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About the Resolvent Kernel of Neutral Linear Fractional System with Distributed Delays

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Abstract: The present work considers the initial problem (IP) for a linear neutral system with derivatives in Caputo's sense of incommensurate order, distributed delay and various kinds of initial functions. For the considered IP, the studied problem of existence and uniqueness of a resolvent kernel under some natural assumptions of boundedness type. In the case when, in the system, the term which describes the outer forces is a locally Lebesgue integrable function and the initial function is continuous, it is proved that the studied IP has a unique solution, which has an integral representation via the corresponding resolvent kernel. Applying the obtained results, we establish that, from the existence and uniqueness of a resolvent kernel, the existence and uniqueness of a fundamental matrix of the homogeneous system and vice versa follows. An explicit formula describing the relationship between the resolvent kernel and the fundamental matrix is proved as well.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; integral representation; resolvent kernel

MSC: 34A08; 34A12

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1. Introduction

In the last few decades, fractional calculus and fractional differential equations have been intensively investigated in view of their application for modeling many phenomena in various fields of science. Practically, it is established that many natural systems can be more accurately modeled via the memory data included through fractional derivative formulation. For more information on fractional calculus theory and fractional differential equations, see the monographs of Kilbas et al. [1], and Podlubny [2]. For distributed order fractional differential equations, we refer to Jiao et al. [3], and an application-oriented exposition is given in Diethelm [4]. The important case of impulsive differential and functional differential equations with fractional derivatives and some applications are considered in the monograph of Stamova and Stamov [5].

Note that the study of fractional differential equations and systems with delay is, generally speaking, more complicated in comparison with fractional differential equations and systems without delay, i.e., we have the same situation as in the case with integer order derivatives. The use of fractional derivatives leads not only to advantages, but also to several complications and new effects. New, for example, is that in the evolution of the processes described by such equations, the dependence on the past history is inspired by two sources. They are the memory source of the fractional derivative and the memory impact caused by the delay, but only the second of them is independent of the derivative type (integer or fractional).

It is well known (from practical experience), that a separate predictable process can be physically realized only if it is stable in some suitable natural sense. So, since the stability properties turn out to be of utmost importance, then the problem of obtaining integral representations of the solutions and existence of fundamental matrix of the studied models appears as a primary tool for the investigation of these properties and is an essential and "evergreen" theme for research. This is probably an explanation why many papers are devoted to these problems. For a good overview concerning these problems for fractional equations without delay, we refer to the works [6–9] and for delayed fractional equations [10–14]. The case of neutral fractional equations is considered in [15,16], the singular case is considered in [17] and fractional equations of distributed order are studied in [18]. As far as we know, there are not many results concerning the initial problem for neutral fractional differential equations with discontinuous initial functions, the results of which allow us to prove the existence and uniqueness of a fundamental matrix of the homogeneous system (see [19] and the references therein).

Generally speaking, integral representations of the solutions of an initial problem (IP) for some given space of initial functions can be obtained in two ways.

The first of them is via direct construction of a fundamental matrix, which leads to solving an auxiliary matrix IP for the considered system with special kinds of discontinuous initial matrix-valued functions. The advantage of this approach is that the requirements are minimal, but in general it is not possible to obtain some information about the asymptotical behaviors of the fundamental matrix. In the non-neutral cases, this problem can be solved via using inequalities of Gronwall's type for obtaining a priori estimates. This approach is used in [14,15,19].

The other one is more complicated and it is used, so far as we know, only for linear systems with integer order derivatives. In the present work, we make an adaption of this approach to neutral fractional systems with Caputo-type derivatives of incommensurate orders for establishing of integral representation of the solutions for the initial problem for these systems. This approach is based on the existence and uniqueness of a corresponding resolvent kernel obtained under some natural conditions. The advantages of this approach are not only the establishing of existence of a fundamental matrix, but also the obtained information about the asymptotical behaviors of this fundamental matrix. As a consequence, some a priori estimates for the solutions can also be obtained. It must be noted that in this approach the requirements are essentially restrictive.

The paper is organized as follows. In Section 2, we recall the definitions of Riemann–Liouville and Caputo fractional derivatives and introduce some needed notations. In this section, the initial problem (IP) for neutral fractional systems with Caputo-type derivatives of incommensurate order, studied in the paper, is also stated. Section 3 is devoted to the problem of existence and uniqueness of a resolvent kernel under natural assumptions of boundedness type. For the case when in the system the term describing the outer forces is a locally Lebesgue integrable function and the initial function is piecewise continuous, it is found that the studied IP has a unique solution, which has an integral representation via the resolvent kernel. In Section 4, as application of the established results in the previous section, we study the relationship between the resolvent kernel and the fundamental matrix of the homogeneous system. Moreover, we obtain a new and more simply integral representation of the unique solution of the studied IP under the same assumptions, via the fundamental matrix as well as an a priori estimation of an arbitrary solution of the studied IP. In Section 5, an interesting open problem for future research is discussed. Section 6 is devoted to some conclusions and comments about the results presented in the previous sections.

2. Preliminaries and Problem Statement

For convenience, below we present the definitions of Riemann–Liouville and Caputo fractional derivatives and some of their properties. More detailed information about this theme can be found in the monographs [1,2].

Let $f \in L_1^{loc}(\mathbb{R}, \mathbb{R})$, where $L_1^{loc}(\mathbb{R}, \mathbb{R})$ is the real linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in (0, 1)$ be an arbitrary number. For $a \in \mathbb{R}$ and each

$t > a$, the left-sided fractional integral operator, the left side Riemann–Liouville and Caputo fractional derivative of order $\alpha \in (0,1)$ are defined by:

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad {}_{RL}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} (D_{a+}^{\alpha-1} f(t)),$$

$${}_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t) = {}_{RL}D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha},$$

respectively.

The following notations will be used: $\mathbb{R}_+ = (0,\infty)$, $\overline{\mathbb{R}}_+ = [0,\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\langle n \rangle = \{1,\ldots,n\}$, $\langle n \rangle_0 = \{0,\ldots,n\}$, $n \in \mathbb{N}$, $J_a = [a,\infty)$, $a \in \mathbb{R}$, $J_{a+M} = [a, a+M]$, $M \in \mathbb{R}_+$, $I, \Theta \in \mathbb{R}^{n\times n}$ are the identity and zero matrix respectively and $\mathbf{0} \in \mathbb{R}^n$ is the zero vector-column. For $W : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $W(t,\theta) = \{w_{kj}(t,\theta)\}_{k,j=1}^n$, for $(t,\theta) \in J_a \times \mathbb{R}$ we denote $|W(t,\theta)| = \sum_{k,j=1}^n |w_{kj}(t,\theta)|$, with $BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ denote the matrix valued functions $W(t,\theta)$ with bounded variation in θ on arbitrary compact interval $K \subset \mathbb{R}$ for every $t \in J_a$, $Var_{\theta\in K} W(t,\theta) = \{Var_{\theta\in K} w_{kj}(t,\theta)\}_{k,j=1}^n$ and $|Var_{\theta\in K} W(t,\theta)| = \sum\limits_{k,j=1}^{n} Var_{\theta\in K} w_{kj}(t,\theta)$.

For $Y(t) = (y_1(t),\ldots,y_n(t))^\top : J_a \to \mathbb{R}^n$ and $\beta = (\beta_1,\ldots,\beta_n)$ with $\beta_k \in [-1,1]$, $k \in \langle n \rangle$ we will use the notation $I_\beta(Y(t)) = \mathrm{diag}(y_1^{\beta_1}(t),\ldots,y_n^{\beta_n}(t))$ and with $Y(t) \in BV_{loc}(J_a, \mathbb{R}^n)$ denote the functions with bounded variation in t on every compact interval $K \subset \mathbb{R}$. It is well known that for $z \in \mathbb{R}_+$ the gamma function $\Gamma(z)$ has a minimum at $z_{min} \approx +1.46$ (truncated) where it attains the value $\Gamma(z_{min}) \approx +0.8856$ (truncated).

