mathematics

Article

Reliability Analysis of the Multi-State *k*-out-of-*n*: F Systems with Multiple Operation Mechanisms

Yanbo Song [1] and Xiaoyue Wang [2,*]

[1] School of Management & Economics, Beijing Institute of Technology, Beijing 100081, China
[2] School of E-Business and Logistics, Beijing Technology and Business University, Beijing 100048, China
* Correspondence: xiaoyue@btbu.edu.cn

Abstract: Modern engineering systems are designed and utilized to realize complicated functions, and their operation mechanisms are becoming more complex. Nevertheless, prior related research mainly focused on the reliability evaluations of the systems with a single operation mechanism, which are not appropriate to depict the operation process of systems with multiple operation mechanisms. Faced with the research gaps and practical needs, this paper establishes a new reliability model for the multi-state k-out-of-n: F system composed of n subsystems, which runs under multiple interactive operation mechanisms, including performance sharing, balanced requirement, and protection strategy. The units in each subsystem can share the performance via a common bus, with the purpose of regulating the performance of all equal units. A new triggering criterion of the protection device in each subsystem is proposed based on the total performance of the units. Due to the protection from the device, the degradation rate of the units between two adjacent states decreases to a lower rate. Each subsystem breaks down when the total performance of the units reaches a critical value. According to the number of failed subsystems, the state of the entire system can be divided into multiple states. The Markov process imbedding method combined with the finite Markov chain imbedding approach is developed to obtain the probabilistic indexes of each subsystem and the entire system. The applicability of the proposed model and the effectiveness of the method can be sufficiently demonstrated by illustrative examples and sensitivity analyses.

Keywords: multi-state k-out-of-n: F system; multiple operation mechanisms; reliability analysis; Markov process imbedding approach; finite Markov chain imbedding approach

MSC: 93E20

Citation: Song, Y.; Wang, X. Reliability Analysis of the Multi-State *k*-out-of-*n*: F Systems with Multiple Operation Mechanisms. *Mathematics* **2022**, *10*, 4615. https://doi.org/10.3390/math10234615

Academic Editor: Manuel Franco

Received: 6 November 2022
Accepted: 2 December 2022
Published: 5 December 2022

1. Introduction

Currently, the operation mechanisms of modern industrial systems become increasingly complicated [1], owing to the great diversity of their designed functions, such as the performance sharing mechanism via the common bus of different multi-unit systems, the balanced requirements during the operation of the balanced systems, and the multi-stage operation process of the systems determined by the functioning state of protective devices. From different research perspectives, many studies have been devoted to depicting the operation mechanisms of engineering systems and analyzing their reliability quantities, based on some practical engineering applications.

The performance sharing mechanism via the common bus of the system was first studied by [2], where only two units were considered and the performance surplus could be transmitted from the reserve unit to the main unit. As an extensive study, Levitin [3] built a system containing n units with the performance sharing mechanism, where a common bus was responsible for transmitting the surplus performance among all units. Subsequently, the reliability of different systems with performance sharing mechanisms through common bus has been widely investigated. For example, Yu et al. [4], Zhao et al. [5], and Peng et al. [6]

studied the performance-sharing mechanisms for the series-parallel repairable binary-state system, multi-state k-out-of-n: G system, and system with two performance sharing groups, respectively. Furthermore, the research on performance-sharing mechanisms has been enriched by considering the multiple stages of operation process [7–9], transmission loss during performance sharing [9–11], related optimization problems [12–14], and so on.

The operation mechanisms of diverse balanced systems have been extensively explored by abundant literature. The previous balanced systems are built according to the specific balanced requirement in their operation mechanisms. The balanced requirements in previous research can be summarized as the following main categories: largest state (capability) difference among all units in the system being at an acceptable level [15–17], no or acceptable difference among the number of working units in different sectors [18–20], simultaneous satisfying of two units in a pair in the same state with the working units symmetric [21–23], and performance (degradation) level of the units in the system identical [24,25].

Protective devices are always installed in many engineering systems to mitigate the system degradation process and enhance system reliability [26]. For example, the cabin pressure control system (CPCS) and power saving system can be regarded as the protective devices for an aircraft system and a battery system, respectively [27–29]. Due to the great practical values in applications, the research on the operation mechanisms of systems with protective devices has aroused strong interest in the domain of reliability. A joint optimization model was constructed by Zhao et al. [30] to optimize the mission abort policy and protective device selection for single-unit systems. References [28,31] analyzed the operation mechanism of a single-unit system with a protective device triggered by system state, by considering the self-exciting shock mechanisms and the shock magnitude thresholds, respectively. Zhao et al. [32] proposed the competing triggering criteria of the device based on the system state and shock numbers. As an extension, Zhao et al. [33] established a k-out-of-n: F system with a multi-state protective device subject to external shocks. By considering the impact of internal degradation, Wang et al. [34] investigated the operation process of the k-out-of-n: F system with m subsystems supported by multiple protective devices. Wang et al. [35] studied a compound operation mechanism containing the protection from devices and balanced requirements of two types of balanced systems.

Through reviewing the literature about the reliability of systems with the abovementioned operation mechanisms, some research gaps can be figured out as follows. First, it can be observed that most research only focused on one specific operation mechanism, and the combination of two or more operation mechanisms has not been thoroughly investigated. Because the actual operation processes of engineering systems are getting more complex, a single operation mechanism, as in previous studies, cannot accurately describe their operation processes. Second, the triggering conditions of protective devices in prior studies included the system state [28,30], competing criteria of the system state and shock numbers, number of failed units [33,34], and the largest state difference among units [35]. There are other triggering criteria of devices worthy to be explored, motivated by real engineering cases.

To fill up the above research gaps, this paper constructs a new reliability model of a multi-state k-out-of-n: F system with multiple operation mechanisms, integrating the performance sharing mechanism via common bus, the balanced condition required for stable operation, and the protection from the devices during operation. The entire system is composed of n subsystems, and each subsystem can be perceived as a performance sharing and balanced subsystem. Each unit in the subsystem has multiple levels of performance. All units in one subsystem are connected via a common bus, which is in charge of transmitting and sharing the surplus performance in order to maintain the same performance among all units. Furthermore, each subsystem is equipped with a protective device, which can be triggered to provide protection for the subsystem when the total performance of the corresponding subsystem is lower than or equal to a threshold. During the functioning of the protective device, the degradation rates of the units degrading to an adjacent lower state in the subsystem decreases. The protective device may break down, and then the degradation rate of the units in the subsystem becomes the original rate with the loss of protection. The insufficient total performance results in the failure of each subsystem, and the entire system has multiple states according to the number of failed subsystems.

The establishment of the proposed model intends to depict the complex operation of the engineering systems, represented by a lithium battery system, whose multiple operation mechanisms are complicated and interactive. The lithium battery system consists of n battery subsystems, and each battery subsystem has multiple battery cells. Each battery cell possesses multiple states of charge (SOC), and different cells may be in different SOCs because of their intrinsic manufacturing tolerances and difference in deterioration process. The surplus SOCs can be transmitted to the cells with lower SOC to ensure the same level of SOC among all units, which is well-known as the cell balancing problem. Meanwhile, once the total SOC of the battery subsystem is insufficient, the power saving device is activated to decrease the degradation process of the cells. The entire lithium battery system can be recognized as different grades of power capability, according to the total number of failed subsystems.

In this paper, a two-stage modeling methodology is applied to derive the probabilistic indexes of the subsystems and the entire system, by combining the Markov process imbedding approach (MPIA) and finite Markov chain imbedding approach (FMCIA). High efficiency can be achieved by employing MPIA to describe the deterioration process of various systems and analyze a series of reliability indexes [36]. Compared with other methods, such as the recursive method and order statistics technique, MPIA shows its great superiority as a powerful tool to determine the reliability indexes with much higher efficiency due to the reduced amount of computation [19]. For example, MPIA has been applied to depict the operation process of balanced systems and formulate their reliability expressions [24,37–39]. FMCIA has become a popular approach in various research fields owing to its effectiveness in addressing complex reliability problems, such as assessing system reliability [40,41], building probabilistic distributions [42–44], and analyzing shock models [45–47]. The developments and advantages of FMCIA have been fully demonstrated by the literature review of the method [48].

To sum up, the remarkable contributions of this paper to current study are listed in the following.

