfrak{z}) = \mathscr{G}(\mathfrak{z})^2 + \varrho \mapsto$ (ImFE method [34]) & $\mathscr{B}'(\mathfrak{z}) = \ln(\mathcal{K})\left(\mathscr{B}(\mathfrak{z})^2 - \mathscr{B}(\mathfrak{z})\right) \mapsto$ (MKud method [35])$\Big]$, where $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, ϱ, a are arbitrary constants, lead to $N = \frac{1}{2}$. Consequently, we have to use another transformation that is given by $\varphi(\mathfrak{z}) = \sqrt{\mathcal{Y}(\mathfrak{z})}$. Thus, Equation (6) transforms into the following formula

$$\mathcal{Y}\mathcal{Y}'' + \mathcal{Y}^4\mathcal{L}_2 - \mathcal{Y}^3\mathcal{L}_3 - \mathcal{Y}^2\mathcal{L}_4 - \mathcal{L}_1\left(\mathcal{Y}'\right)^2 + \mathcal{L}_5 = 0, \tag{7}$$

where $\left[\mathcal{L}_1 = \frac{1}{2}, \mathcal{L}_2 = \frac{3r_2^2}{8(\lambda+r_1)^2}, \mathcal{L}_3 = \frac{r_2\omega}{(\lambda+r_1)^2}, \mathcal{L}_4 = \frac{8\lambda+8r_1-3\omega^2}{4(\lambda+r_1)^2}, \mathcal{L}_5 = -\frac{\omega^4}{8r_2^2(\lambda+r_1)^2}\right]$. Using the homogeneous balance rule on Equation (7), leads to $N = 1$. Thus, the general solutions of the suggested model through the above-mentioned computational schemes are formulated by (for more detial about the implemented schemes; see (Appendix A):

$$\mathcal{Y}(\mathfrak{Z}) = \begin{cases} \sum_{i=0}^{n} a_i\, \mathcal{F}(\mathfrak{Z})^i = a_1\, \mathcal{F}(\mathfrak{Z}) + a_0, \\ \sum_{i=0}^{n} a_i\, (\mathscr{G}(\mathfrak{Z}) + \mu)^i = a_1(\mu + \mathscr{G}(\mathfrak{Z})) + a_0, \\ \sum_{i=0}^{n} a_i\, \mathscr{B}(\mathfrak{Z})^i = a_1\, \mathscr{B}(\mathfrak{Z}) + a_0, \end{cases} \tag{8}$$

where a_0, a_1 are arbitrary constants to be evaluated through the methods' frameworks.

The rest paper's sections are given in the following order; novel solitary wave solutions of the investigated model and their numerical simulations are given in Section 2. The paper's contributions are given in Section 3. The summary of our study and its results are summarized in Section 4.

2. Solitary Wave Solutions

This section aims to derive some novel solitary wave solutions to the $\mathcal{RR}$ equation using the above-suggested computational schemes. Furthermore, the constructed solutions are represented through contour, two-dimensional, and three-dimensional graphs to illustrate the pulses' propagation in optical fibers.

2.1. The ExD Method's Results

Investigating the above-parameters' values through the ExD method's framework, leads to

Set I

$$\mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_2 \to -\frac{3\mathcal{J}_3^2}{2a_1^2}, \mathcal{L}_3 \to \frac{2\mathcal{J}_3(a_1\mathcal{J}_2 - 2a_0\mathcal{J}_3)}{a_1^2}, \mathcal{L}_4 \to \frac{3a_0\mathcal{J}_3(a_0\mathcal{J}_3 - a_1\mathcal{J}_2)}{a_1^2} + \frac{\mathcal{J}_2^2}{2} + \mathcal{J}_1\mathcal{J}_3,$$
$$\mathcal{L}_5 \to \frac{\left(a_0^2\mathcal{J}_3 - a_1a_0\mathcal{J}_2 + a_1^2\mathcal{J}_1\right)^2}{2a_1^2}.$$

Set II

$$a_0 \to 0, a_1 \to i\sqrt{\frac{3}{2}}\frac{\mathcal{J}_3}{\sqrt{\mathcal{L}_2}}, \mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_3 \to -2i\sqrt{\frac{2}{3}}\mathcal{J}_2\sqrt{\mathcal{L}_2}, \mathcal{L}_4 \to \frac{\mathcal{J}_2^2}{2} + \mathcal{J}_1\mathcal{J}_3, \mathcal{L}_5 \to -\frac{3\mathcal{J}_1^2\mathcal{J}_3^2}{4\mathcal{L}_2},$$

where $(\mathcal{L}_2 < 0)$.