For $h > 0$ with $\mathbf{BL} = BL([-h,0],\mathbb{R}^n)$we denote the Banach space of all vector functions $\Phi = (\phi_1,\ldots,\phi_n) : [-h,0] \to \mathbb{R}^n$ which are bounded and Lebesgue measurable on the interval $[-h,0]$ with norm $||\Phi|| = \sum\limits_{k\in\langle n\rangle} \sup_{s\in[-h,0]} |\phi_k(s)| < \infty$. With S^Φ we denote the set of all jumps points of $\Phi \in \mathbf{BL}$. As usual $\mathbf{PC} = PC([-h,0],\mathbb{R}^n)$($\mathbf{C} = C([-h,0],\mathbb{R}^n)$) is the subspace of all piecewise continuous (continuous) functions, $\mathbf{PC}^* = \mathbf{PC} \cap BV([-h,0],\mathbb{R}^n)$ and it is assumed that all these spaces are endowed with the same sup-norm and the functions are right continuous for $t \in S^\Phi$.

Consider for $t > a$ the inhomogeneous neutral linear delayed system with incommensurate type differential orders and distributed delays in the following general form:

$$D_{a+}^{\alpha}\left(X(t) - \int_{-h}^{0} [d_\theta V(t,\theta)] X(t+\theta)\right) = \int_{-h}^{0} [d_\theta U(t,\theta)] X(t+\theta) + F(t), \tag{1}$$

and the corresponding homogeneous neutral linear delayed system

$$D_{a+}^{\alpha}\left(X(t) - \int_{-h}^{0} [d_\theta V(t,\theta)] X(t+\theta)\right) = \int_{-h}^{0} [d_\theta U(t,\theta)] X(t+\theta), \tag{2}$$

where $X : J_a \to \mathbb{R}^n$, $X(t) = (x_1(t),\ldots,x_n(t))^\top$, $F : J_a \to \mathbb{R}^n$, $F \in L_1^{loc}(J_a,\mathbb{R}^n)$,
$V(t,\theta) = \{v_{kj}(t,\theta)\}_{k,j=1}^n = \sum_{l\in\langle r\rangle} V^l(t,\theta)$, $V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$,
$V^l(t,\theta) = \{v_{kj}^l(t,\theta)\}_{k,j=1}^n$, $v_{kj}^l(t,\theta) = \sum_{l\in\langle r\rangle} v_{kj}^l(t,\theta)$, $l \in \langle r \rangle$, $r \in \mathbb{N}$,
$U(t,\theta) = \{u_{kj}(t,\theta)\}_{k,j=1}^n = \sum_{i\in\langle m\rangle_0} U^i(t,\theta)$, $U^i : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$,
$U^i(t,\theta) = \{u_{kj}^i(t,\theta)\}_{k,j=1}^n$, $u_{kj}^i(t,\theta) = \sum_{i\in\langle m\rangle_0} u_{kj}^i(t,\theta)$, $i \in \langle m \rangle_0$, $m \in \mathbb{N}_0$,
$X_t(\theta) = (x_t^1(\theta),\ldots,x_t^n(\theta))^\top$, $X_t(\theta) = X(t+\theta)$, $-h \le \theta \le 0$, $h > 0$,
$\alpha = (\alpha_1,\ldots,\alpha_n)$, $\alpha_k \in (0,1)$, $D_{a+}^{\alpha} X(t) = (D_{a+}^{\alpha_1} x_1(t),\ldots,D_{a+}^{\alpha_n} x_n(t))^\top$ for every $t \in Ja$, where $D_{a+}^{\alpha_k}$ denotes the left side Caputo fractional derivative ${}_{C}D_{a+}^{\alpha_k}$. Let us also denote $\alpha_m = \min\limits_{k\in\langle n\rangle}(\alpha_k)$ and $\alpha_M = \max\limits_{k\in\langle n\rangle}(\alpha_k)$.

For clarity, we can rewrite the system (2) in a more detailed form:

$$D_{a+}^{\alpha_k}\left(x_k(t)-\sum_{l=1}^{r}\left(\sum_{j=1}^{n}\int_{-\tau_l}^{0}x_j(t+\theta)d_\theta v_{kj}^{l}(t,\theta)\right)\right)=\sum_{i=0}^{m}\left(\sum_{j=1}^{n}\int_{-\sigma_i}^{0}x_j(t+\theta)d_\theta u_{kj}^{i}(t,\theta)\right),$$

where $\sigma_i, \tau_l \in \mathbb{R}_+$, $\tau = \max\limits_{l\in\langle r\rangle}\tau_l$, $\sigma = \max\limits_{i\in\langle m\rangle_0}\sigma_i$.

Introduce for arbitrary $\Phi \in \mathbf{BL}$ the following initial condition:

$$X_t(\theta) = X(t+\theta) = \Phi(t-a+\theta) \quad \text{for} \quad t+\theta \le a, \quad \theta \in [-h,0]. \tag{3}$$

where $h = \max(\tau, \sigma)$.

Remark 1. *The initial condition (3) means that the function $t \to X(t)$, $t \in J_a$ is considered as a prolongation of the function $t \to \Phi(t-a)$, $t \in [a-h,a]$.*

Definition 1. *The vector function $X(t) = (x_1(t),\ldots,x_n(t))^\top$ is a solution of the initial problem IP (1), (3) in $J_{a+M}(J_a)$ if $X|_{[a,a+M]} \in C([a,a+M],\mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$ satisfies the system (1) for all $t \in (a,M]$ $(t \in (a,\infty))$ and the initial condition (3) for $t+\theta \le a$, $\theta \in [-h,0]$.*

In our consideration below, we need the following auxiliary system:

$$\begin{aligned} X(t) =& C_{\Phi(0)} + \int_{-h}^{0}[d_\theta V(t,\theta)]X(t+\theta) \\ &+ I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\tau)\left(\int_{-h}^{0}[d_\theta U(t,\theta)]X(\tau+\theta)+F(\tau)\right)\mathrm{d}\tau, \end{aligned} \tag{4}$$

where $C_{\Phi(0)} = \Phi(0) - \int_{-h}^{0}[d_\theta V(a,\theta)]\Phi(\theta)$.

Definition 2. *The vector function $X(t) = (x_1(t),\ldots,x_n(t))^\top$ is a solution of the initial problem IP (4), (3) in $J_{a+M}(J_a)$ if $X|_{[a,a+M]} \in C([a,a+M],\mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$ satisfies the system (4) for all $t \in (a,M]$ $(t \in (a,\infty))$ and the initial condition (3) for $t+\theta \le a$, $-h \le \theta \le 0$.*

We say that for the kernels $U^i, V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, the conditions (**S**) are fulfilled if for every $i \in \langle m\rangle_0$, $l \in \langle r\rangle$ the following conditions hold (see [20,21]):

(**S1**) The functions $(t,\theta) \to U^i(t,\theta)$, $(t,\theta) \to V^l(t,\theta)$ are measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that $U^i(t,\theta) = 0$, $V^l(t,\theta) = 0$ for $\theta \ge 0$, $U^i(t,\theta) = U^i(t,-\sigma_i)$ for $\theta \le -\sigma_i$, $V^l(t,\theta) = V^l(t,-\tau_l)$ for $\theta \le -\tau_l$, for every $t \in J_a$. The kernels $U^i(t,\theta)$ and $V^l(t,\theta)$ are continuous from left in θ on $(-\sigma_i, 0)$ and $(-\tau_l, 0)$, respectively, for $t \in J_a$, and $U^i(t,\cdot), V^l(t,\cdot) \in BV([-h,0],\mathbb{R}^{n\times n})$ for every fixed $t \in J_a$.

