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Stationary Condition for Borwein Proper Efficient Solutions of Nonsmooth Multiobjective Problems with Vanishing Constraints

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Abstract: This paper discusses optimality conditions for Borwein proper efficient solutions of nonsmooth multiobjective optimization problems with vanishing constraints. A new notion in terms of contingent cone and upper directional derivative is introduced, and a necessary condition for the Borwein proper efficient solution of the considered problem is derived. The concept of ε proper Abadie data qualification is also introduced, and a necessary condition which is called a strictly strong stationary condition for Borwein proper efficient solutions is obtained. In view of the strictly strong stationary condition, convexity of the objective functions, and quasi-convexity of constrained functions, sufficient conditions for the Borwein proper efficient solutions are presented. Some examples are given to illustrate the reasonability of the obtained results.

Keywords: Borwein proper efficient solution; nonsmooth multiobjective optimization; stationary condition; Clarke subdifferential

MSC: 90C46

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1. Introduction

Multiobjective optimization plays an important role in management science, operations research, and economics. The reader is referred to the recently published book [1] for more details on vector optimization theory and applications. The classical concept of efficient solution in multiobjective optimization problems was introduced by Pareto [2] under specific preferences. Koopmans [3] proposed the concept of a Pareto efficient solution. After that, many scholars studied Pareto efficiency and obtained a lot of results (see the book [4] and the reference therein). However, the set of all Pareto efficient solutions is large, and part of it cannot be characterized by a scalar minimization problem. To eliminate these abnormal solutions, various kinds of proper efficient solutions have been introduced (see Chapter 4 in [4]), one of which was introduced by Borwein [5] and was called the Borwein proper efficient solution by some later researchers. Since Borwein proper efficiency highlights the geometric property and abandons noneffective decisions in decision making, it has become a standard concept in vector optimization literature (see [6–8]).

In this paper, we consider the following multiobjective mathematical programming with vanishing constraints (MMPVC for short):

$$\begin{aligned} \min\ & (f_1(x), \dots, f_p(x)) \\ \text{s.t. }\ & h_i(x) \geq 0\,, \quad i \in I, \\ & h_i(x) g_i(x) \leq 0\,, \quad i \in I, \end{aligned}$$

where $f_j,\ h_i,\ g_i:\ \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions with $i \in I := \{1, \dots, m\}$, and $j \in J := \{1, \dots, p\}$.

The MMPVC is a complicated programming problem since it involves product function $h_i g_i$ in its constraints with $i \in I$. This complicatedness brings us two difficulties. One is that the feasible set usually is not a convex set; the other is that the constrained property of $g_i(x)$ vanishes in the case $h_i(x) = 0$.

In the special case $p = 1$, MMPVC reduces to the mathematical programming with vanishing constraints (MPVC for short) which was introduced by Achtziger and Kanzow [9]. MPVCs not only play an important role in topology optimization which is a powerful tool in mechanical structures design, but also extend another group of programming problems called mathematical programming with equilibrium constraints (see [9]). For these two reasons, the MPVCs have attracted some researchers' interest. Some stationary conditions of Karush–Kuhn–Tucker-type optimality conditions are given under various qualification condition by the classical subdifferential and normal cones, such as Clarke subdifferential and Clarke normal cone. Readers are referred to the reference [9–12] for smooth MPVCs and [13–15] for nonsmooth MPVCs.

All MPVCs mentioned above concerned a single-objective function. To the best of our knowledge, Mishra, et al. [16] studied MMPVCs involving continuously differentiable functions for the first time. They modified some constraint qualifications such as Cottle constraint qualification, Slater constraint qualification, etc. Then, they established relationships among them and obtained the Karush–Kuhn–Tucker-type necessary optimality conditions for Pareto efficiency solutions. In [17], the MMPVC with its objective functions being continuously differentiable and its constrained function being convex were considered. Two Abadie-type constraint qualifications were introduced and some necessary conditions for Geoffrion properly efficient solutions were given by convex subdifferentials. Recently in [18], for the nonsmooth MMPVCs, some data qualifications characterized by Clarke subdifferential were introduced, and the relationship among them was discussed. Some stationary conditions as necessary or sufficient conditions of weakly efficient and Pareto efficient solutions were also given. Motivated by [18], it is natural for us to consider the stationary condition for Borwein proper efficient solutions of the MMPVC.

The rest of this paper is organized as follows. In Section 2, we introduce some notions and preliminary results which will be needed later. In Section 3, we present our main results. Unlike [18,19] concerning weak efficiency and Pareto efficiency, we consider Borwein proper efficiency. We introduce ε proper Abadie data qualification condition (in short, ε-PADQ) for a given $\varepsilon > 0$. Using ε-PADQ condition, we obtain a strictly strong stationary condition as a necessary condition for the Borwein proper efficient solution of problem MMPVC. Under the assumption of the convexity of objective functions and the ∂-quasi-convexity of constrained functions, we establish a strictly strong stationary condition as a sufficient condition for the Borwein proper efficient solution of problem MMPVC.

2. Preliminaries

Throughout this paper, unless stated otherwise, we always assume that X is a real Banach space, X^* is the dual space of X, Ω is a nonempty subset of X, B_X is the closed unit ball of X, and $\mathbb{R}^p_+ = \{(\xi_1, \dots, \xi_p) \mid \xi_i \geq 0,\ i \in \{1, \dots, p\}\}$. The interior, convex hull, and closure of Ω are denoted by $\mathrm{int}(\Omega)$, $\mathrm{co}(\Omega)$, and $\mathrm{cl}(\Omega)$, respectively.

The set Ω is called a cone if $\lambda x \in \Omega$ for all $x \in \Omega$ and $\lambda \geq 0$. Clearly, a cone Ω is convex if and only if $\Omega + \Omega \subseteq \Omega$. The cone generated by Ω is defined as

$$\mathrm{cone}(\Omega) := \{x \in X \mid x = \lambda y,\ \lambda \geq 0,\ y \in \Omega\}.$$

The negative and strict negative polar cone of Ω are, respectively, defined as

$$\Omega^- := \{x^* \in X^* \mid \langle x^*, a\rangle \leq 0\,,\ \forall\, a \in \Omega\},$$

$$\Omega^\sharp := \{x^* \in X^* \mid \langle x^*, a\rangle < 0\,,\ \forall\, a \in \Omega\}.$$

For $\bar{x} \in \mathrm{cl}(\Omega)$, the contingent cone of Ω at $\bar{x}$ is the set

$$T(\Omega, \bar{x}) := \{v \in X \mid \exists\, v_n \to v, t_n \downarrow 0 \text{ such that } \bar{x} + t_n v_n \in \Omega\}.$$