1. This paper first constructs a multi-state k-out-of-n: F systems composed of n subsystems under multiple and interactive operation mechanisms.
2. This paper proposes an interactive and complex operation mechanism, including the performance sharing mechanism, balanced mechanism, and protection mechanism.
3. This paper put forwards a new triggering criterion of protective devices based on the total performance of the units in each subsystem.
4. This paper formulates a two-stage methodology, including MPIA for each subsystem and FMCIA for the entire system, and the reliability indexes are derived accurately, which makes the complicated reliability problem tractable.

The remainder of this paper is organized as follows. In Section 2, the detailed model descriptions and assumptions are introduced. Section 3 focuses on the reliability analyses for individual subsystems and the entire system, and derives the related reliability indexes efficiently by utilizing the MPIA and FMCIA. Section 4 provides numerical examples based on the lithium battery system to validate the proposed reliability model and applied method. In Section 5, the concluding remarks and future research directions of this paper are presented.

2. Model Assumptions and Descriptions

The system contains n subsystems, and each subsystem can be regarded as a performance-balanced system with a protective device. The total number of units in the i-th subsystem is n_i. Each unit has total H_i levels of performance, represented as $\mathbf{G}_i(t) = \{g_{i,1}, g_{i,2}, \ldots, g_{i,H_i}\}$, where level H_i and 1 denote the maximum and zero performance of the unit, respectively. The state residence times of units in one subsystem follow the exponential distribution, which has been sufficiently proven by existing research that exponential distribution can be used to model the distribution of the state residence times for the units [18–20].

All the units in one subsystem are connected via a common bus, which is used to transmit the shared performance among units. The balance of each subsystem is defined as the same performance of each unit in the subsystem. When the performances of all units are not equal, the units with the performance above the average share the surplus performance to the units with the performance below the average via the common bus in order to regain the performance balance. It is assumed that the time needed for performance rebalancing actions is negligible.

When the total performance of the units in the i-th subsystem is not more than $d_{i,1}$, the protection device is triggered to protect the corresponding subsystem. During the function of the protective device, the degradation rate of the units in state h_i degrading to state $h_i - 1$ in the subsystem i is $\lambda^{i,2}_{h_i(h_i-1)}$. It means that the subsystem runs without the protection and the degradation rate of the unit in state h_i deteriorating to state $h_i - 1$ is $\lambda^{i,1}_{h_i(h_i-1)}$ ($\lambda^{i,1}_{h_i(h_i-1)} > \lambda^{i,2}_{h_i(h_i-1)}$), before the protective device is triggered and after the protective device fails. The protective device may break down due to internal degradation with a rate $\lambda_{i,p}$. When the total performance of the units in subsystem i is not greater than $d_{i,2}$ ($d_{i,2} < d_{i,1}$), the subsystem i fails. Based on the number of failed subsystems in the system, the entire system is divided into $M + 1$ states, where the cases that all subsystems in the system work and fail are recognized as state 0 and M, respectively. State m ($0 < m < M$) is defined as the number of failed subsystems larger than k_{m-1} and not more than k_m ($k_{m-1} < k_m$). The system structure diagram is shown in Figure 1.

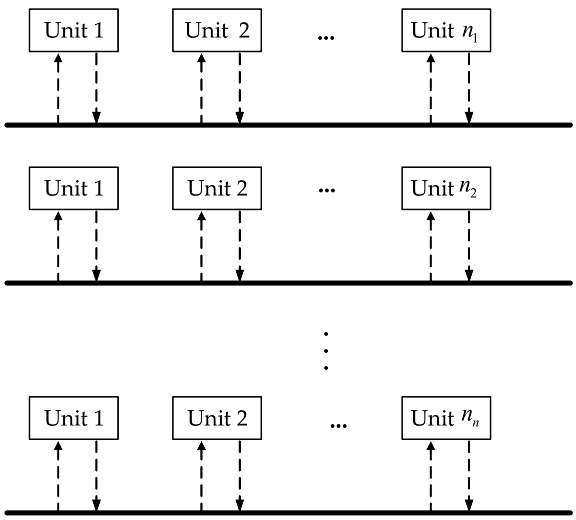

Figure 1. Composition of the system with multiple operation mechanisms.

Example 1. To better understand the operation of a single subsystem, possible operation cases of subsystem i are shown in Figure 2. The subsystem i contains 3 ($n_i = 3$) units, and each unit has 3 ($H_i = 3$) performance levels with corresponding performance values $\mathbf{G}_i(t) = \{0, 3, 6\}$. When the total performance of the units is less than or equal to 12 ($d_{i,1} = 12$), the protective device is triggered to work. When the total performance of the units is not more than 9 ($d_{i,2} = 9$), the subsystem fails. At the initial time, each unit is in the best state with the performance 6, shown in Figure 2a. At $t = 1$, the state of the first unit drops by one with the degradation rate $\lambda^{i,1}_{32}$, and the performance value changes from 6 to 3. At this moment, the subsystem is obviously unbalanced, and the second and third unit transmit 1 unit of performance to the first unit through the common bus. Hence, the subsystem is rebalanced, and works normally until $t = 2$, presented in Figure 2b. At $t = 2$, the state of the second unit is degraded by one level with the rate $\lambda_{32}^{i,1}$ and its performance changes to 3. Then the subsystem can regain the balance through transmitting performance via the common bus. At this time, the total performance of the subsystem is 12, which leads to the triggering of the protective device, as displayed in Figure 2c. At $t = 4$, the second unit becomes failed after it degrades from state 2 to 1 with the rate $\lambda_{21}^{i,2}$, and its performance equals zero. The total performance of the subsystem equals 9, which results in the failure of the subsystem due to reaching the failure criterion of the subsystem.

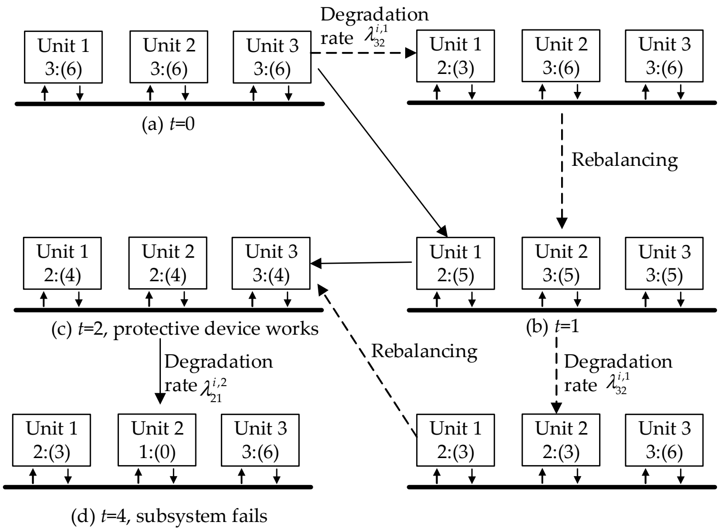

Figure 2. Possible operation cases of the subsystem i.

3. Reliability Evaluations for the System

The first step is applying the Markov process imbedding method to describe the operation of each subsystem and obtain the reliability of each subsystem. Then, as the second step, the finite Markov chain imbedding method is employed to derive the state probability function of the entire system by using the results obtained in the first step. The following Figure 3 summarizes the detailed steps of reliability evaluations for the proposed model.

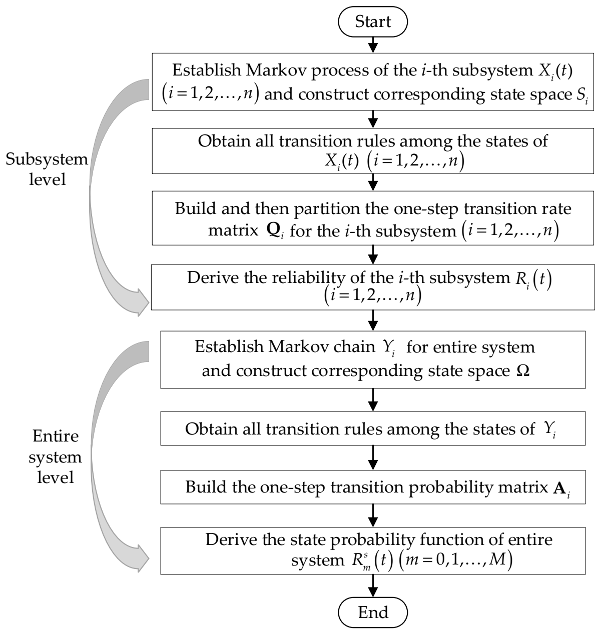

Figure 3. Flow chart of reliability evaluation steps.