(**S2**) For $t \in J_a$ the functions $V^*(t) = Var_{[-h,0]}V(t,\cdot), U^*(t) = Var_{[-h,0]}U(t,\cdot) \in L_1^{loc}(J_a,\mathbb{R}^{n\times n})$ are locally bounded for $t \in J_a$, and the kernel $V(t,\theta)$ is uniformly nonatomic at zero (see [22]) (i.e., for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for each $t \in J_a$ we have that $Var_{[-\delta,0]}V(t,\cdot) < \varepsilon$).

(**S3**) For $t \in J_a$ and $\theta \in [-h,0]$ the Lebesgue decompositions of the kernels have the form:

$$U^i(t,\theta) = U_d^i(t,\theta) + U_c^i(t,\theta),\ \ U_c^i(t,\theta) = U_{ac}^i(t,\theta) + U_s^i(t,\theta)$$
$$V^l(t,\theta) = V_d^l(t,\theta) + V_c^l(t,\theta),\ \ V_c^l(t,\theta) = V_{ac}^l(t,\theta) + V_s^l(t,\theta)$$

where with the indexes d, ac, s are denoted the jump, the absolutely continuous and the singular part, respectively, in the Lebesgue decompositions and $U_c^i(t,\theta), V_c^l(t,\theta)$ are the continuous parts of these decompositions. In addition, we also have

$$U_d^i(t,\theta) = \{a_{kj}^i(t)H(\theta+\sigma_i(t))\}_{k,j=1}^n, \quad V_d^l(t,\theta) = \{\bar{a}_{kj}^l(t)H(\theta+\tau_l(t))\}_{k,j=1}^n,$$

where $H(t)$ is the Heaviside function, $A^i(t) = \{a^i_{kj}(t)\}^n_{k,j=1} \in L_1^{loc}(J_a, \mathbb{R}^{n\times n})$ are locally bounded, $\overline{A}^l(t) = \{\overline{a}^l_{kj}(t)\}^n_{k,j=1} \in C(J_a, \mathbb{R}^{n\times n})$, $\tau_l(t) \in C(J_a, [0, \tau_l]), \sigma_i(t) \in C(J_a, [0, \sigma_i])$, $\sigma_0(t) \equiv 0$ for every $t \in J_a$.

(**S4**) For each $t^* \in J_a$ the relations

$$\lim_{t\to t^*} \int_{-\sigma}^{0} |U^i(t,\theta) - U^i(t^*,\theta)| \mathrm{d}\theta = \lim_{t\to t^*} \int_{-\tau}^{0} |V^l(t,\theta) - V^l(t^*,\theta)| \mathrm{d}\theta = 0$$

hold and there exists $\gamma \in \mathbb{R}_+$ such that the kernels $V^l_{ac}(t,\theta)$ and $V^l_s(t,\theta)$ are continuous in t, when $t \in [a, a+\gamma]$, $\theta \in [-h, 0]$.

(**S5**) The sets $S^\Phi_l = \{t \in J_a | t - \tau_l(t) \in S^\Phi\}$, $\quad S^\Phi_i = \{t \in J_a | t - \sigma_i(t) \in S^\Phi\}$ do not have limit points (see [21]) for every $\Phi \in \mathbf{BL}$.

Remark 2. *Note that in the case when $\Phi \in \mathbf{C}$, the condition (**S5**) is ultimately fulfilled since $S^\Phi = \emptyset$. It is clear also, that from condition (**S5**) it follows that the sets $\bigcup_{i\in\langle m\rangle} S^\Phi_i$ and $\bigcup_{l\in\langle r\rangle} S^\Phi_l$ are finite.*

Everywhere below in our consideration we assume that the conditions (**S**) hold.

Let us, for the arbitrary fixed number $s \in J_a$, define the following matrix valued function:

$$\Phi_C(t,s) = \begin{cases} I, & t = s \\ \Theta, & t < s \end{cases}$$

and consider the following matrix IP:

$$D^\alpha_{a+}(C(t,s) - \int_{-h}^{0} [d_\theta V(t,\theta)] C(t+\theta, s)) = \int_{-h}^{0} [d_\theta U(t,\theta)] C(t+\theta, s) \; t \in (s, \infty), \tag{5}$$

$$C(t,s) = \Phi_C(t,s), \; t \in [s-h, s]. \tag{6}$$

Definition 3. *For each $s \in J_a$ the matrix valued function $C(\cdot, s) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $C(t,s) = \{c_{jk}(t,s)\}^n_{k,j=1}$, is called a solution of the IP (5), (6) for $t \in J_s$, if $C(\cdot, s)$ is continuous in t on J_s and satisfies the matrix Equation (5) for $t \in (s, \infty)$, as well as the initial condition (6) as well.*

Remark 3. *Note that below we will assume that $C(\cdot, s)$ is prolonged in t on $(-\infty, s)$ as Θ.*

3. Main Results

The aim of this section is twofold: First, we study the existence and uniqueness of a resolvent kernel under assumptions for some kind of boundedness of the kernels $U(t,\theta)$ and $V(t,\theta)$. Second, in the case when $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ and $\Phi \in \mathbf{PC}$ is arbitrary, we obtain that the studied IP (1), (3) has a unique solution which has an integral representation via the function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ and the resolvent kernel.

Lemma 1. *Let $\Phi \in \mathbf{C}$ be arbitrary and assume that $X(t)$ is a solution of the initial problem IP (4), (3).*

Then, $X(t)$ satisfies for $t \in J_a$ the Volterra–Stieltjes equation

$$X(t) = \int_{s}^{t} [\mathrm{d}_s K(t,s)] X(s) + f(t), \tag{7}$$

where

$$\begin{aligned} f(t) =& C_{\Phi(0)} + \int_{-h}^{a-t} [d_\theta V(t,\theta)]\Phi(t+\theta-a) \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_{-h}^{a-\tau} [d_\theta U(t,\theta)]\Phi(\tau+\theta-a) \right) \mathrm{d}\tau \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) F(\tau) \mathrm{d}\tau \end{aligned} \tag{8}$$

and vice versa.

Proof. Substituting this solution $X(t)$ in (4) and splitting off that part that explicitly depends on the initial data, we obtain the following system

$$\begin{aligned} X(t) =& \int_{a-t}^0 [d_\theta V(t,\theta)]X(t+\theta) \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_{a-t}^0 [d_\theta U(t,\theta)]X(\tau+\theta) \right) \mathrm{d}\tau + f(t), \end{aligned} \tag{9}$$

where the function $f(t)$ is given via (8). After the substitution $s = t + \theta$ in (9) and (8), we find that

$$\begin{aligned} X(t) =& \int_a^t [d_s V(t,s-t)]X(s) \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_a^\tau [d_s U(\tau,s-\tau)]X(s) \right) \mathrm{d}\tau + f(t) \end{aligned} \tag{10}$$

and

$$\begin{aligned} f(t) =& C_{\Phi(0)} + \int_{t-h}^{a} [d_s V(t,s-t)]\Phi(s-a) \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) \left(\int_{\tau-h}^{a} [d_s U(\tau,s-\tau)]\Phi(s-a) \right) \mathrm{d}\tau \\ &+ I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\tau) F(\tau) \mathrm{d}\tau \end{aligned} \tag{11}$$

