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Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions

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Abstract: We have investigated new Pauli Fibonacci and Pauli Lucas quaternions by taking the components of these quaternions as Gaussian Fibonacci and Gaussian Lucas numbers, respectively. We have calculated some basic identities for these quaternions. Later, the generating functions and Binet formulas are obtained for Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions. Furthermore, Honsberger's identity, Catalan's and Cassini's identities have been given for Pauli Gaussian Fibonacci quaternions.

Keywords: Pauli matrix; Pauli quaternion; Fibonacci quaternion; Pauli Gaussian Fibonacci quaternion; Pauli Gaussian Lucas quaternion

MSC: 11B37; 11B39; 20G20

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1. Introduction

The set of complex numbers with integer coefficients was first described by Carl Friedrich Gauss, and these numbers were called Gaussiannumbers [1]. Then, Horadam gave the definition of the n-th generalized complex Fibonacci quaternion and provided some identities regarding these numbers. Furthermore, he defined Fibonacci quaternions [2]. Gaussian Fibonacci and Gaussian Lucas sequences were introduced by Jordan. Moreover, some basic identities and summation formulas were obtained [3]. The recurrence relation of the Gaussian Fibonacci numbers GF_n for $n > 1$ is definedby

$$GF_n = GF_{n-1} + GF_{n-2}$$

where $GF_0 = i, \quad GF_1 = 1$. These recurrence relations also satisfy the following equality $GF_n = F_n + iF_{n-1}$ where F_n is the n-th Fibonacci number [3,4]. In the same manner, the recurrence relation of the Gaussian Lucas numbers GL_n for $n > 2$ is defined by

$$GL_n = GL_{n-1} + GL_{n-2}$$

where $GL_0 = 2 - i, \quad GL_1 = 1 + 2i$. Again, the following equality can be observed

$$GL_n = L_n + iL_{n-1}$$

where L_n is the n-th Lucas number [3,4].

An extension of the Fibonacci numbers to the complex plane was discussed by Berzsenyi [5].

The algebras of the complex numbers, quaternions, octonions, and sedenions are found by using a doubling procedure. This procedure is called as Cayley–Dickson process. In this regard, we extend the field of real numbers to complex numbers via this process. The complex number system is both commutative and associative. However, the quaternions are not commutative, although they are associative. On the other hand, octonions and sedenions are both non-commutative and non-associative. The main question is, why do we need these expanding number systems? It is because quaternions have applications in quantum mechanics, computer graphics, and vision [6–8]. Octonions are used in quantum information theory and robotics [9,10]. Sedenions are used in neural networks, time series, and traffic forecasting problems [11,12].

The sequences in finite fields whose terms depend in a simple manner on their predecessors are of importance for a variety of applications. Because it is easy to generate by recursive procedures, and these sequences have advantageous features from the computational viewpoint [13]. Thus, mathematicians increased the number of terms to be added at the beginning and turned to studies on number sequences similar to Fibonacci numbers, such as tribonacci, tetranacci, pentanacci, etc. Later, these studies were carried over to Cayley algebras, see [14–32]. Thus, one of the most active research areas of recent years has come to the fore, and studies on Cayley algebras have attracted researchers in various ways.

Complex Fibonacci quaternions have been defined by Halici [33]. Then, Binet's formula, generating functions and the matrix representation of these quaternions have been proven. Recently, n-th quaternion Gaussian Lucas numbers have been introduced to the literature. The Binet formula, some summation formulas, and the Cassini identity have been given by using the matrix representation of Gaussian Lucas numbers [34]. This time, they have expressed the quaternions instead of the Gaussian Lucas numbers by taking the Gaussian Fibonacci coefficients. In this way, the Binet formula, generating function ve some identities regarding the norm of these quaternions have been derived [35].

The Binet formula of the Gaussian Fibonacci sequence and Gaussian Lucas sequence are

$$GF_n = \frac{1}{\alpha-\beta}\{(1-i\beta)\alpha^n - (1-i\alpha)\beta^n\}$$

and

$$GL_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}GL_1 + \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}GL_0$$

respectively, where α and β denote the roots of the characteristic equation for Gaussian Fibonacci sequence and GL_0, GL_1 denote the initial values for the Gaussian Lucas numbers [34,35].

The Pauli matrices that have been introduced by Wolfgang Pauli form a set of 2×2 complex matrices as follows:

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The multiplication rules are given by

$$\begin{aligned} &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, && \sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \\ &\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1, && \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2 \end{aligned}$$

Further, these matrices are Hermitian and unitary. These 2×2 types of Hermitian matrices form a basis for the real vector space and the span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3\}$ is isomorphic to the real algebra of quaternions [36,37].

The Pauli quaternions are defined by Kim as follows:

$$q = a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3,$$

where $1, \sigma_1, \sigma_2$ and σ_3 represent the Pauli matrices.

Let $q = a_0 1 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$ and $p = b_0 1 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3$ be Pauli quaternions, then the product of these quaternions are given by [37]

$$\begin{aligned} q.p = {} & (a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)1 + \{(a_0b_1 + a_1b_0) + i(a_2b_3 - a_3b_2)\}\sigma_1 \\ & + \{(a_0b_2 + a_2b_0) + i(a_3b_1 - a_1b_3)\}\sigma_2 \\ & + \{(a_0b_3 + a_3b_0) + i(a_1b_2 - a_2b_1)\}\sigma_3. \end{aligned}$$

The conjugate and the norm of Pauli quaternions are

$$q^* = a_0 1 - a_1\sigma_1 - a_2\sigma_2 - a_3\sigma_3$$

and

$$N(q) = \sqrt{|q.q^*|} = \sqrt{|a_0^2 - a_1^2 - a_2^2 - a_3^2|},$$

respectively [37].

