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The Gao-Type Constant of Absolute Normalized Norms on $\mathbb{R}^2$

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Abstract: In this paper, we present an extraordinary method to calculate the Gao-type constant of absolute normalized norms on $\mathbb{R}^2$. By using our method, we not only provide the values of the Gao-type constant in new specific spaces, but also improve well-known results from the existing literature.

Keywords: fixed point; Banach space; absolute normalized norm

MSC: 46B20; 46B25

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1. Introduction

The geometric theory of Banach spaces is an important research direction of a nonlinear functional analysis and has been widely applied in many fields of modern mathematics, such as differential equations, economics, optimization, game theory, fixed point theory, dynamic system theory, and so on. In particular, Kirk proved that the Banach space with normal structure has a fixed point property. Since then, research on the existence of fixed points for nonlinear differential equations by using the geometric properties of Banach spaces has rapidly developed. In recent years, many scholars have introduced geometrical constants that could easily describe the geometry properties of Banach spaces (see [1–8]).

Let X be a Banach space with a norm $\|\cdot\|$. The unit sphere of X is denoted by S_X and the unit ball of X is denoted by B_X. The constant $c'_{\mathrm{NJ}}(X)$, which is defined by Gao in [9], is as follows:

$$c'_{\mathrm{NJ}}(X) = \inf\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X \right\}.$$

The Gao constant $c'_{\mathrm{NJ}}(X)$, which plays an important role in [10], was intensively studied by some scholars, in which the famous Tingley problem [11] was partially solved. They gave specific descriptions of the geometric properties, such as uniformly non-square and the normal structure in the context of the fixed point property (see [12–16]). Now, let us collect some properties of constant $c'_{\mathrm{NJ}}(X)$ as follows:

(i) Let X be a Banach space, then $\frac{1}{2} \le c'_{\mathrm{NJ}}(X) \le 1$.
(ii) X is uniformly non-square if and only if $c'_{\mathrm{NJ}}(X) > \frac{1}{2}$.
(iii) X is an inner product space if and only if $c'_{\mathrm{NJ}}(X) = 1$.

It is easy to see that the calculation of the constant $c'_{\mathrm{NJ}}(X)$ for some concrete spaces is important. Sequences on the Gao constant $c'_{\mathrm{NJ}}(X)$ for various spaces were presented, for example, Gao calculated the constant $c'_{\mathrm{NJ}}(X)$ of the spaces ℓ_p (see [9]), Cui and Wang computed the value of $c'_{\mathrm{NJ}}(X)$ for the Lorentz sequence space using the absolute normalized norms (see [17]), Zuo and Cui used the formula to calculate the constant $c'_{\mathrm{NJ}}(X)$ by the modulus of smoothness $\rho_1(\epsilon)$; however, they did not find the exact values of $c'_{\mathrm{NJ}}(X)$. It is hard to compute the values of $\rho_1(\epsilon)$ in some concrete Banach spaces (see [13]).

Motivated by the constant $C_{\mathrm{NJ}}^{(p)}(X)$ from [18], i.e.,

$$C_{\mathrm{NJ}}^{(p)}(X) = \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : \|x\| + \|y\| \neq 0 \right\},$$

Asif et al. [16] considered the following Gao-type constant:

$$c_{\mathrm{NJ}}^{(p)}(X) = \inf\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^p} : x, y \in S_X \right\},$$

where $1 \leq p < +\infty$ is a constant. It is obvious that the Gao-type constant is a generalization of the constant $c_{\mathrm{NJ}}^{'}(X)$ (in fact, $c_{\mathrm{NJ}}^{'}(X) = c_{\mathrm{NJ}}^{(2)}(X)$). Therefore, the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$ is more important than the Gao constant $c_{\mathrm{NJ}}^{'}(X)$, which plays a significant role in the geometry theory of Banach spaces. The exact values of $c_{\mathrm{NJ}}^{'}(X)$ have been calculated for some classical spaces, such as the space ℓ_p, the Lorentz sequence space, the Cesàro space, etc. Naturally, the studies on the values of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$ for these spaces are important. In [16], Asif et al. only obtained $\frac{1}{2^{p-1}} \leq c_{\mathrm{NJ}}^{(p)}(X) \leq 1$ for any Banach space X and $c_{\mathrm{NJ}}^{(p)}(\mathbb{R}^2, \|.\|_1) = \frac{1}{2^{p-1}}$, However, some problems in the existing literature need solving; for instance, how does one compute the values of the constant $c_{\mathrm{NJ}}^{(p)}(X)$ for the absolute normalized norms of some concrete Banach spaces? Can it be used to characterize the inner product space for the value of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$? The main purpose of this paper is to solve the above problems.

2. Preliminaries and Notations

First, we define the general mean and provide an example of the weighted mean of order s.

Definition 1. *Let x, y be real numbers, such that $x < y$. Then, any number $m := m(x, y)$ is called a mean of x and y if it satisfies*

$$x \leq m(x, y) \leq y.$$

One of the most known means is the weighted mean of the order s, which is defined as

$$m^{[s]}(x, y; \omega, 1-\omega) = \begin{cases} (\omega x^s + (1-\omega) y^s)^{\frac{1}{s}}, & s \notin \{0, +\infty, -\infty\}, \\ x^{\omega} y^{1-\omega}, & s = 0, \\ \max\{x, y\}, & s = +\infty, \\ \min\{x, y\}, & s = -\infty, \end{cases}$$

where x, y are positive real numbers and $\omega \in (0, 1)$.

A norm on $\mathbb{R}^2$ is called absolute, if for all $(x, y) \in \mathbb{R}^2$, it satisfies

$$\|(x, y)\| = \|(|x|, |y|)\|.$$

A norm $\|\cdot\|$ is called normalized if

$$\|(1, 0)\| = \|(0, 1)\| = 1.$$

The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by N_{α}^2. Let Φ denote the set of all convex functions on $[0, 1]$ with $\varphi(0) = \varphi(1) = 1$, satisfying

$$\max\{1-t, t\} \leq \varphi(t) \leq 1.$$

Proposition 1 ([19]). *If $\|\cdot\| \in N_\alpha^2$, then $\varphi(t) = \|(1-t,t)\| \in \Phi$. Moreover, if $\varphi(t) \in \Phi$, then*

$$\|(x,y)\|_\varphi = \begin{cases} 0, & (x,y) = (0,0), \\ (|x|+|y|)\varphi\left(\frac{|y|}{|x|+|y|}\right), & (x,y) \neq (0,0) \end{cases}$$

is a norm $\|\cdot\|_\varphi \in N_\alpha^2$.