Let $\psi : X \to \mathbb{R}$, and $\bar{x}, u \in X$. The function ψ is said to be upper directionally differentiable at $\bar{x}$ in the direction u if

$$\psi'(\bar{x}; u) := \limsup_{t \to 0^+} \frac{\psi(\bar{x} + tu) - \psi(\bar{x})}{t}$$

exists, where $t \to 0^+$ means that $t > 0$ and t converges to 0. The function ψ is said to be directionally differentiable at $\bar{x}$ in the direction u, if

$$\nabla\psi(\bar{x}; u) := \lim_{t \to 0^+} \frac{\psi(\bar{x} + tu) - \psi(\bar{x})}{t}$$

exists. Clearly, if ψ is directionally differentiable at $\bar{x}$ in the direction u, then $\nabla\psi(\bar{x}; u) = \psi'(\bar{x}; u)$. If for all $u \in X$,

$$\nabla\psi(\bar{x})(u) := \lim_{t \to 0} \frac{\psi(\bar{x} + tu) - \psi(\bar{x})}{t}$$

exists and $\nabla\psi(\bar{x})$ is a continuous linear mapping, then ψ is said to be Gâteaux differentiable at $\bar{x}$.

Let Y be a Banach space, $z \in X$, a mapping $\varphi : X \to Y$ is said to be locally Lipschitz at z, if there exist $\delta > 0$ and $M > 0$ such that

$$\|\varphi(x) - \varphi(y)\| \le M\|x - y\|, \quad \forall\, x, y \in z + \delta B_X.$$

If φ is locally Lipschitz at each point of Ω, then φ is said to be locally Lipschitz on Ω.

Let $\bar{x}$, $u \in X$ and $\psi : X \to \mathbb{R}$ be locally Lipschitz at $\bar{x}$. The Clarke generalized directional derivative of ψ at $\bar{x}$ in the direction u is defined as

$$\psi^\circ(\bar{x}; u) := \limsup_{x \to \bar{x},\ t \downarrow 0} \frac{\psi(x + tu) - \psi(x)}{t}.$$

The set

$$\partial\psi(\bar{x}) := \{x^* \in X^* \mid \langle x^*, u\rangle \le \psi^\circ(\bar{x}; u)\,,\ \forall\, u \in X\}$$

is called the Clarke subdifferential of ψ at $\bar{x}$.

Lemma 1 ([20])**.** *Let $\psi :\ X \to \mathbb{R}$ be a function, $\bar{x}$, $u \in X$, and ψ be locally Lipschitz at $\bar{x}$. Then:*

(i) $\partial\psi(\bar{x})$ *is a nonempty w^* compact convex set;*
(ii) *There exists $\xi \in \partial\psi(\bar{x})$ such that $\psi^\circ(\bar{x}; u) = \langle \xi, u\rangle$;*
(iii) $\psi'(\bar{x}; u) \le \psi^\circ(\bar{x}; u)$.

Definition 1 ([21])**.** *Let $\psi :\ X \to \mathbb{R}$ be a function, $\bar{x} \in X$ and ψ be locally Lipschitz at $\bar{x}$. The function ψ is called ∂-quasi convex at $\bar{x}$, if for all $x \in X$,*

$$\psi(x) \le \psi(\bar{x}) \Rightarrow \langle \xi, x - \bar{x}\rangle \le 0\,, \quad \forall\, \xi \in \partial\psi(\bar{x}).$$

Definition 2 ([5])**.** *Let A be a nonempty subset of X and Θ be a pointed convex cone of X. A point $\bar{x} \in A$ is called a Borwein proper efficient point of A, if*

$$cl(cone(A - \bar{x})) \cap (-\Theta) = \{0_X\}.$$

The set of all Borwein proper efficient points of A is denoted by $BE(A, \Theta)$.

The following lemma is a standard separation theorem for two convex sets.

Lemma 2 ([4]). *Let A be a nonempty compact convex subset of X and B be a nonempty closed convex subset of X. Then, $A \cap B = \emptyset$ if and only if there exist $x^* \in X^* \setminus \{0_{X^*}\}$ and $\alpha \in \mathbb{R}$ such that*

$$x^*(a) < \alpha < x^*(b)\,, \quad \forall\, a \in A\,,\, b \in B.$$

For convenience of the readers, we give the important notations mentioned above in Table 1.

Table 1. The notations and their explanations throughout the text.

Notation	Description and Explanation of the Notation
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^p$	p-dimensional Euclidean space
I	$I := \{1, 2, \dots, m\}$
J	$J := \{1, 2, \dots, p\}$
0_n	zero vector of $\mathbb{R}^n$
0_p	zero vector of $\mathbb{R}^p$
$\psi'(\bar{x}; u)$	the upper directional derivative ψ at $\bar{x}$ in the direction u
$\nabla\psi(\bar{x}; u)$	the directional derivative ψ at $\bar{x}$ in the direction u
$\nabla\psi(\bar{x})$	the Gâteaux derivative of ψ at $\bar{x}$
$\psi^\circ(\bar{x}; u)$	the Clarke generalized directional derivative of ψ at $\bar{x}$ in the direction u
$\partial\psi(\bar{x})$	the Clarke subdifferential of ψ at $\bar{x}$
Ω^-	the negative polar cone of Ω
$\Omega^\sharp$	the strict negative polar cone of Ω
$f_j^\circ(\bar{x}; u)$	the Clarke generalized directional derivative of f_j at $\bar{x}$ in the direction u
$\left(\partial f_j(\bar{x})\right)^\sharp$	the strict negative polar cone of the Clarke subdifferential $\partial f_j(\bar{x})$

3. Main Results

In this section, we establish necessary and sufficient optimality conditions for the Borwein proper efficient solution of problem MMPVC. The feasible set of problem MMPVC is denoted as follows:

$$S := \{x \in \mathbb{R}^n \mid h_i(x) \geq 0,\ h_i(x)g_i(x) \leq 0,\ i \in I\}.$$

We always assume that $S \neq \emptyset$, and $\bar{x} \in S$ will be fixed in the remainder of this paper. Following [9,13,18], we define the index sets as follows:

$$\begin{aligned}
I_{+0} &:= \{i \in I \mid g_i(\bar{x}) = 0,\ h_i(\bar{x}) > 0\},\\
I_{+-} &:= \{i \in I \mid g_i(\bar{x}) < 0,\ h_i(\bar{x}) > 0\},\\
I_{0+} &:= \{i \in I \mid g_i(\bar{x}) > 0,\ h_i(\bar{x}) = 0\},\\
I_{00} &:= \{i \in I \mid g_i(\bar{x}) = 0,\ h_i(\bar{x}) = 0\},\\
I_{0-} &:= \{i \in I \mid g_i(\bar{x}) < 0,\ h_i(\bar{x}) = 0\}.
\end{aligned}$$