Set III

$$a_1 \to i\sqrt{\frac{3}{2}}\frac{\mathcal{J}_3}{\sqrt{\mathcal{L}_2}}, \mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_3 \to \frac{2}{3}\left(4a_0\mathcal{L}_2 - i\sqrt{6}\mathcal{J}_2\sqrt{\mathcal{L}_2}\right), \mathcal{L}_4 \to i\sqrt{6}a_0\mathcal{J}_2\sqrt{\mathcal{L}_2} - 2a_0^2\mathcal{L}_2 + \frac{\mathcal{J}_2^2}{2} + \mathcal{J}_1\mathcal{J}_3,$$
$$\mathcal{L}_5 \to \frac{1}{12\mathcal{L}_2}\left(4i\sqrt{6}a_0^3\mathcal{J}_2\mathcal{L}_2^{3/2} + 6a_0^2\left(\mathcal{J}_2^2 + 2\mathcal{J}_1\mathcal{J}_3\right)\mathcal{L}_2 - 6i\sqrt{6}a_0\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3\sqrt{\mathcal{L}_2} - 4a_0^4\mathcal{L}_2^2 - 9\mathcal{J}_1^2\mathcal{J}_3^2\right),$$

where $(\mathcal{L}_2 < 0)$.

Thus, the solitary wave solutions of the investigated model are given by;

For $\mathcal{J}_2 = 0$, $\mathcal{J}_1\mathcal{J}_3 < 0$, we obtain

$$\mathscr{R}_{\mathrm{I},1}(x,t) = \Xi\left(a_0 + \frac{a_1\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\tanh\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\mathcal{J}_3}\right)^{\frac{1}{2}}, \tag{9}$$

$$\mathscr{R}_{\mathrm{I},2}(x,t) = \Xi\left(a_0 + \frac{a_1\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\coth\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\mathcal{J}_3}\right)^{\frac{1}{2}}, \tag{10}$$

$$\mathscr{R}_{\mathrm{II},1}(x,t) = \Xi\left(\frac{i\sqrt{\frac{3}{2}}\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\tanh\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}}, \tag{11}$$

$$\mathscr{R}_{\mathrm{II},2}(x,t) = \Xi\left(\frac{i\sqrt{\frac{3}{2}}\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\coth\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}}, \tag{12}$$

$$\mathscr{R}_{\mathrm{III},1}(x,t) = \Xi\left(a_0 + \frac{i\sqrt{\frac{3}{2}}\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\tanh\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}}, \tag{13}$$

$$\mathscr{R}_{\mathrm{III},2}(x,t) = \Xi\left(a_0 + \frac{i\sqrt{\frac{3}{2}}\sqrt{-\mathcal{J}_1\mathcal{J}_3}\,\coth\Big(\sqrt{-\mathcal{J}_1\mathcal{J}_3}(x-\lambda\,t) + \frac{\log(\vartheta)}{2}\Big)}{\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}}. \tag{14}$$

For $\mathcal{J}_1 = 0$, $\mathcal{J}_2 > 0$, we obtain

$$\mathscr{R}_{\mathrm{I},3}(x,t) = \Xi\left(a_0 + \frac{a_1\mathcal{J}_2\,e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}{1-\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}\right)^{\frac{1}{2}}, \tag{15}$$

$$\mathscr{R}_{\mathrm{II},3}(x,t) = \Xi\left(-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2\mathcal{J}_3\,e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}{\sqrt{\mathcal{L}_2}\big(\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}-1\big)}\right)^{\frac{1}{2}}, \tag{16}$$

$$\mathscr{R}_{\mathrm{III},3}(x,t) = \Xi\left(a_0 - \frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2\mathcal{J}_3\,e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}{\sqrt{\mathcal{L}_2}\big(\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}-1\big)}\right)^{\frac{1}{2}}. \tag{17}$$

For $\mathcal{J}_1 = 0$, $\mathcal{J}_2 < 0$, we obtain

$$\mathscr{R}_{\mathrm{I},4}(x,t) = \Xi\left(a_0 - a_1 + \frac{a_1}{\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}+1}\right)^{\frac{1}{2}}, \tag{18}$$

$$\mathscr{R}_{\mathrm{II},4}(x,t)=\Xi\left(-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_3^2\,e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}{\sqrt{\mathcal{L}_2}\left(\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}+1\right)}\right)^{\frac{1}{2}},\tag{19}$$

$$\mathscr{R}_{\mathrm{III},4}(x,t)=\Xi\left(a_0-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_3^2\,e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}}{\sqrt{\mathcal{L}_2}\left(\mathcal{J}_3e^{\mathcal{J}_2(-\lambda\,t+x+\vartheta)}+1\right)}\right)^{\frac{1}{2}}.\tag{20}$$

For $4\mathcal{J}_1\mathcal{J}_3>\mathcal{J}_2^2$, we get

$$\mathscr{R}_{\mathrm{I},5}(x,t)=\Xi\left(-\frac{a_1\mathcal{J}_2}{2\mathcal{J}_3}+a_0+\frac{a_1\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\tan\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\mathcal{J}_3}\right)^{\frac{1}{2}},\tag{21}$$

$$\mathscr{R}_{\mathrm{I},6}(x,t)=\Xi\left(-\frac{a_1\mathcal{J}_2}{2\mathcal{J}_3}+a_0+\frac{a_1\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\cot\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\mathcal{J}_3}\right)^{\frac{1}{2}},\tag{22}$$

$$\mathscr{R}_{\mathrm{II},5}(x,t)=\Xi\left(\frac{i\sqrt{\frac{3}{2}}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\tan\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\sqrt{\mathcal{L}_2}}-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2}{2\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}},\tag{23}$$