The typical example is the ℓ_p norm as follows:

$$\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max\{|x|,|y|\}, & p = \infty. \end{cases}$$

The corresponding convex function $\varphi_p(t)$ is defined as

$$\varphi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max\{1-t,t\}, & p = \infty. \end{cases}$$

It is well known that $\|\cdot\|_\infty \leq \|\cdot\|_\varphi \leq \|\cdot\|_1$ for any $\|\cdot\|_\varphi \in N_\alpha^2$. Moreover, by taking different convex functions $\varphi(t)$, Proposition 1 also enables us to obtain many non-ℓ_p norms. The following lemma will help us utilize our results.

Lemma 1 ([20]). *Let $\varphi(t)$ and $\mu(t)$ be functions of $[a,b]$ with $\varphi(t) \geq \mu(t) > 0$ for all $t \in [a,b]$. If $\varphi(t) - \mu(t)$ attains the maximum at $t = c \in [a,b]$ and the function $\mu(t)$ attains the minimum at $t = c$, then the function $\frac{\varphi(t)}{\mu(t)}$ attains its maximum at $t = c$.*

3. Main Results

Firstly, we will obtain some equivalent definitions of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$ from Proposition 4.3 in [21].

Proposition 2. *For $1 \leq p < +\infty$, then*

$$\begin{aligned} c_{\mathrm{NJ}}^{(p)}(X) &= \inf\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^p} : x,y \in B_X\right\}, \\ &= \inf\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^p \min(\|x\|^p, \|y\|^p)}, \|x\| + \|y\| \neq 0\right\}. \end{aligned}$$

Proposition 3. *Let X be a nontrivial Banach space, then*

$$c_{\mathrm{NJ}}^{(p)}(X) = \inf\left\{c_{\mathrm{NJ}}^{(p)}(Y) : Y \in \Re(X)\right\},$$

where $\Re(X)$ is the set of all two-dimensional subspaces of X.

Proof. Firstly, we have

$$c_{\mathrm{NJ}}^{(p)}(X) \leq \inf\left\{c_{\mathrm{NJ}}^{(p)}(Y) : Y \in \Re(X)\right\}.$$

Secondly, for any $\varepsilon > 0$, there exist x' and y' in S_X, such that

$$c_{\mathrm{NJ}}^{(p)}(X) > \frac{\|x'+y'\|^p + \|x'-y'\|^p}{2^p} - \varepsilon.$$

Let $x', y' \in Y' \in \Re(X)$, then

$$\frac{\|x'+y'\|^p + \|x'-y'\|^p}{2^p} \geq c_{\mathrm{NJ}}^{(p)}(Y') \geq \inf\Big\{ c_{\mathrm{NJ}}^{(p)}(Y') : Y \in \Re(X) \Big\},$$

thus, we obtain

$$c_{\mathrm{NJ}}^{(p)}(X) \geq \inf\Big\{ c_{\mathrm{NJ}}^{(p)}(Y') : Y' \in \Re(X) \Big\} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$c_{\mathrm{NJ}}^{(p)}(X) \geq \inf\Big\{ c_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X) \Big\}.$$

The proof is completed. □

Theorem 1. *Let $\|\cdot\|$ and $|\cdot|$ be two norms, such that*

$$\alpha|\cdot| \leq \|\cdot\| \leq \beta|\cdot|,$$

where α and β are constants with $0 < \alpha < \beta$, then

$$\frac{\alpha^p c_{\mathrm{NJ}}^{(p)}(|\cdot|)}{\beta^p} \leq c_{\mathrm{NJ}}^{(p)}(\|\cdot\|) \leq \frac{\beta^p c_{\mathrm{NJ}}^{(p)}(|\cdot|)}{\alpha^p}.$$

Moreover, if $\|\cdot\| = k|\cdot|$, where $k > 0$ is a constant, then $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|) = c_{\mathrm{NJ}}^{(p)}(|\cdot|)$.

Proof. First, we have

$$\begin{aligned} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|) &= \inf\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^p \min(\|x\|^p, \|y\|^p)} : \|x\| + \|y\| \neq 0 \right\} \\ &\leq \inf\left\{ \frac{\beta^p \{|x+y|^p + |x-y|^p\}}{\alpha^p 2^p \min(|x|^p, |y|^p)} : \|x\| + \|y\| \neq 0 \right\} \\ &= \frac{\beta^p}{\alpha^p} \inf\left\{ \frac{\{|x+y|^p + |x-y|^p\}}{2^p \min(|x|^p, |y|^p)} : \|x\| + \|y\| \neq 0) \right\} \\ &\leq \frac{\beta^p}{\alpha^p} c_{\mathrm{NJ}}^{(p)}(|\cdot|). \end{aligned}$$

Similarly, we can obtain the following inequality:

$$\frac{\alpha^p c_{\mathrm{NJ}}^{(p)}(|\cdot|)}{\beta^p} \leq c_{\mathrm{NJ}}^{(p)}(\|\cdot\|).$$

The proof is completed. □

Now, let us denote

$$M_1 = \max_{0 \leq t \leq 1} \frac{\varphi(t)}{\mu(t)} \text{ and } M_2 = \max_{0 \leq t \leq 1} \frac{\mu(t)}{\varphi(t)}.$$

Theorem 2. *Let $\varphi(t), \mu(t) \in \Phi$ and $\varphi(t) \geq \mu(t)$ for all $t \in [0,1]$. Suppose that $\frac{\varphi(t)}{\mu(t)}$ attains its maximum at $t = \frac{1}{2}$ and $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = \frac{1}{2^{p-1}\mu^p(\frac{1}{2})}$, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{2^{p-1}\varphi^p(\frac{1}{2})}.$$

Proof. From the condition of $\varphi(t) \geq \mu(t)$ and the definition of M_1, one has

$$\|\cdot\|_\mu \leq \|\cdot\|_\varphi \leq M_1\|\cdot\|_\mu.$$

By taking $\alpha = 1$ and $\beta = M_1$ in Theorem 1, we obtain the following inequality:

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \geq \frac{1}{M_1^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu).$$

Since the function $\frac{\varphi(t)}{\mu(t)}$ attains its maximum at $t = \frac{1}{2}$, i.e., $M_1 = \frac{\varphi(\frac{1}{2})}{\mu(\frac{1}{2})}$, and note that $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = \frac{1}{2^{p-1}\mu^p(\frac{1}{2})}$, then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \geq \frac{1}{M_1^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = \frac{1}{2^{p-1}\varphi^p(\frac{1}{2})}. \tag{1}$$