Let $I_+ := I_{+0} \cup I_{+-}$ and $I_0 := I_{0+} \cup I_{00} \cup I_{0-}$. Obviously, $I = I_{+0} \cup I_{+-} \cup I_{0+} \cup I_{00} \cup I_{0-}$. For each $k \in J$, set $J_k := J \backslash \{k\}$, and define

$$A_k(\bar{x}) := \begin{cases} T(S, \bar{x}) \bigcap \left\{ u \in \mathbb{R}^n \mid f_j'(\bar{x}; u) \leq 0, j \in J_k \right\}, & p > 1, \\ T(S, \bar{x})\,, & p = 1. \end{cases}$$

Let $f := (f_1, \dots, f_p)$. Now, using Definition 2, we can define a Borwein proper efficient solution of problem MMPVC.

Definition 3. *A point $\bar{x} \in S$ is said to be a Borwein proper efficient solution of problem MMPVC, if $f(\bar{x}) \in BE(f(S), \mathbb{R}^p_+)$, that is,*

$$\text{cl}(\text{cone}(f(S) - f(\bar{x}))) \bigcap (-\mathbb{R}^p_+) = \{0_p\}.$$

The set of all Borwein proper efficient solutions is denoted by Ξ_B. If

$$(f(S) - f(\bar{x})) \bigcap (-\mathbb{R}^p_+) = \{0_p\},$$

then $\bar{x}$ is called a Pareto efficient solution of problem MMPVC. The set of all Pareto efficient solutions is denoted by Ξ_E.

Lemma 3. *Suppose that $\bar{x} \in \Xi_B$, and f_j is locally Lipschitz at $\bar{x}$ for all $j \in J$. Then,*

$$\left(\bigcup_{j \in J} (\partial f_j(\bar{x}))^\sharp \right) \bigcap \left(\bigcap_{j \in J} A_j(\bar{x}) \right) = \emptyset. \tag{1}$$

Proof. We divide p into two cases: $p = 1$ and $p > 1$.

Case 1: $p = 1$. In this case, (1) equals

$$(\partial f_1(\bar{x}))^\sharp \bigcap T(S, \bar{x}) = \emptyset.$$

Suppose to the contrary that there exists some $d \in (\partial f_1(\bar{x}))^\sharp \bigcap T(S, \bar{x})$, then there exist $\{t_n\} \subseteq \mathbb{R}_+$ with $t_n \downarrow 0$, $\{d_n\} \subseteq \mathbb{R}^n$ with $d_n \to d$ such that $\bar{x} + t_n d_n \in S$ for all n. Since f_1 is locally Lipschitz at $\bar{x}$ and $d \in (\partial f_1(\bar{x}))^\sharp$, it follows from Lemma 1 that there exists $\xi \in \partial_c f_1(\bar{x})$ such that

$$\langle \xi, d \rangle = f_1^\circ(\bar{x}; d) < 0.$$

Since f_1 is locally Lipschitz at $\bar{x}$, there exist $L > 0$ and $\delta > 0$ such that for all $u, v \in \bar{x} + \delta B_{\mathbb{R}^n}$,

$$|f_1(u) - f_1(v)| \leq L\|u - v\|.$$

Since $\bar{x} + t_n d_n \to \bar{x}$, $\bar{x} + t_n d \to \bar{x}$, there exists a positive integer number N such that for all $n > N$,

$$|f_1(\bar{x} + t_n d_n) - f_1(\bar{x} + t_n d)| \leq L t_n ||d_n - d||.$$

By (iii) of Lemma 1, we have

$$\begin{aligned} v_1 &:= \limsup_{n \to \infty} \frac{f_1(\bar{x} + t_n d_n) - f_1(\bar{x})}{t_n} \\ &\leq \limsup_{n \to \infty} \frac{f_1(\bar{x} + t_n d) - f_1(\bar{x})}{t_n} + \limsup_{n \to \infty} \frac{L t_n \|d_n - d\|}{t_n} \\ &\leq f_1'(\bar{x}; d) \leq f_1^\circ(\bar{x}; d) < 0. \end{aligned}$$

As

$$\frac{f_1(\bar{x} + t_n d_n) - f_1(\bar{x})}{t_n} \in \text{cone}(f(S) - f(\bar{x})),$$

we obtain

$$v_1 \in \text{cl}(\text{cone}(f(S) - f(\bar{x}))) \cap (-\mathbb{R}_+),$$

which contradicts $\bar{x} \in \Xi_B$ since $v_1 \neq 0$.

Case 2: $p > 1$. To verify (1), it suffices to prove that

$$(\partial f_j(\bar{x}))^\sharp \bigcap A_j(\bar{x}) = \emptyset, \quad j \in J.$$

Without loss of generality, we only need to show that

$$(\partial f_1(\bar{x}))^{\sharp} \bigcap A_1(\bar{x}) = \emptyset\,.$$

Suppose to the contrary that there exists $d \in (\partial f_1(\bar{x}))^{\sharp} \bigcap A_1(\bar{x})$. Since $d \in A_1(\bar{x}) \subseteq T(S, \bar{x})$, there exist $\{t_n\} \subseteq \mathbb{R}_+$ with $t_n \downarrow 0$, $\{d_n\} \subseteq \mathbb{R}^n$ with $d_n \to d$ such that $\bar{x} + t_n d_n \in S$. Using the same proof of case $p = 1$, we obtain

$$v_1 := \limsup_{n\to\infty} \frac{f_1(\bar{x} + t_n d_n) - f_1(\bar{x})}{t_n} < 0,$$

By the definition of $A_1(\bar{x})$, we have

$$f_j'(\bar{x}; d) \leq 0\,, \quad j = 2, \ldots, p.$$

Since f_j is locally Lipschitz at $\bar{x}$, we have

$$\begin{aligned} v_j &:= \limsup_{n\to\infty} \frac{f_j(\bar{x} + t_n d_n) - f_j(\bar{x})}{t_n} \\ &\leq \limsup_{n\to\infty} \frac{f_j(\bar{x} + t_n d) - f_j(\bar{x})}{t_n} \\ &\leq f_j'(\bar{x}; d) \leq 0\,, \quad j = 2, \ldots, p. \end{aligned}$$