$$\mathscr{R}_{\mathrm{II},6}(x,t)=\Xi\left(\frac{i\sqrt{\frac{3}{2}}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\cot\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\sqrt{\mathcal{L}_2}}-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2}{2\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}},\tag{24}$$

$$\mathscr{R}_{\mathrm{III},5}(x,t)=\Xi\left(a_0-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2}{2\sqrt{\mathcal{L}_2}}+\frac{i\sqrt{\frac{3}{2}}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\tan\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}},\tag{25}$$

$$\mathscr{R}_{\mathrm{III},6}(x,t)=\Xi\left(a_0-\frac{i\sqrt{\frac{3}{2}}\mathcal{J}_2}{2\sqrt{\mathcal{L}_2}}+\frac{i\sqrt{\frac{3}{2}}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}\,\cot\left(\frac{1}{2}\sqrt{4\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2^2}(-\lambda\,t+x+\vartheta)\right)}{2\sqrt{\mathcal{L}_2}}\right)^{\frac{1}{2}}.\tag{26}$$

where $\Xi=\exp\left(\frac{i}{4\,r_2\,(\lambda+r_1)}\int\frac{\left(\omega-r_2\,\varphi^2\right)^2}{\varphi^2}\,d\,\mathfrak{z}-i\,t\omega\right)$.

2.2. The ImFE Method's Results

Investigating the above-parameters' values through the ImFE method's framework, leads to

Set I

$$\mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_2 \to -\frac{3}{2a_1^2}, \mathcal{L}_3 \to -\frac{4(a_1\mu + a_0)}{a_1^2}, \mathcal{L}_4 \to \frac{3a_0(2a_1\mu + a_0)}{a_1^2} + 3\mu^2 + \varrho,$$
$$\mathcal{L}_5 \to \frac{\left(a_1^2(\mu^2+\varrho) + 2a_1a_0\mu + a_0^2\right)^2}{2a_1^2}.$$

Set II

$$a_0 \to -\frac{1}{4}a_1(a_1\mathcal{L}_3 + 4\mu), \mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_2 \to -\frac{3}{2a_1^2}, \mathcal{L}_4 \to \frac{1}{8}a_1^2\mathcal{L}_3^2 - \frac{\sqrt{2}\sqrt{\mathcal{L}_5}}{a_1}, \varrho \to -\frac{1}{16}a_1^2\mathcal{L}_3^2 - \frac{\sqrt{2}\sqrt{\mathcal{L}_5}}{a_1}.$$

Thus, the solitary wave solutions of the investigated model are given by
For $\varrho \neq 0$, we obtain

$$\mathscr{R}_{\mathrm{I},1}(x,t) = \Xi\left(a_1(\mu + \sqrt{\varrho}\tan(\sqrt{\varrho}\,(x-\lambda\,t))) + a_0\right)^{\frac{1}{2}}, \tag{27}$$

$$\mathscr{R}_{\mathrm{I},2}(x,t) = \Xi\left(a_1(\mu - \sqrt{\varrho}\cot(\sqrt{\varrho}\,(x-\lambda\,t))) + a_0\right)^{\frac{1}{2}}, \tag{28}$$

$$\mathscr{R}_{\mathrm{II},1}(x,t) = \Xi\left(-\frac{1}{4}a_1\left(a_1\mathcal{L}_3 - 4\sqrt{\varrho}\tan(\sqrt{\varrho}\,(x-\lambda\,t))\right)\right)^{\frac{1}{2}}, \tag{29}$$

$$\mathscr{R}_{\mathrm{II},2}(x,t) = \Xi\left(-\frac{1}{4}a_1\left(a_1\mathcal{L}_3 + 4\sqrt{\varrho}\cot(\sqrt{\varrho}\,(x-\lambda\,t))\right)\right)^{\frac{1}{2}}. \tag{30}$$

For $\varrho = 0$, we obtain

$$\mathscr{R}_{\mathrm{I},3}(x,t) = \Xi\left(a_1\left(\mu + \frac{1}{\lambda\,t - x}\right) + a_0\right)^{\frac{1}{2}}, \tag{31}$$

$$\mathscr{R}_{\mathrm{II},4}(x,t) = \Xi\left(\frac{1}{4}a_1\left(-a_1\mathcal{L}_3 - \frac{4}{x-\lambda\,t}\right)\right)^{\frac{1}{2}}. \tag{32}$$

where $\Xi = \exp\left(\frac{i}{4\,r_2\,(\lambda+r_1)}\int\frac{(\omega - r_2\,\varphi^2)^2}{\varphi^2}\,d\,\mathfrak{z} - i\,t\omega\right)$.

2.3. The MKud Method's Results

Investigating the above-parameters' values through the ImFE method's framework, leads to

$$\mathcal{L}_1 \to \frac{1}{2}, \mathcal{L}_2 \to -\frac{3\ln^2(\mathcal{K})}{2a_1^2}, \mathcal{L}_3 \to -\frac{2(2a_0+a_1)\ln^2(\mathcal{K})}{a_1^2}, \mathcal{L}_4 \to \frac{(6a_0^2 + 6a_1a_0 + a_1^2)\ln^2(\mathcal{K})}{2a_1^2},$$
$$\mathcal{L}_5 \to \frac{a_0^2(a_0+a_1)^2\ln^2(\mathcal{K})}{2a_1^2}.$$

Thus , the solitary wave solutions of the investigated model are given by

$$\mathscr{R}(x,t) = \Xi\left(\frac{a_1}{1 \pm \mathcal{K}^{(x-\lambda\, t)}} + a_0\right)^{\frac{1}{2}}. \tag{33}$$

where $\Xi = \exp\left(\frac{i}{4\, r_2\,(\lambda + r_1)} \int \frac{(\omega - r_2\,\varphi^2)^2}{\varphi^2}\, d\,\mathfrak{z} - i\, t\omega\right)$.