Let us put $x_1 = \left(\frac{1}{2\varphi(\frac{1}{2})}, \frac{1}{2\varphi(\frac{1}{2})}\right)$ and $y_1 = \left(\frac{1}{2\varphi(\frac{1}{2})}, -\frac{1}{2\varphi(\frac{1}{2})}\right)$, then

$$\|x_1\|_\varphi = \|y_1\|_\varphi = 1,$$

$$\|x_1 + y_1\|_\varphi = \|x_1 - y_1\|_\varphi = \frac{1}{\varphi(\frac{1}{2})},$$

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \leq \frac{\|x_1 + y_1\|^p + \|x_1 - y_1\|^p}{2^p} = \frac{1}{2^{p-1}\varphi^p(\frac{1}{2})}. \tag{2}$$

From (1) and (2), we obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_1^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = \frac{1}{2^{p-1}\varphi^p(\frac{1}{2})}.$$

□

Theorem 3. *Let $\varphi(t) \in \Phi$ and $\varphi(t) \geq \varphi_p(t)$ $(1 < p \leq 2)$ for all $t \in [0,1]$, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_1^p}.$$

Proof. Let $x, y \in S_X$. By the condition that $\varphi(t) \geq \varphi_p(t)$ $(1 < p \leq 2)$ for all $t \in [0,1]$ and the Clarkson inequality in [22], we have

$$\begin{aligned}
\|x+y\|_\varphi^p + \|x-y\|_\varphi^p &\geq (\|x+y\|_p^p + \|x-y\|_p^p) \\
&\geq 2^{p-1}(\|x\|_p^p + \|y\|_p^p) \\
&\geq \frac{2^{p-1}}{M_1^p}(\|x\|_\varphi^p + \|y\|_\varphi^p) \\
&= \frac{2^p}{M_1^p}.
\end{aligned}$$

The definition of $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi)$ implies that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \geq \frac{1}{M_1^p}. \tag{3}$$

On the other hand, note that $\frac{\varphi(t)}{\varphi_p(t)}$ attains its maximum $M_1 = \frac{\varphi(t_1)}{\varphi_p(t_1)}$. Let us put $x_2 = (1-t_1, t_1)$ and $y_2 = (1-t_1, -t_1)$, then

$$\|x_2\|_\varphi = \|y_2\|_\varphi = \varphi(t_1).$$

$$\begin{aligned}\|x_2+y_2\|_\varphi^p + \|x_2-y_2\|_\varphi^p &= 2^p[(1-t_1)^p + (t_1)^p], \\ &= 2^p \varphi_p^p(t_1), \\ &= \frac{2^p}{M_1^p}\varphi^p(t_1).\end{aligned}$$

Therefore, we can obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \le \frac{\|x_2+y_2\|_\varphi^p + \|x_2-y_2\|_\varphi^p}{2^p \min(\|x_2\|_\varphi^p, \|y_2\|_\varphi^p)} = \frac{1}{M_1^p}. \tag{4}$$

From inequalities (3) and (4), we infer that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_1^p}.$$

We complete the proof. □

Theorem 4. *Let $\varphi(t), \mu(t) \in \Phi$ and $\varphi(t) \le \mu(t)$ for all $t \in [0,1]$. Suppose that $\frac{\mu(t)}{\varphi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = 2\mu^p(\frac{1}{2})$, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = 2\varphi^p\left(\frac{1}{2}\right).$$

Proof. From the condition that $\varphi(t) \le \mu(t)$ and the definition of M_2, we can obtain

$$\frac{1}{M_2}\|\cdot\|_\mu \le \|\cdot\|_\varphi \le \|\cdot\|_\mu.$$

Taking $\alpha = \frac{1}{M_2}$ and $\beta = 1$ in Theorem 1, we have

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \ge \frac{1}{M_2^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu).$$

Since $M_2 = \frac{\mu(\frac{1}{2})}{\varphi(\frac{1}{2})}$ and $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = 2\mu^p(\frac{1}{2})$, then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \ge \frac{1}{M_2^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = 2\varphi^p\left(\frac{1}{2}\right). \tag{5}$$

On the other hand, let us put $x_3 = (1,0)$ and $y_3 = (0,1)$, then

$$\|x_3\|_\varphi = \|y_3\|_\varphi = 1,$$

$$\|x_3+y_3\|_\varphi = \|x_3-y_3\|_\varphi = 2\varphi\left(\frac{1}{2}\right),$$

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \le \frac{\|x_3+y_3\|_\varphi^p + \|x_3-y_3\|_\varphi^p}{2^p} = \frac{2^{p+1}\varphi^p(\frac{1}{2})}{2^p} = 2\varphi^p\left(\frac{1}{2}\right). \tag{6}$$

By the inequalities (5) and (6), we can obtain that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_2^p} c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\mu) = 2\varphi^p\Big(\frac{1}{2}\Big).$$

We end the proof. □

Theorem 5. *Let $\varphi(t) \in \Psi$ and $\varphi(t) \le \varphi_p(t)$ $(2 \le p < \infty)$, if the maximum $M_2 = \max_{0\le t\le 1} \frac{\varphi_p(t)}{\varphi(t)}$ attains at $t = \frac{1}{2}$, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{2^{2-p}}{M_2^p}.$$

Proof. Let $x, y \in S_X$, from the condition of $\varphi(t) \le \varphi_p(t)$ $(2 \le p < \infty)$ for all $t \in [0,1]$, and the Clarkson inequality in [22], we have

$$\begin{aligned}
\|x+y\|_\varphi^p + \|x-y\|_\varphi^p &\ge \frac{1}{M_2^p}(\|x+y\|_p^p + \|x-y\|_p^p) \\
&\ge \frac{2}{M_2^p}(\|x\|_p^p + \|y\|_p^p) \\
&\ge \frac{2}{M_2^p}(\|x\|_\varphi^p + \|y\|_\varphi^p) \\
&= \frac{2^2}{M_2^p},
\end{aligned}$$

which implies that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \ge \frac{2^{2-p}}{M_2^p}. \tag{7}$$

On the other hand, since $M_2 = \frac{\varphi_p(\frac{1}{2})}{\varphi(\frac{1}{2})}$, let us put $x_4 = (\frac{1}{2}, 0)$ and $y_4 = (0, \frac{1}{2})$, then

$$\|x_4\|_\varphi^p = \|y_4\|_\varphi^p = \Big(\frac{1}{2}\Big)^p,$$

$$\|x_4 + y_4\|_\varphi = \|x_4 - y_4\|_\varphi = \varphi\Big(\frac{1}{2}\Big),$$

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \le \frac{\|x_4+y_4\|_\varphi^p + \|x_4-y_4\|_\psi^p}{2^p \min(\|x_4\|_\varphi^p, \|y_4\|_\varphi^p)} = 2^{2-p}\frac{\varphi^p(\frac{1}{2})}{\varphi_p^p(\frac{1}{2})} = \frac{2^{2-p}}{M_2^p}. \tag{8}$$

From inequalities (7) and (8), we infer that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{2^{2-p}}{M_2^p}.$$

We obtain the desired result. □

In the following, let us state the conclusion about the general mean $m(t)$.