Therefore,

$$(v_1, \ldots, v_p) \in \mathrm{cl}(\mathrm{cone}(f(S) - f(\bar{x}))) \cap (-\mathbb{R}^p_+),$$

which contradicts $\bar{x} \in \Xi_B$ since $(v_1, \ldots, v_p) \neq (0, \ldots, 0)$. Therefore,

$$(\partial f_1(\bar{x}))^{\sharp} \bigcap A_1(\bar{x}) = \emptyset.$$

In conclusion, Equation (1) is verified. □

Remark 1. *In [19] (Lemma 5.1) (also see [18] (Lemma 2)), Li proved that if $\bar{x} \in \Xi_E$, then*

$$\left(\bigcup_{j\in J} (\partial f_j(\bar{x}))^{\sharp} \right) \bigcap \left(\bigcap_{j\in J} T(Q_j, \bar{x}) \right) = \emptyset,$$

where

$$Q_j = \begin{cases} S \bigcap \{x \in \mathbb{R}^n \mid f_\kappa(x) \leq f_\kappa(\bar{x}),\ \kappa \in J_j\}\,, & p > 1, \\ S\,, \quad p = 1. \end{cases}$$

It is known that a Borwein proper efficient solution is a Pareto efficient solution, but the converse is not true. To illustrate that Lemma 3 sharpens Li's result, it suffices to give an example that $\bigcap_{j\in J} T(Q_j, \bar{x})$ is a strict subset of $\bigcap_{j\in J} A_j(\bar{x})$. See the following example.

Example 1. *In problem MMPVC, we take $I = \{1\}$, $J = \{1, 2\}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$f(x) = (f_1(x), f_2(x)) = (x_1 + x_2, x_1^2 + x_2^2)\,, \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2,$$

$$h_1(x) = x_1 + x_2\,, \quad g_1(x) = x_2\,, \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

Clearly, $\bar{x} = (0, 0)$ is a Borwein proper efficient solution. We calculate that

$$\begin{aligned} S &= \{x \in \mathbb{R}^2 \mid h_1(x) \geq 0,\ h_1(x) g_1(x) \leq 0\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq -x_2 \geq 0\} \bigcup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0,\ x_2 \geq 0\}, \end{aligned}$$

$$Q_1 = S\bigcap\{x \in \mathbb{R}^2 \mid f_2(x) \leq f_2(\bar{x})\} = \{(0,0)\},$$
$$Q_2 = S\bigcap\{x \in \mathbb{R}^2 \mid f_1(x) \leq f_1(\bar{x})\} = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0\},$$
$$A_1(\bar{x}) = T(S,\bar{x})\bigcap\{d \in \mathbb{R}^2 \mid f_2'(\bar{x};d) \leq 0\} = S,$$
$$A_2(\bar{x}) = T(S,\bar{x})\bigcap\{d \in \mathbb{R}^2 \mid f_1'(\bar{x};d) \leq 0\} = Q_2,$$
$$\bigcap_{j=1}^{2} T(Q_j,\bar{x}) = \{(0,0)\}\,,\ \bigcap_{j=1}^{2} A_j(\bar{x}) = Q_2.$$

Therefore, $\bigcap\limits_{j=1}^{2} T(Q_j,\bar{x})$ *is a strict subset of* $\bigcap\limits_{j=1}^{2} A_j(\bar{x})$.

Under some mild conditions, $\bigcap\limits_{j\in J} T(Q_j,\bar{x})$ is a subset of $\bigcap\limits_{j\in J} A_j(\bar{x})$.

Proposition 1. *Suppose that* $\bar{x} \in S$; f_j *is locally Lipschitz at* $\bar{x}$ *and directionally differentiable at* $\bar{x}$ *in any direction for all* $j \in J$. *Then,*

$$\bigcap_{j\in J} T(Q_j,\bar{x}) \subseteq \bigcap_{j\in J} A_j(\bar{x}). \tag{2}$$

Proof. To verify Equation (2), it suffices to show that

$$T(Q_j,\bar{x}) \subseteq A_j(\bar{x})\,, \quad j = 1,\ldots,p.$$

Without loss of generality, we only need to show that

$$T(Q_1,\bar{x}) \subseteq A_1(\bar{x}). \tag{3}$$

We divide p into two cases: $p = 1$ and $p > 1$.
Case 1: $p = 1$. In this case, $Q_1 = S$, and hence

$$T(Q_1,\bar{x}) = T(S,\bar{x}) = A_1(\bar{x}),$$

Equation (3) is verified.
Case 2: $p > 1$. Let $d \in T(Q_1,\bar{x})$; then, there exist $\{t_n\} \subseteq \mathbb{R}_+$ with $t_n \downarrow 0$, $\{d_n\} \subseteq \mathbb{R}^n$ with $d_n \to d$ such that $\bar{x} + t_n d_n \in Q_1$. By the definition of Q_1, we obtain $\bar{x} + t_n d_n \in S$ and

$$f_j(\bar{x} + t_n d_n) \leq f_j(\bar{x})\,, \quad j = 2,\ldots,p,$$

and hence $d \in T(S,\bar{x})$. Since f_j is locally Lipschitz at $\bar{x}$ and directionally differentiable at $\bar{x}$ in any direction, we obtain

$$\begin{aligned} f_j'(\bar{x};d) :&= \limsup_{t\to 0^+} \frac{f_j(\bar{x}+td) - f_j(\bar{x})}{t} = \nabla f_j(\bar{x};d) \\ &= \lim_{t\to 0^+} \frac{f_j(\bar{x}+td) - f_j(\bar{x})}{t} = \lim_{n\to\infty} \frac{f_j(\bar{x}+t_n d) - f_j(\bar{x})}{t_n} \\ &= \lim_{n\to\infty} \frac{f_j(\bar{x}+t_n d_n) - f_j(\bar{x})}{t_n} \leq 0\,, \quad j = 2,\ldots,p. \end{aligned}$$

This implies that $d \in A_1(\bar{x})$, and so Equation (3) is verified. □

Before we give necessary conditions for the Borwein proper efficient solution of problem MMPVC, we introduce the following two definitions.

Definition 4. *Let* $\Lambda := \{1, \ldots, p\}$, *for each* $\lambda \in \Lambda$, D_Λ *be a nonempty convex set of* $\mathbb{R}^n$, $0 < \varepsilon < \frac{1}{p}$, *and* $D := \operatorname{co}\left(\bigcup_{\lambda\in\Lambda} D_\lambda\right)$. *The set*

$$\operatorname{core}_\varepsilon(D) := \left\{ \sum_{\lambda\in\Lambda} \theta_\lambda x_\lambda \mid x_\lambda \in D_\lambda\,,\ \sum_{\lambda\in\Lambda} \theta_\lambda = 1\,,\ \lambda \in \Lambda\,,\ \theta_\lambda \geq \varepsilon \right\}$$

is called the ε-core of D.