Torunbalcı has presented Pauli Fibonacci and Pauli Lucas quaternions by taking the real coefficients of Pauli quaternion as the Fibonacci number sequence. Honsberger's, d'Ocagne's, Catalan, Cassini identities, generating function and Binet formula and the matrix representation have been given for the Pauli Fibonacci quaternions [38].

In a recent paper [39] on a base of quaternions, the families of associated sequences of real polynomials and numbers were defined. Quaternion equivalents for quasi-Fibonacci numbers (shortly quaternaccis) were introduced. In OEIS, there is a number of quaternacci sequences connected with generalized Gaussian Fibonacci integers. We are also interested in quaternions with coefficients of Gaussian Fibonacci numbers.

Especially, in this work, our aim is to introduce Pauli Fibonacci quaternions and Pauli Lucas quaternions whose coefficients consist of Gaussian Fibonacci numbers and Gaussian Lucas numbers, respectively. We called these numbers the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas numbers, respectively. Then, some algebraic properties of these quaternions have been shown. Moreover, some identities and formulas for these quaternions have been obtained.

2. The Pauli Gaussian Fibonacci Quaternions

In this section, the definition of Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions will be given. Then, some algebraic properties, identities and theorems are given for Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions.

Definition 1. *The Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions are defined by*

$$Q_pGF_n = GF_n 1 + GF_{n+1}\sigma_1 + GF_{n+2}\sigma_2 + GF_{n+3}\sigma_3$$

and

$$Q_pGL_n = GL_n 1 + GL_{n+1}\sigma_1 + GL_{n+2}\sigma_2 + GL_{n+3}\sigma_3$$

respectively, where GF_n and GL_n are the n-th Gaussian Fibonacci and Gaussian Lucas numbers.

Furthermore, these numbers are related to Pauli Fibonacci and Pauli Lucas quaternions as follows

$$Q_pGF_n = Q_pF_n + iQ_pF_{n-1}$$

and

$$Q_pGL_n = Q_pL_n + iQ_pL_{n-1}$$

respectively.

In order to obtain the recursive relations for Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions, we will consider the relations $GF_{n+2} = GF_n + GF_{n+1}$ and $GL_{n+2} = GL_n + GL_{n+1}$ for Gaussian Fibonacci and Gaussian Lucas numbers, respectively.

Hence, for $n \geq 0$

$$Q_pGF_{n+2} = Q_pGF_n + Q_pGF_{n+1}$$

and

$$Q_pGL_{n+2} = Q_pGL_n + Q_pGL_{n+1}$$

respectively.

Definition 2. *The conjugates of the Pauli Gaussian Fibonacci quaternion Q_pGF_n and the Pauli Gaussian Lucas quaternion Q_pGL_n are defined by*

$$\overline{Q_pGF_n} = GF_n 1 - GF_{n+1}\sigma_1 - GF_{n+2}\sigma_2 - GF_{n+3}\sigma_3$$

and

$$\overline{Q_pGL_n} = GL_n 1 - GL_{n+1}\sigma_1 - GL_{n+2}\sigma_2 - GL_{n+3}\sigma_3$$

respectively.

The addition, subtraction and multiplication of two Pauli Gaussian Fibonacci quaternions Q_pGF_n and Q_pGF_m aregiven by

$$\begin{aligned} Q_pGF_n \pm Q_pGF_m = {} & (GF_n \pm GF_m).1 + (GF_{n+1} \pm GF_{m+1})\sigma_1 \\ & + (GF_{n+2} \pm GF_{m+2})\sigma_2 + (GF_{n+3} \pm GF_{m+3})\sigma_3 \end{aligned} \tag{1}$$

and

$$\begin{aligned} Q_pGF_n \times Q_pGF_m = {} & (GF_n.GF_m + GF_{n+1}.GF_{m+1} + GF_{n+2}.GF_{m+2} + GF_{n+3}.GF_{m+3}).1 \\ & +(GF_{n+1}.GF_m + GF_n.GF_{m+1} - iGF_{n+3}.GF_{m+2} + iGF_{n+2}.GF_{m+3})\sigma_1 \\ & +(GF_{n+2}.GF_m + iGF_{n+3}.GF_{m+1} + GF_n.GF_{m+2} - iGF_{n+1}.GF_{m+3})\sigma_2 \\ & +(GF_{n+3}.GF_m - iGF_{n+2}.GF_{m+1} + iGF_{n+1}.GF_{m+2} + GF_n.GF_{m+3})\sigma_3 \end{aligned} \tag{2}$$

respectively.

Note that $Q_pGF_n \times Q_pGF_m \neq Q_pGF_m \times Q_pGF_n$.