Corollary 1. *Let $\omega(t) \le \mu(t) \in \Phi$ for all $t \in [0,1]$, $m(t) := m(\omega(t), \mu(t))$ is the mean convex function of the functions $\omega(t)$ and $\mu(t)$.*

(i) If $\frac{m(t)}{\omega(t)}$ attains its maximum at $t = \frac{1}{2}$ and $c_{\rm NJ}^{(p)}(\|\cdot\|_\omega) = \frac{1}{2^{p-1}\omega^p(\frac{1}{2})}$, then

$$c_{\rm NJ}^{(p)}(\|\cdot\|_m) = \frac{1}{2^{p-1}m^p(\frac{1}{2})}.$$

(ii) If $\frac{\mu(t)}{m(t)}$ attains its maximum at $t = \frac{1}{2}$ and $c_{\rm NJ}^{(p)}(\|\cdot\|_\mu) = 2\mu^p(\frac{1}{2})$, then

$$c_{\rm NJ}^{(p)}(\|\cdot\|_m) = 2m^p\Big(\frac{1}{2}\Big).$$

Proof. It is well known that

$$\omega(t) \le m(t) \le \mu(t)$$

for all $t \in [0,1]$. It is easy to check that $m(t) \in \Phi$. Since the function $m(t)$ is convex, we can obtain the result from Theorem 2 and Theorem 4, respectively. □

Next, we give the lower bound and upper bound of the constant $c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi)$ for the general case $\varphi(t) \in \Phi$.

Theorem 6. *Let $\varphi(t) \in \Phi$ for $t \in [0,1]$, $M_1 = \max\limits_{0\le t\le 1} \frac{\varphi(t)}{\varphi_p(t)}$, $M_2 = \max\limits_{0\le t\le 1} \frac{\varphi_p(t)}{\varphi(t)}$.*

(i) If $1 < p \le 2$, then

$$\frac{1}{(M_1M_2)^p} \le c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi) \le \frac{1}{M_1^p},$$

(ii) If $2 < p < \infty$, then

$$\frac{2^{2-p}}{(M_1M_2)^p} \le c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi) \le \frac{2^{2-p}}{M_2^p}.$$

Proof. (i) If $1 \le p \le 2$, it is easy to obtain the right inequality from (4), then

$$c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi) \le \frac{1}{M_1^p}.$$

Let $x, y \in S_X$. By the definition of M_1, M_2, and the Clarkson inequality, we have

$$\begin{aligned}
\|x+y\|_\varphi^p + \|x-y\|_\psi^p &\ge \frac{1}{M_2^p}(\|x+y\|_p^p + \|x-y\|_p^p) \\
&\ge \frac{2^{p-1}}{M_2^p}(\|x\|_p^p + \|y\|_p^p) \\
&\ge \frac{2^{p-1}}{(M_1M_2)^p}(\|x\|_\varphi^p + \|y\|_\varphi^p) \\
&= \frac{2^p}{(M_1M_2)^p}.
\end{aligned}$$

The inequality implies that

$$c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi) \ge \frac{1}{(M_1M_2)^p}.$$

(ii) Let $2 < p \le \infty$. The right inequality is obvious from inequality (8), then

$$c_{\rm NJ}^{(p)}(\|\cdot\|_\varphi) \le \frac{2^{2-p}}{M_2^p}.$$

Let $x, y \in S_X$. From the Clarkson inequality, we can obtain

$$\begin{aligned} \|x+y\|_\varphi^p + \|x-y\|_\varphi^p &\geq \frac{1}{M_2^p}(\|x+y\|_p^p + \|x-y\|_p^p) \\ &\geq \frac{2}{M_2^p}(\|x\|_p^p + \|y\|_p^p) \\ &\geq \frac{2}{(M_1 M_2)^p}(\|x\|_\varphi^p + \|y\|_\varphi^p) \\ &= \frac{2^2}{(M_1 M_2)^p} \end{aligned}$$

The definition of $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi)$ implies that the left inequality is as follows:

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) \geq \frac{2^{2-p}}{(M_1 M_2)^p}.$$

We complete the proof. □

From Theorem 3 and Theorem 5, the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi)$ coincides with the upper bound. In the following, we only give some conditions under which the Gao constant $c'_{\mathrm{NJ}}(\|\cdot\|_\varphi)$ coincides with the lower bound.

Theorem 7. *Let $\varphi(t) = \varphi(1-t)$ for all $t \in [0,1]$. If there exist two points $t_1, t_2 \in [0, \frac{1}{2}]$, such that*

$$M_1 = \frac{\varphi(t_1)}{\varphi_2(t_1)}, \ \ M_2 = \frac{\varphi_2(t_2)}{\varphi(t_2)} \ \ \textit{and} \ \ (1-t_1)(1-t_2) = \frac{1}{2},$$

then

$$c'_{\mathrm{NJ}}(\|\cdot\|_\varphi) = \frac{1}{(M_1 M_2)^2}.$$

Proof. Firstly, take $\alpha = \frac{1}{M_1}$ and $\beta = M_2$ in Theorem 1. Since $c'_{\mathrm{NJ}}(\|\cdot\|_2) = 1$, then

$$c'_{\mathrm{NJ}}(\|\cdot\|_\varphi) \geq c'_{\mathrm{NJ}}(\|\cdot\|_2)\frac{1}{(M_1 M_2)^2} = \frac{1}{(M_1 M_2)^2}. \tag{9}$$