3. Results and Discussion

This section investigates the paper's results and the employed computational schemes. Three analytical (ExD, ImFE, and MKud) techniques have been applied to the $\mathcal{RR}$ equation and many soliton wave solutions have been constructed in various formulas. The applied methods' auxiliary equations are similar. The ExD and ImFE methods are equivalent $\mapsto$ $(\mathcal{J}_3 = 1,\, \mathcal{J}_2 = \varrho,\, \mathcal{J}_1 = 0)$, the ExD, and MKud methods are equivalent $\mapsto$ $(\mathcal{J}_3 = 1,\, \mathcal{J}_2 = -1,\, \mathcal{J}_1 = 0,\, \mathcal{K} = 10)$, and the ExD, and MKud methods are equivalent $\mapsto$ $(\mathcal{J}_3 = 1,\, \mathcal{J}_2 = -1,\, \mathcal{J}_1 = 0,\, \mathcal{K} = 10)$. This equivalence between the applied computational schemes' auxiliary equations leads to similarity of their methods' solutions, such as Equations (9)–(14), (27)–(30), which are similar to each other but at the same time are different from the previously published articles. Comparing our constructed solutions with those that have been published in [36–38] leads to distinguishing the novelty of our results. The obtained results of Equations (9), (11), (13), (27), and (29) are represented in some graphs (Figures 1–6 where each of these figures contains three distinct representations of the tested solutions and these representations are given, respectively, (a) 3D, (b) 2D, and (c) contour graphs.). These represented figures have the following parameter: Figure 1 uses $(a_0 = 7,\, a_1 = 6,\, \mathcal{J}_1 = -3,\, \mathcal{J}_3 = 27,\, \mathcal{J}_3 = 5,\, \lambda = 9,\, \Xi = 1,\, \vartheta = 1)$, Figure 2 is given under $(\mathcal{J}_1 = -1,\, \mathcal{J}_3 = 9,\, \lambda = 5,\, \Xi = 1,\, \mathcal{L}_2 = -4,\, \vartheta = 1)$, Figure 3 is illustrated for $(a_0 = 7,\, \mathcal{J}_1 = -1,\, \mathcal{J}_3 = 4,\, \lambda = 8,\, \Xi = 1,\, \mathcal{L}_2 = -8,\, \vartheta = 1)$, Figure 4 is demonstrated by $(a_0 = 3,\, a_1 = 6,\, \lambda = 8,\, \mu = 5,\, \Xi = 1,\, \varrho = -4)$, Figure 5 is numerically given by $(a_0 = 3,\, a_1 = 6,\, \lambda = 5,\, \Xi = 1,\, \mathcal{L}_3 = 5,\, \varrho = -9)$, and Figure 6 is represented by $(a_0 = 3,\, a_1 = 6,\, \lambda = 7,\, \Xi = 1)$. The represented graphs explains some distinct characterizations of the investigated model, respectively: kink wave, singular wave, periodic wave, anti-kink wave, bright, and drak-wave. The interactions between the obtained soliton wave solutions are represented through (Figure 7).

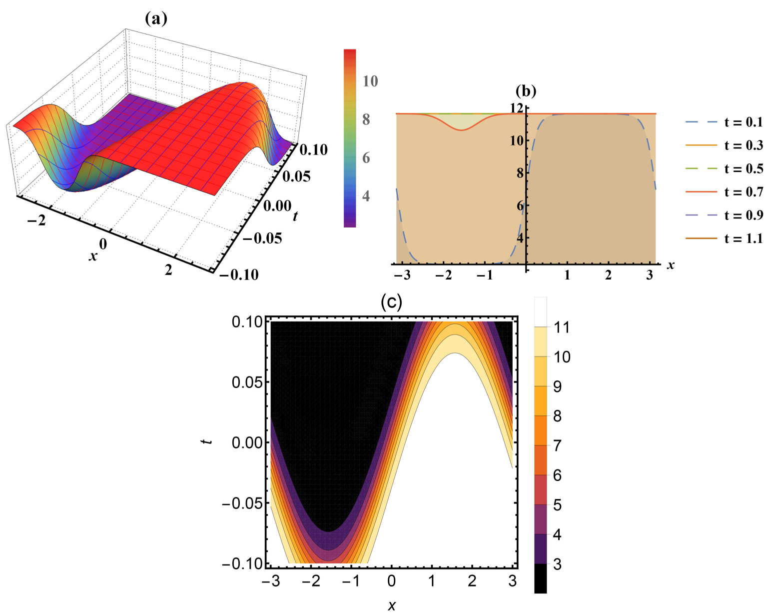

Figure 1. Numerical demonstration of kink waves for Equation (9) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

