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Article

A Two-Dimensional port-Hamiltonian Model for Coupled Heat Transfer

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Abstract: In this paper, we construct a highly simplified mathematical model for studying the problem of conjugate heat transfer in gas turbine blades and their cooling ducts. Our simple model focuses on the relevant coupling structures and aims to reduce the unrelated complexity as much as possible. Then, we apply the port-Hamiltonian formalism to this model and its subsystems and investigate the interconnections. Finally, we apply a simple spatial discretization to the system to investigate the properties of the resulting finite-dimensional port-Hamiltonian system and to determine whether the order of coupling and discretization affect the resulting semi-discrete system.

Keywords: port-Hamiltonian system; conjugate heat transfer; coupled system; thermodynamics

MSC: 80A19; 93A99

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1. Introduction

The role of gas turbines in the power grid will most likely change as the share of renewable energy sources continues to rise. Their short start-up times and high efficiency make them well suited as backup power plants, and leading manufacturers are working on new technologies to make them suitable for use in an energy storage system, in which they would run on hydrogen or synthetic methane. This brings new requirements and places new demands on the design process. While there is an almost overwhelming amount of engineering research and development, surprisingly little has been conducted on the mathematics for gas turbines. A mathematical approach to the turbine blade design is being attempted as part of the GivEn project [1], which aims to combine multiphysics simulations with multicriteria shape optimization.

One of the physical processes that must be considered to obtain useful results from shape optimization is heat transfer within the turbine blade. Since gas turbines operate at extreme temperatures for efficiency reasons—often close to or even above the nominal melting point of the alloy used for the turbine blades—measures must be taken to protect the blade from the 1200 °C to 1500 °C of the combustion gas surrounding it. One method is to insert small cooling channels into the turbine blade, filled with a continuous stream of relatively cool air, to cool the blade by convection cooling from the inside. The shape, arrangement, and wall structure of these channels are themselves the subject of extensive engineering research, as described, for example, in [2,3]. Because the flow in these cooling channels is intentionally kept highly turbulent to optimize heat transfer, it is difficult to simulate the flow explicitly. While it is possible, it is usually too sensitive and costly to do so as part of a multiphysics simulation. Instead, in most cases, a parametric one-dimensional model is used. Although quite dated, [4,5] gives a reasonable overview of the basics of such a one-dimensional model. For more details on the background of port-Hamiltonian systems and their variety of applications, we refer the interested reader to [6–9].

Combining these cooling channels with the heat transfer inside the turbine blade, we obtain the so-called *conjugate heat transfer* (CHT) problem, i.e., strong thermal interactions between solids and fluids. Although [10] focuses on the coupling with the hot fluid surrounding the blade and not with the internal cooling fluid, both problems belong to the same large group. Alternatively, [11] considers both the external and internal fluids, but places little emphasis on coupling.

In this paper, we present a highly simplified model of conjugate heat transfer involving the turbine blade and a cooling channel. While this model is too simplified to be useful for actual engineering purposes, it is intended to represent the coupling structure between the turbine blade and a cooling channel and to allow us to study this coupling without having to deal with other engineering difficulties that might cloud the results. We extend and improve on the work conducted in [12,13], in which we had considered a one-dimensional model for heat conduction within the blade metal, which led to strange and undesirable behaviors and properties of the system. Instead, we will consider a two-dimensional heat equation and investigate whether this eliminates the problems of the one-dimensional model. We then formulate the model system as an infinite-dimensional *port-Hamiltonian system* (*pHS*) and apply a spatial discretization to obtain a finite-dimensional pHS. Port-Hamiltonian systems are closely related to the Hamiltonian formalism, which was originally developed in theoretical physics, and are therefore well suited for modeling physical systems. The formalism makes conservation laws, a property central to virtually all physical systems, explicit and allows the construction of new port-Hamiltonian systems by connecting two pHSs with a suitable coupling. For a formal discussion of infinite-dimensional pHSs with boundary flow, see [14]. It also allows time discretization schemes that preserve the conservation laws of the continuous system [15]. The proposed discretization is a discrete form of the port-Hamiltonian system and hence we immediately obtain a discrete form of the dissipativity of the scheme and hence the stability.

The present paper is organized as follows. In Section 2, we introduce and motivate the mathematical model of the coupled system we study. In Section 3, we present a port-Hamiltonian formulation for each of the subsystems and study the coupling structure of their connection to determine whether the connection of these two subsystems forms a pHS for the overall system. Section 4 will contain a spatial discretization of the pHS formulated in Section 3. Here, we will investigate whether the resulting semi-discrete systems form finite-dimensional port Hamiltonian systems and whether there is a difference between the coupling of the discretized systems and the discretization of the coupled system. Finally, we will summarize the results in Section 5, outline open questions, and make some concluding remarks.

2. The Model System

Here, we introduce the mathematical model of the coupled system under investigation. Let $\Omega_m = (0,1) \times (0,1) \subset \mathbb{R}^2$ denote the spatial domain of the blade metal. The heat equation on Ω_m is given by

$$\frac{\partial T}{\partial t}(x,y,t) = \frac{1}{c_m \rho_m} \operatorname{div}\big(\lambda \operatorname{grad} T(x,y,t)\big) \quad \text{for } (x,y) \in \Omega_m, \tag{1}$$

where λ is the material's conductivity and ρ_m is the density of the blade metal. Here, c_m denotes the *isochoric specific heat capacity* of the metal, i.e., the specific heat capacity at constant volume. Analogous definitions hold for ρ_c, c_c and the cooling channel. In Figure 1, we give a rough sketch of the model setting.