Secondly, note that $(1-t_1)(1-t_2) = \frac{1}{2}$. Put $x = \frac{1}{\varphi(t_1)}(1-t_1, t_1)$, $y = \frac{1}{\varphi(t_1)}(t_1, t_1 - 1)$, then

$$\|x\|_\varphi = 1, \ \ \|y\|_\varphi = 1.$$

From the conditions that $\varphi(t) = \varphi(1-t)$ and $(1-t_1)(1-t_2) = \frac{1}{2}$, we have

$$\|x+y\|_\varphi = \frac{(2-2t_1)}{\varphi(t_1)}\varphi\left(\frac{1-2t_1}{2-2t_1}\right) = \frac{(2-2t_1)\varphi(t_2)}{\varphi(t_1)},$$

$$\|x-y\|_\varphi = \frac{(2-2t_1)}{\varphi(t_1)}\varphi\left(\frac{1}{2-2t_1}\right) = \frac{(2-2t_1)\varphi(t_2)}{\varphi(t_1)}.$$

It is well known that

$$\varphi_2(t) = \sqrt{2}(1-t)\varphi_2\left(\frac{1}{2-2t}\right),$$

then

$$\varphi_2^2(t_1) = 2(1-t_1)^2\varphi_2^2\left(\frac{1}{2-2t_1}\right) = 2(1-t_1)^2\varphi_2^2(t_2).$$

Consequently, we obtain

$$\begin{aligned}
c'_{\mathrm{NJ}}(\|\cdot\|_{\varphi}) &\leq \frac{\|x+y\|_{\varphi}^2 + \|x-y\|_{\varphi}^2}{4} \\
&= \frac{2(1-t_1)^2\varphi^2(t_2)}{\varphi^2(t_1)} \\
&= \frac{1}{(M_1M_2)^2} \cdot \frac{2(1-t_1)^2\varphi_2^2(t_2)}{\varphi_2^2(t_1)} \\
&= \frac{1}{(M_1M_2)^2}.
\end{aligned} \tag{10}$$

From inequalities (9) and (10), we infer that

$$c'_{\mathrm{NJ}}(\|\cdot\|_{\varphi}) = \frac{1}{(M_1M_2)^2}.$$

Thus, the claim holds. □

4. Some Examples

In this section, we compute the values of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$ on some specific spaces. We give the exact value of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(X)$ under the absolute normalized norms in $\mathbb{R}^2$, and provide examples to show that the value of the Gao-type constant cannot characterize the inner product space in a general case.

Example 1. *For the usual ℓ_p^2 space, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p) = \begin{cases} 1, & 1 < p \leq 2, \\ 2^{2-p}, & 2 < p < \infty. \end{cases}$$

In particular, $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_1) = c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\infty) = 2^{1-p}$.

Proof. Let $1 < p \leq 2$ and $x, y \in S_X$. From the Clarkson inequality, we have

$$(\|x+y\|_p^p + \|x-y\|_p^p) \geq 2^{p-1}(\|x\|_p^p + \|y\|_p^p),$$

which means that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p) \geq 1. \tag{11}$$

Let us put $x = (\frac{1}{2}, \frac{1}{2})$ and $y = (\frac{1}{2}, -\frac{1}{2})$, then

$$\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2^p \min(\|x\|_p^p, \|y\|_p^p)} = 1. \tag{12}$$

From inequalities (11) and (12), we can obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p) = 1.$$

Let $2 < p < \infty$ and $x, y \in S_X$. From the Clarkson inequality, we obtain

$$(\|x+y\|_p^p + \|x-y\|_p^p) \geq 2(\|x\|_p^p + \|y\|_p^p),$$

then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p) \geq 2^{2-p}. \tag{13}$$

Taking $x = (1,0)$, $y = (0,1)$, then

$$\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2^p \min(\|x\|_p^p, \|y\|_p^p)} \leq 2^{2-p}. \tag{14}$$

The definition of $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p)$ from (13) and (14) implies that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_p) = 2^{2-p}.$$

Since $\|\cdot\|_1 \geq \|\cdot\|_p (1 < p \leq 2)$, it is well known that $\frac{\varphi_1(t)}{\varphi_p(t)} = \frac{1}{[(1-t)^p + t^p]^{\frac{1}{p}}}$ attains the maximum at $t = \frac{1}{2}$, then

$$M_1 = \max_{0 \leq t \leq 1} \frac{\varphi_1(t)}{\varphi_p(t)} = 2^{1-\frac{1}{p}}.$$

By Theorem 3, we obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_1) = 2^{1-p}.$$

It is well known that $\|\cdot\|_\infty \leq \|\cdot\|_p (2 \leq p \leq \infty)$ and

$$\varphi_\infty(t) = \begin{cases} 1-t, & 0 \leq t \leq \frac{1}{2}, \\ t, & \frac{1}{2} < t < 1. \end{cases}$$

(i) Let $0 \leq t \leq \frac{1}{2}$, $\frac{\varphi_p(t)}{\varphi_\infty(t)} = \frac{((1-t)^p + t^p)^{\frac{1}{p}}}{1-t} = l(t)$, then $l'(t) > 0$ and $M_2 = l(\frac{1}{2}) = 2^{\frac{1}{p}}$.

(ii) Let $\frac{1}{2} \leq t \leq 1$, $\frac{\varphi_p(t)}{\varphi_\infty(t)} = \frac{((1-t)^p + t^p)^{\frac{1}{p}}}{t} = m(t)$, then $m'(t) < 0$ and $M_2 = m(\frac{1}{2}) = 2^{\frac{1}{p}}$.

Therefore, by Theorem 5, we have $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\infty) = \frac{2^{2-p}}{M_2^p} = 2^{1-p}$. □

Remark 1.

(i) *Since the Gao-type constant has two-dimensional characters and the concept of an absolute normalized norm concerns spaces with bases, we can first consider the examples as norms in* $\mathbb{R}^2$*, from Proposition 3 and Example 1, we have*

$$c_{\mathrm{NJ}}^{(p)}(\ell_p) = \begin{cases} 1, & 1 < p \leq 2, \\ 2^{2-p}, & 2 < p < \infty. \end{cases}$$

This method can be helpful for us to deal with the values of $c_{\mathrm{NJ}}^{(p)}(X)$ *for the general spaces* X.