From (10) applying Fubini's theorem, we obtain the system

$$\begin{aligned} X(t) &= \int_a^t [d_s \left(V(t,s-t)] + I_{-1}(\Gamma(\alpha)) \int_s^t U(\tau,s-\tau) I_{\alpha-1}(t-\tau) \mathrm{d}\tau \right)]X(s) + f(t) \\ &= \int_s^t [\mathrm{d}_s K(t,s)]X(s) + f(t), \end{aligned}$$

from which it follows the Volterra–Stieltjes Equation (7), where the kernel $K(t,s)$ has the form

$$K(t,s) = V(t,s-t) + I_{-1}(\Gamma(\alpha)) \int_s^t U(\tau,s-\tau) I_{\alpha-1}(t-\tau) \mathrm{d}\tau. \tag{12}$$

The vice versa statement can be proved in a reverse way if we assume that the solution of (7) is continuous on J_a. □

Definition 4 (see [23], Ch. 10)**.** *The function $K(t,s) : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is called Stieltjes–Volterra type $\mathbf{B}^\infty$ kernel on $J_a \times \mathbb{R}$ ($K \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$), if the following conditions (**K**) hold:*

*(**K1**) The function $(t,s) \to K(t,s)$ is measurable in t for each fixed s, right continuous in s on (a,t) and $K(t,s) = \Theta$ for $s > t$.*

*(**K2**) $K(t,s)$ is bounded and the total variation in s of $K(t,s)$ for every fixed t is uniformly bounded in t on J_a too.*

With $SVB^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$, we denote the functions whose restrictions to an arbitrary compact subset $J, J \subset J_a$ belong to $\mathbf{B}^{\infty}$.

Below, we will need the following lemma which clarifies the relation between the conditions (**S**) and (**K**).

Lemma 2. *Let the following conditions be fulfilled:*

(1) *The conditions (**S**) hold.*
(2) *The function $V^*(t)$ is uniformly bounded for $t \in J_a$.*
(3) *The function $U^*(t) = O(t^{-\alpha_M})$.*

Then, the kernel $K(t,s)$ defined via (12) is a Volterra–Stieltjes type $\mathbf{B}^{\infty}$ kernel on $J_a \times \mathbb{R}$, i.e., $K \in SVB^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$.

Proof. From conditions (**S1**) and (**S2**), it follows that $K(t,s)$ is locally bounded, measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and since $U(t,\theta) = V(t,\theta) = 0$ for $\theta = s - t \geq 0$, then $K(t,s) = \Theta$ even for $s \geq t$. Condition (**S1**) implies that $K(t,s)$ is right continuous in s on $(0,t)$ for every fixed t, and condition 2 implies that the total variation in s of the first addend in the right side of (12) is uniformly bounded in t on J_a. Taking into account condition (**S1**) from (12), we obtain (see (21) for details) that

$$
\begin{aligned}
K(t,s) &= V(t,s-t) + I_{-1}(\Gamma(\alpha)) \int_a^t U(\tau,s-\tau) I_{\alpha-1}(t-\tau)\mathrm{d}\tau \\
&= V(t,s-t) + I_{-1}(\Gamma(1+\alpha)) \int_a^t U(\tau,s-\tau)\mathrm{d}_\tau I_{\alpha}(t-\tau).
\end{aligned} \tag{13}
$$

From condition (**S1**) it follows that $U(t,\cdot) \in BV_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ for every fixed $t \in J_a$ and hence $\frac{\partial U(t,\theta)}{\partial \theta}$ exists almost everywhere and is locally integrable for $\theta \in \mathbb{R}$. Thus, it follows that the second addend in the right side of (13) has bounded variation in s on every compact interval. Taking into account condition 3, for the second addend in the right side of (13) we obtain that

$$
\left| \int_a^t U(\tau,s-\tau)\mathrm{d}_\tau I_{\alpha}(t-\tau) \right| \leq U^*(t)(t-a)^{\alpha_M} \tag{14}
$$

and hence from (14) it follows that the second addend in the right side of (13) is uniformly bounded in t on J_a, which completes the proof. □

Remark 4. *When we consider a kernel $K(t,s)$ defined via (12) and satisfying the conditions (**S**) only in a domain of the type $J_{a+T} \times \mathbb{R}$ for arbitrary $T \in \mathbb{R}_+$, then obviously $K \in SVB^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ and it is simple to see that conditions 2 and 3 of Lemma 2 are unnecessarily for establishing the boundedness of $K(t,s)$, since in this case t is varying in the compact interval J_{a+T} and the necessary boundedness follows from conditions (**S**).*

The example given below, shows that if t varies in a non-compact interval for example as $\overline{\mathbb{R}}_+$, then the kernel $K(t,s)$ can satisfy condition (**S**) and be unbounded in contrast with the case when t varies in a compact interval. Thus, we will show that in the case of unbounded interval it is possible the kernels $V(t,\theta), U(t,\theta)$ and $K(t,s)$ to satisfy conditions (**S**), but $K(t,s)$ does not satisfy conditions (**K**), without additional conditions like conditions 2 and 3 of Lemma 2.

The next example also shows that the kernel $K(t,s)$ can be unbounded even in the case when both kernels $V(t,\theta)$ and $U(t,\theta)$ are uniformly bonded in t. Thus, the conditions 2 and 3 of Lemma 2 are essential, and are not a corollary from the chosen technique for the proof. It must be noted that this situation is not noticed in the case with integer order derivatives ($\alpha = 1$).

Example 1. *Let* $a = 0, I_\alpha(t) = t^\alpha, \alpha \in (0,1), V(t,\theta) = G(\theta), U(t,\theta) = G(\theta)$, *where*

$$G(\theta) = \begin{cases} 0, & \theta > 0, \\ \theta, & \theta \in [-h, 0], \\ -h, & \theta < -h. \end{cases}$$

Consider via (12) *the kernel*

$$K(t,s) = G(s-\tau) + \Gamma^{-1}(\alpha) \int_s^t G(s-\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau \tag{15}$$

Then, for the second addend in (15) *for arbitrary fixed* $s \in (a,t)$, *via the substitution* $t - \tau = z$, *we obtain*

$$\begin{aligned}
&\Gamma^{-1}(\alpha) \int_s^t G(s-\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau = -\Gamma^{-1}(1+\alpha) \int_s^t G(s-\tau) \mathrm{d}(t-\tau)^\alpha \\
&= -\Gamma^{-1}(1+\alpha) \int_{t-s}^0 G(z+s-t) \mathrm{d}z^\alpha \\
&= \Gamma^{-1}(1+\alpha)(\int_0^h G(z+s-t) \mathrm{d}z^\alpha + \int_h^{t-s} G(z+s-t) \mathrm{d}z^\alpha) \\
&= \Gamma^{-1}(1+\alpha) \int_0^h G(z+s-t) \mathrm{d}z^\alpha - h\Gamma^{-1}(1+\alpha) \int_h^{t-s} \mathrm{d}z^\alpha) \\
&= \Gamma^{-1}(1+\alpha)(h^\alpha(h+s-t) - \int_0^h z^\alpha \mathrm{d}z) - h\Gamma^{-1}(2+\alpha)((t-s)^{1+\alpha} - h^{1+\alpha}) \\
&= \Gamma^{-1}(1+\alpha)h^\alpha(h+s-t) - \Gamma^{-1}(2+\alpha)h^{1+\alpha} - h\Gamma^{-1}(2+\alpha)((t-s)^{1+\alpha} - h^{1+\alpha})
\end{aligned} \tag{16}$$

which implies that the second addend for every fixed $s < t$ *is unbounded, since the right side of* (16) *tends to* $-\infty$ *when* $t \to \infty$ *and hence the kernel* $K(t,s)$ *defined via* (15) *is unbounded too.*