Definition 5. *Let* $0 < \varepsilon < \frac{1}{p}$. *We say that problem MMPVC satisfies ε proper Abadie data qualification (in short, ε-PADQ) at* $\bar{x} \in S$, *if*

$$\left(\operatorname{core}_\varepsilon\left(\operatorname{co}\left(\bigcup_{j\in J} \partial f_j(\bar{x})\right)\right)\right)^\sharp \bigcap \mathcal{L}^- \subseteq \bigcap_{j\in J} A_j(\bar{x}),$$

where

$$\mathcal{L} := \left(\bigcup_{i\in I_{0+}} \partial h_i(\bar{x})\right) \bigcup \left(-\bigcup_{i\in I_0} \partial h_i(\bar{x})\right) \bigcup \left(\bigcup_{i\in I_{+0}} \partial g_i(\bar{x})\right).$$

Remark 2. *Assume that* $0 < \varepsilon_1 < \varepsilon_2$. *Since*

$$\operatorname{core}_{\varepsilon_2}\left(\operatorname{co}\left(\bigcup_{j\in J} \partial f_j(\bar{x})\right)\right) \subseteq \operatorname{core}_{\varepsilon_1}\left(\operatorname{co}\left(\bigcup_{j\in J} \partial f_j(\bar{x})\right)\right),$$

we deduce that if problem MMPVC satisfies ε_2*-PADQ, then it satisfies* ε_1*-PADQ. If we take* $\varepsilon = 0$, *replace* $A_j(\bar{x})$ *and "*$\sharp$*" with* $T(Q_j, \bar{x})$ *and "*$-$*", respectively, in Definition 5, then ε-PADQ reduces to EADQ introduced in [18] (Definition 2). The meaning of* $\mathcal{L}$ *was introduced in [13].*

Here, ε-PADQ reveals the relationship between the subdifferentials of the objective functions and the constrained functions and the feasible set of problem MMPVC. It is somewhat abstract, which leads to the difficulty of verifying it for a general problem MMPVC. However, under some mild conditions, ε-PADQ is easy to verify.

Proposition 2. *Let* $\bar{x} \in S$, $\varepsilon \in \left(0, \dfrac{1}{p}\right)$ *with* $p > 1$. *Suppose that* f_j $(j \in J)$, h_i, *and* g_i $(i \in I)$ *are locally Lipschitz at* $\bar{x}$. *If one of the following conditions holds:*

(i) $0_n \in \operatorname{int}(\operatorname{core}_\varepsilon(\operatorname{co}(\bigcup_{j\in J} \partial f_j(\bar{x}))) \bigcup \mathcal{L})$;
(ii) $0_n \in \operatorname{int}(\mathcal{L})$;
(iii) $0_n \in \operatorname{core}_\varepsilon(\operatorname{co}(\bigcup_{j\in J} \partial f_j(\bar{x})))$;
(iv) f_j $(j \in J)$ *is Gâteaux differentiable at* $\bar{x}$ *and*

$$0_n \in \left\{ \sum_{j\in J} \theta_j \nabla f_j(\bar{x}) \mid \sum_{j\in J} \theta_j = 1,\ j \in J,\ \theta_j \geq \varepsilon \right\},$$

then problem MMPVC satisfies ε-PADQ at $\bar{x}$.

Proof. Let

$$H := \operatorname{core}_\varepsilon\left(\operatorname{co}\left(\bigcup_{j\in J} \partial f_j(\bar{x})\right)\right).$$

To verify problem MMPVC satisfying ε-PADQ at $\bar{x}$, it suffices to show that

$$H^\sharp \cap \mathcal{L}^- \subseteq \bigcap_{j\in J} A_j(\bar{x}). \tag{4}$$

Assume that (i) holds. Then, we have $0_n \in \mathrm{int}(H \cup \mathcal{L})$. By the definition of negative polar cone, we have $(H \cup \mathcal{L})^- = \{0_n\}$. By the definition of $A_j(\bar{x})$, we obtain $0_n \in \bigcap_{j\in J} A_j(\bar{x})$. To verify (4), it suffices to prove that

$$H^\sharp \cap \mathcal{L}^- \subseteq (H \cup \mathcal{L})^-. \tag{5}$$

Obviously, (5) holds true if $H^\sharp \cap \mathcal{L}^- = \emptyset$. Now, let $v \in H^\sharp \cap \mathcal{L}^-$. Then, we have

$$\langle v, x\rangle < 0, \quad \forall\, x \in H,$$

$$\langle v, x\rangle \leq 0, \quad \forall\, x \in \mathcal{L}.$$

and so,

$$\langle v, x\rangle \leq 0, \quad \forall\, x \in H \cup \mathcal{L}.$$

Therefore, $v \in (H \cup \mathcal{L})^-$, which verifies Equation (5).

Assume that (ii) holds. Then, (i) also holds since $0_n \in \mathrm{int}(\mathcal{L}) \subseteq \mathrm{int}(H \cup \mathcal{L})$. Therefore, problem MMPVC satisfies ε-PADQ at $\bar{x}$.

Assume that (iii) holds. Then, $0_n \in H$. By the definition of strict negative polar cone, we have $H^\sharp = \emptyset$, and so Equation (4) is verified. Therefore, we deduce that problem MMPVC satisfies ε-PADQ at $\bar{x}$.

Finally, assume that (iv) holds. Since f_j $(j \in J)$ is Gâteaux differentiable at $\bar{x}$, we have

$$H = \left\{ \sum_{j\in J} \theta_j \nabla f_j(\bar{x}) \mid \sum_{j\in J} \theta_j = 1,\ j \in J,\ \theta_j \geq \varepsilon \right\}.$$

Here, (iv) implies that $0_n \in H$. By (iii), problem MMPVC satisfies ε-PADQ at $\bar{x}$. $\square$

Now, we give stationary conditions for the Borwein proper efficient solution of problem MMPVC.