(ii) *Since* $c_{\mathrm{NJ}}^{(p)}(\ell_p) = 1$ *for any* $1 < p \leq 2$*; therefore, the exact value of the Gao-type constant* $c_{\mathrm{NJ}}^{(p)}(X)$ *cannot characterize the inner product space in a general case.*

Example 2. *Let* $X_{p,q,\mu}$ *be the space* $\mathbb{R}^2$ *with the norm*

$$\|\cdot\|_{p,q,\mu} = \max\{\|\cdot\|_p, \mu\|\cdot\|_q\},$$

where $\mu \in [2^{\frac{1}{p}-\frac{1}{q}}, 1]$ *and* $1 \leq q \leq p \leq \infty$*, then*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\mu}) = \begin{cases} \frac{2^{1-\frac{p}{q}}}{\mu^p}, & \text{if } 1 \leq q < p \leq 2, \\ \mu^p 2^{\frac{p}{q}-p+1}, & \text{if } 2 \leq q < p \leq \infty. \end{cases}$$

Proof. Firstly, the norm $\|\cdot\|_{p,q,\mu} = \max\{\|\cdot\|_p, \mu\|\cdot\|_q\} \in \mathbb{N}_\alpha^2$, the corresponding function is

$$\varphi(t) = \|(1-t,t)\|_{p,q,\mu} = \max\{\varphi_p(t), \mu\varphi_q(t)\}.$$

In fact, since $\varphi(t)$ is symmetric with respect to $t=\frac{1}{2}$, we can only consider the function $\varphi(t)$ on the interval $[0,\frac{1}{2}]$. Let $t_0\in[0,\frac{1}{2}]$ be a point such that $\varphi_p(t_0)=\mu\varphi_q(t_0)$, then

$$\varphi(t)=\begin{cases}\varphi_p(t), & t\in[0,t_0],\\ \mu\varphi_q(t), & t\in[t_0,\frac{1}{2}].\end{cases}$$

(i) Let $1\le q<p\le 2$, since $\varphi(t)\ge\varphi_p(t)$ and the function

$$\frac{\varphi(t)}{\varphi_p(t)}=\begin{cases}1, & t\in[0,t_0]\cup[1-t_0,1],\\ \frac{\mu\varphi_q(t)}{\varphi_p(t)}, & t\in[t_0,1-t_0]\end{cases}$$

attains the maximum at $t=\frac{1}{2}$. By Theorem 3, we obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\mu})=\frac{1}{M_1^p}=\frac{2^{1-\frac{p}{q}}}{\mu^p}.$$

(ii) Let $2\le q<p\le\infty$. Since for any $t\in[0,1]$, $\varphi_p(t)\le\varphi_q(t)$ and $\mu\varphi_q(t)\le\varphi_q(t)$, then $\varphi(t)\le\varphi_q(t)$, and the function

$$\frac{\varphi_q(t)}{\varphi(t)}=\begin{cases}\frac{\varphi_q(t)}{\varphi_p(t)}, & t\in[0,t_0]\cup[1-t_0,1],\\ \frac{1}{\mu}, & t\in[t_0,1-t_0]\end{cases}$$

also attains the maximum at $t=\frac{1}{2}$. By Theorem 5, we obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\mu})=\frac{2^{2-p}}{M_2^p}=\mu^p2^{\frac{p}{q}-p+1}.$$

This completes the process. □

Example 3. *Let $1\le p<q\le\infty$, $1\le k<\infty$ and $\mu>0$ be constants. The Banach space $Z_{\mu,p,q,k}$ and its corresponding norm is*

$$\|\cdot\|_{\mu,p,q,k}=(1+\mu)^{-\frac{1}{k}}(\|\cdot\|_p^k+\mu\|\cdot\|_q^k)^{\frac{1}{k}}.$$

Therefore, the corresponding function $\varphi_{\mu,p,q,k}(t)$ is defined by

$$\varphi_{\mu,p,q,k}(t)=(1+\mu)^{-\frac{1}{k}}(\varphi_p^k(t)+\mu\varphi_q^k(t))^{\frac{1}{k}},$$

then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\mu,p,q,k})=\begin{cases}2(1+\mu)^{\frac{p}{k}}(2^{\frac{k}{q}}+\mu2^{\frac{k}{q}})^{\frac{-p}{k}}, & \text{if } 1\le p<q\le 2,\\ 2(1+\mu)^{\frac{-p}{k}}(2^{\frac{k}{p}}+\mu2^{\frac{k}{q}})^{\frac{p}{k}}, & \text{if } 2\le p<q\le\infty.\end{cases}$$

Proof. Since $\varphi_{\mu,p,q,k}(t)$ is the weighted mean of order k of the functions $\varphi_p(t)$ and $\varphi_q(t)$, then

$$\varphi_q(t)\le\varphi_{\mu,p,q,k}(t)\le\varphi_p(t).$$

(i) If $1\le p<q\le 2$, then by the simple calculations, $\varphi_{\mu,p,q,k}(t)\ge\varphi_q(t)$ and the function $\frac{\varphi_{\mu,p,q,k}(t)}{\varphi_q(t)}$ attains the maximum at $t=\frac{1}{2}$. Take $\omega(t)=\varphi_q(t)$ and $\mu(t)=\varphi_p(t)$ in Corollary 1 (i), then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\mu,p,q,k})=\frac{1}{2^{p-1}\varphi_{\mu,p,q,k}^p(\frac{1}{2})}=2(1+\mu)^{\frac{p}{k}}(2^{\frac{k}{q}}+\mu2^{\frac{k}{q}})^{\frac{-p}{k}}.$$

(ii) If $2 \leq p < q \leq \infty$, then $\varphi_{\mu,p,q,k}(t) \leq \varphi_p(t)$ and the function $\frac{\varphi_p(t)}{\varphi_{\mu,p,q,k}(t)}$ attains its maximum at $t = \frac{1}{2}$. Similarly, take $\omega(t) = \varphi_q(t)$ and $\mu(t) = \varphi_p(t)$ in Corollary 1 (ii), then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\mu,p,q,k}) = 2\varphi^p_{\mu,p,q,k}(\frac{1}{2}) = 2(1+\mu)^{\frac{-p}{k}}(2^{\frac{k}{p}} + \mu 2^{\frac{k}{q}})^{\frac{p}{k}}.$$

We obtain the desired result. □

Remark 2.

(i) *In fact, take $p = 2$, $q = 1$ or $p \geq 2$, $q = \infty$ in Example 2, the Gao constant $c'_{\mathrm{NJ}}(X)$ was calculated in [17,23,24]. Now, Example 2 calculates the values of the constant $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\mu})$ for the general case $\mu \in [2^{\frac{1}{p}-\frac{1}{q}}, 1]$ and $1 \leq q \leq p \leq \infty$.*

(ii) *In fact, the concrete Banach space $Z_{\mu,2,\infty,k}$ in Example 3 was studied in some papers (see [24–26]. However, the exact value of $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\mu,p,q,k})$ for the general case remains undiscovered. Example 3 gives the value of the constant $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\mu,p,q,k})$ for the general case $\mu > 0$, $1 \leq p < q \leq \infty$ and $1 \leq k < \infty$.*

From Examples 1, 2, 3, the maximum value M_1 is always attained at $t = \frac{1}{2}$. However, we give some examples to show that M_1 does not attain at $t = \frac{1}{2}$.