Definition 5 ([22,23])**.** *A kernel* $R \in SVB^\infty_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ $(R \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n}))$ *is called a Stieltjes–Volterra resolvent of type* $\mathbf{B}^\infty$ *corresponding to a kernel* $K \in SVB^\infty_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ $(K \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n}))$ *if for* $s \in \mathbb{R}, t \in J \cap \{t \geq s\}(t \in J_a \cap \{t \geq s\})$ *satisfies the system*

$$R(t,s) = -K(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)]R(\eta,s) = -K(t,s) + \int_s^t \mathrm{d}_\eta[R(t,\eta)]K(\eta,s), \tag{17}$$

where the integrals in (17) *are understood in the sense of Lebesgue–Stieltjes and* $J \subset J_a$ *is an arbitrary compact subset.*

Introduce the sets

$$\mathfrak{I}_T = \{R(t,s) : J_{a+T} \times \mathbb{R} \to \mathbb{R}^{n\times n} \mid R \in SVB^\infty_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n}), T \in \mathbb{R}_+\}$$

for arbitrary $T \in \mathbb{R}_+$ and

$$\mathfrak{I}^\infty = \{R(t,s) : J_{a+T} \times \mathbb{R} \to \mathbb{R}^{n\times n} \mid R \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})\}.$$

Obviously, every one of them is a real Banach space under the corresponding norms

$$||R(t,s)||_T = \sup_{t\in[a,a+T]} Var_{s\in[a,a+T]}|R(t,s)| \text{ and } ||R(t,s)||_\infty = \sup_{t\in J_a} Var_{s\in J_a}|R(t,s)|.$$

The next theorem presents a statement, well-known for the case of integer order derivatives. Note that in the case of fractional derivatives, the Volterra–Stieltjes integral system given by (17) is weak singular in comparison to the case of derivatives with integer order, where the corresponding system is non-singular.

Theorem 1. *Let the following conditions be fulfilled:*

(1) *The conditions (**S**) hold.*

(2) *The function $K(t,s) : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined via (12).*

Then, the resolvent Equation (17) for every $T \in \mathbb{R}_+$ has a unique solution $R(t,s) \in \Im_T$ with $R(t,t) = \Theta, t \in [a, a+T]$.

Proof. The idea of this proof is similar to the proof's idea applied in the case with integer derivatives, but the details are different.

The conditions (**S**) and (12) imply that the conditions of Lemma 2 hold locally and then $K \in SVB^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$, i.e., $K(t,s)$ is a $\mathbf{B}^{\infty}$ type Stieltjes–Volterra kernel for arbitrary $T \in \mathbb{R}_+$ on each subset $[a, a+T] \times \mathbb{R}$.

Applying the D. Morgenstern method [24] for each $R(t,s) \in \Im_T$, we define for arbitrary $\lambda \in \mathbb{R}_+$

$$\bar{R}(t,s) = e^{-\lambda(t-s)}R(t,s) \text{ and } \bar{K}(t,s) = e^{-\lambda(t-s)}K(t,s). \tag{18}$$

For every $T \in \mathbb{R}_+$ from Lemma 2, it follows that if $R(t,s) \in \Im_T$, then from (18) it follows that $\bar{R}(t,s), \bar{K}(t,s) \in \Im_T$ too, for every $\lambda \in \mathbb{R}_+$.

For every $\lambda \in \mathbb{R}_+$, consider Equation (17), where $\bar{R}(t,s), \bar{K}(t,s)$ are defined via (18):

$$\bar{R}(t,s) = -\bar{K}(t,s) + \int_s^t \mathrm{d}_\eta[\bar{R}(t,\eta)]\bar{K}(\eta,s). \tag{19}$$

It is clear that if $||\bar{K}(t,s)||_T < 1$ for some $\lambda \in \mathbb{R}_+$, then for these $\lambda \in \mathbb{R}_+$ there exists a unique solution of (19) $\bar{R}(t,s)$. Using (19), we define for arbitrary $\lambda \in \mathbb{R}_+$ the following operator:

$$(\mathfrak{R}\bar{R})(t,s) = -\bar{K}(t,s) + \int_s^t \mathrm{d}_\eta[\bar{R}(t,\eta)]\bar{K}(\eta,s). \tag{20}$$

It is not so hard to check that from (12) and Lemma 2 it follows that $\mathfrak{R}\Im_T \subseteq \Im_T$.

Let $\bar{R}(t,s), \bar{R}^*(t,s)$ be arbitrary, and then from (20) it follows that

$$\begin{aligned} ||(\mathfrak{R}\bar{R})(t,s) - (\mathfrak{R}\bar{R}^*)(t,s)||_T &\le \sup_{t\in[a,a+T]} Var_{s\in[0,T]} \int_s^t \mathrm{d}_\eta[\bar{R}(t,\eta) - \bar{R}^*(t,\eta)]\bar{K}(\eta,s) \\ &\le ||\bar{K}(t,s)||_T ||\bar{R}(t,s) - \bar{R}^*(t,s)||_T. \end{aligned} \tag{21}$$

As was mentioned above, we will prove that $||\bar{K}(t,s)||_T < 1$ for arbitrary fixed $T \in \mathbb{R}_+$ for a sufficiently large value of $\lambda \in \mathbb{R}_+$. From (12) and (18), it follows that

$$\begin{aligned} \bar{K}(t,s) &= e^{-\lambda(t-s)}K(t,s) \\ &= e^{-\lambda(t-s)}V(t,s-t) + I_{-1}(\Gamma(\alpha))e^{-\lambda(t-s)} \int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau. \end{aligned} \tag{22}$$

Let $q \in (0,1)$ and $\varepsilon \in (0, \frac{q}{4})$ be arbitrary and in virtue of condition (**S2**), there exists $\delta \in (0, \frac{q}{4})$ such that $Var_{-\delta\le s-t\le 0}V(t.\cdot) < \varepsilon$ holds uniformly in t.

Since $|V(t,s-t)| \le |V(t,0)| + Var_{-\delta\le s-t\le 0}V(t,s-t) < \varepsilon$, then $\sup\limits_{s\in[a,a+T]} |V(t,t-s) \le \varepsilon$ for $t-\delta \le s \le t$. Taking into account that $|V(t,s-t)| \le Var_{-h\le s-t\le -\delta}V(t,s-t)$, then we have $V_{sup} = \max(1, \sup\limits_{t\in[a,a+T]} \sup\limits_{t-h\le s\le t-\delta} |V(t,t-s)|) < \infty$. Thus, for the first addend in the right side of (22) for $\lambda \ge \delta^{-1}\ln\frac{4V_{sup}}{q}$ (note that $\frac{4V_{sup}}{q} > 1$) and taking into account conditions (**S**), we find that

$$
\begin{aligned}
&\sup_{t\in[a,a+T]} Var_{s\in[a,a+T]} e^{-\lambda(t-s)}V(t,s-t) \le \sup_{t\in[a,a+T]} Var_{-h\le s-t\le 0}\, e^{-\lambda(t-s)}V(t,s-t)\\
&\le \sup_{t\in[a,a+T]} Var_{-h\le s-t\le -\delta}\, e^{-\lambda(t-s)}V(t,s-t)\\
&+ \sup_{t\in[a,a+T]} Var_{-\delta\le s-t\le 0}\, e^{-\lambda(t-s)}V(t,s-t)\\
&\le V_{sup}(e^{-\lambda\delta}-e^{-\lambda h})+\varepsilon \le V_{sup}e^{-\lambda\delta}+\frac{q}{4}\le\frac{q}{2}.
\end{aligned}
\tag{23}
$$

