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New Applications of Perov's Fixed Point Theorem

Sorin Mureşan [1], Loredana Florentina Iambor [1,*] and Omar Bazighifan [2,3,4,*]

[1] Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania
[2] Department of Mathematics, Faculty of Science, Hadhramout University, Mukalla 50512, Yemen
[3] Department of Mathematics, Faculty of Education, Seiyun University, Seiyun 50512, Yemen
[4] Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Roma, Italy
* Correspondence: iambor.loredana@gmail.com (L.F.I.); o.bazighifan@gmail.com (O.B.)

Abstract: The goal of this paper is to consider a differential equation system written as an interesting equivalent form that has not been used before. Using Perov's fixed point theorem in generalized metric spaces, the existence and uniqueness of the solution are obtained for the proposed system. The approximation of the solution is given, and as a novelty, the approximation of its derivative is also obtained using the same iteration steps.

Keywords: fixed point; Perov's fixed point theorem; differential equation system

MSC: 58C30

1. Introduction

The concept of a metric space is a very important tool in many scientific fields and particulary in the fixed point theory. The fundamental result of a metric fixed-point theory is the Banach contraction principle. This result has many applications that are not only observed in different branches of mathematics, such as ordinary differential equations, partial differential equations, integral equations, optimization, and variational analysis, but it is also used as an effective tool in other subjects, such as economics, game theory, and biology.

One of the worthwhile generalization of these results was provided by Perov [1] in 1964. In [1], Perov extended the Banach contraction principle to a space with a vector-valued metric. This result helps us study the existence of different solutions for different types of differential and integral equations.

Some interesting contributions to the development of fixed point theories and its applications in this context were obtained over the years by I.A. Rus [2,3], A.M. Bica, S. Mureşan [4,5], A.M. Bica [6], A. Bucur, L. Guran, A. Petruşel [7], A.D. Filip, A. Petruşel [8], M.U. Ali, J.K. Kim [9,10], M. Abbas, V. Rakocevic, A. Iqbal [11], and I. Altun et al. [12],

A very recent application of Perov's results can be seen in the studies of L. Guran, M. Bota [13], N. Mirkov, S. Radojevic [14], A. Petruşel, G. Petruşel [15], and Y. Almalki et al. [16]. Moreover, several researchers studied the common fixed problem in different spaces, for instance, the common fixed problem that exists in a cone metric space [17,18], the fixed point with a Taylor expansion [19], and the fixed point in a metric space and the Bielecki metric [20,21].

In this paper, we show that the Perov's fixed point theorem is also applicable to the following system.

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \\ x(t_0) = x_0, y(t_0) = y_0 \end{cases}, t \in [t_0, t_1]. \quad (1)$$

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By applying $\int_{t_0}^{t} ds$ in both members of the differential equations (see [22]), we obtain the following equivalent Volterra integral system.

$$\begin{cases} x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), y(s))ds \\ y(t) = y_0 + \int_{t_0}^{t} g(s, x(s), y(s))ds \end{cases}, t \in [t_0, t_1]. \tag{2}$$

Starting from Burton's idea in paper [23], we will use Perov's theorem in generalized metric spaces to prove the existence and uniqueness of the solution for the system.

The study begins by presenting an equivalent form of the differential equation system (1).

$$\begin{cases} x'(t) = f\left(t, x_0 + \int_{t_0}^{t} x'(s)ds, y_0 + \int_{t_0}^{t} y'(s)ds\right) \\ y'(t) = g\left(t, x_0 + \int_{t_0}^{t} x'(s)ds, y_0 + \int_{t_0}^{t} y'(s)ds\right) \end{cases}, t \in [t_0, t_1]. \tag{3}$$