Theorem 1. *Let $\varepsilon \in \left(0, \dfrac{1}{p}\right)$ with $p > 1$. Suppose that $\bar{x} \in \Xi_B$, f_j $(j \in J)$, h_i, and g_i $(i \in I)$ are locally Lipschitz at $\bar{x}$, cone(co($\mathcal{L}$)) is a closed set, and problem MMPVC satisfies ε-PADQ at $\bar{x}$ and (Y): for all $v_n \in \left(\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j\in J} \partial f_j(\bar{x})\right)\right)\right)^\sharp$ with $\|v_n\| = 1$, $v_n \to v \Rightarrow v \in \bigcup_{j\in J}\left(\partial f_j(\bar{x})\right)^\sharp$. Then, there exist $\zeta_j^f, \zeta_i^h, \zeta_i^g$ $(j \in J,\ i \in I)$ such that*

$$0_n \in \sum_{j\in J} \zeta_j^f \partial f_j(\bar{x}) + \sum_{i\in I} (\zeta_i^g \partial g_i(\bar{x}) - \zeta_i^h \partial h_i(\bar{x})), \tag{6}$$

$$\zeta_i^h = 0\ (i \in I_{+0} \cup I_{+-}),\quad \zeta_i^h \geq 0\ (i \in I_{0-} \cup I_{00}),\quad \zeta_i^h \in \mathbb{R}\ (i \in I_{0+}), \tag{7}$$

$$\zeta_i^g = 0\ (i \in I_0 \cup I_{+-}),\quad \zeta_i^g \geq 0\ (i \in I_{+0}), \tag{8}$$

and

$$(\zeta_1^f, \ldots, \zeta_p^f) > 0_p,\quad \sum_{j=1}^{p} \zeta_j^f = 1. \tag{9}$$

Proof. Since f_j, h_i, and g_i are locally Lipschitz at $\bar{x}$, with Lemma 1, $\partial f_j(\bar{x})$, $\partial_c h_i(\bar{x})$, and $\partial_c g_i(\bar{x})$ are nonempty compact convex sets. Hence, $\mathrm{core}_\varepsilon(\mathrm{co}(\bigcup_{j\in J} \partial f_j(\bar{x})))$ is a compact convex set.

Firstly, we will prove that there exists $\eta \in (0, \varepsilon)$ such that

$$\mathrm{core}_{\eta}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right)\bigcap(-\mathrm{cone}(\mathrm{co}(\mathcal{L}))) \neq \emptyset. \tag{10}$$

Suppose to the contrary that for all positive integer numbers n with $\frac{1}{n} < \varepsilon$,

$$\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right)\bigcap(-\mathrm{cone}(\mathrm{co}(\mathcal{L}))) = \emptyset.$$

Since $\mathrm{cone}(\mathrm{co}(\mathcal{L}))$ is a closed convex set, applying Lemma 2 to the above equation, there exists $v_n \in \mathbb{R}^n$ with $\|v_n\| = 1$ such that

$$\langle x, v_n\rangle < 0, \quad \forall\, x \in \mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right), \tag{11}$$

$$\langle x, v_n\rangle \geq 0, \quad \forall\, x \in -\mathrm{cone}(\mathrm{co}(\mathcal{L})). \tag{12}$$

Since problem MMPVC satisfies ε-PADQ at $\bar{x}$, with (11), (12), and Remark 2, we obtain

$$\begin{aligned} v_n &\in \left(\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right)\right)^{\sharp}\bigcap(\mathrm{cone}(\mathrm{co}(\mathcal{L})))^{-} \\ &\subseteq \left(\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right)\right)^{\sharp}\bigcap\mathcal{L}^{-} \\ &\subseteq \left(\mathrm{core}_{\varepsilon}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right)\right)^{\sharp}\bigcap\mathcal{L}^{-} \subseteq \bigcap_{j\in J} A_j(\bar{x}), \end{aligned} \tag{13}$$

whenever $\frac{1}{n} < \varepsilon$. Since $\{\|v_n\|\}$ is bounded, we may assume that $v_n \to v$. Since $A_j(\bar{x})$ $(j \in J)$ is a closed set, we obtain

$$v \in \bigcap_{j\in J} A_j(\bar{x}). \tag{14}$$

The condition (Y) and Equation (13) imply that

$$v \in \bigcup_{j\in J}(\partial f_j(\bar{x}))^{\sharp}. \tag{15}$$

From Equations (14) and (15), we obtain

$$v \in \left(\bigcap_{j\in J} A_j(\bar{x})\right)\bigcap\left(\bigcup_{j\in J}(\partial f_j(\bar{x}))^{\sharp}\right),$$

which contradicts Lemma 3. Therefore, (10) is justified and

$$0_n \in \mathrm{core}_{\eta}\left(\mathrm{co}\left(\bigcup_{j\in J}\partial f_j(\bar{x})\right)\right) + \mathrm{cone}(\mathrm{co}(\mathcal{L})).$$

This implies that there exist nonnegative real numbers $\zeta_j^f \geq \eta$ $(j \in J)$, α_i $(i \in I_{0+})$, β_i $(i \in I_{0+})$, γ_i $(i \in I_{00} \cup I_{0-})$, δ_i $(i \in I_{+0})$ with $\sum_{j=1}^{p} \zeta_j^f = 1$ such that

$$0_n \in \sum_{j \in J} \zeta_j^f \partial f_j(\bar{x}) + \sum_{i \in I_{0+}} \alpha_i \partial_c h_i(\bar{x}) + \sum_{i \in I_{0+}} (-\beta_i \partial_c h_i(\bar{x})) \\ + \sum_{i \in I_{00} \cup I_{0-}} (-\gamma_i \partial_c h_i(\bar{x})) + \sum_{i \in I_{+0}} \delta_i \partial_c g_i(\bar{x}).$$

Using the same approach of Theorem 4.1 in [13], we let

$$\zeta_i^h := \begin{cases} -(\alpha_i - \beta_i), & i \in I_{0+}, \\ \gamma_i, & i \in I_{00} \cup I_{0-}, \\ 0, & i \in I_{+-} \cup I_{+0}, \end{cases}$$

$$\zeta_i^g := \begin{cases} \delta_i, & i \in I_{+0}, \\ 0, & i \in I_{+-} \cup I_0, \end{cases}$$

and the conclusion is proved. □

Remark 3. *In Theorem 4 of [18], Sadeghieh et al. established the following result. Suppose that* $\bar{x} \in \Xi_E$*, and*

$$\left(\bigcup_{j \in J} \partial f_j(\bar{x}) \right)^- \bigcap \mathcal{L}^- \subseteq \bigcap_{j \in J} T(Q_j, \bar{x}).$$

If $\partial f_j(\bar{x})$*,* $\partial h_i(\bar{x})$*, and* $\partial g_i(\bar{x})$ *are polyhedron, then Equations (6)–(9) hold. In [18], Equations (6)–(9) are called "strong strongly stationary conditions" and by "strong* $S - SC$*" in short. In this paper, we call the conditions (6)–(9) "strictly strong stationary conditions" only from grammar angles.*