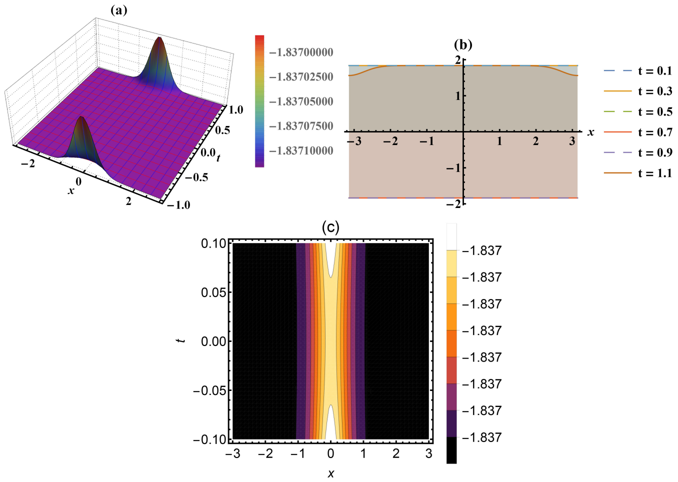

Figure 2. A numerical explanation of kink waves for Equation (11) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

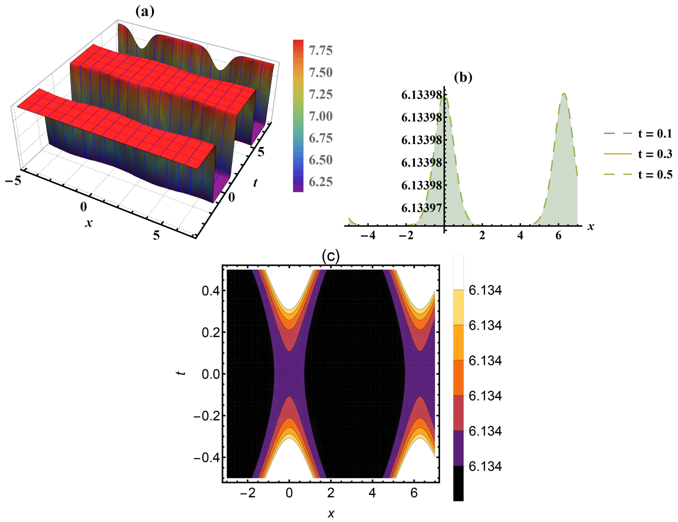

Figure 3. An illustration of periodic kink waves for Equation (13) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

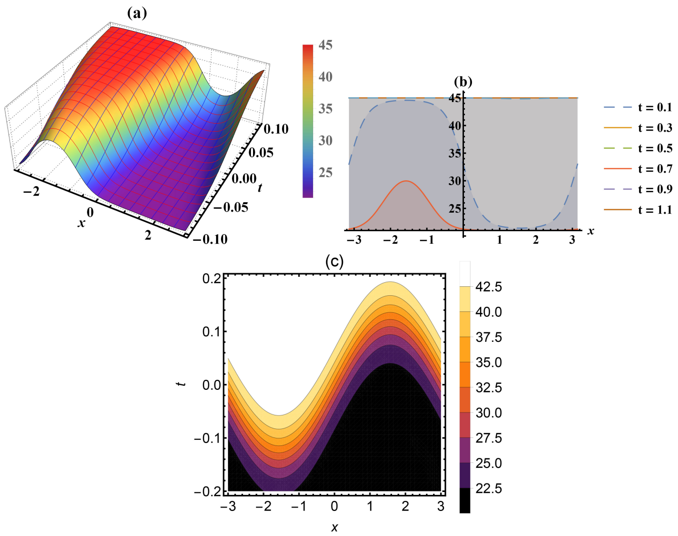

Figure 4. An illustration of kink waves for Equation (27) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

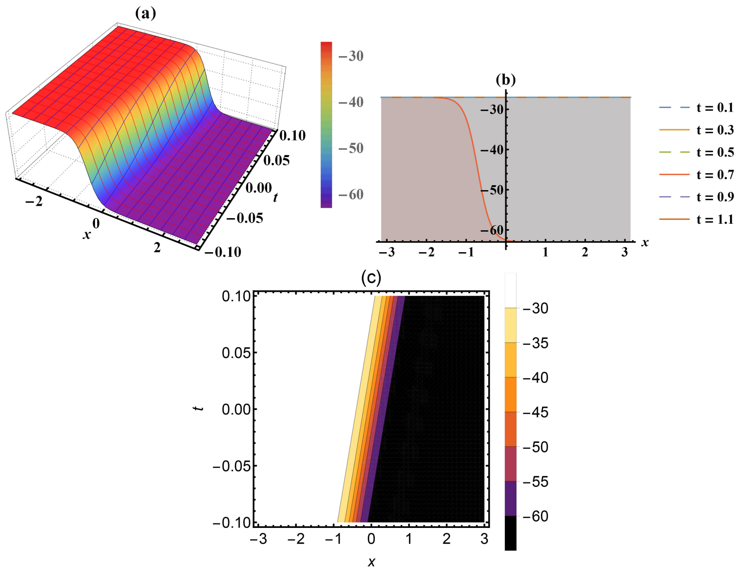

Figure 5. A numerical demonstration of kink waves for Equation (29) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