3.1. Reliability Evaluation for an Individual Subsystem

The state residence times of units in one subsystem follow the exponential distribution. Therefore, the behavior of a single subsystem can be characterized by applying a Markov process and n Markov processes can be applied to describe the operation of n subsystems, respectively. The Markov process of the i-th subsystem $\{X_i(t), t \geq 0\}$ is established as follows:

$$X_i(t) = \mathbf{u}_{i,c_i}, c_i = 1, 2, \ldots, N_i$$

where $\mathbf{u}_{i,c_i}$ represents a state in the state space S_i of the i-th subsystem, and $|S_i| = N_i$. $S_i = W_i \cup F_i$, where W_i and F_i represent the working and the failed state space of the i-th subsystem, respectively.

Suppose that $\mathbf{u}_{i,c_i} = (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,1}, x_{i,p})$, where $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,1})$ represents the total number of units in each performance state in the i-th subsystem, and define

$$x_{i,p} = \begin{cases} 0, \text{ protective device is untriggered;} \\ 1, \text{ protective device is working;} \\ 2, \text{ protective device is failed.} \end{cases}$$

For convenience,$I(x)$ is used as an indicator function, such that $I(x) = \begin{cases} 1 \text{ if } x \text{ is true} \\ 0 \text{ if } x \text{ is false} \end{cases}$. The transition rates between all states of the i-th subsystem are given as follows.

(1) Condition: $\sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} > d_{i,1}$, $\sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - g_{i,h_i} + g_{i,h_i-1} > d_{i,1}$, and $x_{i,h_i} > 0$;
Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 0) \rightarrow (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i} - 1, x_{i,h_i-1} + 1, \ldots, x_{i,1}, 0)$;
Transition rate: $x_{i,h_i} \lambda^{i,1}_{h_i(h_i-1)}$.

(2) Condition: $\sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} > d_{i,1}$, $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - g_{i,h_i} + g_{i,h_i-1} \leq d_{i,1}$, and $x_{i,h_i} > 0$;
Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 0) \rightarrow (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i} - 1, x_{i,h_i-1} + 1, \ldots, x_{i,1}, 1)$;
Transition rate: $x_{i,h_i} \lambda^{i,1}_{h_i(h_i-1)}$.

(3) Condition: $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} \leq d_{i,1}$, $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - g_{i,h_i} + g_{i,h_i-1} \leq d_{i,1}$, and $x_{i,h_i} > 0$;
Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 1) \rightarrow (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i} - 1, x_{i,h_i-1} + 1, \ldots, x_{i,1}, 1)$;
Transition rate: $x_{i,h_i} \lambda^{i,2}_{h_i(h_i-1)}$.

(4) Conditions: $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} \leq d_{i,1}$;
Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 1) \rightarrow (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 2)$;
Transition rate: $\lambda_{i,p}$.

(5) Condition: $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} \leq d_{i,1}$, $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - g_{i,h_i} + g_{i,h_i-1} \leq d_{i,1}$, and $x_{i,h_i} > 0$;
Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 2) \rightarrow (x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i} - 1, x_{i,h_i-1} + 1, \ldots, x_{i,1}, 2)$;
Transition rate: $x_{i,h_i} \lambda^{i,1}_{h_i(h_i-1)}$.

(6) Condition: $d_{i,2} < \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i}$, $\sum\limits_{h_i=1}^{H_i} I\left((g_{i,h_i} - g_{i,h_i-1}) I(x_{i,h_i} > 0) \geq \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - d_{i,2} \right) > 0$;
(6.1) Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 1) \rightarrow F_i$;
(6.1) Transition rate: $\sum\limits_{h_i=1}^{H_i} I\left((g_{i,h_i} - g_{i,h_i-1}) I(x_{i,h_i} > 0) \geq \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - d_{i,2} \right) x_{i,h_i} \lambda^{i,2}_{h_i(h_i-1)}$,
(6.2) Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 0) \rightarrow F_i$;
(6.2) Transition rate: $\sum\limits_{h_i=1}^{H_i} I\left((g_{i,h_i} - g_{i,h_i-1}) I(x_{i,h_i} > 0) \geq \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - d_{i,2} \right) x_{i,h_i} \lambda^{i,1}_{h_i(h_i-1)}$;
(6.3) Transition: $(x_{i,H_i}, x_{i,H_i-1}, \ldots, x_{i,h_i}, x_{i,h_i-1}, \ldots, x_{i,1}, 2) \rightarrow F_i$;
(6.3) Transition rate: $\sum\limits_{h_i=1}^{H_i} I\left((g_{i,h_i} - g_{i,h_i-1}) I(x_{i,h_i} > 0) \geq \sum\limits_{h_i=1}^{H_i} g_{i,h_i} x_{i,h_i} - d_{i,2} \right) x_{i,h_i} \lambda^{i,1}_{h_i(h_i-1)}$.

After identifying the one-step transition rules, the transition rate matrix $\mathbf{Q}_i$ of the i-th subsystem can be obtained as

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{Q}_{W_iW_i} & \mathbf{Q}_{W_iF_i} \\ \mathbf{Q}_{F_iW_i} & \mathbf{Q}_{F_iF_i} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{W_iW_i} & \mathbf{Q}_{W_iF_i} \\ 0 & 0 \end{bmatrix} \tag{1}$$

The matrix $\mathbf{Q}_{W_iW_i}$ represents the transitions between the working states, and $\mathbf{Q}_{W_iF_i}$ is composed of the transition rates from the working states to the failed state. In addition, $\mathbf{Q}_{F_iW_i}$ and $\mathbf{Q}_{F_iF_i}$ denote the matrix containing the transition rates from the failed state to the working states and from the failed state to the failed state, respectively.

Example 2. An illustration is provided to construct the Markov process for the i-th subsystem. There are 3 ($n_i = 3$) units in the i-th subsystem, and each unit has 4 ($H_i = 4$) states with corresponding performance $\mathbf{G}_i(t) = \{0, 2, 4, 8\}$. Once the total performance is less than or equals 14 ($d_{i,1} = 14$), the protection device is activated to work. When the total performance is not more than 12 ($d_{i,2} = 12$), the subsystem fails. The degradation rate of units from state g to $g-1$ ($g = 3, 2, 1$) is $\lambda_{g(g-1)}^{i,1}$ and $\lambda_{g(g-1)}^{i,2}$, without and under the protection from the device, respectively. The failure rate of the protective device is denoted as $\lambda_{i,p}$.

Based on the above analyses, the state space of the Markov process for subsystem i can be obtained as

$$S_i = W_i \cup F_i = \left\{ \begin{array}{l} (3,0,0,0,0), (2,1,0,0,0), (1,2,0,0,0), (2,0,1,0,0), \\ (1,1,1,0,1), (1,1,1,0,2), (2,0,0,1,0) \end{array} \right\} \cup F_i.$$

Based on the obtained transition rules, the state transition diagram of the Markov process for the subsystem i is presented in Figure 4 as follows.

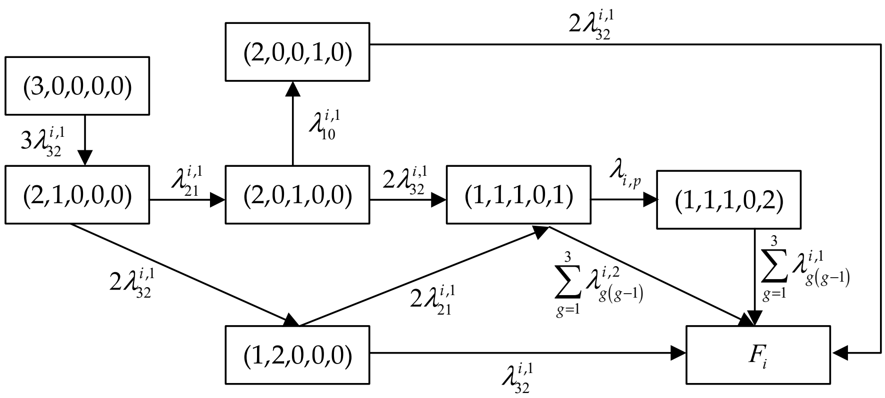

Figure 4. State transition diagram of the Markov process for the subsystem i.