Denoted by $u(t) = x'(t)$ and $v(t) = y'(t)$, the final form will be

$$\begin{cases} u(t) = f\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds\right) \equiv F_1(u(t), v(t)) \\ v(t) = g\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds\right) \equiv F_2(u(t), v(t)) \end{cases}, t \in [t_0, t_1]. \tag{4}$$

which is the following fixed point problem:

$$(u(t), v(t)) = F(u(t), v(t)), \tag{5}$$

where

$$\begin{aligned} F &: C[t_0, t_1] \times C[t_0, t_1] \to C[t_0, t_1] \times C[t_0, t_1], \\ F(u(t), v(t)) &= (F_1(u(t), v(t)), F_2(u(t), v(t))). \end{aligned} \tag{6}$$

In this paper, we will prove that the pair $(u^*(t), v^*(t))$ is the solution of the system.

2. Preliminaries

In this section, we recall the following important concepts; papers [1,24–33] contain more details.

2.1. Generalized Metric Space

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from $\mathbb{R}^n$, we have $x \leq y \Leftrightarrow x_i \leq y_i$, $i = \overline{1, n}$.

Definition 1. *Let X be a nonempty set and $n \in \mathbb{N}$, $n \geqslant 1$. A mapping $d : X \times X \to \mathbb{R}^n$ is called a vector-valued metric on X if the following statements are satisfied for all $x, y, z \in X$:*
(d1) $d(x, y) \geqslant 0_n$, where $0_n = (0, \ldots, 0) \in \mathbb{R}^n$ and $d(x, y) = 0_n$; iff $x = y$;
(d2) $d(x, y) = d(y, x)$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$.

We mention that if $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, and $c \in \mathbb{R}$, then by $\alpha \leqslant \beta$ (respectively, $\alpha < \beta$), we mean that $\alpha_i \leqslant \beta_i$ (respectively $\alpha_i < \beta_i$) for all $i = \overline{1, m}$, and by $\alpha \leqslant c$, we mean that $\alpha_i \leqslant c$ for all $i = \overline{1, m}$.

A set, X, equipped with a vector-valued metric d is called a generalized metric space. We will denote such a space with (X, d).

Example 1. *For $X = \mathbb{R}^n$, we can consider $d(x,y) = (|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|)$.*

Example 2. *We now examine $X = C([a,b], \mathbb{R}^n) = \{f : [a,b] \to \mathbb{R}^n, f \text{ continuous}\}$. For $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n) \in X$ and any positive τ, we can consider the general Bielecki metric.*

$$d_B(f,g) = \left(\max_{t\in[a,b]} |f_1(t) - g_1(t)| e^{-\tau(t-a)}, \ldots, \max_{t\in[a,b]} |f_n(t) - g_n(t)| e^{-\tau(t-a)} \right). \tag{7}$$

Remark 1. *For generalized metric spaces, the notions of a convergent sequence, Cauchy (fundamental) sequence, completeness, open subset, and closed subset are similar to those for usual metric spaces.*

2.2. Lipschitz Condition

Definition 2. *For a general metric space (X,d) ($d(x,y) \in \mathbb{R}^n$), we say that the operator $f : X \to X$ satisfies a Lipschitz condition iff there exists a matrix $L \in M_{nn}(\mathbb{R})$ such that*

$$d(f(x), f(y)) \leq L d(x,y), \forall x, y \in X. \tag{8}$$

Definition 3 ([34])**.** *A matrix $L \in M_{nn}(\mathbb{R}_+)$ is said to be convergent toward zero if and only if $L^n \to \Theta$ as $n \to \infty$, where Θ is the zero $n \times n$ matrix and the identity $n \times n$ matrix is I_n.*

Proposition 1. *For matrix $L \in M_{nn}(\mathbb{R})$, the following statements are equivalent:*
(i) L converges at zero;
(ii) The eigenvalues of L lies within the open unit disc, i.e., $|\lambda| < 1$, for all $\lambda \in \mathbb{C}$ with $\det(A - \lambda I_n) = 0$;
(iii) The matrix $I_n - L$ is non-singular and $(I_n - L)^{-1} = I_n + L + \ldots + L^k + \ldots$.