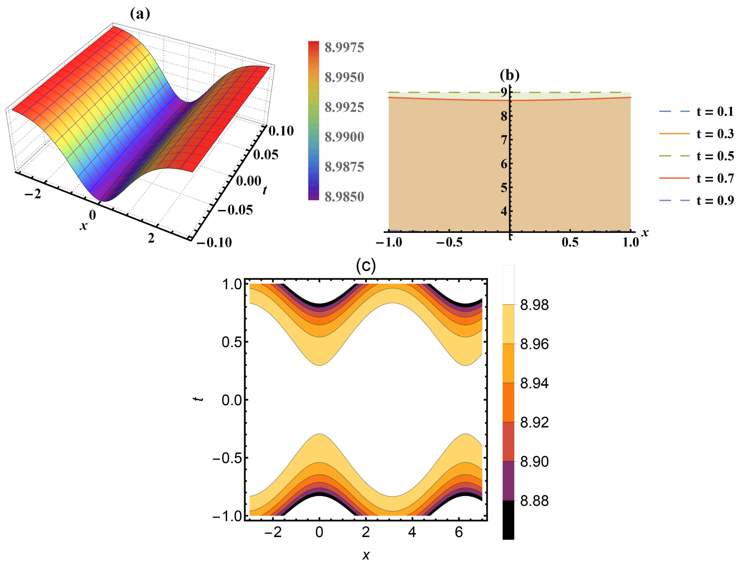

Figure 6. A numerical illustration of periodic kink waves for Equation (33) in some distinct graphs ((**a**) 3D, (**b**) 2D, and (**c**) density graph).