Then the one-step transition rate matrix can be derived in the following,

$$\mathbf{Q}_i = \begin{array}{c} (3,0,0,0,0) \\ (2,1,0,0,0) \\ (1,2,0,0,0) \\ (2,0,1,0,0) \\ (1,1,1,0,1) \\ (1,1,1,0,2) \\ (2,0,0,1,0) \\ F_i \end{array} \left[\begin{array}{ccccccc|c} -3\lambda_{32}^{i,1} & 3\lambda_{32}^{i,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\lambda_{32}^{i,1} - \lambda_{21}^{i,1} & 2\lambda_{32}^{i,1} & \lambda_{21}^{i,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\lambda_{21}^{i,1} - \lambda_{32}^{i,1} & 0 & 2\lambda_{21}^{i,1} & 0 & 0 & \lambda_{32}^{i,1} \\ 0 & 0 & 0 & -2\lambda_{32}^{i,1} - \lambda_{10}^{i,1} & 2\lambda_{32}^{i,1} & 0 & \lambda_{10}^{i,1} & 0 \\ 0 & 0 & 0 & 0 & -\sum_{g=1}^{3}\lambda_{g(g-1)}^{i,2} - \lambda_{i,p} & \lambda_{i,p} & 0 & \sum_{g=1}^{3}\lambda_{g(g-1)}^{i,2} \\ 0 & 0 & 0 & 0 & 0 & -\sum_{g=1}^{3}\lambda_{g(g-1)}^{i,1} & 0 & \sum_{g=1}^{3}\lambda_{g(g-1)}^{i,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda_{32}^{i,1} & 2\lambda_{32}^{i,1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

According to the one-step transition rate matrix for the subsystem i, the reliability, the lifetime distribution, and the lth moments of the lifetime of the subsystem i can be expressed respectively as,

$$R_i(t) = \boldsymbol{\alpha}_i \exp(\mathbf{Q}_{W_iW_i}t)\mathbf{I}_i, \tag{2}$$

$$F_i(t) = 1 - R_i(t) = 1 - \boldsymbol{\alpha}_i \exp(\mathbf{Q}_{W_iW_i}t)\mathbf{I}_i, \tag{3}$$

$$E\left[T_i{}^l\right] = (-1)^l l! \boldsymbol{\alpha}_i \mathbf{Q}_{W_iW_i}^{-1} I_i{}^T,\ (l = 1, 2, \ldots), \tag{4}$$

where $\boldsymbol{\alpha}_i = (1, 0, 0, \ldots, 0)_{1\times|W_i|}$ represents the initial state at $t = 0$, and $\mathbf{I}_i = (1, 1, \ldots, 1)^T_{1\times|W_i|}$.

3.2. Reliability Evaluation for Entire System

After acquiring knowledge of the reliability of a single subsystem, the finite Markov chain imbedding method is applied to formulate the state probability function of the entire system. Define a random variable N_i^s to represent the number of failed subsystems in the first i subsystems of the entire system. A Markov chain with the random variable N_i^s is built as

$$Y_i = N_i^s, i = 1, 2, \ldots, n,$$

where the initial state of the Markov chain is $Y_0 = 0$ at $t = 0$.

The state space of the defined Markov chain is

$$\Omega = W \cup F = \{n_i^s : 0 \le n_i^s \le n-1\} \cup \{n_i^s : n_i^s = n\},$$

where W is the set of the working states of the system and F denotes the failed state of the system with a meaning of no working subsystem. Therefore, this Markov chain has $n+1$ states in total. The transient rules among the states of the Markov chain for the entire system are listed as follows.

(1) If $n_i^s \in \{0, 1, \ldots, n-1\}$, $P\{Y_{i+1} = n_i^s | Y_i = n_i^s\} = R_i(t)$.
(2) If $n_i^s \in \{0, 1, \ldots, n-2\}$, $P\{Y_{i+1} = n_i^s + 1 | Y_i = n_i^s\} = 1 - R_i(t)$.
(3) $P\{Y_{i+1} = F | Y_i = n-1\} = 1 - R_i(t)$.
(4) $P\{Y_{i+1} = F | Y_i = F\} = 1$.
(5) All other transition probabilities are zeros.

After figuring out the transition rules, the one-step transition probability matrix can be formulated as:

$$\mathbf{A}_i = \begin{array}{c} 0 \\ 1 \\ \vdots \\ n-2 \\ n-1 \\ F \end{array} \left[\begin{array}{cccccc|c} R_i(t) & 1-R_i(t) & \cdots & 0 & 0 & & 0 \\ 0 & R_i(t) & \cdots & 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_i(t) & 1-R_i(t) & & 0 \\ 0 & 0 & \cdots & 0 & R_i(t) & & 1-R_i(t) \\ \hline 0 & 0 & \cdots & 0 & 0 & & 1 \end{array}\right]$$

Subsequently, the state probability of the Markov chain can be expressed as follows:

$$\mathbf{P}(t) = \boldsymbol{\alpha} \prod_{i=1}^{n} \mathbf{A}_i \tag{5}$$

where $\boldsymbol{\alpha} = [1, 0, \ldots, 0]_{1\times(n+1)}$. Then the state probability function of the entire system is represented as:

$$R_m^s(t) = \begin{cases} \mathbf{P}(t, 1), & m = 0 \\ \sum\limits_{j=k_{n-1}+1}^{k_m} \mathbf{P}(t, j), & m = 1, 2, \ldots, M-1 \\ \mathbf{P}(t, n+1), & m = M \end{cases} \tag{6}$$

where $\mathbf{P}(t, j)$ represents the value of the j-th element in the vector $\mathbf{P}(t)$.

Example 3. The application of the finite Markov chain imbedding method is illustrated as follows to derive the state probability function of the entire system. The system consists of 5 ($n = 5$) subsystems, and the reliability of each subsystem is r_1, r_2, r_3, r_4, r_5. State 1 of the entire system indicates that the number of failed subsystems in the system is greater than 0 ($k_0 = 0$) and less than or equal to 2 ($k_1 = 2$). State 2 denotes that the number of failed subsystems in the system is greater than 2 ($k_1 = 2$) and not more than 4 ($k_2 = 4$). Then the Markov chain is constructed as,

$$Y_i = N_i^s, i = 1, 2, 3, 4, 5$$

with the initial state $Y_0 = 0$ at $t = 0$.

The state space of the Markov chain is obtained as

$$\Omega = W \cup F = \{0, 1, 2, 3, 4\} \cup \{5\}$$

The one-step transition probability matrix $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$ can be written as

$$\mathbf{A}_i = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ F \end{array} \left[\begin{array}{ccccc|c} r_i & 1-r_i & 0 & 0 & 0 & 0 \\ 0 & r_i & 1-r_i & 0 & 0 & 0 \\ 0 & 0 & r_i & 1-r_i & 0 & 0 \\ 0 & 0 & 0 & r_i & 1-r_i & 0 \\ 0 & 0 & 0 & 0 & r_i & 1-r_i \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

The state probability vector of the Markov chain is derived as:

$$\mathbf{P}(t) = [1, 0, 0, 0, 0, 0] \prod_{i=1}^{5} \mathbf{A}_i = \begin{bmatrix} \mathbf{P}(t,1) & \mathbf{P}(t,2) & \mathbf{P}(t,3) & \mathbf{P}(t,4) & \mathbf{P}(t,5) & \mathbf{P}(t,6) \end{bmatrix},$$

where $\mathbf{P}(t,1) = r_1 r_2 r_3 r_4 r_5$, $\begin{array}{l} \mathbf{P}(t,2) = -r_5(r_4(r_3(r_1(r_2-1)+r_2(r_1-1))+r_1r_2(r_3-1)) \\ \quad +r_1r_2r_3(r_4-1)) - r_1r_2r_3r_4(r_5-1) \end{array}$,

$$\begin{array}{l} \mathbf{P}(t,3) = r_5((r_4-1)(r_3(r_1(r_2-1)+r_2(r_1-1))+r_1r_2(r_3-1)) \\ +r_4((r_1(r_2-1)+r_2(r_1-1))(r_3-1)+r_3(r_1-1)(r_2-1))) \\ +(r_4(r_3(r_1(r_2-1)+r_2(r_1-1))+r_1r_2(r_3-1))+r_1r_2r_3(r_4-1))(r_5-1) \end{array},$$

$$\begin{array}{l} \mathbf{P}(t,4) = -(r_5-1)((r_4-1)(r_3(r_1(r_2-1)+r_2(r_1-1))+r_1r_2(r_3-1)) \\ +r_4((r_1(r_2-1)+r_2(r_1-1))(r_3-1)+r_3(r_1-1)(r_2-1))) \\ -r_5(((r_1(r_2-1)+r_2(r_1-1))(r_3-1)+r_3(r_1-1)(r_2-1))(r_4-1)+r_4(r_1-1)(r_2-1)(r_3-1)) \end{array},$$

$$\begin{array}{l} \mathbf{P}(t,5) = (((r_1(r_2-1)+r_2(r_1-1))(r_3-1)+r_3(r_1-1)(r_2-1))(r_4-1)+r_4(r_1-1)(r_2-1)(r_3-1))(r_5-1) \\ +r_5(r_1-1)(r_2-1)(r_3-1)(r_4-1) \end{array}$$

and $\mathbf{P}(t,6) = -(r_1-1)(r_2-1)(r_3-1)(r_4-1)(r_5-1)$.