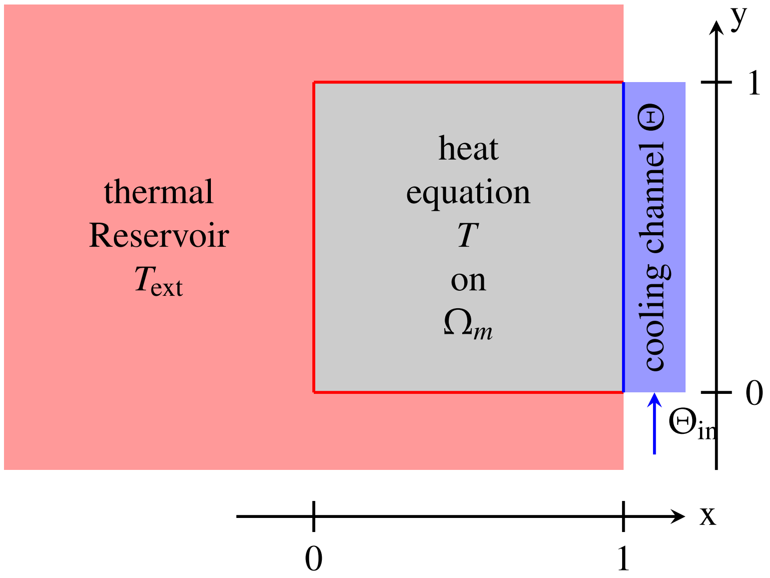

Figure 1. Schematic of the 2D model system with $\partial\Omega_{ext}$ marked as a red line and $\partial\Omega_c$ as a blue line.

For alternative (variational) formulations of the heat equation, we refer the reader to the classical work of Goodman [16] and especially the approach of Biot [17].

The left, upper and lower boundary ($x = 0$, $y = 0$ and $y = 1$) should be in contact with a thermal reservoir with a given temperature T_{ext}, leading to a Robin-BC:

$$-\lambda \frac{\partial T}{\partial x}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big) \quad \text{for } x = 0,\ y \in [0,1] \tag{2a}$$
$$-\lambda \frac{\partial T}{\partial y}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big) \quad \text{for } x \in (0,1),\ y = 0 \tag{2b}$$
$$\lambda \frac{\partial T}{\partial y}(x,y,t) = h_0\big(T_{\text{ext}}(t) - T(x,y,t)\big) \quad \text{for } x \in (0,1),\ y = 1, \tag{2c}$$

where h_0 denotes the heat flux coefficient across the boundary to the thermal reservoir. In the following, we denote these parts of the boundary with $\partial\Omega_{ext}$. The right boundary ($x = 1$), which is in contact with the cooling channel, will be denoted by $\partial\Omega_c$, so that $\partial\Omega_m = \partial\Omega_{ext} \cup \partial\Omega_c$.

For $\partial\Omega_c$, we then have the boundary condition

$$-\lambda \frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big) \quad \text{for } (1,y) \in \partial\Omega, \tag{3}$$

where h_1 is the heat flux coefficient of the interface to the cooling channel and Θ denotes the temperature of the cooling channel and is governed by a transport equation with constant velocity v and an additional source term describing the heat flux into the channel:

$$\frac{\partial \Theta}{\partial t}(y,t) = -v\frac{\partial \Theta}{\partial y}(y,t) + \frac{h_1}{c_c\rho_c}\big(T(1,y,t) - \Theta(y,t)\big), \tag{4}$$
$$\Theta(0,t) = \Theta_{\text{in}}(t). \tag{5}$$

3. Port-Hamiltonian Formulation

In this section, we formulate port-Hamiltonian systems for each of the two subsystems. To this end, we will use quadratic Hamiltonians (referred to as the *Lyapunov formulation* in [18]) rather than physical (thermodynamic) energy for two reasons. First, the resulting boundary ports of the heat equation will involve measurable quantities relevant in practice. Second, and more importantly, the transport equation causes problems with a non-quadratic Hamiltonian.

3.1. Heat Equation

For the heat equation in the metal, we choose the Hamiltonian

$$H(t) = \frac{1}{2} \int_{\Omega_m} \frac{1}{\rho(z) c_m(z)} q(t,z)^2 \, \mathrm{d}z \tag{6}$$

with $q(t,z)$ being the internal energy density, at point $z = (x,y)^\top$ and time t. We assume a Dulong-Petit-like model for q, i.e. $q(t,z) = \rho(z) c_m(z) T(t,z)$ with $T(t,z)$ the temperature. For further thermodynamic details, we refer the reader to [19]. These assumptions and choice of Hamiltonian are similar to those made in [18], called the "Lyapunov formulation".

Please note that the chosen Hamiltonian is neither the physical energy nor the physical entropy. While it is not a physically meaningful quantity, it still satisfies all properties required for a Hamiltonian. It also has a few advantages for our use case, such as nice boundary port variables and compatibility with the transport equation, as observed later on.

Choosing the Lebesgue space $L^2(\Omega_m)$ as state space and q as our state variable, we obtain the flow and effort variables

$$e_T = \delta_q H = T, \qquad f_T = \partial_t q = \rho c_m \partial_t T, \tag{7}$$

with δ_q denoting the variational derivative with regards to the internal energy density, q.