In this case, the norm of any Pauli Gaussian Fibonacci quaternion can be written as

$$N^2_{Q_pGF_n} = Q_pGF_n \times \overline{Q_pGF_n} = \left| GF_n^2 - GF_{n+1}^2 - GF_{n+2}^2 - GF_{n+3}^2 \right|.$$

So, the scalar and vectorial part of any Pauli Gaussian Fibonacci quaternion is represented by

$$S_{Q_pGF_n} = GF_n, \qquad V_{Q_pGF_n} = GF_{n+1}\sigma_1 + GF_{n+2}\sigma_2 + GF_{n+3}\sigma_3.$$

In addition, Equation (2) can be rewritten in terms of the scalar and vector parts of the Pauli Gaussian Fibonacci quaternion as follows.

$$\begin{aligned} Q_pGF_n \times Q_pGF_m = {} & S_{Q_pGF_n}S_{Q_pGF_m} + \left\langle V_{Q_pGF_n}, V_{Q_pGF_m} \right\rangle + S_{Q_pGF_n}V_{Q_pGF_m} \\ & + S_{Q_pGF_m}V_{Q_pGF_n} + V_{Q_pGF_n} \wedge V_{Q_pGF_m}. \end{aligned}$$

With the aid of Equation (2), the following Pauli Gaussian Fibonacci quaternion can be expressed as a matrix form

$$Q_pGF_n \times Q_pGF_m = \begin{bmatrix} GF_n & GF_{n+1} & GF_{n+2} & GF_{n+3} \\ GF_{n+1} & GF_n & -iGF_{n+3} & iGF_{n+2} \\ GF_{n+2} & iGF_{n+3} & GF_n & -iGF_{n+1} \\ GF_{n+3} & -iGF_{n+2} & iGF_{n+1} & GF_n \end{bmatrix} \begin{bmatrix} GF_m \\ GF_{m+1} \\ GF_{m+2} \\ GF_{m+3} \end{bmatrix}.$$

Theorem 1. *Let Q_pGF_n, Q_pGL_n and GF_n denote the Gaussian Fibonacci number, Gaussian Lucas number and the Gaussian Fibonacci number, respectively. For $n \geq 1$, we get the following relations*

(i)

$$Q_pGF_{n+1} + Q_pGF_{n-1} = Q_pGL_n$$

(ii)

$$Q_pGF_n + Q_pGF_{n-1} = Q_pGF_{n+1}$$

(iii)

$$Q_pGF_{n+2} - Q_pGF_{n-2} = Q_pGL_n$$

(iv)

$$Q_pGF_n - Q_pGF_{n+1}\sigma_1 - Q_pGF_{n+2}\sigma_2 - Q_pGF_{n+3}\sigma_3 = GF_n - GF_{n+2} - GF_{n+4} - GF_{n+6}$$

Proof. **(i)** Considering Equation (1) and using the identity $GF_{n+1} + GF_{n-1} = GL_n$ [3], we have the proof as follows

$$\begin{aligned} Q_pGF_{n+1} + Q_pGF_{n-1} &= (GF_{n+1} + GF_{n-1}).1 + (GF_{n+2} + GF_n)\sigma_1 \\ &\quad +(GF_{n+3} + GF_{n+1})\sigma_2 + (GF_{n+4} + GF_{n+2})\sigma_3 \\ &= GL_n 1 + GL_{n+1}\sigma_1 + GL_{n+2}\sigma_2 + GL_{n+3}\sigma_3 \\ &= Q_pGL_n. \end{aligned}$$

(ii) If we use Equation (1) and the recurrence relation of the Gaussian Fibonacci numbers, the proof can be easily seen.

(iii) Using Equation (1) and the recurrence relation of the Gaussian Fibonacci numbers, we obtain

$$\begin{aligned} Q_pGF_{n+2} - Q_pGF_{n-2} &= (GF_{n+1} + GF_{n-1}).1 + (GF_{n+2} + GF_n)\sigma_1 \\ &\quad +(GF_{n+3} + GF_{n+1})\sigma_2 + (GF_{n+4} + GF_{n+2})\sigma_3. \end{aligned}$$

By substituting the identity $GF_{n+1} + GF_{n-1} = GL_n$ [3] into the previous equation we get

$$Q_pGF_{n+2} - Q_pGF_{n-2} = Q_pGL_n.$$

(iv) Multiplying both sides of the Pauli Gaussian Fibonacci quaternions Q_pGF_{n+1}, Q_pGF_{n+2}, Q_pGF_{n+3} by $-\sigma_1$, $-\sigma_2$ and $-\sigma_3$ respectively gives

$$\begin{aligned} -Q_pGF_{n+1}\sigma_1 - Q_pGF_{n+2}\sigma_2 - Q_pGF_{n+3}\sigma_3 &= -GF_{n+1}\sigma_1 - GF_{n+2}1 + iGF_{n+3}\sigma_3 - iGF_{n+4}\sigma_2 \\ &\quad - GF_{n+2}\sigma_2 - iGF_{n+3}\sigma_3 - GF_{n+4}1 + iGF_{n+5}\sigma_1 \\ &\quad - GF_{n+3}\sigma_3 + iGF_{n+4}\sigma_2 - iGF_{n+5}\sigma_1 - GF_{n+6}1 \\ &= -GF_{n+1}\sigma_1 - GF_{n+2}\sigma_2 - GF_{n+3}\sigma_3 \\ &\quad - (GF_{n+2} + GF_{n+4} + GF_{n+6})1. \end{aligned}$$

Then, adding the above equation with Q_pGF_n yields

$$\begin{aligned} &Q_pGF_n - Q_pGF_{n+1}\sigma_1 - Q_pGF_{n+2}\sigma_2 - Q_pGF_{n+3}\sigma_3 \\ &= GF_n1 + GF_{n+1}\sigma_1 + GF_{n+2}\sigma_2 + GF_{n+3}\sigma_3 - GF_{n+1}\sigma_1 - GF_{n+2}\sigma_2 - GF_{n+3}\sigma_3 \\ &\quad -(GF_{n+2} + GF_{n+4} + GF_{n+6})1 \\ &= (GF_n - GF_{n+2} - GF_{n+4} - GF_{n+6})1. \end{aligned}$$