2.3. Perov Theorem

Theorem 1 ([1])**.** *Let (X,d) be a complete general metric space ($d \in \mathbb{R}^n$) and an operator $f{:}X \to X$, which satisfies Lipschitz condition (8). If matrix L converges at zero, then the following is the case:*
(i) f has a unique fixed point x^*;
(ii) For any $x_0 \in X$, the successive approximations sequence $(f^m(x_0))_m$ converges at x^*;
(iii) For any $m \in \mathbb{N}^*$, we have the following estimation.

$$d(f^m(x_0), x^*) \leq L^m (I_n - L)^{-1} d(x_0, x_1). \tag{9}$$

3. Main Results

In this section, we prove the existence and uniqueness of the solution for the system defined by (1) by applying Perov's fixed point theorem in the case of the equivalent form of the system given in (3).

Theorem 2. *Let X be a nonempty set and $F : C[t_0,t_1] \times C[t_0,t_1] \to C[t_0,t_1] \times C[t_0,t_1]$ be an operator, which is defined by*

$$F(u(t), v(t)) \quad = \quad (F_1(u(t), v(t)), F_2(u(t), v(t))) \tag{10}$$

where functions $F_1, F_2 : X \to \mathbb{R}$ are given by

$$\begin{cases} F_1(u(t), v(t)) = f\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds \right) \\ F_2(u(t), v(t)) = g\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds \right) \end{cases}, t \in [t_0, t_1]. \tag{11}$$

If functions $f, g : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following Lipschitz conditions*

$$\begin{aligned} |f(t,u,v) - f(t,\overline{u},\overline{v})| \leq \quad & a|u-\overline{u}| + b|v-\overline{v}| \text{ and} \\ |g(t,u,v) - g(t,\overline{u},\overline{v})| \leq \quad & c|u-\overline{u}| + d|v-\overline{v}|, \end{aligned} \tag{12}$$

where a, b, c, *and* d *are positive constants,* $t \in [t_0, t_1]$ *and* $u, \overline{u}, v, \overline{v} \in \mathbb{R}$, *then operator* F *has a unique fixed point* $(u^*(t), v^*(t))$, *which is the unique solution of (11).*

Proof. Let $X = C[t_0, t_1] \times C[t_0, t_1]$ be the general complete metric space endowed with the vector-valued Bielecki-type metric.

$$d_B(f,g) = \left(\max_{t\in[t_0,t_1]} |f_1(t) - g_1(t)| e^{-\tau(t-t_0)}, \max_{t\in[t_0,t_1]} |f_2(t) - g_2(t)| e^{-\tau(t-t_0)} \right).$$

We can use denotation $d_B(f,g) = (d(f_1,g_1), d(f_2,g_2))$ for the sake of simplicity.

Next, we prove that F_1 and F_2 are contractions.

We have $|F_1(u(t), v(t)) - F_1(\overline{u}(t), \overline{v}(t))| =$

$$= \left| f\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds\right) - f\left(t, x_0 + \int_{t_0}^{t} \overline{u}(s)ds, y_0 + \int_{t_0}^{t} \overline{v}(s)ds\right) \right|$$

$$\overset{(hyp)}{\leq} a\left| \int_{t_0}^{t} [u(s) - \overline{u}(s)]ds \right| + b\left| \int_{t_0}^{t} [v(s) - \overline{v}(s)]ds \right| \leq$$

$$\leq a\int_{t_0}^{t} |u(s) - \overline{u}(s)|ds + b\int_{t_0}^{t} |v(s) - \overline{v}(s)|ds =$$

$$= a\int_{t_0}^{t} |u(s) - \overline{u}(s)| e^{-\tau(s-t_0)} e^{\tau(s-t_0)} ds + b\int_{t_0}^{t} |v(s) - \overline{v}(s)| e^{-\tau(s-t_0)} e^{\tau(s-t_0)} ds \leq$$