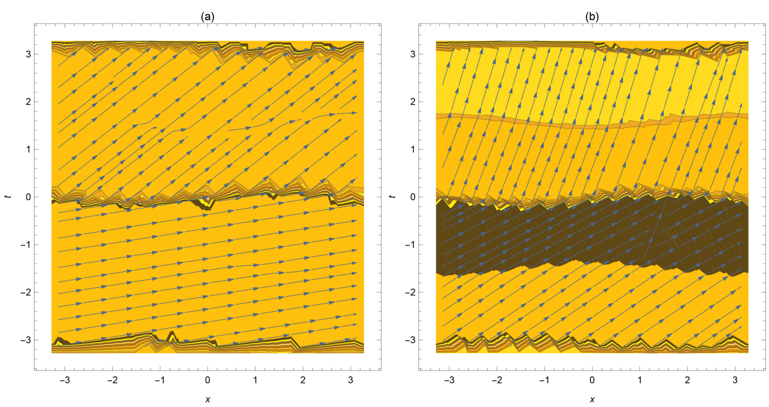

Figure 7. *Cont.*

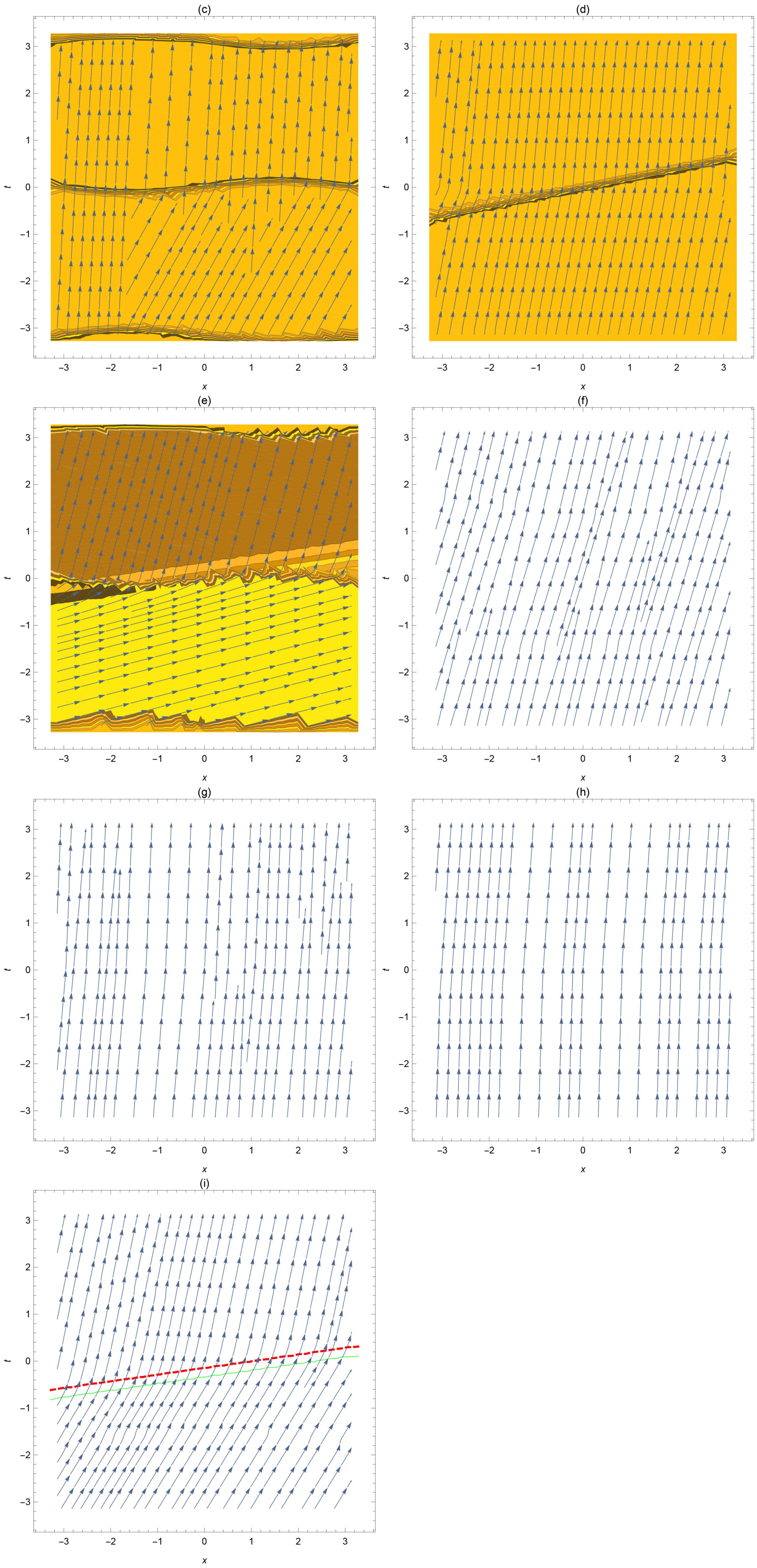

Figure 7. *Cont*.

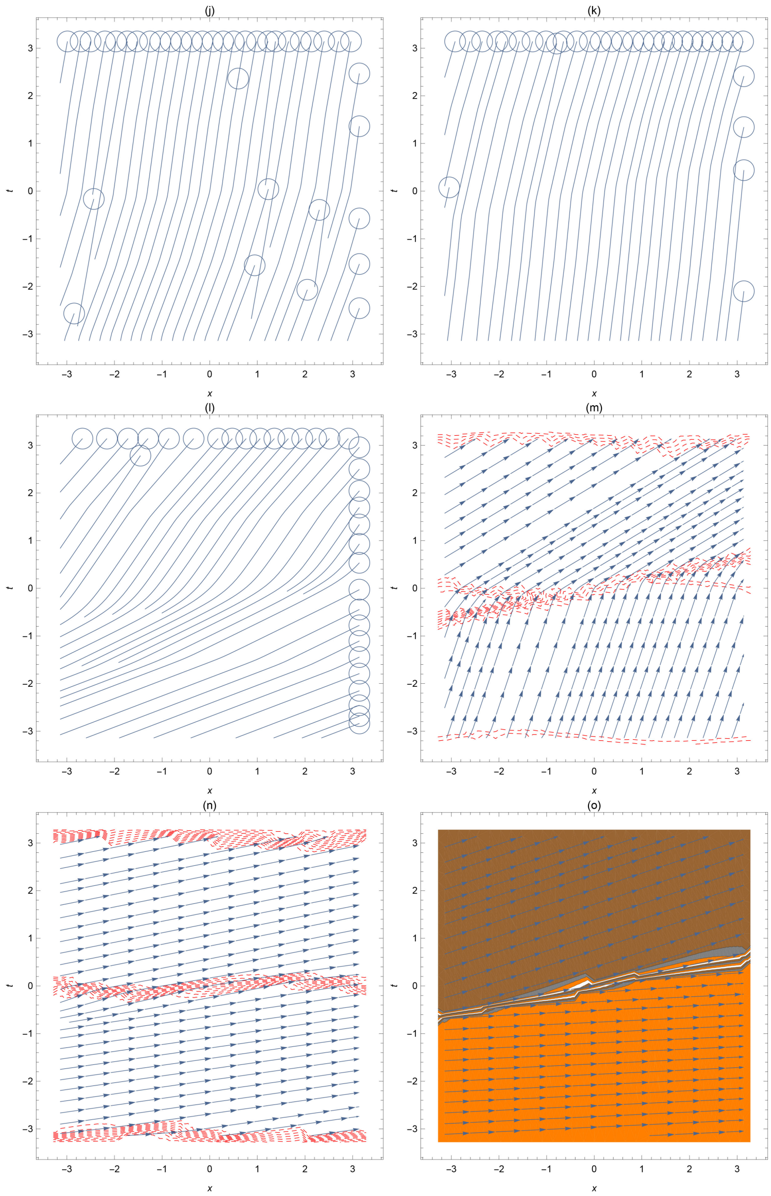

Figure 7. Propagation of pulses regarding the dispersion effect in optical fibers ((**a**) Equation (9), (**b**) Equation (11)). Propagation of pulses regarding the dispersion effect in optical fibers ((**c**) Equation (9), (**d**) Equation (11), (**e**) Equation (13), (**f**) Equation (27), (**g**) Equation (29), (**h**) Equation (33), (**i**) Equation (9)). Propagation of pulses regarding the dispersion effect in optical fibers ((**j**) Equation (9), (**k**) Equation (11), (**l**) Equation (13), (**m**) Equation (27), (**n**) Equation (29), (**o**) Equation (33)).