Example 4. *Let $1 < q \leq 2$ and the function be defined by*

$$\varphi(t) = \begin{cases} \varphi_q(t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2-\sqrt{2})t + \sqrt{2} - 1, & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \max_{\frac{1}{2} < t \leq 1} \frac{\varphi_q^p(t)}{\varphi^p(t)}.$$

Proof. It is obvious that $\varphi(t) \in \Phi$, and the norm of $\varphi(t)$ is

$$\|(x,y)\|_\varphi = \begin{cases} (|x|^q + |y|^q)^{\frac{1}{q}} & (|x| \geq |y|), \\ (\sqrt{2}-1)|x| + |y| & (|x| \leq |y|). \end{cases}$$

Since $\varphi(t) \geq \varphi_q(t)$, then from Theorem 3, it follows that

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_1^p} = \max_{0 \leq t \leq 1} \frac{\varphi_q^p(t)}{\varphi^p(t)} = \max_{\frac{1}{2} < t \leq 1} \frac{\varphi_q^p(t)}{\varphi^p(t)}.$$

This proof is completed. □

Example 5. *If $1 < q \leq 2$ and $ces_q^{(2)}$ be two-dimensional Cesàro space, then*

$$c_{\mathrm{NJ}}^{(p)}(ces_q^{(2)}) = \max_{0 \leq t \leq 1} \frac{\varphi_2^p(t)}{\varphi^p(t)},$$

where

$$\varphi(t) = \left[\frac{2^q(1-t)^q}{1+2^q} + \left(\frac{1-t}{(1+2^q)^{1/q}} + t\right)^q\right]^{\frac{1}{q}}.$$

Proof. Firstly, we define a norm

$$|x,y| = \left\| \left(\frac{2x}{(1+2^q)^{\frac{1}{q}}}, 2y \right) \right\|_{ces_q^{(2)}}$$

for all $(x,y) \in \mathbb{R}^2$. Obviously, $(\mathbb{R}^2, |\cdot|)$ is the absolute and normalized norm space, the corresponding convex function is as

$$\varphi(t) = \left[\frac{2^q(1-t)^q}{1+2^q} + \left(\frac{1-t}{(1+2^q)^{1/q}} + t \right)^q \right]^{\frac{1}{q}}.$$

It has been proved that $ces_q^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2, |\cdot|)$. Note that

$$\left(\frac{1-t}{(1+2^q)^{1/q}} + t \right)^q \geq \frac{(1-t)^q}{1+2^q} + t^q.$$

Consequently, $\varphi(t) \geq \varphi_q(t)$ $(1 < q \leq 2)$. By using Theorem 3, we obtain

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \frac{1}{M_1^p} = \max_{0 \leq t \leq 1} \frac{\varphi_q^p(t)}{\varphi^p(t)}.$$

The proof is completed. □

Remark 3.

(i) *In particular, take $q = 2$ in Example 4, some classical constants were calculated in [15,20]. Now, we obtain*

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi) = \max_{\frac{1}{2} < t \leq 1} \frac{\varphi_2^p(t)}{\varphi^p(t)} = \frac{\varphi_2^p(\frac{\sqrt{2}}{2})}{\varphi^p(\frac{\sqrt{2}}{2})} = \frac{1}{(4-2\sqrt{2})^{\frac{p}{2}}}.$$

Moreover, Example 4 gives the exact value of the Gao-type constant $c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_\varphi)$ for the general case $1 < q \leq 2$.

(ii) *In Example 5, the function $\frac{\varphi_q(t)}{\varphi(t)}$ attains the maximum M_1 at $t = \frac{1}{2}$ if and only if $q = 2$ for the Cesàro space $ces_q^{(2)}$.*

Example 6. *The Lorentz sequence space $l^{(2)}(\omega', r)$ is $\mathbb{R}^2$ with the norm:*

$$\|(x,y)\|_{\omega',r} = ((x')^r + \omega'(y')^r)^{\frac{1}{r}},$$

where $0 < \omega' < 1$ and $2 \leq r < \infty$, (x', y') is the rearrangement of $(|x|, |y|)$ satisfying $x' \geq y'$, then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\omega',r}) = 2^{1-p}(1+\omega')^{\frac{p}{r}}.$$

Proof. The norm $\|(x,y)\|_{\omega',r}$ is absolute and normalized on $\mathbb{R}^2$, and the corresponding convex function is as

$$\varphi_{\omega',r}(t) = \begin{cases} ((1-t)^r + \omega' t^r)^{\frac{1}{r}}, & 0 \leq t \leq \frac{1}{2}, \\ (t^r + \omega'(1-t)^r)^{\frac{1}{r}}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $0 < \omega' < 1$, it is obvious that $\varphi_{\omega',r}(t) \leq \varphi_r(t)$. We can only consider $\frac{\varphi_r(t)}{\varphi_{\omega',r}(t)}$ for $t \in [0, \frac{1}{2}]$, in which the function is symmetric with respect to $t = \frac{1}{2}$. For any $t \in [0, \frac{1}{2}]$, let $h(t) = \frac{\varphi_r^r(t)}{\varphi_{\omega',r}^r(t)}$, then

$$h'(t) = \frac{r(1-\omega')[t(1-t)]^{r-1}}{[(1-t)^r + \omega' t^r]^2},$$

therefore, $h'(t) \geq 0$ for $0 \leq t \leq \frac{1}{2}$, and the function $\frac{\varphi_r(t)}{\varphi_{\omega',r}(t)}$ attains its maximum at $t = \frac{1}{2}$. By Theorem 5, then

$$c_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\omega',r}) = \frac{2^{2-p}}{M_2^p} = 2^{1-p}(1+\omega')^{\frac{p}{r}}.$$