For the second addend in (22), we find that

$$
\begin{aligned}
&\sup_{t\in[a,a+T]} Var_{s\in[a,a+T]}\left(e^{-\lambda(t-s)}I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau\right)\\
&\le \sup_{t\in[a,a+T]} Var_{-\delta\le s-t\le 0}\left(e^{-\lambda(t-s)}I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau\right)\\
&+ \sup_{t\in[a,a+T]} Var_{-h\le s-t\le -\delta}\left(e^{-\lambda(t-s)}I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau\right)
\end{aligned}
\tag{24}
$$

For every $k,j\in\langle n\rangle$ we have

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha_k)}\left|\int_s^t (t-\tau)^{\alpha_k-1}u_{kj}(\tau,s-\tau)\mathrm{d}\tau\right| = \frac{1}{\alpha_k\Gamma(\alpha_k)}\left|\int_s^t u_{kj}(\tau,s-\tau)\mathrm{d}(t-\tau)^{\alpha_k}\right|\\
&=\frac{1}{\Gamma(1+\alpha_k)}\left|\int_s^t u_{kj}(\tau,s-\tau)\mathrm{d}(t-\tau)^{\alpha_k}\right|\\
&=\frac{(t-s)^{\alpha_k}}{\Gamma(1+\alpha_k)}\sup_{t\in[a,a+T]} Var_{\theta\in[-h,0]}u_{kj}(t,\theta)\\
&\le\frac{(t-s)^{\alpha_k}}{\Gamma(z_{min})}\sup_{t\in[a,a+T]} Var_{\theta\in[-h,0]}u_{kj}(t,\theta)
\end{aligned}
\tag{25}
$$

Taking into account that $|I_\alpha(\delta)|\le n\delta^{\alpha_m}$, $|I_{-1}(\Gamma(1+\alpha))|\le n\Gamma^{-1}(z_{min})$ and (25), for the first addend in the right hand of (24), we have the estimation

$$
\begin{aligned}
&\sup_{t\in[a,a+T]} Var_{-\delta\le s-t\le 0}\left(e^{-\lambda(t-s)}I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau\right)\\
&\le (1-e^{-\lambda\delta})|I_{-1}(\alpha)|\,|I_{-1}(\Gamma(\alpha))|\,|I_\alpha(\delta)|\,||U||_T\\
&+\sup_{t\in[a,a+T]} Var_{-\delta\le s-t\le 0}\left(I_{-1}(\alpha)I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)\mathrm{d}I_\alpha(t-\tau)\right)\\
&\le n\delta^{\alpha_m}|I_{-1}(\Gamma(1+\alpha))|\,||U||_T+|I_\alpha(\delta)|\,|I_{-1}(\Gamma(1+\alpha))|\,||U||_T\\
&\le \delta^{\alpha_m}||U||_T n^2\Gamma^{-1}(z_{min})
\end{aligned}
\tag{26}
$$

Since conditions (**K**) imply that

$$
M=\sup_{t\in[a,a+T]} Var_{-h\le s-t\le -\delta}\left|\int_s^t U(\tau,s-\tau)\mathrm{d}I_\alpha(t-\tau)\right|<\infty,
$$

for the second addend in (24), the following estimation holds:

$$\begin{aligned}
&\sup_{t\in[a,a+T]} Var_{-h\leq s-t\leq-\delta}\left(e^{-\lambda(t-s)}I_{-1}(\Gamma(\alpha))\int_s^t U(\tau,s-\tau)I_{\alpha-1}(t-\tau)\mathrm{d}\tau\right)\\
&\leq (e^{-\lambda\delta}-e^{-\lambda h})|I_{-1}(\Gamma(1+\alpha))|\,|I_\alpha(T)|||U||_T\\
&+e^{-\lambda\delta}|I_{-1}(\Gamma(1+\alpha))| \sup_{t\in[a,a+T]} Var_{-h\leq s-t\leq-\delta}\left|\int_s^t U(\tau,s-\tau)\mathrm{d}I_\alpha(t-\tau)\right|\\
&\leq n\Gamma^{-1}(z_{min})e^{-\lambda\delta}(|I_\alpha(T)|\,||U||_T+M).
\end{aligned} \tag{27}$$

Let

$$\delta<\frac{q\Gamma(z_{min})}{8n^2||U||_T}\ \text{ and }\ \lambda\geq\delta^{-1}\ln\frac{q\Gamma(z_{min})}{n(|I_\alpha(T)|\,||U||_T+M)}. \tag{28}$$

Then, from (22)–(24), (27) and (28), it follows that $||\bar{K}(t,s)||_T<1$. Since $\mathfrak{R}\mathfrak{I}_T\subseteq\mathfrak{I}_T$ and the operator $\mathfrak{R}$ is a contraction in $\mathfrak{I}_T$, then $\mathfrak{R}$ possesses a unique fixed point in $\mathfrak{I}_T$.

In addition, if $R(t,s)$ is a solution of (17), and since the kernel $K(t,s)$ is defined via (12), then from (17) it follows that $R(t,t)=-K(t,t)=\Theta$ for arbitrary $t\in[a,a+T]$, which completes the proof. □

Corollary 1. *Let the conditions of Theorem 1 hold.*

Then, the system (7) has a unique solution $X(t)$ for arbitrary $\Phi\in\mathbf{PC}$ and every $T\in\mathbb{R}_+$ with the following representation:

$$X(t)=f(t)-\int_a^t[\mathrm{d}_\eta R(t,\eta)]f(\eta), \tag{29}$$

Proof. Let $R(t,s)\in\mathfrak{I}_T$ be the unique solution of (17) for arbitrary $\Phi\in\mathbf{PC}$ and some $T\in\mathbb{R}_+$, existing in virtue of Theorem 1. Then, according to Theorem 2.5 in Chapter 10 of [23] the system (7) has a unique solution $X(t)$ for this T, which possess the representation (29).

It remains to be proved that $X(t)$ is a continuous function on J_{a+T} for every $T\in\mathbb{R}_+$. Since $\Phi\in\mathbf{PC}$ and the conditions (**S**) hold, then in virtue of Lemma 1 in [20], the integrals $\int_{t-h}^a[\mathrm{d}_sV(t,s-t)]\Phi(s-a)$ and $\int_{t-h}^a[\mathrm{d}_sU(t,s-t)]\Phi(s-a)$ are continuous functions in t on J_{a+T}. Thus, from (11), it follows that $f(t)\in C(J_{a+T},\mathbb{R}^n)$ for every $T\in\mathbb{R}_+$. Applying again the same lemma to the integral in the right side of (29), we obtain that the unique solution $X(t)$ of the system (7) given with (26) is also a continuous function. □

Remark 5. *It is clear that if $t\in J_a$ general speaking without some additional information about the asymptotic behavior of the kernel $K(t,s)$, nothing can be established about the asymptotic behavior of the resolvent $R(t,s)$ defined in the same interval J_a. Moreover, it is not possible to prolong the solution $X(t)$ from J_{a+T} to J_a via the Kuratowski–Zorn's lemma applying the technique of the nested intervals of existence using the uniqueness of the solutions in the intersection of their existence intervals. This occurs since, for different choices of $T_1,T_2\in\mathbb{R}_+$, the corresponding scaling numbers $\lambda_1,\lambda_2\in\mathbb{R}_+$ in general will be different. Normally, we will find that $\lambda_1\leq\lambda_2$, and if the kernel $K(t,s)$ is unbounded, then the set of the corresponding scaling numbers $\lambda\in\mathbb{R}_+$ will be unbounded too, which excludes the prolongation.*

The next theorem considers the case when $K(t,s)$ is defined via (12) and $K\in SVB^\infty(J_a\times\mathbb{R},\mathbb{R}^{n\times n})$.