Finally, studying the above-solutions' stability using the Hamiltonian system characterizations leads to find the momentum of Equation (9) in the following structure

$$\begin{aligned}\mathcal{M} =& \frac{1}{162\lambda}\Big(32400\lambda + 30\lambda\Big(\tanh^{-1}(\tanh(15(\lambda+1))) + \tanh^{-1}(\tanh(15-15\lambda))\Big) \\ &+ \log(1-\tanh(15(\lambda+1))) + \log(\tanh(15(\lambda+1))+1) \\ &- \log(1-\tanh(15-15\lambda)) - \log(\tanh(15-15\lambda)+1)\Big).\end{aligned} \tag{34}$$

Thus, we find

$$\left.\frac{\partial\mathcal{M}}{\partial\lambda}\right|_{\lambda=5} = 0.0925925926. \tag{35}$$

Consequently, Equation (9) is table solution in $x \in [-5,5]$, $t \in [-5,5]$. With same technique, we can study the other-obtained solutions' stability conditions.

4. Conclusions

This research paper has constructed many soliton wave solutions of the Rangwala–Rao equation in some distinct formulas such as rational, hyperbolic, and trigonometric. These solutions explain the effect of the integrability conditions on the electric field's slowly varying envelope. Moreover, it impacts the slowly changing electric field envelope and the propagation of pulses in optical fibers regarding the dispersion effect. All these characterizations have been explained through some distinguished graphs. The paper's novelty and scientific contributions have been discussed by comparing our results with previously published articles. All solutions' accuracy has been checked by putting them back into the original model by Mathematica 13.1.

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Appendix A

This section gives the methods' headlines as follows:

Assume the following form for the equation of nonlinear evolution:

$$\mathcal{E}(\mathfrak{U}, \mathfrak{U}_x, \mathfrak{U}_t, \mathfrak{U}_{x,t}, \ldots) = 0, \tag{A1}$$

where $\mathcal{E} = \mathcal{E}(x,t)$ is a polynomial of $\mathfrak{U}(x,t)$ and its partial derivatives wherein the highest order derivatives and nonlinear terms are concerned. The main steps of the employed method [33–35] are as follows

Step 1. The traveling wave transformation

$$\mathfrak{U}(x,\, t) = \nu(\mathfrak{z}), \quad \mathfrak{z} = x + c\, t, \tag{A2}$$

converting Equation (A1) into the following ODE

$$\mathcal{E}\big(\nu,\, \nu', \nu'', \ldots\big) = 0, \tag{A3}$$

where $\mathcal{E}$ is a polynomial in $\nu(\mathfrak{z})$ and its total derivatives, wherein $\nu'(\mathfrak{z}) = \frac{d\nu}{d\mathfrak{z}}$.

Step 2. We suppose the solution of (A3) is of the form

$$\nu(\mathfrak{z}) = \begin{cases} \sum\limits_{i=0}^{N} a_i\, \mathcal{F}(\mathfrak{z})^i, \\ \\ \sum\limits_{i=0}^{N} a_i\, (\mathscr{G}(\mathfrak{z}) + \mu)^i, \\ \\ \sum\limits_{i=0}^{N} a_i\, \mathscr{B}(\mathfrak{z})^i, \end{cases} \tag{A4}$$

where a_k $(k = 0, 1, 2, 3, \ldots, N)$ are arbitrary constants to be determined, such that $a_N \neq 0$, and $\Phi(\mathfrak{z})$ is an unidentified function to be determined afterwards. This function satisfies the following equation

$$\begin{cases} \mathcal{F}'(\mathfrak{z}) = \mathcal{J}_3\,\mathcal{F}(\mathfrak{z})^2 + \mathcal{J}_2\,\mathcal{F}(\mathfrak{z}) + \mathcal{J}_1, \\ \mathscr{G}'(\mathfrak{z}) = \mathscr{G}(\mathfrak{z})^2 + \varrho, \\ \mathscr{B}'(\mathfrak{z}) = \ln(\mathcal{K})\big(\mathscr{B}(\mathfrak{z})^2 - \mathscr{B}(\mathfrak{z})\big), \end{cases} \tag{A5}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, ϱ, a are arbitrary constants

Step 3. We determine the positive integer N come out in (A4) by considering the homogeneous balance between the highest order derivatives and the highest order nonlinear terms occurring in (A3).

Step 4. We compute all the required derivatives $\nu', \nu'', \ldots$, and substitute (A4) and the derivatives into (A3) and then we account for the function $\mathcal{F}(\mathfrak{z})$, $\mathscr{G}(\mathfrak{z})$, $\mathscr{B}(\mathfrak{z})$. As a result of this substitution, we obtain a polynomial of $\mathcal{F}^j(\mathfrak{z})$, $\mathscr{G}^j(\mathfrak{z})$, $\mathscr{B}^j(\mathfrak{z})$ and its derivatives. In this polynomial, we equate all the coefficients to zero. This procedure yields a system of equations whichever can be solved to find a_k and $\mathcal{F}(\mathfrak{z})$, $\mathscr{G}(\mathfrak{z})$, $\mathscr{B}(\mathfrak{z})$.

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