□

Theorem 2 (Honsberger's Identity). *For $n, m \geq 0$ and GF_n, the Honsberger identity for the Pauli Gaussian Fibonacci quaternions is given by*

$$Q_pGF_n \times Q_pGF_m + Q_pGF_{n+1} \times Q_pGF_{m+1} = \big(2Q_pGF_{n+m} + 9F_{n+m+1} + 5F_{n+m+2}\big)(1+2i).$$

Proof. By the Equations (1) and (2), weget

$$\begin{aligned} &Q_pGF_n \times Q_pGF_m + Q_pGF_{n+1} \times Q_pGF_{m+1} \\ &= [(GF_n.GF_m + GF_{n+1}.GF_{m+1}) + (GF_{n+2}.GF_{m+2} + GF_{n+3}.GF_{m+3}) \\ &\qquad +(GF_{n+1}.GF_{m+1} + GF_{n+2}.GF_{m+2}) + (GF_{n+3}.GF_{m+3} + GF_{n+4}.GF_{m+4})]1 \\ &\quad +[(GF_{n+1}.GF_m + GF_{n+2}.GF_{m+1}) + (GF_n.GF_{m+1} + GF_{n+1}.GF_{m+2}) \\ &\qquad +i(GF_{n+2}.GF_{m+3} + GF_{n+3}.GF_{m+4}) - i(GF_{n+3}.GF_{m+2} + GF_{n+4}.GF_{m+3})]\sigma_1 \\ &\quad +[(GF_{n+2}.GF_m + GF_{n+3}.GF_{m+1}) + (GF_n.GF_{m+2} + GF_{n+1}.GF_{m+3}) \\ &\qquad +i(GF_{n+3}.GF_{m+1} + GF_{n+4}.GF_{m+2}) - i(GF_{n+1}.GF_{m+3} + GF_{n+2}.GF_{m+4})]\sigma_2 \\ &\quad +[(GF_{n+3}.GF_m + GF_{n+4}.GF_{m+1}) + (GF_n.GF_{m+3} + GF_{n+1}.GF_{m+4}) \\ &\qquad +i(GF_{n+1}.GF_{m+2} + GF_{n+2}.GF_{m+3}) - i(GF_{n+2}.GF_{m+1} + GF_{n+3}.GF_{m+2})]\sigma_3. \end{aligned}$$

Using the identity $GF_nGF_m + GF_{n+1}GF_{m+1} = F_{n+m}(1+2i)$ [3], we obtain

$$\begin{aligned}&Q_pGF_n \times Q_pGF_m + Q_pGF_{n+1} \times Q_pGF_{m+1}\\&= ((F_{n+m} + F_{n+m+2} + F_{n+m+4} + F_{n+m+6}).1\\&+2(F_{n+m+1}\sigma_1 + F_{n+m+2}\sigma_2 + F_{n+m+3}\sigma_3))(1+2i).\end{aligned}$$

If the necessary arrangements are made in the last equation, we have

$$Q_pGF_n \times Q_pGF_m + Q_pGF_{n+1} \times Q_pGF_{m+1} = \left(2Q_pGF_{n+m} + 9F_{n+m+1}1 + 5F_{n+m+2}1\right)(1+2i).$$

Thus, the claim is verified. □

Theorem 3 (Generating Function)**.** *The generating functions of the Pauli Gaussian Fibonacci quaternions and Pauli Gaussian Lucas quaternions are as follows:*

$$g(x) = \sum_{n=0}^{\infty} Q_pGF_n.x^n = \frac{Q_pGF_0 + (Q_pGF_1 - Q_pGF_0)x}{1-x-x^2}$$

and

$$h(x) = \sum_{n=0}^{\infty} Q_pGL_n.x^n = \frac{Q_pGL_0 + (Q_pGL_1 - Q_pGL_0)x}{1-x-x^2}$$

respectively.

Proof. Let us use the definition of a generating function of Q_pGF_n as follows

$$g(x) = Q_pGF_0 + Q_pGF_1.x + Q_pGF_2.x^2 + \cdots + Q_pGF_n.x^n + \ldots \quad (3)$$

Multiplying both sides of the Equation (3) by $-x$ and $-x^2$ gives

$$-xg(x) = -Q_pGF_0.x - Q_pGF_1.x^2 - Q_pGF_2.x^3 - \cdots + Q_pGF_n.x^{n+1} - \ldots \quad (4)$$

and

$$-x^2g(x) = -Q_pGF_0.x^2 - Q_pGF_1.x^3 - Q_pGF_2.x^4 - \cdots + Q_pGF_n.x^{n+2} - \ldots \quad (5)$$

If we add the Equations (3)–(5) and use Theorem 1, we conclude that

$$\left(1-x-x^2\right)g(x) = Q_pGF_0 + (Q_pGF_1 - Q_pGF_0)x$$

Then, we write

$$g(x) = \sum_{n=0}^{\infty} Q_pGF_n.x^n = \frac{Q_pGF_0 + (Q_pGF_1 - Q_pGF_0)x}{1-x-x^2}.$$

Let us write the generating function of Q_pGL_n as follows

$$h(x) = Q_pGL_0 + Q_pGL_1.x + Q_pGL_2.x^2 + \cdots + Q_pGL_n.x^n + \ldots \quad (6)$$

The proof can be easily seen if we apply a similar method used to prove the generating function for the Pauli Gaussian Fibonacci quaternions to the Equation (6). □

Now, we will obtain the Binet formulas, which give us the n-th Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions, respectively.