Theorem 2. *Let the following conditions be fulfilled:*

(1) *The conditions (**S**) hold and the function $V^*(t)$ is uniformly bounded on J_a.*
(2) *The kernel $K(t,s):J_a\times\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined via (12) and $K\in SVB^\infty(J_a\times\mathbb{R},\mathbb{R}^{n\times n})$.*

Then, the resolvent Equation (17) has a unique solution $R\in SVB^\infty(J_a\times\mathbb{R},\mathbb{R}^{n\times n})$.

Proof. Since the details of the proof are very similar to the proof of Theorem 1, we will only sketch the differences. The analysis of the proof of the Theorem 1 leads to following: If formally everywhere we replace the norm $||\cdot||_T$, where $T \in \mathbb{R}_+$ is an arbitrary fixed number with the norm $||\cdot||_\infty$, then instead to prove for enough large $\lambda \in \mathbb{R}_+$ that $||K(t,s)||_T < 1, T \in \mathbb{R}_+$, we must prove that for enough large $\lambda \in \mathbb{R}_+$ the inequality $||K(t,s)||_\infty < 1$ holds.

Since $K(t,s)$ is defined via (12) and $\sup\limits_{T\in\mathbb{R}_+} ||V||_T = ||V||_\infty < \infty$, from condition 2 it follows that $\sup\limits_{T\in\mathbb{R}_+} ||U||_T = C_U < \infty$ too. Thus, all calculation remains valid if we replace in them $||U||_T$ with C_U and everywhere instead to take $\sup\limits_{t\in[a,a+T]}(\cdot)$ we must take $\sup\limits_{t\in J_a}(\cdot)$. Then, everywhere in the inequalities (28) we can replace $||U||_T$ with C_U too. Thus, we can choose from the inequalities (29) the numbers δ and λ independent from $T \in \mathbb{R}_+$ in such a way that $||K(t,s)||_\infty < 1$. □

Corollary 2. *Let the conditions of Theorem 2 hold.*

*Then, the system (7) has a unique solution $X(t)$ with interval of existence J_a for arbitrary $\Phi \in$ **PC**, which has the representation (29).*

The proof is the same as the proof of Corollary 1 (it must only be applied instead Theorem 2.5 its Corollary 2.7 in Chapter 10 of [23]), and that is why it will be omitted.

4. Applications

In this section, we apply the obtained results to study the relationship between the resolvent kernel and the fundamental matrix of the system (2). Moreover, we obtain a new and more simply integral representation of the unique solution of the studied IP (1), (3) with the assumptions that $\Phi \in \mathbf{PC}$ and $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded, via the fundamental matrix as well as an a priori estimation of the arbitrary solution in the studied IP (1), (3) under the same assumptions.

The goal of the next theorem is to establish a relation between the fundamental matrix and the resolvent kernel. In partial we show that the problem of existence of a unique resolvent kernel is equivalent to the problem of existence of a unique fundamental matrix of the system (2).

Let introduce the matrix function

$$H(t,s) = \begin{cases} I, & t \geq s \\ \Theta, & t < s \end{cases}$$

for arbitrary fixed $s \in J_a$, $R \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ and define the matrix function $C(t,s)$ via the relation

$$C(t,s) = H(t,s) + R(t,s). \tag{30}$$

Theorem 3. *Let the conditions of Theorem 2 hold.*

Then, the relation (30) holds if and only if $R \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ is the unique solution of the resolvent Equation (17) and the function $C(t,s)$ defined via (30) is a fundamental matrix of (2) and $C \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$.

Proof. Let $R \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ be the unique solution of the resolvent Equation (17). Then, from (30) it follows that $C \in SVB^\infty(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ and for $J_a \cap \{t \geq s\}$ from (17) it follows that

$$
\begin{aligned}
I + R(t,s) &= I - K(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)](R(\eta,s) + I - I) \\
&= I - K(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)](I + R(\eta,s)) - \int_s^t \mathrm{d}_\eta[K(t,\eta)] \\
&= I - K(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)](I + R(\eta,s)) + K(t,s) \\
&= I + \int_s^t \mathrm{d}_\eta[K(t,\eta)](I + R(\eta,s))
\end{aligned}
$$

and hence we obtain

$$
H(t,s) + R(t,s) = H(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)](H(\eta,s) + R(\eta,s)) \tag{31}
$$

From (31) since $t \in J_a \cap \{t \geq s\}$ it follows that $C(t,s)$ satisfies the matrix system

$$
C(t,s) = H(t,s) + \int_s^t \mathrm{d}_\eta[K(t,\eta)]C(\eta,s) = I + \int_s^t \mathrm{d}_\eta[K(t,\eta)]C(\eta,s) \tag{32}
$$

for $t, s \in J_a$ and $s \leq t$.

Moreover, since $R(t,s) = \Theta$ and $H(t,s) = \Theta$ for $s > t$ then $C(t,s) = \Theta$ for $s > t$, and from (32) it follows that $C(s,s) = I$. Thus, $C(t,s)$ satisfies the IP (5), (6) for $t, s \in J_a$ and $s \leq t$.

The vice versa statement can be proved in the reverse way. □

Theorem 4. *Let the following conditions be fulfilled:*

(1) *The conditions of Theorem 2 hold.*

(2) *The function $F(t) \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded in J_a.*

Then, for every initial function $\Phi \in \mathbf{C}$ the unique solution of IP (4), (3) has the following integral representation

$$
X(t) = C(t,a)\Phi(0) + \int_a^t C(t,\eta)\mathrm{d}_\eta f(\eta), \tag{33}
$$

where the function $f(t)$ is defined via the equality (8) and $C(t,s)$ is the fundamental matrix of (2).

Proof. Since the kernel $K(t,s)$ in the system (2) has the same form as in equality (12) and satisfies conditions (**K**), then from Theorem 3 it follows that there exists a unique solution $R(t,s)$ of the resolvent Equation (17), which satisfies the conditions (**K**) and hence from (30) it follows that there exists a unique fundamental matrix $C(t,s)$ of (2) which satisfies the conditions (**K**) too. Then, according to condition 1 of Theorem 2, Lemma 1 in [20] and equality (11), it follows that the function $f(t)$ is continuous in J_a, and hence the integral in the right side of (33) is correctly defined. Moreover, from the integrating by parts formula (see [25]), we can conclude that the integral in (33) is correctly defined for each $t, s \in J_a$. Then, from (29) and integrating by parts, we find that

$$
\begin{aligned}
X(t) &= f(t) - \int_a^t [\mathrm{d}_\eta R(t,\eta)]f(\eta) = f(t) - \int_a^t [\mathrm{d}_\eta C(t,\eta)]f(\eta) \\
&= f(t) - C(t,t)f(t) + C(t,a)f(a) + \int_a^t C(t,\eta)\mathrm{d}_\eta f(\eta) \\
&= C(t,a)\Phi(0) + \int_a^t C(t,\eta)\mathrm{d}_\eta f(\eta),
\end{aligned} \tag{34}
$$

which completes the proof. □

As an application of the obtained representation, we also obtain an a priori estimation of the solutions of IP (1) and (3) in the case when the kernel $K(t,s) : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined via (12) and $K \in SVB^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$.