With the above mentioned assumptions, the first law of thermodynamics gives us

$$\partial_t q(t,z) = -\operatorname{div} \Phi_Q(t,z), \tag{8}$$

with the heat flux Φ_Q. From the (isotropic) Fourier's law, we have

$$\Phi_Q(t,z) = -\lambda \operatorname{grad} T(t,z). \tag{9}$$

Note that an anisotropic thermal conductivity would also be possible, as in [18], but would not add anything interesting to the model, while complicating the coupling formulation. Therefore, we introduce the additional flow and effort variables similar to [18]

$$e_Q = \Phi_Q, \quad f_Q = -\operatorname{grad} T, \tag{10}$$

to obtain the system of equations

$$\begin{pmatrix} f_T \\ f_Q \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} e_T \\ e_Q \end{pmatrix}, \tag{11}$$

$$e_Q = \lambda f_Q. \tag{12}$$

We can now calculate the time derivative of the Hamiltonian:

$$\begin{aligned} d_t H &= \int_{\Omega_m} \rho(z) c_m(z) \partial_t T(t,z) T(t,z) \, \mathrm{d}x \\ &= -\int_{\Omega_m} \operatorname{div}(\Phi_Q) T \, \mathrm{d}x \\ &= \int_{\Omega_m} \Phi_Q \operatorname{grad}(T) \, \mathrm{d}x - \int_{\partial\Omega_m} T \Phi_Q n \, \mathrm{d}\gamma \\ &= -\int_{\Omega_m} e_Q f_Q \, \mathrm{d}x - \int_{\partial\Omega_m} e_T(e_Q n) \, \mathrm{d}\gamma, \end{aligned} \tag{13}$$

recovering the same boundary port variables as [18], i.e., the temperature T and the heat flux *into* the system $-\Phi_Q \cdot n$. Both of these quantities are commonly used as boundary conditions in engineering applications. In the following, we present the special case of a Robin boundary condition involving both of them.

Since $\begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{pmatrix}$ is skew-adjoint on the space $L^2(\Omega_m) \times L^2(\Omega_m)^2$ with suitable boundary conditions, cf. e.g., [20], we can now formulate the Dirac structure $\mathcal{D}_m \subset (L^2(\Omega_m) \times L^2(\Omega_m)^2 \times L^2(\partial\Omega_m))^2$ for our system:

$$\mathcal{D}_m = \left\{ \left(\begin{pmatrix} f_T \\ f_Q \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_T \\ e_Q \\ e_\partial \end{pmatrix} \right) \in (L^2(\Omega_m) \times L^2(\Omega_m)^2 \times L^2(\partial\Omega_m))^2 \,\middle|\, e_T \in H^1(\Omega_m), \right.$$
$$\left. e_Q \in H(\operatorname{div}) \text{ and } \begin{pmatrix} f_T \\ f_Q \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} e_T \\ e_Q \end{pmatrix}, \begin{array}{l} e_\partial = e_T|_{\partial\Omega_m}, \\ f_\partial = e_Q \cdot n|_{\partial\Omega_m} \end{array} \right\}.$$

Here, $H^1(\Omega_m)$ denotes the classical Sobolev space and $H(\operatorname{div})$ consists of all functions f in $L^2(\Omega_m)$, which are weakly differentiable with $\operatorname{div} f \in L^2(\Omega_m)$ and $f \cdot n|_{\partial\Omega_m} \in L^2(\partial\Omega_m)$.

Since our model system from Section 2 has different boundary conditions on different parts of the boundary, we first split the single boundary port into two ports, one each for $\partial\Omega_{ext}$ and $\partial\Omega_c$, giving us

$$f_{\partial,e} = f_\partial|_{\partial\Omega_{ext}}, \quad e_{\partial,e} = e_\partial|_{\partial\Omega_{ext}}, \quad f_{\partial,c} = f_\partial|_{\partial\Omega_c}, \quad e_{\partial,c} = e_\partial|_{\partial\Omega_c}. \tag{14}$$

The latter part will then be coupled to the cooling channel later. To replicate the Robin boundary conditions (2), we set

$$e_Q n = \Phi_Q n = h_0(T - T_{\text{ext}}) \quad \text{on } \partial\Omega_{ext} \tag{15}$$

which is equivalent to the Dirac structure

$$\mathcal{D}_e = \left\{ \left(\begin{pmatrix} f_{\partial,e} \\ w \end{pmatrix}, \begin{pmatrix} e_{\partial,e} \\ u \end{pmatrix} \right) \in (L^2(\partial\Omega_{ext})^2)^2 \,\middle|\, \begin{pmatrix} f_{\partial,e} \\ w \end{pmatrix} = \begin{pmatrix} h_0 & -h_0 \\ h_0 & 0 \end{pmatrix} \begin{pmatrix} e_{\partial,e} \\ u \end{pmatrix} \right\}$$

with the new input $u = T_{\text{ext}}$ and the new output $w = h_0 T$.

The power balance of Equation (13) becomes

$$\begin{aligned} d_t H &= -\int_{\Omega_m} e_Q f_Q \,\mathrm{d}x - \int_{\partial\Omega_m} e_T(e_Q n) \,\mathrm{d}\gamma \\ &= -\int_{\Omega_m} e_Q f_Q \,\mathrm{d}x - \int_{\partial\Omega_{ext}} h_0 e_T^2 \,\mathrm{d}\gamma + \int_{\partial\Omega_{ext}} h_0 e_T T_{\text{ext}} \,\mathrm{d}\gamma - \int_{\partial\Omega_c} e_T(e_Q n) \,\mathrm{d}\gamma \\ &= -\int_{\Omega_m} e_Q f_Q \,\mathrm{d}x - \int_{\partial\Omega_{ext}} h_0 e_{\partial,e}^2 \,\mathrm{d}\gamma + \int_{\partial\Omega_{ext}} uw \,\mathrm{d}\gamma - \int_{\partial\Omega_c} e_{\partial,c} f_{\partial,c} \,\mathrm{d}\gamma, \end{aligned} \tag{16}$$

presenting an additional dissipative term on the boundary. We can also observe the new inputs and outputs $u = T_{\text{ext}}$ and $y = h_0 T$ for the external boundary, as well as $f_{\partial,c}$ and $e_{\partial,c}$ for the coupling boundary.