Remark 4. *In Theorem 1, if there exists a* $j_0 \in J$ *such that*

$$\partial f_{j_0}(\bar{x}) \subseteq \{u \in \mathbb{R}^n \mid u = (\eta_1, \eta_2, \ldots, \eta_n),\ \eta_i \neq 0,\ i = 1, 2, \ldots, n\}, \tag{16}$$

then the condition (Y) *holds true. In fact, for all* $v_n \in \left(\mathrm{core}_{\frac{1}{n}} \left(\mathrm{co} \left(\bigcup_{j \in J} \partial f_j(\bar{x}) \right) \right) \right)^\sharp$ *with* $\|v_n\| = 1$ *and* $v_n \to v$*, we have* $\|v\| = 1$*. Let* $u \in \partial f_{j_0}(\bar{x})$ *be arbitrarily given. Take* $w_j \in \partial f_{j_0}(\bar{x})$ $(j \neq j_0, j \in J)$*. Then, we have*

$$\left(1 - \frac{p-1}{n}\right) u + \sum_{j \neq j_0,\ j \in J} \frac{1}{n} w_j \in \mathrm{core}_{\frac{1}{n}} \left(\mathrm{co} \left(\bigcup_{j \in J} \partial f_j(\bar{x}) \right) \right)$$

and

$$\left\langle v_n, \left(1 - \frac{p-1}{n}\right) u + \sum_{j \neq j_0,\ j \in J} \frac{1}{n} w_j \right\rangle < 0. \tag{17}$$

Letting $n \to \infty$ *in (17), we have* $\langle v, u \rangle \leq 0$*. Combining* $\|v\| = 1$ *and (16), we obtain* $\langle v, u \rangle < 0$*. This inequality implies that* $v \in (\partial f_{j_0}(\bar{x}))^\sharp$*, and so the condition* (Y) *holds.*

In problem MMPVC, condition (Y) does not always hold true. See the following Example 2.

Example 2. *Let $f_1,\ f_2,\ h_1,\ g_1 : \mathbb{R}^2 \to \mathbb{R}$. Consider the following problem MMPVC1:*

$$\begin{aligned} &\min\,(x_1,\ x_1 + x_2^2) \\ &\text{s.t. } x_1 \geq 0, \\ &\quad x_1 x_2 \leq 0, \end{aligned}$$

where

$$\begin{aligned} &f_1(x_1,x_2) = x_1,\ f_2(x_1,x_2) = x_1 + x_2^2, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2, \\ &h_1(x_1,x_2) = x_1,\ g_1(x_1,x_2) = x_2, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2. \end{aligned}$$

We can verify that $\bar{x} = (0,0)$ is a Borwein proper efficient solution of problem MMPVC1. Now, we prove that condition (Y) does not hold true. We calculate that

$$\partial f_1(\bar{x}) = \{(1,0)\}, \quad \partial f_2(\bar{x}) = \{(1,0)\},$$

$$\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j=1}^{2} \partial f_j(\bar{x})\right)\right) = \{(\lambda_1 + \lambda_2, 0) \mid \lambda_1 + \lambda_2 = 1,\ \lambda_1,\ \lambda_2 \geq \frac{1}{n}\}.$$

Let $v_n := \left(-\frac{1}{n}, -\sqrt{1 - \frac{1}{n^2}}\right)$. Then, $v_n \in \left(\mathrm{core}_{\frac{1}{n}}\left(\mathrm{co}\left(\bigcup_{j=1}^{2} \partial f_j(\bar{x})\right)\right)\right)^{\sharp}$, $\|v_n\| = 1$, and $v_n \to v := (0,-1)$. However, $v \notin (\partial f_1(\bar{x}))^{\sharp} \bigcup (\partial f_2(\bar{x}))^{\sharp}$.

In the following, we give an example to illustrate Theorem 1.

Example 3. *Let $f_1,\ f_2,\ h_1,\ g_1 : \mathbb{R}^2 \to \mathbb{R}$. Consider the following problem MMPVC2:*

$$\begin{aligned} &\min\left(|x_1| + x_2\,,\ x_2 + \sqrt{x_1^2 + x_2^2}\right) \\ &\text{s.t. } x_2 \geq 0, \\ &\quad x_2(-2|x_1| + x_2) \leq 0, \end{aligned}$$

where

$$\begin{aligned} &f_1(x_1,x_2) = |x_1| + x_2\,,\ f_2(x_1,x_2) = x_2 + \sqrt{x_1^2 + x_2^2}, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2, \\ &h_1(x_1,x_2) = x_2\,,\ g_1(x_1,x_2) = -2|x_1| + x_2, \quad \forall\,(x_1,x_2) \in \mathbb{R}^2. \end{aligned}$$

We can verify that $\bar{x} = (0,0)$ is a Borwein proper efficient solution of problem MMPVC2. We calculate that

$$\partial f_1(\bar{x}) = \{(x_1,1) \mid -1 \leq x_1 \leq 1\},$$

$$\partial f_2(\bar{x}) = \left\{(x_1,\ 1 + x_2) \mid \sqrt{x_1^2 + x_2^2} \leq 1\right\} = \{(0,1)\} + B_{\mathbb{R}^2},$$

$$\partial h_1(\bar{x}) = \{(0,1)\}, \quad \partial g_1(\bar{x}) = \{(x_1,1) \mid -2 \leq x_1 \leq 2\},$$

$$S = \{(x_1,x_2) \ \mid\ 0 \leq x_2 \leq 2|x_1|\},$$

$$\begin{aligned} A_1(\bar{x}) &= T(S,\bar{x}) \cap \{(d_1,d_2) \mid d = (d_1,d_2) \in \mathbb{R}^2\,,\ f_2'(\bar{x};d) \leq 0\} \\ &= S \cap \{(0,0)\} = \{(0,0)\}, \end{aligned}$$

$$\begin{aligned} A_2(\bar{x}) &= T(S,\bar{x}) \cap \{(d_1,d_2) \mid d = (d_1,d_2) \in \mathbb{R}^2\,,\ f_1'(\bar{x};d) \leq 0\} \\ &= S \bigcap \{(d_1,d_2) \mid \ \mid d_1 \mid + d_2 \leq 0\} = \{(0,0)\}, \end{aligned}$$