$$\leq (ad(u,\overline{u}) + bd(v,\overline{v})) \int_{t_0}^{t} e^{\tau(s-t_0)} ds =$$

$$= (ad(u,\overline{u}) + bd(v,\overline{v})) \frac{1}{\tau}\left(e^{\tau(t-t_0)} - 1\right) \leq \left(\frac{a}{\tau} d(u,\overline{u}) + \frac{b}{\tau} d(v,\overline{v})\right) e^{\tau(t-t_0)},$$

where a and b are positive constants.

Using the general Bielecki metric, we finally obtain

$$|F_1(u(t), v(t)) - F_1(\overline{u}(t), \overline{v}(t))| e^{-\tau(t-t_0)} \leq \frac{a}{\tau} d(u,\overline{u}) + \frac{b}{\tau} d(v,\overline{v}) \text{ which gives}$$

$$d(F_1(u,v), F_1(\overline{u},\overline{v})) \leq \frac{a}{\tau} d(u,\overline{u}) + \frac{b}{\tau} d(v,\overline{v}), \tag{13}$$

where a and b are positive constants.

Using the same deduction method, we obtain

$$|F_2(u(t), v(t)) - F_2(\overline{u}(t), \overline{v}(t))| =$$

$$= \left| g\left(t, x_0 + \int_{t_0}^{t} u(s)ds, y_0 + \int_{t_0}^{t} v(s)ds\right) - g\left(t, x_0 + \int_{t_0}^{t} \overline{u}(s)ds, y_0 + \int_{t_0}^{t} \overline{v}(s)ds\right) \right| \leq$$

$$\leq c\left| \int_{t}^{t_0} [u(s) - \overline{u}(s)ds] \right| + d\left| \int_{t}^{t_0} [v(s) - \overline{v}(s)ds] \right| \leq$$

$$\leq c\int_{t_0}^{t} |u(s) - \overline{u}(s)|ds + d\int_{t_0}^{t} |v(s) - \overline{v}(s)|ds$$

$$\leq \left[\frac{c}{\tau} d(u,\overline{u}) + \frac{d}{\tau} d(v,\overline{v})\right] e^{\tau(t-t_0)},$$

where c and d are positive constants.

Using the general Bielecki metric, we obtain

$$|F_2(u(t),v(t)) - F_2(\overline{u}(t),\overline{v}(t))|e^{-\tau(t-t_0)} \leq \frac{c}{\tau}d(u,\overline{u}) + \frac{d}{\tau}d(v,\overline{v})$$

which provides

$$d(F_2(u,v),F_2(\overline{u},\overline{v})) \leq \frac{c}{\tau}d(u,\overline{u}) + \frac{d}{\tau}d(v,\overline{v}), \tag{14}$$

where c and d are positive constants.

Using relations (13) and (14), the operator $F : X \to X, F(u,v) = (F_1(u,v), F_2(u,v))$ can be defined by the following.

$$\begin{aligned} d_B(F(u,v),F(\overline{u},\overline{v})) &= (d(F_1(u,v),F_1(\overline{u},\overline{v})), d(F_2(u,v),F_2(\overline{u},\overline{v}))) \\ &\leq \left(\frac{a}{\tau}d(u,\overline{u}) + \frac{b}{\tau}d(v,\overline{v}), \frac{c}{\tau}d(u,\overline{u}) + \frac{d}{\tau}d(v,\overline{v})\right) \\ &= \begin{pmatrix} \frac{a}{\tau} & \frac{b}{\tau} \\ \frac{c}{\tau} & \frac{d}{\tau} \end{pmatrix} \begin{pmatrix} d(u,\overline{u}) \\ d(v,\overline{v}) \end{pmatrix}. \end{aligned}$$