Remark 1. *Had we used the physical energy as a Hamiltonian, the additional term would be linear in the temperature instead of quadratic. We would then have to ensure the temperature always stays positive for it to be dissipative. While that is a reasonable requirement from a physical point of view, it restricts the choice of numerical schemes.*

Remark 2. *The above model, including the splitting of the boundary into two parts with separate boundary conditions, is not restricted to the square shape used as an example. It also holds for more general shapes and can be extended to higher dimensions, so long as the boundary is sufficiently well-behaved.*

3.2. Cooling Channel

We remind the reader that the cooling channel is modelled by the following PDE:

$$\frac{\partial \Theta}{\partial t}(t,y) = -v\frac{\partial \Theta}{\partial y}(t,y) + \frac{h_1}{c_c\rho_c}\big(T(t,1,y) - \Theta(t,y)\big), \tag{17}$$

with the temperature of the cooling fluid $\Theta(t,y)$ in the cooling channel, the heat transfer coefficient h_1 and assuming that ρ_c and c_c are constant. To model this kind of PDE as a pHS, we require a Hamiltonian quadratic in Θ. For consistency with the previous sub-system, we choose the Hamiltonian

$$H = \frac{1}{2}\int_0^1 \frac{1}{\rho_c c_c} q_c^2(t,y)\,\mathrm{d}y, \tag{18}$$

with the internal energy $q_c(t,y) = \rho_c c_c \Theta(t,y)$. We then have

$$e_\Theta = \delta_{q_c} H = \Theta, \qquad f_\Theta = \frac{\partial q_c}{\partial t} = \rho_c c_c \frac{\partial \Theta}{\partial t}. \tag{19}$$

In regard to the proper spaces of these variables, as well as the ones below, we refer the reader to [21], where this question has been considered in detail. We choose

$$\begin{aligned}
J &= -v\rho_c c_c \frac{\partial}{\partial y}, & R &= h_1,\\
B &= 0, & P &= -h_1,\\
S &= h_1, & &\\
f_d &= T(t,1,y), & e_d &= h_1(T(t,1,y) - \Theta(t,y),\\
e_\partial &= \frac{1}{\sqrt{2}}\big(\Theta(t,1) + \Theta(t,0)\big), & f_\partial &= -\frac{1}{\sqrt{2}} v\rho_c c_c\big(\Theta(t,1) - \Theta(t,0)\big),
\end{aligned}$$

resulting in the system, cf. [22]

$$f_\Theta = (J - R)e_\Theta + (B - P)f_d, \tag{20}$$
$$e_d = (B + P)^\top e_\Theta + S f_d, \tag{21}$$
$$e_\partial = \frac{1}{\sqrt{2}}\big(\Theta(1,t) + \Theta(0,t)\big), \tag{22}$$
$$f_\partial = -\frac{1}{\sqrt{2}} v\rho_c c_c\big(\Theta(1,t) - \Theta(0,t)\big). \tag{23}$$

This gives us the Dirac structure for our cooling channel, cf. [15,22]

$$\mathcal{D}_c = \left\{ \left(\begin{pmatrix} f_\Theta \\ f_d \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_\Theta \\ e_d \\ e_\partial \end{pmatrix} \right) \in (L^2((0,1))^2 \times \mathbb{R})^2 \,\middle|\, e_\Theta \in H^1((0,1)), \right.$$
$$\left. \begin{pmatrix} f_\Theta \\ e_d \end{pmatrix} = \begin{pmatrix} J - R & B - P \\ (B+P)^\top & S \end{pmatrix} \begin{pmatrix} e_\Theta \\ f_d \end{pmatrix}, \begin{array}{ll} e_\partial & = \frac{1}{\sqrt{2}}\big(e_\Theta(1,t) + e_\Theta(0,t)\big) \\ f_\partial & = -\frac{1}{\sqrt{2}} v\rho_c c_c \big(e_\Theta(1,t) - e_\Theta(0,t)\big) \end{array} \right\}.$$

To check that this Dirac structure, combined with our Hamiltonian of Equation (18), does actually form a port-Hamiltonian system, we calculate the power balance

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_0^1 e_\Theta f_\Theta \,\mathrm{d}y = \int_0^1 \rho_c c_c \Theta \frac{\partial \Theta}{\partial t}\,\mathrm{d}y \\
&= \int_0^1 \Theta\Big(-v\rho_c c_c \frac{\partial \Theta}{\partial y} - h_1\Theta + h_1 T\Big)\,\mathrm{d}y \\
&= \underbrace{-v\rho_c c_c \int_0^1 \Theta \frac{\partial \Theta}{\partial y}\,\mathrm{d}y}_{=-\frac{v}{2}\rho_c c_c [\Theta^2]_{y=0}^{y=1}} + \int_0^1 \Theta h_1 (T-\Theta)\,\mathrm{d}y.
\end{aligned} \tag{24}$$

For the system to be dissipative, we need

$$\frac{\mathrm{d}H}{\mathrm{d}t} \le e_\partial f_\partial + \int_0^1 e_d f_d \,\mathrm{d}y = -\frac{v}{2}\rho_c c_c [\Theta^2]_{y=0}^{y=1} + \int_0^1 h_1(T-\Theta)T\,\mathrm{d}y. \tag{25}$$

A comparison with Equation (24) shows that the equality holds for the first term. For the second term, we require

$$\int_0^1 \Theta h_1(T-\Theta)\,\mathrm{d}y \overset{!}{\le} \int_0^1 T h_1(T-\Theta)\,\mathrm{d}y. \tag{26}$$

Since

$$\Theta h_1(T-\Theta) - T h_1(T-\Theta) = -h_1(T-\Theta)^2 \le 0 \tag{27}$$

holds, this is satisfied. We therefore have the distributed output $e_d = h_1(T(1,y,t) - \Theta(y,t))$ and the distributed input $f_d = T(1,y,t)$.