Subsequently, the state probability function of the entire system can be gained as follows:

$$R_m^s(t) = \begin{cases} \mathbf{P}(t,1), & m = 0 \\ \mathbf{P}(t,2) + \mathbf{P}(t,3), & m = 1 \\ \mathbf{P}(t,4) + \mathbf{P}(t,5), & m = 2 \\ \mathbf{P}(t,6), & m = 3 \end{cases}$$

4. Numerical Examples

A lithium battery system is used to demonstrate the applicability of the proposed model and the validity of the applied method. In a lithium battery pack, there are 6 ($n = 6$) subsystems, each of which consists of several battery cells. During operation, the batteries in each subsystem should maintain the SOC balance, that is, the batteries belonging to the same subsystem should maintain the same level of SOC. Sharing the SOC via the common bus is one of the commonly used methods for the SOC balance of the lithium battery system. It refers to the fact that the degradation paths of individual cells are not the same, which may lead to unequal SOC between batteries. The common bus transmits the SOC to the cells with a SOC below the average value, thereby making the SOC equal between the cells in real time. Since this process happens instantaneously, the time is negligible. When the total performance of the lithium battery subsystem reduces to a certain value, the battery protection will be automatically turned on to lower the degradation rate and slow down the deterioration of the batteries. Meanwhile, the battery protection may also fail over time. Once the total performance of the lithium battery subsystem is not greater than a certain threshold, the subsystem is regarded as failed. The entire battery system is divided into multiple states according to the number of failed subsystems, as shown in Table 1. The parameters of the batteries in each subsystem are presented in Table 2.

Table 1. Multiple states of entire system and corresponding number of failed subsystems.

State of entire System	**0**	**1**	**2**	**3**	**4**
Number of failed subsystems	0	1, 2	3, 4	5	6
Parameters of state division	-	$k_0 = 0,\ k_1 = 2$	$k_2 = 4$	$k_3 = 5$	-

Table 2. Model parameters in the operation mechanism of each subsystem.

Subsystem *i*	n_i	$(\lambda^{i,1}_{h_i(h_i-1)}, \lambda^{i,2}_{h_i(h_i-1)})$, h_i=3,2,1	$\lambda_{i,p}$	G_i	$d_{i,1}, d_{i,2}$
1	3	(0.02, 0.01)	0.15	$\{0,2,4,8\}$	14, 12
2	3	(0.02, 0.01)	0.1	$\{0,2,4,8\}$	14, 10
3	3	(0.02, 0.01)	0.1	$\{0,2,4,8\}$	14, 12
4	3	(0.02, 0.01)	0.15	$\{0,2,4,8\}$	16, 12
5	3	(0.02, 0.005)	0.1	$\{0,2,4,8\}$	14, 12
6	4	(0.15, 0.01)	0.1	$\{0,2,4,8\}$	14, 12

When the proposed model is utilized in practice, the confidence intervals of the reliability indexes and a set of model parameters can be estimated based on the data. For example, the literature [49–51] can provide reference values for the confidence intervals of the reliability indexes. Estimation methods, such as the moment estimation and maximum likelihood estimation, can be employed to estimate the model parameters [52–54].

Based on the above analysis processes, the reliability of a single subsystem and the state probability function of the entire system can be obtained. The reliability curves of each subsystem are presented in Figure 5. As seen in Figure 5, the reliability of each subsystem decreases along with time until it converges to 0. This is because the batteries in the subsystem degrade over time, without considering maintenance, which includes the activities to improve the battery performance.

Figure 6 gives multiple comparisons of the reliability between two subsystems, with the aim of carrying out the sensitivity analysis of the model parameters of the subsystems. By comparing subsystem 1 and 3, the reliability of these two subsystems is not obviously different with different failure rates of the protective device, as shown in Figure 6a. Figure 6b shows the comparison of subsystem 1 and 4, which examine the effect of varying triggering parameters of protective devices on the subsystem reliability. The reliability of subsystem 4 is slightly larger than that of subsystem 1 because its device can be triggered earlier. In Figure 6c, the parameter difference between subsystem 2 and 3 is the different failure criteria of the entire system. It can be observed in Figure 6c that the parameter of the failure criterion of the entire system exerts great influence on the system reliability, and the reliability of subsystem 2, with a more stringent failure criterion, is much greater than that of subsystem 3. In Figure 6d, the failure rate and triggering parameters of protective device are all different, but the difference in the reliability of subsystem 3 and 4 is not large. As seen in Figure 6e, the reliability curve of subsystem 5 is in a higher place than that of subsystem 3, resulting from the smaller degradation rate of the units during the operation of the protective device. Figure 6f reveals that the reliability of subsystem 3 is significantly higher than that of subsystem 6, owing to the much smaller degradation rates of the units during operation without protection from the device. To sum up, it can be determined that the failure rates of the units and the failure threshold of the subsystem have a greater impact on the reliability of the subsystem.

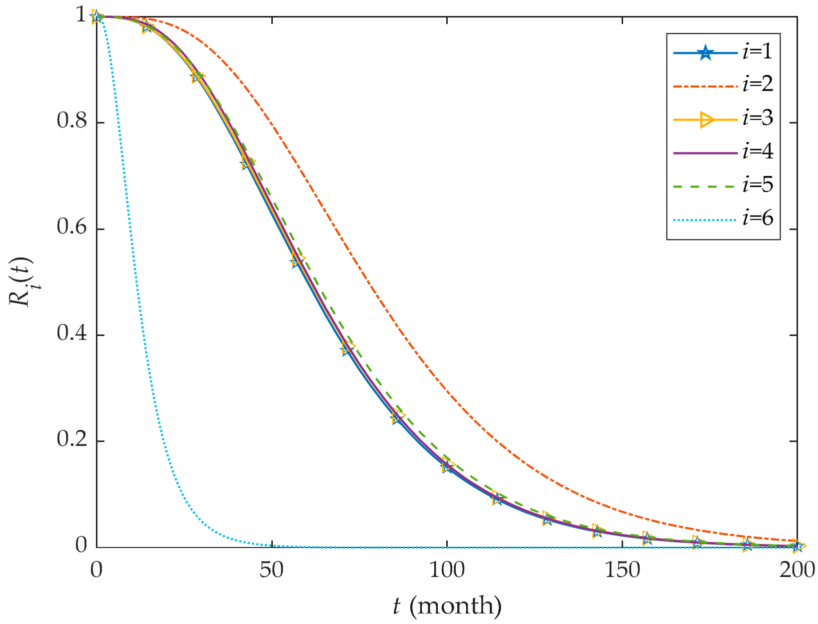

Figure 5. Reliability of each subsystem in the system.