The Lipschitz condition is satisfied by F with

$$d_B(F(u,v),F(\overline{u},\overline{v})) \leq \begin{pmatrix} \frac{a}{\tau} & \frac{b}{\tau} \\ \frac{c}{\tau} & \frac{d}{\tau} \end{pmatrix} d_B((u,v),(\overline{u},\overline{v})). \tag{15}$$

To obtain the unique fixed point of F, we must see if matrix $L = \begin{pmatrix} \frac{a}{\tau} & \frac{b}{\tau} \\ \frac{c}{\tau} & \frac{d}{\tau} \end{pmatrix}$ converges at zero. We use Proposition 1(ii) from the Preliminaries section.

The eigenvalues of L are the solutions of

$$\begin{vmatrix} \frac{a}{\tau} - \lambda & \frac{b}{\tau} \\ \frac{c}{\tau} & \frac{d}{\tau} - \lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - \left(\frac{a}{\tau} + \frac{d}{\tau}\right)\lambda + \frac{ad-bc}{\tau^2} = 0.$$

We observe the following.

$$\lambda_1, \lambda_2 = \frac{1}{2\tau}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right),$$

Thus, we can obtain a positive value of τ such that $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

All the required conditions are now satisfied, and by applying Perov's theorem, the proof is completed. The existence and uniqueness result is assured for the solution of the system defined by (1) and that is investigated in its equivalent form in (3). □

4. Remarks

From what was obtained above, the following remarks can be made.

4.1. a

The solution $(u^*(t), v^*(t))$ is, in fact, $(x'^*(t), y'^*(t))$, which is a solution of (3).

4.2. b

The solution $(x^*(t), y^*(t))$ for initial problem (1) will now be

$$\begin{aligned} x^*(t) &= x_0 + \int_{t_0}^{t} x'^*(s)ds = x_0 + \int_{t_0}^{t} u^*(s)ds, \\ y^*(t) &= y_0 + \int_{t_0}^{t} y'^*(s)ds = y_0 + \int_{t_0}^{t} v^*(s)ds. \end{aligned} \tag{16}$$

4.3. c

If we want to apply (9) for obtaining an approximation for $(u^*(t), v^*(t))$ (and $(x^*(t), y^*(t))$ by using relation 16), we must find L^m and $(I_2 - L)^{-1}$.

The form $L = \begin{pmatrix} \frac{a}{\tau} & \frac{b}{\tau} \\ \frac{c}{\tau} & \frac{d}{\tau} \end{pmatrix}$ is too complicated for the process of approximation.

By denoting $\alpha = \max\{a, b, c, d\}$, for any $k > 2$, we find τ such that $\frac{\alpha}{\tau} < \frac{1}{2^{k+1}}$.

We can consider matrix $M = \frac{1}{2^{k+1}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

It is true that $d_B(F(u,v), F(\overline{u}, \overline{v})) \leq M d_B((u,v), (\overline{u}, \overline{v}))$ and M converges at zero; thus, we can approximate $(u^*(t), v^*(t))$ using $M^m = \frac{1}{2^{mk+1}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\forall m \in \mathbb{N}^*$, and $(I_2 - M)^{-1} = \frac{1}{2^{k+1}-2} \begin{pmatrix} 2^{k+1} - 1 & 1 \\ 1 & 2^{k+1} - 1 \end{pmatrix}$ obtained after applying elementary calculus.