For the boundary input, we set $u(t) = W_B\binom{f_\partial}{e_\partial}$, and for the output $w(t) = W_C\binom{f_\partial}{e_\partial}$. Then, according to [8,23], we have a port-Hamiltonian system if $W_B \Sigma W_B^\top$ is positive semi-definite and $\binom{W_B}{W_C}$ has full rank. If we want to use the inlet temperature as the input and the outlet temperature as output, this setting results in

$$W_B = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{v\rho_c c_c} & 1 \end{pmatrix}, \tag{28}$$

$$W_C = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{1}{v\rho_c c_c} & 1 \end{pmatrix}, \tag{29}$$

which satisfies the above criteria.

3.3. Coupling

To recover a port-Hamiltonian formulation of the model system discussed in Section 2 by coupling the port-Hamiltonian systems discussed in Sections 3.1 and 3.2, we need the following equality:

$$-\lambda \frac{\partial T}{\partial x}(1,y,t) = h_1\big(T(1,y,t) - \Theta(y,t)\big). \tag{30}$$

Considering the relevant inputs and outputs of the two systems, we find that the 'gyrative' interconnection (cf. [24]), i.e., the Dirac structure

$$\mathcal{D}_i = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\middle|\, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right\} \tag{31}$$

is power conserving, and with the choices

$$e_1 = T(1,y,t), \qquad f_1 = -\Phi_Q n = \lambda \frac{\partial T}{\partial x}(1,y,t), \tag{32}$$

$$e_2 = h_1\big(T(1,y,t) - \Theta(y,t)\big), \qquad f_2 = T(1,y,t). \tag{33}$$

obviously satisfies (30). Therefore, the combined system is again a port-Hamiltonian system.

4. Finite Difference Discretization

In this section, we present a finite difference discretization in space of the port-Hamiltonian systems from the previous section. The resulting system of ODEs or DAEs can then be discretized in time, which is omitted here. A finite difference discretization is certainly not the only possible space discretization. Other methods, such as the *Partitioned Finite Element Method (PFEM)* presented in [25–27] and applied to the heat equation in [28,29] and [30] (Section 3.4), might provide better results from a numerical point of view. However, the numerical performance of different discretizations is deliberately left outside of the scope of this work, as the reason for this section's existence is different: In a previous model, we have found curious effects of space-discretization on the system's behavior [12]. As we wanted to investigate whether similar effects also appear in this model, we decided to forego numerical performance and focus on a discretization scheme that could be easily analyzed and where we could be as explicit as possible. For an alternative approach to the heat equation based on the finite difference, we refer the reader to [31]. Here, the variables considered on the spatial grid are indicated by an underscore, e.g., $\underline{x}$.

4.1. Heat Equation

We assume that ρ and c_m are constant. We consider a uniform grid in x-direction with $N+1$ equidistant discretization points $\underline{x}_0 = 0$, $\underline{x}_1 = \Delta x, \dots, \underline{x}_N = 1$ and similarly for y with $M+1$ points, leading to the grid variable $\underline{y}$ and a space step Δy. We define $\underline{T} \in \mathbb{R}^{N \cdot M}$, such that $\underline{T}$ is defined on an offset grid, i.e. $\underline{T}_{i+jN} \approx T(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_j + \frac{\Delta y}{2}), \quad i = 0, \dots, N-1$, $j = 0, \dots, M-1$.

We discretize the Hamiltonian (6) with regards to the space using the midpoint rule

$$\underline{H} = \frac{1}{2} \rho c_m \Delta x \Delta y \, \underline{T}^\top \underline{T}, \tag{34}$$

with the time derivative

$$\frac{\mathrm{d}\underline{H}}{\mathrm{d}t} = \rho c_m \Delta x \Delta y \, \underline{T}^\top \frac{\partial \underline{T}}{\partial t}, \tag{35}$$

giving us the internal energy change as flow variable and the temperature as effort variable:

$$\underline{f}^{(T)} = \rho c_m \Delta x \Delta y \, \frac{\partial \underline{T}}{\partial t}, \quad \underline{e}^{(T)} = \underline{T}. \tag{36}$$

Using central differences (with half step sizes) to discretize Equation (9), we obtain the following approximation for the heat fluxes in the interior

$$\Phi_x(\underline{x}_i, \underline{y}_j + \tfrac{\Delta y}{2}) = -\lambda \frac{1}{\Delta x} \big(\underline{T}_{i+Nj} - \underline{T}_{i-1+Nj} \big) + \mathcal{O}(\Delta x^2),$$
$$\Phi_y(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_j) = -\lambda \frac{1}{\Delta y} \big(\underline{T}_{i+Nj} - \underline{T}_{i+N(j-1)} \big) + \mathcal{O}(\Delta y^2).$$

On the boundary, we use one-sided difference quotients to approximate the heat fluxes, since this accuracy is sufficient according to Gustafsson [32]. Together with the boundary conditions, we then have for the left boundary

$$\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) = h\big(T_{ext}(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) - T(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2})\big)$$
$$\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) = -\lambda \frac{2}{\Delta x} \big(T(\underline{x}_0 + \frac{\Delta x}{2}, \underline{y}_j + \tfrac{\Delta y}{2}) - T(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) \big) + \mathcal{O}(\Delta x)$$

Solving both for $T(\underline{x}_0, \underline{y}_j + \frac{\Delta y}{2})$ and combining them, we find

$$\Phi_x(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) \approx \frac{2h\lambda}{2\lambda + h\Delta x}\big(T_{ext}(\underline{x}_0, \underline{y}_j + \tfrac{\Delta y}{2}) - \underline{T}_{0+Nj}\big).$$