Theorem 4 (Binet's Formula). ***(i)*** *For $n \geq 1$, Binet formula of the Pauli Gaussian Fibonacci quaternions is given by*

$$Q_pGF_n = c\alpha^n\hat{\alpha} + d\beta^n\hat{\beta}$$

where $c = \frac{1-\beta i}{\alpha-\beta}$, $d = \frac{-1+\alpha i}{\alpha-\beta}$, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
(ii) *For $n \in \mathbb{N}$, the Binet formula of the Pauli Gaussian Lucas quaternions is given by*

$$Q_pGL_n = Q_PF_nGL_1 + Q_PF_{n-1}GL_0.$$

This last formula gives us the relationship between Pauli Fibonacci quaternions and Gaussian Lucas numbers.

Proof. **(i)** Applying the Binet's formula of the Gaussian Fibonacci to Q_pGF_n, we get

$$\begin{aligned} Q_pGF_n = &\left(c\alpha^n\hat{\alpha} + d\beta^n\hat{\beta}\right)1 + \left(c\alpha^{n+1}\hat{\alpha} + d\beta^{n+1}\hat{\beta}\right)\sigma_1 \\ &+ \left(c\alpha^{n+2}\hat{\alpha} + d\beta^{n+2}\hat{\beta}\right)\sigma_2 + \left(c\alpha^{n+3}\hat{\alpha} + d\beta^{n+3}\hat{\beta}\right)\sigma_3. \end{aligned} \tag{7}$$

If Equation (7) is arranged, we have

$$\begin{aligned} Q_pGF_n &= c\alpha^n\left(1 + \alpha\sigma_1 + \alpha^2\sigma_2 + \alpha^3\sigma_3\right) + d\beta^n\left(1 + \beta\sigma_1 + \beta^2\sigma_2 + \beta^3\sigma_3\right) \\ Q_pGF_n &= c\alpha^n\hat{\alpha} + d\beta^n\hat{\beta}. \end{aligned}$$

such that

$$\hat{\alpha} = 1 + \alpha\sigma_1 + \alpha^2\sigma_2 + \alpha^3\sigma_3$$

and

$$\hat{\beta} = 1 + \beta\sigma_1 + \beta^2\sigma_2 + \beta^3\sigma_3.$$

(ii) Applying Binet's formula of the Gaussian Lucas to Q_pGL_n, we get

$$\begin{aligned} Q_pGL_n = &\left(\tfrac{\alpha^n-\beta^n}{\alpha-\beta}GL_1 + \tfrac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}GL_0\right)1 \\ &+ \left(\tfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}GL_1 + \tfrac{\alpha^n-\beta^n}{\alpha-\beta}GL_0\right)\sigma_1 \\ &+ \left(\tfrac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}GL_1 + \tfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}GL_0\right)\sigma_2 \\ &+ \left(\tfrac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}GL_1 + \tfrac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}GL_0\right)\sigma_3. \end{aligned} \tag{8}$$

Equation (8) can be stated in terms of Fibonacci numbers as follows:

$$\begin{aligned} Q_pGL_n &= (F_n1 + F_{n+1}\sigma_1 + F_{n+2}\sigma_2 + F_{n+3}\sigma_3)GL_1 \\ &\quad + (F_{n-1}1 + F_n\sigma_1 + F_{n+1}\sigma_2 + F_{n+2}\sigma_3)GL_0 \\ &= Q_pF_nGL_1 + Q_pF_{n-1}GL_0. \end{aligned}$$

□

Example 1. *Let Q_pGF_2 be Pauli Gaussian Fibonacci quaternion. Applying Theorem 4 for $n = 2$, we get*

$$\begin{aligned} Q_pGF_2 &= \tfrac{(1-\beta i)\alpha^2\left(1+\alpha\sigma_1+\alpha^2\sigma_2+\alpha^3\sigma_3\right)+(-1+\alpha i)\beta^2\left(1+\beta\sigma_1+\beta^2\sigma_2+\beta^3\sigma_3\right)}{\sqrt{5}} \\ &= (1+i)1 + (2+i)\sigma_1 + (3+2i)\sigma_2 + (5+3i)\sigma_3. \end{aligned}$$

Furthermore, the above Pauli Gaussian Fibonacci quaternion is written by

$$\begin{aligned} Q_pGF_2 &= (11 + 2\sigma_1 + 3\sigma_2 + 5\sigma_3) + i(11 + 1\sigma_1 + 2\sigma_2 + 3\sigma_3) \\ Q_pGF_2 &= Q_pF_2 + iQ_pF_1. \end{aligned}$$

Notice that the real and the imaginary parts of the Pauli Gaussian Fibonacci quaternion correspond Pauli Fibonacci quaternions for $n = 2$ and $n = 1$.