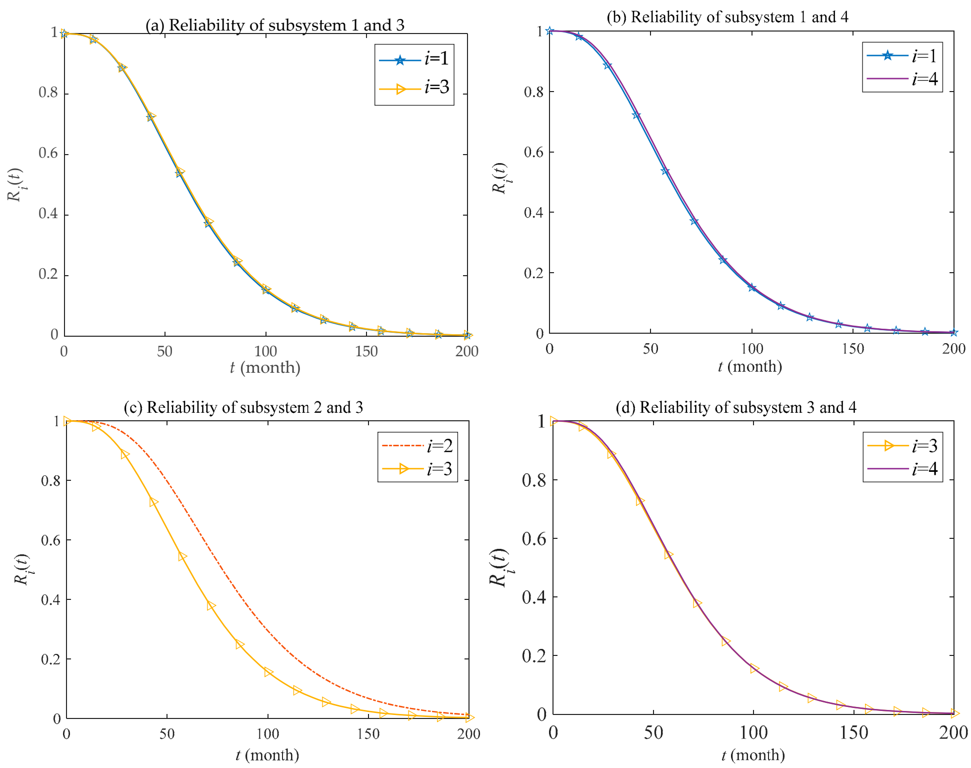

Figure 6. *Cont.*

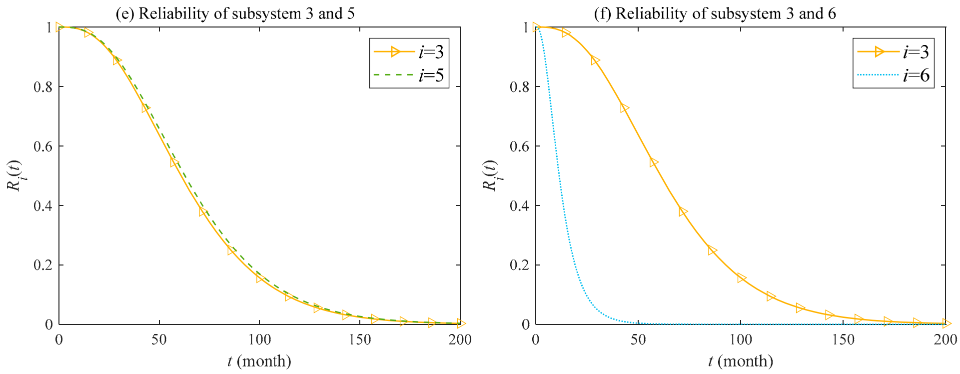

Figure 6. Comparisons of the reliability of different subsystems.

Figure 7 presents the state probability function of the entire system reliability varying with time. The system is in state 0 at the initial time, and then the probability of the system in state 0 gradually decreases until converging to 0 at about $t = 40$ months. Along with the time, the probability of the system in the intermediate states (states 1, 2, and 3) rises first and then decreases until they drop to 0. The likelihood that the system is in state 4 equals 0 at the initial moment and gradually gets larger until reaching the value 1. It can be interpreted that the number of failed subsystems gradually increases with the time, and the conditions of being in the intermediate states will be satisfied at first, and eventually all subsystems will fail, which leads to the system staying in state 4.

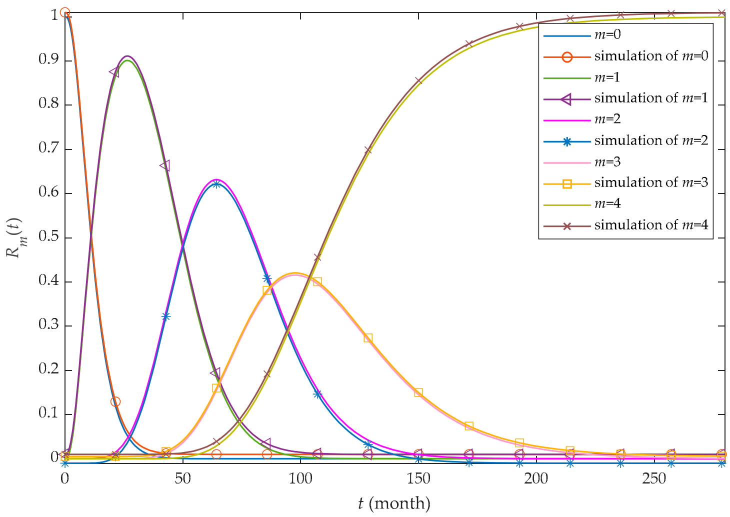

Figure 7. Analytic and simulative results of state probability function of an entire system.

Additionally, this paper constructs a Monte Carlo simulation-based algorithm to derive the state probability function of the entire system, presented in Figure 8. As displayed in Figure 7, the curves of the simulative and analytical results fit perfectly, which demonstrates the correctness of the state probability function of the system obtained by the proposed method. In the simulation procedure, it is applicable that the state residence times of units and the lifetime of protective devices follow other appropriate stochastic distributions, which are not limited to exponential distributions.

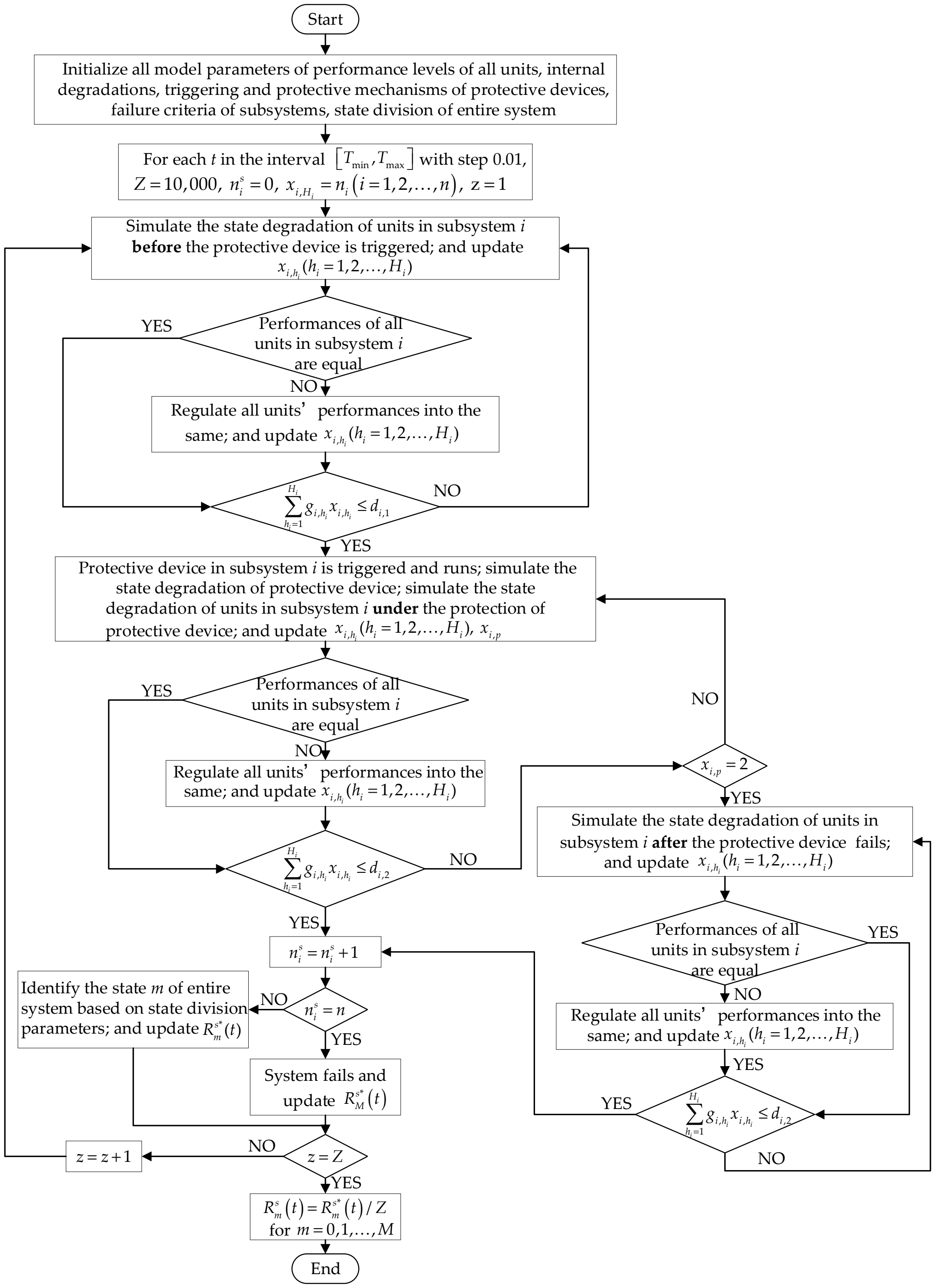

Figure 8. Simulation algorithm for deriving the state probability function of an entire system.