Similarly, for the right, lower and upper boundary, we find

$$\begin{aligned}
\Phi_x(\underline{x}_N, \underline{y}_j + \tfrac{\Delta y}{2}) &\approx \frac{2h\lambda}{2\lambda + h\Delta x}\big(\underline{T}_{N-1+Nj} - \Theta(\underline{y}_j + \tfrac{\Delta y}{2})\big)\\
\Phi_y(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_0) &\approx \frac{2h\lambda}{2\lambda + h\Delta y}\big(T_{ext}(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_0) - T_i\big)\\
\Phi_y(\underline{x}_i + \frac{\Delta x}{2}, \underline{y}_M) &\approx \frac{2h\lambda}{2\lambda + h\Delta y}\big(T_{i+(M-1)N} - T_{ext}(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_M)\big)
\end{aligned}$$

Let $\underline{\Phi_x} \in \mathbb{R}^{(N+1)M}$ and $\underline{\Phi_y} \in \mathbb{R}^{N(M+1)}$ with

$$\begin{aligned}
\underline{\Phi_x}_{i+(N)j} &\approx \Phi_x(\underline{x}_i, \underline{y}_j + \tfrac{\Delta y}{2}) &&\text{for } i = 0, \dots, N &&\text{and } j = 0, \dots, M-1\\
\underline{\Phi_y}_{i+Nj} &\approx \Phi_y(\underline{x}_i + \tfrac{\Delta x}{2}, \underline{y}_j) &&\text{for } i = 0, \dots, N-1 &&\text{and } j = 0, \dots, M.
\end{aligned}$$

Using central differences again to discretize (8), we obtain

$$\rho c_m \Delta x \Delta y \frac{\partial \underline{T}_{i+Nj}}{\partial t} = \Delta y\big(\underline{\Phi_x}_{i+(N+1)j} - \underline{\Phi_x}_{i+1+(N+1)j}\big) + \Delta x\big(\underline{\Phi_y}_{i+Nj} - \underline{\Phi_y}_{i+N(j+1)}\big)$$

Let $J_{x,1} \in \mathbb{R}^{N\times(N+1)}$, $J_x \in \mathbb{R}^{(N*M)\times((N+1)*M)}$, $J_y \in \mathbb{R}^{(N*M)\times(N*(M+1))}$ and $I_{N\times N}$ the $N \times N$ unit matrix, with

$$J_{x,1} = \begin{pmatrix} 1 & -1 & & \\ & \ddots & -1 & \\ & & 1 & -1 \end{pmatrix}, \quad J_x = \Delta y \begin{pmatrix} J_{x,1} & & \\ & \ddots & \\ & & J_{x,1} \end{pmatrix}, \tag{37}$$

$$J_y = \Delta x \begin{pmatrix} I_{N\times N} & -I_{N\times N} & & \\ & \ddots & \ddots & \\ & & I_{N\times N} & -I_{N\times N} \end{pmatrix}. \tag{38}$$

Also, let $R_x \in \mathbb{R}^{(N+1)M\times(N+1)M}$, $R_{x,1} \in \mathbb{R}^{(N+1)\times(N+1)}$ and $R_y \in \mathbb{R}^{N(M+1)\times N(M+1)}$ with

$$R_{x,1} = \Delta y \begin{pmatrix} \frac{2\lambda+h\Delta x}{2h\lambda} & & & & \\ & \frac{\Delta x}{\lambda} & & & \\ & & \ddots & & \\ & & & \frac{\Delta x}{\lambda} & \\ & & & & \frac{2\lambda+h\Delta x}{2h\lambda} \end{pmatrix}, \quad R_x = \begin{pmatrix} R_{x,1} & & \\ & \ddots & \\ & & R_{x,1} \end{pmatrix}, \tag{39}$$

$$R_y = \Delta x \begin{pmatrix} \frac{2\lambda+h\Delta y}{2h\lambda} I_{N\times N} & & & & \\ & \frac{\Delta y}{\lambda} I_{N\times N} & & & \\ & & \ddots & & \\ & & & \frac{\Delta y}{\lambda} I_{N\times N} & \\ & & & & \frac{2\lambda+h\Delta y}{2h\lambda} I_{N\times N} \end{pmatrix}. \tag{40}$$

Finally let

$$b_{x,0} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^\top \in \mathbb{R}^{N+1}, \qquad b_{x,N} = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^\top \in \mathbb{R}^{N+1},$$

$$B_{x,0} = \Delta y \begin{pmatrix} b_{x,0} & & \\ & \ddots & \\ & & b_{x,0} \end{pmatrix} \in \mathbb{R}^{(N+1)M \times M}, \qquad B_{y,0} = \Delta x \begin{pmatrix} I_{N \times N} \\ 0 \end{pmatrix} \in \mathbb{R}^{N*(M+1) \times N},$$

$$B_{x,N} = -\Delta y \begin{pmatrix} b_{x,N} & & \\ & \ddots & \\ & & b_{x,N} \end{pmatrix} \in \mathbb{R}^{(N+1)M \times M}, \quad B_{y,M} = -\Delta x \begin{pmatrix} 0 \\ I_{N \times N} \end{pmatrix} \in \mathbb{R}^{N*(M+1) \times N}.$$