Applying Perov's theorem, we obtained the following estimation for $m \in \mathbb{N}^*$.

$$d\left(\begin{pmatrix} v_m \\ u_m \end{pmatrix}, \begin{pmatrix} v^* \\ u^* \end{pmatrix}\right) \leqslant \frac{1}{2^{k(m-1)+1}(2^k - 1)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} d\left(\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}\right).$$

5. Application

5.1. a

Let the following Cauchy problem be stated as follows.

$$\begin{cases} x''(t) = f(t, x'(t), x(t)) \\ x(t_0) = x_0,\ x'(t_0) = x_1 \end{cases}, t \in [t_0, t_1].$$

If we make notations

$$x''(t) = u(t), x'(t) = v(t)$$

and knowing that

$$x(t) = \int_{t_0}^{t} x'(s)ds + x_0$$

$$x'(t) = \int_{t_0}^{t} x''(s)ds + x_1$$

then we can obtain the following system:

$$\begin{cases} u(t) = f\left(t, \int_{t_0}^{t} u(s)ds + x_1, \int_{t_0}^{t} v(s)ds + x_0\right) \equiv F_1(u(t), v(t)) \\ v(t) = g\left(t, \int_{t_0}^{t} u(s)ds + x_1, 0\right) \equiv F_2(u(t), v(t)) \end{cases}, t \in [t_0, t_1]$$

where $f, g : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following Lipschitz conditions:

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant a|u - \overline{u}| + b|v - \overline{v}|$$

$$|g(t,u,v) - g(t,\overline{u},\overline{v})| \leqslant c|u - \overline{u}| + d|v - \overline{v}|$$

where a, b, c, and d are positive constants and $t \in [t_0, t_1]$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$. In this case, we have

$$|F_1(t,u,v) - F_1(t,\overline{u},\overline{v})| \leqslant a\int_t^{t_0} |u(s) - \overline{u}(s)|ds + b\int_t^{t_0} |v(s) - \overline{v}(s)|ds \leqslant ,$$
$$\leqslant \left(\frac{a}{\tau}d(u,\overline{u}) + \frac{b}{\tau}d(v,\overline{v})\right)e^{\tau(t-t_0)}$$

where a and b are positive constants.

Using the general Bielecki metric, we obtain

$$d(F_1(u,v), F_1(\overline{u},\overline{v})) \leqslant \tfrac{a}{\tau}d(u,\overline{u}) + \tfrac{b}{\tau}d(v,\overline{v}).$$

Using the same deduction, we obtain

$$d(F_2(u,v), F_2(\overline{u},\overline{v})) \leqslant \tfrac{1}{\tau}d(u,\overline{u}) \leqslant \tfrac{1}{\tau}d(u,\overline{u}) + \tfrac{1}{\tau}d(v,\overline{v}).$$

In this case, we have the following matrix.

$$L = \begin{pmatrix} \frac{a}{\tau} & \frac{b}{\tau} \\ \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix}$$

With notation $\alpha = \max\left\{\frac{a}{\tau}, \frac{b}{\tau}, \frac{1}{\tau}\right\}$, for any $k > 2$, we find τ such that $\alpha < \frac{1}{2^{k+1}}$.
Thus, we can consider the following matrix.

$$M = \tfrac{1}{2^{k+1}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

After applying elementary calculus, we obtain

$$M^m = \tfrac{1}{2^{mk+1}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \forall m \in \mathbb{N}^* \text{ and } (I_2 - M)^{-1} = \tfrac{1}{2^{k+1}-2}\begin{pmatrix} 2^{k+1}-1 & 1 \\ 1 & 2^{k+1}-1 \end{pmatrix}.$$

Applying the Perov's Theorem, for $m \in \mathbb{N}^*$, we obtain the following estimation.

$$d\left(\begin{pmatrix} v_m \\ u_m \end{pmatrix}, \begin{pmatrix} v^* \\ u^* \end{pmatrix}\right) \leqslant \frac{1}{2^{k(m-1)+1}(2^k-1)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} d\left(\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}\right)$$

5.2. b

We consider the following system of equations:

$$\begin{cases} x'(t) = x(t) + 2t \cdot y(t) \\ y'(t) = -2t \cdot x(t) + y(t) \end{cases}$$

where we have

$$\begin{cases} x'(t) = \int_0^t x'(s)ds + 2t\left(\int_0^t y'(s)ds + 1\right) \\ y'(t) = -2t\int_0^t x'(s)ds + \left(\int_0^t y'(s)ds + 1\right) \end{cases}$$