Based on the above results, it can be summarized that the degradation rates of the units, the failure rate of the protective device, the triggering conditions of the protective device and the failure threshold of the subsystem all affect the reliability of the system. Nevertheless, various influencing factors exert different degrees of effects on the system reliability. Therefore, under the condition of limited cost, sensitivity analyses of various influencing factors can be conducted to determine the key factors, and then feasible suggestions can be put forward for the engineers.

5. Conclusions

This work builds a reliability model that comprehensively considers the balancing mechanism, performance sharing mechanism, and protection mechanism. The entire system contains several subsystems, and each subsystem consists of multiple units. In one subsystem, each unit has multiple levels of performance, and the equal performance among all units is required for the smooth operation of the subsystem. When some units are in different performance states and the subsystem is unbalanced, the performance can be transmitted through the common bus to make the subsystem regain the performance balance. When the total performance of the subsystem reaches a certain threshold, the protective device is triggered and starts to work on reducing the degradation rates of the units. In addition, the protective device may also fail, and the degradation rates of the units return to the original values after the device failure. The subsystem fails once the total performance of the units is lower than or equals a certain threshold. Based on the number of failed subsystems, the state of the system is divided into multiple states. The research results show that the system reliability is affected by many factors, such as the unit degradation rates, protection device triggering conditions, degradation rate of the protective device, system failure threshold, and so on. Nevertheless, the above factors all have limits to their influence on the system reliability.

Despite this research fruit regarding the reliability of the multi-state k-out-of-n system with multiple operation mechanisms, some extensive works can be conducted in the following three aspects. First, the common bus for the performance rebalancing actions is considered to have infinite capacity. Future study can consider the capacity of the common bus as having a maximum value, which may result in the failure of performance rebalancing actions, causing system imbalance. Second, it is assumed that the protective device has binary states in this paper. It is worth studying the case where the protective device has multiple states in the future, which is more general. Third, deriving the confidence intervals of the reliability functions of the proposed model could be another interesting topic in the future.

Author Contributions: Conceptualization and writing—original draft preparation, Y.S.; Methodology, funding acquisition and writing—review and editing, X.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Beijing Social Science Foundation (Grant No. 20GLC052).

Data Availability Statement: All data is included in the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

n	Total number of subsystems in entire system
n_i	Total number of units in subsystem i
$\lambda^{i,1}_{h_i(h_i-1)}$	Degradation rate of the units deteriorating from state h_i to $h_i - 1$ in subsystem i without the protection from protective device
$\lambda^{i,2}_{h_i(h_i-1)}$	Degradation rate of the units deteriorating from state $h_i$to $h_i - 1$ in subsystem i when the protective device works
$\lambda_{i,p}$	Degradation rate of the protective device in subsystem i when the protective device works
$d_{i,1}$	Critical threshold of the total performance of the subsystem i leading to the trigger of its protective device
$d_{i,2}$	Critical value of the total performance of the subsystem i leading to its failure
$M + 1$	Total number of system states
$R_i(t)$	Reliability function of the subsystem i
$R^s_m(t)$	State probability function of the entire system

References

1. Dui, H.Y.; Tian, T.Z.; Zhao, J.B.; Wu, S.M. Comparing with the joint importance under consideration of consecutive-*k*-out-of-*n* system structure changes. *Reliab. Eng. Syst. Saf.* **2022**, *219*, 108255. [CrossRef]
2. Lisnianski, A.; Ding, Y. Redundancy analysis for repairable multi-state system by using combined stochastic processes methods and universal generating function technique. *Reliab. Eng. Syst. Saf.* **2009**, *94*, 1788–1795. [CrossRef]
3. Levitin, G. Reliability of multi-state systems with common bus performance sharing. *IIE Trans.* **2011**, *43*, 518–524. [CrossRef]
4. Yu, H.; Yang, J.; Mo, H. Reliability analysis of repairable multi-state system with common bus performance sharing. *Reliab. Eng. Syst. Saf.* **2014**, *132*, 90–96. [CrossRef]
5. Zhao, X.; Wu, C.S.; Wang, S.Q.; Wang, X.Y. Reliability analysis of multi-state *k*-out-of-*n*: G system with common bus performance sharing. *Comput. Ind. Eng.* **2018**, *124*, 359–369. [CrossRef]
6. Peng, R.; Liu, H.L.; Xie, M. A study of reliability of multi-state systems with two performance sharing groups. *Qual. Reliab. Eng. Int.* **2016**, *32*, 2623–2632. [CrossRef]
7. Yu, H.; Yang, J.; Lin, J.; Zhao, Y. Reliability evaluation of non-repairable phased-mission common bus systems with common cause failures. *Comput. Ind. Eng.* **2017**, *111*, 445–457. [CrossRef]
8. Yu, H.; Yang, J.; Zhao, Y. Reliability of nonrepairable phased-mission systems with common bus performance sharing. *Proc. Inst. Mech. Eng. O J. Risk Reliab.* **2018**, *232*, 647–660. [CrossRef]
9. Cheng, C.; Yang, J.; Li, L. Reliability assessment of multi-state phased mission systems with common bus performance sharing considering transmission loss and performance storage. *Reliab. Eng. Syst. Saf.* **2020**, *199*, 106917. [CrossRef]
10. Yi, K.; Xiao, H.; Kou, G.; Peng, R. Trade-off between maintenance and protection for multi-state performance shairng system with transmission loss. *Comput. Ind. Eng.* **2019**, *136*, 305–315. [CrossRef]
11. Qiu, S.; Ming, H.X.G. Reliability evaluation of multi-state series-parallel systems with common bus performance sharing considering transmission loss. *Reliab. Eng. Syst. Saf.* **2019**, *189*, 406–415. [CrossRef]
12. Xiao, H.; Shi, D.; Ding, Y.; Peng, R. Optimal loading and protection of multi-state systems considering performance sharing mechanism. *Reliab. Eng. Syst. Saf.* **2016**, *149*, 88–95. [CrossRef]
13. Xiao, H.; Zhang, Y.; Xiang, Y.; Peng, R. Optimal design of a linear sliding window system with consideration of performance sharing. *Reliab. Eng. Syst. Saf.* **2020**, *198*, 106900. [CrossRef]
14. Peng, R. Optimal component allocation in a multi-state system with hierarchical performance sharing groups. *J. Oper. Res. Soc.* **2019**, *70*, 581–587. [CrossRef]
15. Cui, L.R.; Chen, J.H.; Li, X.C. Balanced reliability systems under Markov processes. *IISE Trans.* **2019**, *51*, 1025–1035. [CrossRef]
16. Zhao, X.; Wang, S.Q.; Wang, X.Y.; Fan, Y. Multi-state balanced systems in a shock environment. *Reliab. Eng. Syst. Saf.* **2020**, *193*, 106592. [CrossRef]
17. Wang, X.Y.; Ning, R.; Zhao, X.; Zhou, J. Reliability analyses of *k*-out-of-*n*: F capability-balanced systems in a multi-source shock environment. *Reliab. Eng. Syst. Saf.* **2022**, *227*, 108733. [CrossRef]
18. Zhao, X.; Wu, C.S.; Wang, X.Y.; Sun, J.L. Reliability analysis of *k*-out-of-*n*: F balanced systems with multiple functional sectors. *Appl. Math. Model.* **2020**, *82*, 108–124. [CrossRef]
19. Cui, L.R.; Gao, H.D.; Mo, Y.C. Reliability for *k*-out-of-*n*: F balanced systems with *m* sectors. *IISE Trans.* **2018**, *50*, 381–393. [CrossRef]
20. Wang, X.; Zhao, X.; Wu, C.; Lin, C. Reliability assessment for balanced systems with restricted rebalanced mechanisms. *Comput. Ind. Eng.* **2020**, *149*, 106801. [CrossRef]
21. Hua, D.G.; Elsayed, E.A. Reliability estimation of *k*-out-of-*n* pairs:G balanced systems with spatially distributed units. *IEEE Trans. Reliab.* **2016**, *65*, 886–900. [CrossRef]
22. Hua, D.G.; Elsayed, E.A. Degradation analysis of *k*-out-of-*n* pairs:G balanced system with spatially distributed units. *IEEE Trans. Reliab.* **2016**, *65*, 941–956. [CrossRef]
23. Hua, D.G.; Elsayed, E.A. Reliability approximation of *k*-out-of-*n* pairs: G balanced systems with spatially distributed units. *IISE Trans.* **2018**, *50*, 616–626. [CrossRef]
24. Wu, C.S.; Zhao, X.; Wang, X.Y.; Wang, S.Q. Reliability analysis of performance-based balanced systems with common bus performance sharing. *Reliab. Eng. Syst. Saf.* **2021**, *215*, 107865. [CrossRef]
25. Zhao, X.J.; Wang, Z.Y. Maintenance policies for two-unit balanced systems subject to degradation. *IEEE Trans. Reliab.* **2022**, *71*, 1116–1126. [CrossRef]
26. Dui, H.Y.; Zhang, C.; Tian, T.Z.; Wu, S.M. Different costs-informed component preventive maintenance with system lifetime changes. *Reliab. Eng. Syst. Saf.* **2022**, *228*, 108755. [CrossRef]
27. Dui, H.; Xu, Z.; Chen, L.; Xing, L.; Liu, B. Data-driven maintenance priority and resilience evaluation of performance loss in a main coolant system. *Mathematics* **2022**, *10*, 563. [CrossRef]
28. Zhao, X.; Li, R.; Fan, Y.; Qiu, Q.G. Reliability and optimal triggering policy for multi-state systems subject to shocks and supported by a protective device. *Comput. Ind. Eng.* **2021**, *156*, 107232. [CrossRef]
29. Chaurasiya, K.L.; Bhattacharya, B.; Varma, A.K.; Rastogi, S. Dynamic modeling of a cabin pressure control system. *Proc. Inst. Mech. Eng. Part G J. Aerosp. Eng.* **2020**, *234*, 401–415. [CrossRef]
30. Zhao, X.; Chai, X.; Sun, J.; Qiu, Q. Joint optimization of mission abort and protective device selection policies for multistate systems. *Risk Anal.* **2022**. [CrossRef]