We can now recover the discretized version of (11) in the form of a port-Hamiltonian descriptor system (pHDAE) [6]:

$$\begin{pmatrix} \underline{f}^{(T)} \\ 0 \\ 0 \end{pmatrix} = (J - R) \begin{pmatrix} \underline{e}^{(T)} \\ \underline{\Phi}_x \\ \underline{\Phi}_y \end{pmatrix} + B \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ \Theta(\underline{y} + \frac{\Delta y}{2}) \end{pmatrix}, \tag{41}$$

$$\underline{w} = B^\top \begin{pmatrix} \underline{e} \\ \underline{\Phi}_x \\ \underline{\Phi}_y \end{pmatrix} \approx \begin{pmatrix} \Delta y \, \Phi_x(x_0, \underline{y} + \frac{\Delta y}{2}) \\ \Delta x \, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_0) \\ -\Delta x \, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_M) \\ -\Delta y \, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \end{pmatrix}, \tag{42}$$

with

$$J = \begin{pmatrix} 0 & J_x & J_y \\ -J_x^\top & 0 & 0 \\ -J_y^\top & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_x & 0 \\ 0 & 0 & R_y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & B_{x,N} \\ 0 & B_{y,0} & B_{y,M} & 0 \end{pmatrix}. \tag{43}$$

4.2. Transport Equation

We choose the $M+1$ equidistant discretization points $\underline{y}_0 = 0, \underline{y}_1 = \Delta y, \ldots, \underline{y}_M = 1$ with $\Delta y = \frac{1}{M}$. Then, we set $\underline{\Theta} = (\Theta(\underline{y}_0 + \frac{\Delta y}{2}), \ldots, \Theta(\underline{y}_{M-1} + \frac{\Delta y}{2}))^\top \in \mathbb{R}^M$, matching the discretization scheme of the heat equation above. Discretizing Equation (18) with the midpoint rule results in the semi-discrete Hamiltonian

$$\underline{H} = \frac{1}{2} \Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i^2, \tag{44}$$

with the time derivative

$$\frac{\mathrm{d}\underline{H}}{\mathrm{d}t} = \Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i \frac{\partial \underline{\Theta}_i}{\partial t}, \tag{45}$$

allowing us to set the flow and effort variables

$$\underline{f}_i = \Delta y \, \rho_c c_c \frac{\partial \underline{\Theta}_i}{\partial t}, \qquad \underline{e}_i = \underline{\Theta}_i, \qquad \forall i = 0, \ldots, M-1. \tag{46}$$

Using an upwind discretization for the spatial derivative, i.e.,

$$\frac{\partial \Theta}{\partial y}(\underline{y}_i) = \frac{\underline{\Theta}_i - \underline{\Theta}_{i-1}}{\Delta y} + \mathcal{O}(\Delta y), \tag{47}$$

we obtain the following discretized version of Equations (20)–(23)

$$\underline{f} = (J - R)\underline{e} + (B - P)\underline{u}, \qquad \underline{w} = (B + P)^\top \underline{e} + (S - N)\underline{u}, \tag{48}$$

with

$$J = \frac{1}{2} v\rho c_c \begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}, \quad R = \frac{1}{2} v\rho c_c \begin{pmatrix} 2 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 2 \end{pmatrix} \tag{49}$$

$$B = \begin{pmatrix} 1 & & & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}, \qquad P = 0, \quad S = 0, \quad N = 0, \tag{50}$$

as well as the input $\underline{u}$ and output $\underline{w}$

$$\underline{u} = \begin{pmatrix} \Delta y \, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ v\rho c_c \Theta_{in} \end{pmatrix}, \qquad \underline{w} = \begin{pmatrix} \underline{\Theta} \\ \underline{\Theta}_0 \end{pmatrix}. \tag{51}$$

Note that we could also use $\underline{\Theta}_{in}$ as input by moving the preceding factors into B, which would make those factors also appear in the output. If an output at the end of the cooling channel is desired, this requires an artificial feed-through between the input and output, as in [12]. With matrices chosen as above, we easily find that

$$W = \begin{pmatrix} R & P \\ P^\top & S \end{pmatrix} \tag{52}$$

is positive semi-definite according to the Gershgorin circle theorem.

4.3. Coupling the Discretized Systems

We consider two finite-dimensional port-Hamiltonian (descriptor) systems of the form

$$Ef = (J - R)e + (B - P)u, \qquad w = (B + P)^\top e + (S - F)u, \tag{53}$$

and Hamiltonian H. According to [15], the system resulting from an interconnection of these two systems is again a PHDAE, if there are matrices M and N, so that

$$Mu + Nw = 0, \tag{54}$$

with $u = \binom{u_1}{u_2}$ the combined inputs of both systems and $w = \binom{w_1}{w_2}$ their combined outputs. If $Mu + Nw = 0$ defines a Dirac structure for (w, u), the system can usually be made smaller through index reduction and row operations. The coupled system takes the form

$$\begin{pmatrix} Ef \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} J - R & B - P & 0 & 0 \\ -(B+P)^\top & S - F & I & -M^\top \\ 0 & -I & 0 & -N^\top \\ 0 & M & N & 0 \end{pmatrix} \begin{pmatrix} e \\ \hat{u} \\ \hat{w} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix} u, \tag{55}$$

$$w = \hat{w}, \tag{56}$$

with I being the identity, and $E = \operatorname{diag}(E_1, E_2)$, $J = \operatorname{diag}(J_1, J_2)$ and R, B, P, S, F similar. The form given here is equivalent to the one given in [15], but without requiring the (unspecified) permutation matrices present in their formulation.