The equivalent form of the system is

$$\begin{cases} x'(t) = \int_0^t u(s)ds + 2t\int_0^t v(s)ds + 2t \equiv F_1(t,u,v) \\ y'(t) = -2t\int_0^t u(s)ds + \int_0^t v(s)ds + 1 \equiv F_2(t,u,v) \end{cases}$$

Now, we have

$$|F_1(t,u,v)-F_1(t,\overline{u},\overline{v})| \leqslant \int_0^t |u(s)-\overline{u}(s)|ds + 2t\int_0^t |v(s)-\overline{v}(s)|ds \leqslant$$
$$\leqslant \int_0^t e^{\tau s}ds\|u-\overline{u}\|_\tau + 2\pi\int_0^t e^{\tau s}ds\|v-\overline{v}\|_\tau =$$
$$= \tfrac{1}{\tau}e^{\tau s}\Big|_0^t\|u-\overline{u}\|_\tau + \tfrac{2\pi}{\tau}e^{\tau s}|_0^t\|v-\overline{v}\|_\tau \leqslant$$
$$\leqslant \tfrac{1}{\tau}e^{\tau t}\|u-\overline{u}\|_\tau + \tfrac{2\pi}{\tau}e^{\tau t}\|v-\overline{v}\|_\tau.$$

Using the general Bielecki metric, we obtain

$$\|F_1(t,u,v)-F_1(t,\overline{u},\overline{v})\|_\tau \leqslant \frac{1}{\tau}\|u-\overline{u}\|_\tau + \frac{2\pi}{\tau}\|v-\overline{v}\|_\tau.$$

In the same manner, we obtain

$$|F_2(t,u,v)-F_2(t,\overline{u},\overline{v})| \leqslant 2t\int_0^t |u(s)-\overline{u}(s)|ds + \int_0^t |v(s)-\overline{v}(s)|ds \leqslant$$
$$\leqslant 2\pi\int_0^t e^{\tau s}ds\|u-\overline{u}\|_\tau + \int_0^t e^{\tau s}ds\|v-\overline{v}\|_\tau \leqslant$$
$$= 2\pi\tfrac{1}{\tau}e^{\tau s}\Big|_0^t\|u-\overline{u}\|_\tau + \tfrac{1}{\tau}e^{\tau s}\Big|_0^t\|v-\overline{v}\|_\tau \leqslant$$
$$\leqslant \tfrac{2\pi}{\tau}e^{\tau t}\|u-\overline{u}\|_\tau + \tfrac{1}{\tau}e^{\tau t}\|v-\overline{v}\|_\tau.$$

Finally, using the Bielecki metric, we obtain

$$\|F_2(t,u,v)-F_2(t,\overline{u},\overline{v})\|_\tau \leqslant \frac{2\pi}{\tau}\|u-\overline{u}\|_\tau + \frac{1}{\tau}\|v-\overline{v}\|_\tau.$$

Now, we have

$$\|F(t,u,v)-F(t,\overline{u},\overline{v})\|_\tau \leqslant \begin{pmatrix} \frac{1}{\tau} & \frac{2\pi}{\tau} \\ \frac{2\pi}{\tau} & \frac{1}{\tau} \end{pmatrix}\begin{pmatrix} \|u-\overline{u}\|_\tau \\ \|v-\overline{v}\|_\tau \end{pmatrix}.$$

Taking $\alpha = \max\{1, 2\pi\} = 2\pi$, for any $k > 2$, we obtain τ such that $\frac{2\pi}{\tau} < \frac{1}{2^{k+1}}$.
Now, we can choose a positive value of τ in order to obtain matrix M for Section 4.3.

6. Conclusions

In this paper, a differential equation system was studied by using a new equivalent form that has not been used before and by using Perov's theorem in generalized metric spaces to find the unique solution of the system.

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