Corollary 3. *Let the conditions of Theorem 4 hold.*

Then, for every initial function $\Phi \in \mathbf{C}$, the unique solution of the IP (4), (3) has the following a priori estimation:

$$|X(t)| = C_X e^{\gamma_* t}(||\Phi|| + |f(t)| + \int_a^t |f(\eta)| \mathrm{d}_\eta). \tag{35}$$

Proof. Let $\Phi \in \mathbf{C}$ be arbitrary and $X(t)$ is the corresponding unique solution of IP (4) and (3) which has the integral representation (33). Since $K \in SVB^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ then there exists $\gamma_* \in \mathbb{R}$, such that $||\bar{K}(t,s)||_\infty < 1$, where $\bar{K}(t,s)$ is defined via (18). Then, the system (19) according to Theorem 2 has a unique solution $\bar{R} \in SVB^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ and hence there exists $C_{\bar{R}} > 0$, such that $\sup_{(t,s)\in J_a\times\mathbb{R}} |\bar{R}(t,s)| \le C_{\bar{R}}$. Then, from (18) it follows that $R(t,s) = e^{\gamma_*(t-s)}\bar{R}(t,s)$ and therefore in virtue of Theorem 3 from (30) we obtain that for $(t,s) \in J_a \times \mathbb{R}$ the following estimation holds

$$|C(t,s)| \le |H(t,s) + R(t,s)| \le C_{\bar{R}} e^{\gamma_*(t-s)}. \tag{36}$$

Without loss of generality, we can assume that $\gamma_* > 1$, and then from (33) and (36) we obtain that

$$\begin{aligned}
|X(t)| &= |C(t,a)\Phi(0) + \int_a^t C(t,\eta)\mathrm{d}_\eta f(\eta)| \le |C(t,a)\Phi(0)| + |\int_a^t |C(t,\eta)|\mathrm{d}_\eta |f(\eta)|| \\
&\le C_{\bar{R}} e^{\gamma_* t}||\Phi|| + C_{\bar{R}} e^{\gamma_* t}|\int_a^t e^{\gamma_* t}\mathrm{d}_\eta |f(\eta)|| \\
&\le C_{\bar{R}} e^{\gamma_* t}(||\Phi|| + ||f(t)| - e^{-\gamma_* a}|\Phi(0)| + \gamma_* \int_a^t |f(\eta)|e^{-\gamma_* \eta}\mathrm{d}_\eta|) \\
&\le C_X e^{\gamma_* t}(||\Phi|| + |f(t)| + \int_a^t |f(\eta)|\mathrm{d}_\eta),
\end{aligned} \tag{37}$$

where in (37) $C_X = \gamma_*^{-1} C_{\bar{R}}$ and then (35) follows from (37). □

Remark 6. *Here we would like to note some possibilities of application of the systems under consideration in modeling real natural processes.*

Controller's design

In the book [21] (p. 13) is presented an overview of systems with intentional delays in the controller's design. It has been shown that the cost functional may be improved by judicious use of time-delay actions. On the basis of the overviewed works, a conclusion is made that a controller with time delays can eliminate overshoot and quench the oscillation, yielding a smooth and fast transient response, and hence for certain systems, the proportional minus delay controller may replace the proportional plus derivative regulator (PD regulator). A controller is proposed for general multivariate nonlinear systems in which distributed delays are included in the feedback loop in view that a controller with time delays can eliminate overshoot and quench the oscillation, yielding a smooth and fast transient response. As a result, the closed feedback system can be described either by retarded or by neutral differential equations.

As an example, consider a plant with transfer function $s \to W(s) := K_0(Ts)^{-1}s^{-s}$, where K_0 is the amplification, T is a time constant and the delay of the plant equals 1. The plant is regulated by a PD type controller. If the regulator transfer function $s \to W_1(s) := K_1$ is constant, then the fractional equation of the closed system takes the form

$$T(D_0^\alpha x)(t) + x(t) + Kx(t-1) = K_0 y(t-1), \ \ K := K_0 K_1, \tag{38}$$

where $\alpha \in (0,1]$. For PD regulator with transfer function $s \to W_1(s) := K_1(1+sd)$, the closed system is described by the neutral equation

$$TD_0^{\alpha}(x(t) + T^{-1}dKx(t-1)) + x(t) + Kx(t-1) = K_0y(t-1), \tag{39}$$

where d is a constant and $y(t)$ is the output signal from the controller (note that the examples in [21] are considered only in the case when $\alpha = 1$).

*It is simple to be seen that the considered examples (38) and (39) are partial cases of the system (1) in the case $n = 1$, when $V(t,\theta) \equiv 0$ and $V_d(t,\theta) = KH(t-1), V_{ac}(t,\theta) = V_s(t,\theta) \equiv 0$, respectively. Moreover, the conditions (**S**) are fulfilled, and hence all statements proved in our article hold.*

The systems studied in our article also include a generalization of the systems used in the following two other models:

Model of coexistence of competitive micro-organisms

The considered model (see [21] (p. 25)) describes the competing micro-organisms surviving on a single nutrient with delays in birth and death processes.

Model of cancer chemotherapy

The model (see [21] (p. 30)) is devoted to the effect of anti-tumor drugs on the kinetics of the cell cycle. Historically, it was assumed that drugs act instantaneously and without aftereffects, but this assumption is far from reality. In reality, to determine the time-varying effects of drugs on the cell cycle, one has to take into account the empirically established fact that the action of drugs has a finite duration. It is assumed that the interaction process between drugs and tumor cells under the following empirical assumptions: first, that the entire cell population grows at a constant rate, and second, that the time for a cell cycle remains constant. Note that if we replace the system of [21] with system (1), then the first assumption is unnecessary.

5. Discussion

Note that all results after Theorem 2 are proved under the essential assumptions that the kernel $K(t,s)$ is defined via (12) and $K \in SVB^{\infty}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$. This fact leads to some interesting open problems:

1. To prove that condition 3 of Theorem 2 in [19] is unnecessary and it follows from the conditions (**S**) even in the case when the lower terminal a is a critical point or critical jump point for some delays (see Definition 4 in [19]). From this fact will follow that, according Theorems 1–3 in [19], there exists a unique fundamental matrix $C(t,s)$ of the system (2), when conditions (**S**) hold without additional conditions for $K(t,s)$ defined via (12).
2. Using the obtained result for existence of a fundamental matrix $C(t,s)$ via the relation (30) to define the corresponding to $K(t,s)$ resolvent kernel $R(t,s)$ and to prove that $R(t,s)$ satisfies Equation (17) and $R \in SVB^{\infty}_{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ only with the assumption, that the conditions (**S**) hold.

6. Conclusions and Comments

As a first result, the existence and uniqueness of a resolvent kernel under assumptions for some kind of boundedness of the kernels $U(t,\theta)$ and $V(t,\theta)$ is studied. Second, in the case when $F(t) \in L_1^{loc}(J_a, \mathbb{R}^n)$ and the initial function $\Phi \in \mathbf{C}$ is arbitrary, we obtain that the studied IP (1), (3) has a unique solution, which has an integral representation via the function $F(t)$ and the resolvent kernel. The obtained results are based on an adaption of an approach used, so far as we know, only for linear neutral systems with integer order derivatives for establishing integral representation of their solutions, to the case of fractional neutral systems with Caputo-type derivatives and distributed delays. As an application of the obtained result, we establish a new and more simply integral representation of the unique solution of the studied IP (1), (3) in the case when $F(t) \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded and $\Phi \in \mathbf{C}$, via the corresponding resolvent kernel. Then we study the relationship between the resolvent kernel and the fundamental matrix of the system (2) and prove an explicit formula for them. Moreover, under the same assumptions, we obtain another simpler integral representation of the solutions of the studied IP (1), (3) via the fundamental matrix as well as an a priory estimation of the solutions of this IP. Note that the conditions obtained are similar to those that guarantee the same result in the case of linear neutral systems with distributed delays and integer order of differentiation. Finally, we would like to note that in Remark 6, some models describing natural processes, corresponding to the investigated in the article differential systems, are mentioned, and in Section 5, an interesting open problem for future research is formulated.

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