Theorem 5 (d'Ocagne's Identity). *For $n, m \geq 0$, the following identity holds*

$$Q_pGF_m \times Q_pGF_{n+1} - Q_pGF_{m+1} \times Q_pGF_n = \left(\frac{i-2}{\sqrt{5}}\right)\left[\beta^m\alpha^n\hat{\beta}\hat{\alpha} - \beta^m\alpha^n\hat{\alpha}\hat{\beta}\right].$$

Proof. Considering the Binet formula in Theorem 4 and making some necessary calculations, the following expression is obtained.

$$\begin{aligned}&Q_pGF_m \times Q_pGF_{n+1} - Q_pGF_{m+1} \times Q_pGF_n \\ &= (cd)\left[\left(\beta^m\alpha^{n+1} - \beta^{m+1}\alpha^n\right)\hat{\beta}\hat{\alpha} + \left(\alpha^m\beta^{n+1} - \alpha^{m+1}\beta^n\right)\hat{\alpha}\hat{\beta}\right] \\ &= (cd)\left[\beta^m\alpha^n(\alpha-\beta)\hat{\beta}\hat{\alpha} - \beta^m\alpha^n(\alpha-\beta)\hat{\alpha}\hat{\beta}\right].\end{aligned}$$

To achieve our purpose, we now put the values $cd = dc = \frac{i-2}{5}$, $\alpha - \beta = \sqrt{5}$ in the above equality. Thus, the proof is completed. □

Theorem 6. *(Catalan's Identity) For $n \geq 1$, the Catalan identity for the Pauli Gaussian Fibonacci quaternions is*

$$\begin{aligned}&Q_pGF_n^2 - Q_pGF_{n+r} \times Q_pGF_{n-r} \\ &= (-1)^{n+1}\tfrac{(2-i)}{5}\left[\left(1 - \left(\tfrac{-3+\sqrt{5}}{2}\right)^r\right)\hat{\beta}\hat{\alpha} + \left(1 - \left(\tfrac{-3-\sqrt{5}}{2}\right)^r\right)\hat{\alpha}\hat{\beta}\right].\end{aligned}$$

Proof. Using the Binet formula for Pauli Gaussian Fibonacci quaternions, we have

$$\begin{aligned}&Q_pGF_n^2 - Q_pGF_{n+r} \times Q_pGF_{n-r} \\ &= \left(c\alpha^n\hat{\alpha} + d\beta^n\hat{\beta}\right)\left(c\alpha^n\hat{\alpha} + d\beta^n\hat{\beta}\right) - \left(c\alpha^{n+r}\hat{\alpha} + d\beta^{n+r}\hat{\beta}\right)\left(c\alpha^{n-r}\hat{\alpha} + d\beta^{n-r}\hat{\beta}\right) \\ &= dc\left((\beta\alpha)^n - \beta^{n+r}\alpha^{n-r}\right)\hat{\beta}\hat{\alpha} + cd\left((\alpha\beta)^n - \alpha^{n+r}\beta^{n-r}\right)\hat{\alpha}\hat{\beta}.\end{aligned}$$

Note that we have the following identities

$$\begin{aligned}\hat{\alpha} &= 1 + \left(\tfrac{1+\sqrt{5}}{2}\right)\sigma_1 + \left(\tfrac{3+\sqrt{5}}{2}\right)\sigma_2 + \left(2+\sqrt{5}\right)\sigma_3, \\ \hat{\beta} &= 1 + \left(\tfrac{1-\sqrt{5}}{2}\right)\sigma_1 + \left(\tfrac{3-\sqrt{5}}{2}\right)\sigma_2 + \left(2-\sqrt{5}\right)\sigma_3, \\ \hat{\alpha}\hat{\beta} &= \left(1-\sqrt{5}i\right)\sigma_1 + \left(3-\sqrt{5}i\right)\sigma_2 + \left(4+\sqrt{5}i\right)\sigma_3, \\ \hat{\beta}\hat{\alpha} &= \left(1+\sqrt{5}i\right)\sigma_1 + \left(3+\sqrt{5}i\right)\sigma_2 + \left(4-\sqrt{5}i\right)\sigma_3.\end{aligned}$$

□

It is well-known that if $r = 1$ the Cassini identity corresponds to the Cassini identity. Thus, the following corollary can be given.

Corollary 1. *For $n \geq 1$, the Cassini identity for the Pauli Gaussian Fibonacci quaternions is*

$$Q_pGF_n^2 - Q_pGF_{n+1} \times Q_pGF_{n-1} = (-1)^{n+1}\frac{(2-i)}{5}\left[\left(\frac{5-\sqrt{5}}{2}\right)\hat{\beta}\hat{\alpha} + \left(\frac{5+\sqrt{5}}{2}\right)\hat{\alpha}\hat{\beta}\right].$$

Example 2. *Let Q_pGF_5, Q_pGF_3 and Q_pGF_1 be Pauli Gaussian Fibonacci quaternions. If we consider Theorem 6 for $n = 3$ and $r = 2$, the calculations give the following equality*

$$\begin{aligned}&Q_pGF_3^2 - Q_pGF_5 \times Q_pGF_1 \\ &= (-1)^4\tfrac{(2-i)}{5}\Big[\left(\tfrac{-5+3\sqrt{5}}{2}\right)\left(\left(1+\sqrt{5}i\right)\sigma_1 + \left(3+\sqrt{5}i\right)\sigma_2 + \left(4-\sqrt{5}i\right)\sigma_3\right) \\ &\qquad + \left(\tfrac{-5-3\sqrt{5}}{2}\right)\left(\left(1-\sqrt{5}i\right)\sigma_1 + \left(3-\sqrt{5}i\right)\sigma_2 + \left(4+\sqrt{5}i\right)\sigma_3\right)\Big] \\ &= \tfrac{(2-i)}{5}[(-5+15i)\sigma_1 + (-15+15i)\sigma_2 + (-20-15i)\sigma_3] \\ &= (2-i)[(-1+3i)\sigma_1 + (-3+3i)\sigma_2 + (-4-3i)\sigma_3].\end{aligned}$$