31. Wu, Y.G.; Qiu, Q.A. Optimal triggering policy of protective devices considering self-exciting mechanism of shocks. *Mathematics* **2022**, *10*, 2732. [CrossRef]
32. Zhao, X.; Li, R.; Fan, Y.; Qiu, Q.G. Reliability modeling for multi-state systems with a protective device considering multiple triggering mechanism. *Proc. Inst. Mech. Eng. O J. Risk Reliab.* **2022**, *236*, 173–193. [CrossRef]
33. Zhao, X.; Dong, B.B.; Wang, X.Y.; Song, Y.B. Reliability assessment for a k-out-of-n: F system supported by a multi-state protective device in a shock environment. *Comput. Ind. Eng.* **2022**, *171*, 108426. [CrossRef]
34. Wang, X.Y.; Ning, R.; Zhao, X.; Zhou, J. Reliability evaluations for a multi-state k-out-of-n: F system with m subsystems supported by multiple protective devices. *Comput. Ind. Eng.* **2022**, *171*, 108409. [CrossRef]
35. Wang, X.Y.; Ning, R.; Zhao, X.; Wu, C.S. Reliability assessments for two types of balanced systems with multi-state protective devices. *Reliab. Eng. Syst. Saf.* **2022**, *229*, 108852. [CrossRef]
36. Zhu, M.M.; Pham, H. A novel system reliability modeling of hardware, software, and interactions of hardware and software. *Mathematics* **2019**, *7*, 1049. [CrossRef]
37. Wu, C.S.; Zhao, X.; Qiu, Q.G.; Sun, J.L. Optimal mission abort policy for k-out-of-n: F balanced systems. *Reliab. Eng. Syst. Saf.* **2021**, *208*, 107398. [CrossRef]
38. Wang, S.Q.; Zhao, X.; Tian, Z.G.; Zuo, M.J. Optimum component reassignment for balanced systems with multi-state components operating in a shock environment. *Reliab. Eng. Syst. Saf.* **2021**, *210*, 107514. [CrossRef]
39. Wu, C.S.; Zhao, X.; Wang, S.Q.; Song, Y.B. Reliability analysis of consecutive-k-out-of-r-from-n subsystems: F balanced systems with load sharing. *Reliab. Eng. Syst. Saf.* **2022**, *228*, 108776. [CrossRef]
40. Wu, C.S.; Pan, R.; Zhao, X. Reliability assessment of multi-state performance sharing systems with transmission loss and random shocks. *IISE Trans.* **2022**, *54*, 1060–1071. [CrossRef]
41. Yin, J.; Cui, L.R.; Sun, Y.D.; Balakrishnan, N. Reliability modelling for linear and circular k-out-of-n: F systems with shared components. *Reliab. Eng. Syst. Saf.* **2022**, *219*, 108172. [CrossRef]
42. Eryilmaz, S. Compound Markov negative binomial distribution. *J. Comput. Appl. Math.* **2016**, *292*, 1–6. [CrossRef]
43. Wang, X.Y.; Zhao, X.; Sun, J.L. A compound negative binomial distribution with mutative termination conditions based on a change point. *J. Comput. Appl. Math.* **2019**, *351*, 237–249. [CrossRef]
44. Eryilmaz, S. A new class of lifetime distributions. *Stat. Probab. Lett.* **2016**, *112*, 63–71. [CrossRef]
45. Eryilmaz, S. Assessment of a multi-state system under a shock model. *Appl. Math. Comput.* **2015**, *269*, 1–8. [CrossRef]
46. Eryilmaz, S. Discrete time shock models involve ing runs. *Stat. Probab. Lett.* **2015**, *107*, 93–100. [CrossRef]
47. Wang, X.Y.; Zhao, X.; Wu, C.S.; Wang, S.Q. Mixed shock model for multi-state weighted k-out-of-n: F systems with degraded resistance against shocks. *Reliab. Eng. Syst. Saf.* **2022**, *217*, 108098. [CrossRef]
48. Cui, L.R.; Xu, Y.; Zhao, X.A. Developments and applications of the finite Markov chain imbedding approach in reliability. *IEEE Trans. Reliab.* **2010**, *59*, 685–690. [CrossRef]
49. Wang, Z.Y.; Shafieezadeh, A. On confidence intervals for failure probability estimates in Kriging-based reliability analysis. *Reliab. Eng. Syst. Saf.* **2020**, *196*, 106758. [CrossRef]
50. Wang, Z.Z.; Bai, X.C.; Li, J.J. Modified generalized confidence interval for the stress-strength reliability from exponentiated Weibull distribution. *Concurr. Comp.-Pract. E* **2022**, *34*, e6969. [CrossRef]
51. Wang, B.X.; Geng, Y.P.; Zhou, J.X. Inference for the generalized exponential stress-strength model. *Appl. Math. Model.* **2018**, *53*, 267–275. [CrossRef]
52. Lawless, J.F. *Statistical Models and Methods for Lifetime Data*; Wiley-Interscience: New York, NY, USA, 1982.
53. Tian, Y.Z.; Zhu, Q.Q.; Tian, M.Z. Inference for mixed generalized exponential distribution under progressively type-II censored samples. *J. Appl. Stat.* **2014**, *41*, 660–676. [CrossRef]
54. Jiang, T.; Liu, Y. Parameter inference for non-repairable multi-state system reliability models by multi-level observation sequences. *Reliab. Eng. Syst. Saf.* **2017**, *166*, 3–15. [CrossRef]