We gain the conclusion. □

Example 7. *Let $2 \leq p < \infty$ and V_p be the space $\mathbb{R}^2$ with the norm:*

$$\|(x_1, x_2)\|_{V_p} = \max\left\{\left(\left|\frac{x_1}{2}\right|^p + |x_2|^p\right)^{\frac{1}{p}}, \left(|x_1|^p + \left|\frac{x_2}{2}\right|^p\right)^{\frac{1}{p}}\right\},$$

then

$$c_{\mathrm{NJ}}^{(p)}(V_p) = \frac{2^{2-p}}{M_2^p} = 2^{1-p}(1+2^{-p}).$$

Proof. Firstly, the norm $\|(x_1, x_2)\|_{V_p}$ is absolute and normalized on $\mathbb{R}^2$, and the corresponding convex function is

$$\varphi_{V_p}(t) = \begin{cases} \left((\frac{t}{2})^p + (1-t)^p\right)^{\frac{1}{p}}, & 0 \leq t \leq \frac{1}{2}, \\ \left(t^p + (\frac{1-t}{2})^p\right)^{\frac{1}{p}}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is obvious that $\varphi_{V_p}(t) \leq \varphi_p(t)$, and we can consider $\frac{\varphi_p^p(t)}{\varphi_{V_p}^p(t)}$ for $t \in [0, \frac{1}{2}]$. The function $\varphi_p^p(t) - \varphi_{V_p}^p(t) = (1 - \frac{1}{2^p})t^p$ attains the maximum at $t = \frac{1}{2}$ and $\varphi_{V_p}(t)$ attains its minimum at $t = \frac{1}{2}$. By Lemma 1, we obtain that function $\frac{\varphi_p(t)}{\varphi_{V_p}(t)}$ attains the maximum at $t = \frac{1}{2}$; hence, it follows immediately from Theorem 5 that

$$c_{\mathrm{NJ}}^{(p)}(V_p) = \frac{2^{2-p}}{M_2^p} = 2^{1-p}(1+2^{-p}).$$

We complete the proof. □

We can discuss something similar, such as in Example 7, and obtain Example 8 as follows.

Example 8. *Let Y_p $(2 \leq p < \infty)$ be the space $\mathbb{R}^2$ endowed with the norm:*

$$\|(x_1, x_2)\|_{Y_p} = \max\left\{\left(\frac{|x_1|^p}{2} + |x_2|^p\right)^{\frac{1}{p}}, \left(|x_1|^p + \frac{|x_2|^p}{2}\right)^{\frac{1}{p}}\right\},$$

then

$$c_{\mathrm{NJ}}^{(p)}(Y_p) = 2^{2-p}\frac{\varphi_{Y_p}^p(\frac{1}{2})}{\varphi_p^p(\frac{1}{2})} = \frac{3}{2^p},$$

in which the corresponding convex function $\varphi_{Y_p}(t)$ has the form:

$$\varphi_{Y_p}(t) = \begin{cases} \left(\frac{t^p}{2} + (1-t)^p\right)^{\frac{1}{p}}, & 0 \le t \le \frac{1}{2}, \\ \left(t^p + \frac{(1-t)^p}{2}\right)^{\frac{1}{p}}, & \frac{1}{2} \le t \le 1. \end{cases}$$

Remark 4.

(i) *Taking $r = 2$ and $\omega' = 2^{\frac{2}{p}-1} \in (0,1)(2 \le p < \infty)$ in Example 6, we obtain the Lorentz sequence space $l_{p,2}$, which were studied in [20,23,27], and the exact value of Gao's constant $c'_{\mathrm{NJ}}(X)$ was given in [17]. Now, we obtain the exact value of the Gao-type constant $c^{(p)}_{\mathrm{NJ}}(l^{(2)}(\omega', r))$ for the general case $0 < \omega' < 1$ and $2 \le r < \infty$ in Example 6.*

(ii) *The Banach spaces V_2 and Y_2 were studied widely in [24,28], where some classical constants were calculated. Now, the values of $c^{(p)}_{\mathrm{NJ}}(X)$ are calculated for the general Banach spaces V_p, Y_p in Examples 7 and 8 by Theorem 5.*

Finally, we present a practical example that satisfies the conditions of Theorem 7; thus, the exact value of Gao's constant $c'_{\mathrm{NJ}}(X)$ coincides with the lower bound $\frac{1}{(M_1 M_2)^2}$.

Example 9. *Let $\sqrt{3} - 1 < a \le 1$. The corresponding convex function is given by*

$$\varphi_a(t) = \max\{1 - at, 1 - a + at, 1 - \frac{a^2}{2}\} \ for \ 0 \le t \le 1,$$

then

$$c'_{\mathrm{NJ}}(\| \cdot \|_{\varphi_a}) = \frac{2 - a^2}{2(a^2 - 2a + 2)^2}.$$

Proof. It is easy to check that $\varphi_a(t) \in \Phi$ and $\varphi_a(t) = \varphi_a(1-t)$ for all $t \in [0,1]$. If $\sqrt{3} - 1 < a \le 1$, simple calculations show that

$$M_1 = \frac{\varphi_a(t_2)}{\varphi_2(t_2)} = \sqrt{a^2 - 2a + 2}, \quad M_2 = \frac{\varphi_2(t_1)}{\varphi_a(t_1)} = \sqrt{\frac{2(a^2 - 2a + 2)}{(2 - a^2)}},$$

where $t_1 = \frac{a}{2}$, $t_2 = \frac{1-a}{2-a}$ satisfy the condition $(1 - t_1)(1 - t_2) = \frac{1}{2}$ in Theorem 7. Then we have

$$c'_{\mathrm{NJ}}(\| \cdot \|_{\varphi_a}) = \frac{1}{(M_1 M_2)^2} = \frac{2 - a^2}{2(a^2 - 2a + 2)^2}.$$

Therefore, we finish the proof. □

5. Conclusions

In this paper, we present a general method to calculate the Gao-type constant $c^{(p)}_{\mathrm{NJ}}(X)$ for some Banach spaces with the absolute normalized norms, which can help us compute the values of the Gao-type constant on some new specific spaces. Furthermore, we also present an example to show that the value of the Gao-type constant cannot characterize the inner product space in a general case. However, some problems remain unsolved, i.e., the precise lower bound and upper bound of the constant $c^{(p)}_{\mathrm{NJ}}(\| \cdot \|_{\varphi})$ for the general case $\varphi(t) \in \Phi$. The values of the Gao-type constant $c^{(p)}_{\mathrm{NJ}}(X)$ on some specific spaces are not yet known, such as the Banach space $Z_{\mu,p,q,k}$ in the case of $1 \le p \le k \le 2 < q \le \infty$, the Lorentz sequence space $l^{(2)}(\omega', r)$ for the case $1 \le r < 2$, etc. We will investigate these questions in the future.

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