Catalan identity for $n = 3$ and $r = 2$ obtained from Aydın (see [38]), we found

$$\begin{aligned} Q_pF_3^2 - Q_pF_5 \times Q_pF_1 &= (-1)^1 F_2[(1-i)\sigma_1 + (3-i)\sigma_2 + (4+i)\sigma_3] \\ Q_pF_3^2 - Q_pF_5 \times Q_pF_1 &= -[(1-i)\sigma_1 + (3-i)\sigma_2 + (4+i)\sigma_3]. \end{aligned}$$

where Q_pF_1, Q_pF_3 and Q_pF_5 are Pauli Fibonacci quaternions.

Example 3. *Let Q_pGF_3, Q_pGF_2 and Q_pGF_1 be Pauli Gaussian Fibonacci quaternions. If we consider Corollary for $n = 3$ and $r = 1$, the calculations give the following equality*

$$\begin{aligned} & Q_pGF_3^2 - Q_pGF_4 \times Q_pGF_2 \\ & = (-1)^4 \frac{(2-i)}{5}\Big[\Big(\frac{5-\sqrt{5}}{2}\Big)\Big(\Big(1+\sqrt{5}i\Big)\sigma_1 + \Big(3+\sqrt{5}i\Big)\sigma_2 + \Big(4-\sqrt{5}i\Big)\sigma_3\Big) \\ & \qquad + \Big(\frac{5+\sqrt{5}}{2}\Big)\Big(\Big(1-\sqrt{5}i\Big)\sigma_1 + \Big(3-\sqrt{5}i\Big)\sigma_2 + \Big(4+\sqrt{5}i\Big)\sigma_3\Big)\Big] \\ & = \frac{(2-i)}{5}[(5-5i)\sigma_1 + (15-5i)\sigma_2 + (20+5i)\sigma_3] \\ & = (2-i)[(1-i)\sigma_1 + (3-i)\sigma_2 + (4+i)\sigma_3]. \end{aligned}$$

On the other hand, if we consider the Cassini identity for $n = 3$ and $r = 1$ obtained from Aydın (see [38]), we found

$$\begin{aligned} Q_pF_3^2 - Q_pF_4 \times Q_pF_2 &= (-1)^2 F_1[(1-i)\sigma_1 + (3-i)\sigma_2 + (4+i)\sigma_3] \\ Q_pF_3^2 - Q_pF_4 \times Q_pF_2 &= (1-i)\sigma_1 + (3-i)\sigma_2 + (4+i)\sigma_3. \end{aligned}$$

where Q_pF_1, Q_pF_2 and Q_pF_3 are Pauli Fibonacci quaternions.

3. Conclusions

In this research, we have extended the quaternions in [38] to the complex case by taking the components of Gaussian Fibonacci and Gaussian Lucas numbers. We have obtained some identities and formulas for these special quaternions, which are specific to Fibonacci quaternions. On the other hand, Pauli matrices and Pauli quaternions have applications in many areas, including quantum mechanics and quantum field theory. We believe that it will be a resource for researchers working in these fields.

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References

1. Gauss, C.F. *Theoria Residuorum Biquadraticorum*; Commentatio Prima Sumtibus Dieterichtianis, Gottingae; Cambridge University Press: Cambridge, UK, 1832.
2. Horadam, A.F. Complex Fibonacci Numbers and Fibonacci Quaternions. *Am. Math. Mon.* **1963**, *70*, 289–291. [CrossRef]
3. Jordan, J.H. Gaussian Fibonacci and Lucas Numbers. *Fibonacci Quart.* **1965**, *3*, 315–318.
4. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Wiley-Interscience Publication: New York, NY, USA, 2001.
5. Berzsenyi, G. Gaussian Fibonacci Numbers. *Fibonacci Quart.* **1977**, *15*, 233–236.
6. Mc Carthy, J.M. *Introduction to Theoretical Kinematics*; MIT Press: Cambridge, MA, USA, 1990.
7. Shoemake, K. Animation Rotation with Quaternion Curves. In Proceedings of the SIGGRAPH '85: 12th Annual Conference on Computer Graphics and Interactive Techniques, San Francisco, CA, USA, 22–26 July 1985; Volume 19, pp. 245–254.
8. Silberstein, L. LXXVI. Quaternionic Form of Relativity. *Lond. Edinb. Dubl. Phil. Mag.* **1912**, *23*, 790–809. [CrossRef]
9. Blake, C.S. Sporadic SICs and the Normed Division Algebras. *Found. Phys.* **2017**, *47*, 1060–1064.
10. Wu, J.; Sun, Y.; Wang, M.; Liu, M. Hand-eye Calibration: 4-D procrustes Analysis Approach. *IEEE Trans. Instrum. Meas.* **2019**, *69*, 2966–2981. [CrossRef]
11. Kopp, M.; Kreil, D.; Neun, M.; Jonietz, D.; Martin, H.; Herruzo, P.; Gruca, A.; Soleymani, A.; Wu, F.; Liu, Y.; et al. Traffic4cast at Neurips 2020—Yet More on the Unreasonable Effectiveness of Gridded Geo-spatial Processes. In Proceedings of the NeurIPS 2020 Competition and Demonstration Track, Virtual Event, 6–12 December 2021; Escalante, H.J., Hofmann, K., Eds.; Volume 133, pp. 325–343.