Setting the heat equation to be the first system and the transport equation to be the second system, we can choose

$$M = \begin{pmatrix} 0 & 0 & 0 & I_{M\times M} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{M\times M} & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 & -I_{M\times M} & 0 \\ 0 & 0 & 0 & I_{M\times M} & 0 & 0 \end{pmatrix}, \tag{57}$$

which defines a Dirac structure for those parts of the input and output, that are involved in the interconnection. It should therefore be possible to shrink the system. Writing down the vectors and matrices for the coupled system,

$$J = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & 0 & 0 & 0 \\ -J_y^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & J_\Theta \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R_x & 0 & 0 \\ 0 & 0 & R_y & 0 \\ 0 & 0 & 0 & R_\Theta \end{pmatrix}, \tag{58}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & B_{x,N} & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_\Theta \end{pmatrix}, \tag{59}$$

$$u = \hat{u} = \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ \underline{\Theta} \\ \Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ v\rho c_c \Theta_{in} \end{pmatrix}, \qquad w = \hat{w} = \begin{pmatrix} \Delta y\, \Phi_x(x_0, \underline{y} + \frac{\Delta y}{2}) \\ \Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_0) \\ -\Delta x\, \Phi_y(\underline{x} + \frac{\Delta x}{2}, y_M) \\ -\Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2}) \\ \underline{\Theta} \\ \underline{\Theta}_0 \end{pmatrix}, \tag{60}$$

we immediately observe that $\underline{\Theta}$ and $\Delta y\, \Phi_x(x_N, \underline{y} + \frac{\Delta y}{2})$ appear in both the input and output, and from the previous sections, we also know that they occur in e as well. We can therefore eliminate them from the input and output and move the relevant terms into J, resulting in the—significantly more compact—condensed system

$$\begin{pmatrix} f^{(T)} \\ 0 \\ 0 \\ f^{(\Theta)} \end{pmatrix} = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & -R_x & 0 & B_{x,N} \\ -J_y^\top & 0 & -R_y & 0 \\ 0 & -B_{x,N}^\top & 0 & J_\Theta - R_\Theta \end{pmatrix} \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix} \tag{61}$$

$$+ \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 \\ 0 & 0 & 0 & B_\Theta \end{pmatrix}}_{\widetilde{B}} \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ v\rho c_c \Theta_{in} \end{pmatrix}, \tag{62}$$

$$w = \widetilde{B}^\top \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix}, \qquad \text{with} \quad B_\Theta = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^\top. \tag{63}$$

4.4. Discretizing the Coupled System

Coupling the two systems from Sections 3.1 and 3.2 results in a system with the Hamiltonian

$$H = \frac{1}{2} \int_\Omega \rho(z) c_m(z)\, T(t,z)^2 \,\mathrm{d}z + \frac{1}{2} \int_0^1 \rho_c c_c\, \Theta^2(y,t) \,\mathrm{d}y. \tag{64}$$

Discretizing T and Θ each with an appropriate midpoint rule as in Sections 4.1 and 4.2, results in

$$\underline{H} = \frac{1}{2} \rho c_m \Delta x \Delta y\, \underline{T}^\top \underline{T} + \frac{1}{2} \Delta y \sum_{i=0}^{M-1} \rho_c c_c \underline{\Theta}_i^2. \tag{65}$$

Proceeding as in the previous sections, we then obtain the following system:

$$\begin{pmatrix} f^{(T)} \\ 0 \\ 0 \\ f^{(\Theta)} \end{pmatrix} = \begin{pmatrix} 0 & J_x & J_y & 0 \\ -J_x^\top & -R_x & 0 & B_{x,N} \\ -J_y^\top & 0 & -R_y & 0 \\ 0 & -B_{x,N}^\top & 0 & J_\Theta - R_\Theta \end{pmatrix} \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix} + B \begin{pmatrix} T_{ext}(\underline{x}_0, \underline{y} + \frac{\Delta y}{2}) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_0) \\ T_{ext}(\underline{x} + \frac{\Delta x}{2}, \underline{y}_M) \\ v\rho c_c \Theta_{in} \end{pmatrix},$$

$$\underline{\widetilde{w}} = B^\top \begin{pmatrix} e^{(T)} \\ \underline{\Phi_x} \\ \underline{\Phi_y} \\ e^{(\Theta)} \end{pmatrix},$$

with

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{x,0} & 0 & 0 & 0 \\ 0 & B_{y,0} & B_{y,M} & 0 \\ 0 & 0 & 0 & B_\Theta \end{pmatrix}, \qquad B_\Theta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{M \times 1},$$

all other matrices and vectors containing Θ as in Section 4.2 and the remaining quantities as in Section 4.1. As we can observe, this is the same system we obtain by eliminating superfluous variables from the system in Section 4.3.

5. Conclusions and Outlook

In this work, we proposed a model consisting of two subsystems for a simplified conjugate heat transfer in a turbine blade. We were then able to demonstrate that each of these subsystems is a port-Hamiltonian system and their interconnection defines a Dirac structure. Therefore, the entire model is also a port-Hamiltonian system. While the one-dimensional model previously proposed in [13] had constraints on the physical parameters, this two-dimensional model no longer has constraints beyond those that are physically meaningful. However, the question of the existence and uniqueness of the solution is still open. Results regarding the existence and uniqueness of solutions exist for the wave equation in two-dimensions, see [20]. However, it is non-trivial to generalize these to systems with a closure relation. While we are confident this is possible, it will certainly exceed the scope of this work. We therefore plan to revisit this interesting question in a future publication dedicated to this topic.

In Section 4, it was then demonstrated that the application of an appropriate but very simple spatial discretization leads to a finite-dimensional port-Hamiltonian system. The finite-dimensional port-Hamiltonian system resulting from the discretization of the subsystems and the coupling of the resulting finite-dimensional subsystems is equivalent to the system resulting from the coupling of the subsystems and the subsequent discretization of the complete coupled system (using the same discretization scheme). It might be worthwhile to investigate whether this is a peculiarity of the system and discretization method under consideration, or whether it is a general result that holds for all port-Hamiltonian systems.

While the intuitive choices for quadrature of the Hamiltonian and the spatial discretization worked quite well for this system, it remains an open question whether the same holds for the general case. It would also be interesting to investigate whether a particular choice of quadrature for the Hamiltonian uniquely determines the spatial discretization of the differential equations and vice versa.

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