12. Saoud, L.S.; Al-Marzouqi, H. Metacognitive Sedenion-Valued Neural Network and Its Learning Algorithm. *IEEE Access* **2020**, *8*, 144823–144838. [CrossRef]
13. Lidl, R.; Niederreiter, H. *Introduction to Finite Fields and Their Applications*; Cambridge University Press: Cambridge, UK, 1994.
14. Akkus, I.; Kızılaslan, G. On Some Properties of Tribonacci Quaternions. *An. Şiinţ. Univ. Ovidius Constanta* **2018**, *26*, 5–20. [CrossRef]
15. Akyiğit, M.; Hidayet, H.K.; Tosun, M. Fibonacci Generalized Quaternions. *Adv. Appl. Clifford Algebr.* **2014**, *24*, 631–641. [CrossRef]
16. Bilgici, G.; Tokeşer, Ü.; Ünal, Z. Fibonacci and Lucas Sedenions. *J. Integer Seq.* **2017**, *20*, 1–11.
17. Catarino, P.; Campos, H. From Fibonacci Sequence to More Recent Generalisations. In *Mathematics and Its Applications in Science and Engineering*; Springer: Cham, Swiztherland, 2022.
18. Cerda-Morales, G. On a Generalization for Tribonacci Quaternions. *Mediterr. J. Math.* **2017**, *14*, 239. [CrossRef]
19. Cerda-Morales, G. The Unifying Formula for All Tribonacci-type Octonions Sequences and Their Properties. *Konuralp J. Math.* **2019**, *7*, 292–299.
20. Cereceda, J.L. Binet's formula for generalized tribonacci numbers. *Int. J. Math. Edu. Sci. Technol.* **2015**, *46*, 1235–1243. [CrossRef]
21. Flaut, C.; Shpakivskyi, V. On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions. *Adv. Appl. Clifford Algebr.* **2013**, *23*, 673–688. [CrossRef]
22. Halici, S.; Karatas, A. On a Generalizaton for Fibonacci Quaternions. *Chaos Solitons Fractals* **2017**, *98*, 178–182. [CrossRef]
23. Halici, S. On Fibonacci Quaternions. *Adv. Appl. Clifford Algebr.* **2017**, *22*, 321–327. [CrossRef]
24. Iyer, M.R. Some Results on Fibonacci Quaternions. *Fibonacci Quart.* **1969**, *7*, 201–210.
25. Karataş, A.; Halıcı, S. Horadam Octonions. *An. Şiinţ. Univ. Ovidius Constanta* **2017**, *25*, 97–106. [CrossRef]
26. Kızılateş, C.; Kirlak, S. A New Generalization of Fibonacci and Lucas Type Sedenions. *J. Discret. Math. Sci. Cryptogr.* **2022**, 1–12. [CrossRef]
27. Savin, D. Some Properties of Fibonacci Numbers, Fibonacci Octonions, and Generalized Fibonacci-Lucas Octonions. *Adv. Differ. Equ.* **2015**, *2015*, 298. [CrossRef]
28. Soykan, Y. Tribonacci and Tribonacci-Lucas Sedenions. *Mathematics* **2019**, *7*, 74. [CrossRef]
29. Soykan, Y. Tetranacci and Tetranacci-Lucas Quaternions. *Asian Res. J. Math.* **2019**, *15*, 1–24. [CrossRef]
30. Soykan, Y.; Okumuş, İ.; Taşdemir, E. On Generalized Tribonacci Sedenions. *Sarajevo J. Math.* **2020**, *16*, 103–122.
31. Soykan, Y.; Özmen, N.; Göcen, M. On Generalized Pentanacci Quaternions. *Tbil. Math. J.* **2020**, *13*, 169–181. [CrossRef]
32. Spickerman, W.R. Binet's Formula for the Tribonacci Sequence. *Fibonacci Quart.* **1982**, *20*, 118–120.
33. Halici, S. On Complex Fibonacci Quaternions. *Am. Math. Mon.* **2013**, *23*, 105–112. [CrossRef]
34. Halici, S. On Quaternion-Gaussian Lucas Numbers. *Math. Meth. Appl. Sci.* **2021**, *44*, 7601–7606. [CrossRef]
35. Halici, S.; Cerda-Morales, G. On Quaternion-Gaussian Fibonacci Numbers and Their Properties. *An. Şiinţ. Univ. Ovidius Constanta* **2021**, *29*, 71–82. [CrossRef]
36. Condon, E.U.; Morse, P.M. *Quantum Mechanics*; McGraw-Hill: New York, NY, USA, 1929.
37. Kim, J.E. A Representation of de Moivre's Formula Over Pauli Quaternions. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **2017**, *9*, 145–151.
38. Aydın, F.T. Pauli-Fibonacci Quaternions. *Notes Number Theory Discret. Math.* **2021**, *27*, 184–193. [CrossRef]
39. Bajorska-Harapińska, B.; Smoleń, B.; Wituła, R. On Quaternion Equivalents for Quasi-Fibonacci Numbers, Shortly Quaternaccis. *Adv. Appl. Clifford Algebr.* **2019**, *29*, 54. [CrossRef]