$$\bigcap_{j=1}^{2} A_j(\bar{x}) = \{(0,0)\},$$

$$\begin{aligned} I = \{1\},\ \ I_{+0} &:= \{i \in I \mid g_i(\bar{x}) = 0,\ h_i(\bar{x}) > 0\} = \emptyset, \\ I_{+-} &:= \{i \in I \mid g_i(\bar{x}) < 0,\ h_i(\bar{x}) > 0\} = \emptyset, \\ I_{0+} &:= \{i \in I \mid g_i(\bar{x}) > 0,\ h_i(\bar{x}) = 0\} = \emptyset, \\ I_{00} &:= \{i \in I \mid g_i(\bar{x}) = 0,\ h_i(\bar{x}) = 0\} = \{1\}, \\ I_{0-} &:= \{i \in I \mid g_i(\bar{x}) < 0,\ h_i(\bar{x}) = 0\} = \emptyset, \\ I_{+} = I_{+0} \cup I_{+-} = \emptyset,\ \ I_0 &:= I_{0+} \cup I_{00} \cup I_{0-} = \{1\}. \end{aligned}$$

$$\begin{aligned} \mathcal{L} &= \left(\bigcup_{i \in I_{0+}} \partial h_i(\bar{x}) \right) \bigcup \left(- \bigcup_{i \in I_0} \partial h_i(\bar{x}) \right) \bigcup \left(\bigcup_{i \in I_{+0}} \partial g_i(\bar{x}) \right) \\ &= -\partial h_1(\bar{x}) = \{(0,-1)\}, \end{aligned}$$

$$\mathcal{L}^- = \{(d_1, d_2) \mid d_1 \in \mathbb{R}, d_2 \geq 0\}.$$

Clearly, cone$(co(\mathcal{L}))$ is a closed set. For $\varepsilon = \frac{1}{10}$, *we calculate that*

$$\begin{aligned} &\left(\mathrm{core}_{\frac{1}{10}} \left(\mathrm{co} \left(\bigcup_{j=1}^{2} \partial f_j(\bar{x}) \right) \right) \right)^{\sharp} \bigcap \mathcal{L}^- \\ &= \left\{ (d_1, d_2) \mid \lambda |d_1| + d_2 + (1-\lambda)\sqrt{d_1^2 + d_2^2} < 0,\ \frac{1}{10} \leq \lambda \leq 1, \frac{1}{10} \leq 1 - \lambda \leq 1 \right\} \bigcap \mathcal{L}^- \\ &= \emptyset \subseteq \bigcap_{j=1}^{2} A_j(\bar{x}). \end{aligned}$$

Hence, problem MMPVC2 satisfies $\frac{1}{10}$*-PADQ condition at* $\bar{x}$.

In the following, we will verify that condition (Υ) *holds. Assume that*

$$(d_1^n, d_2^n) \in \left(\mathrm{core}_{\frac{1}{n}} \left(\mathrm{co} \left(\bigcup_{j=1}^{2} \partial f_j(\bar{x}) \right) \right) \right)^{\sharp} \ \text{with}\ \|(d_1^n, d_2^n)\| = 1, (d_1^n, d_2^n) \to (d_1, d_2).$$

Then, we have

$$\lambda d_1^n + d_2^n + (1-\lambda) < 0, \quad \forall\, \lambda, 1-\lambda \in \left[\frac{1}{n}, 1\right]. \tag{18}$$

By sending $n \to \infty$ *in (18), we have*

$$\lambda d_1 + d_2 + (1-\lambda) \leq 0, \quad \forall\, \lambda, 1-\lambda \in [0,1]. \tag{19}$$

Taking $\lambda = 0$ *in (19), we obtain* $d_2 \leq -1$. *Combining* $\|(d_1, d_2)\| = 1$, *we obtain* $d_2 = -1$, *and so*

$$(d_1, d_2) = (0, -1) \in (\partial f_1(\bar{x}))^{\sharp} \subseteq \bigcup_{j=1}^{2} (\partial f_j(\bar{x}))^{\sharp}.$$

Hence, the condition (Υ) *is verified. All conditions of Theorem 1 are satisfied. By Theorem 1, Equations (6)–(9) hold. In fact, take* $\zeta_1^f = \frac{1}{2}$, $\zeta_2^f = \frac{1}{2}$, $\zeta_1^h = \frac{1}{2}$, $\zeta_1^g = 0$, *then*

$$0_2 \in \frac{1}{2}\partial f_1(\bar{x}) + \frac{1}{2}\partial f_2(\bar{x}) + 0\partial g_1(\bar{x}) - \frac{1}{2}\partial h_1(\bar{x}).$$

However, since $\partial f_2(\bar{x})$ *is not a polyhedron, Theorem 4 of [18] cannot be applied to MMPVC2.*

For $\bar{x} \in S$, we suppose that problem MMPVC satisfies the strictly strong stationary condition at $\bar{x}$, that is, satisfies (6)–(9). Motivated by [18,22], we define the index sets as follows:

$$I_{00}^{+} := \left\{ i \in I_{00} \mid \zeta_i^h > 0 \right\}, \quad I_{00}^{0} := \left\{ i \in I_{00} \mid \zeta_i^h = 0 \right\},$$

$$I_{0-}^{+} := \left\{ i \in I_{0-} \mid \zeta_i^h > 0 \right\}, \quad I_{0-}^{0} := \left\{ i \in I_{0-} \mid \zeta_i^h = 0 \right\},$$

$$I_{0+}^{+} := \left\{ i \in I_{0+} \mid \zeta_i^h > 0 \right\}, \quad I_{0+}^{-} := \left\{ i \in I_{0+} \mid \zeta_i^h < 0 \right\},$$

$$I_{0+}^{0} := \left\{ i \in I_{0+} \mid \zeta_i^h = 0 \right\}, \quad I_{+0}^{0+} := \left\{ i \in I_{+0} \mid \zeta_i^h = 0,\ \zeta_i^g > 0 \right\},$$

$$I_{+0}^{00} := \left\{ i \in I_{+0} \mid \zeta_i^h = 0,\ \zeta_i^g = 0 \right\}.$$

Clearly,

$$I_{00} = I_{00}^{+} \cup I_{00}^{0}\,, \quad I_{0-} = I_{0-}^{+} \cup I_{0-}^{0}, \tag{20}$$

$$I_{+0} = I_{+0}^{0+} \cup I_{+0}^{00}\,, \quad I_{0+} = I_{0+}^{+} \cup I_{0+}^{-} \cup I_{0+}^{0}. \tag{21}$$

Now, we give sufficient optimality conditions for the Borwein proper efficient solution of problem MMPVC in terms of the strictly strong stationary condition.

Theorem 2. *Suppose that problem MMPVC satisfies (6)–(9) at $\bar{x} \in S$, f_j $(j \in J)$, h_i, and g_i $(i \in I)$ are locally Lipschitz at $\bar{x}$, f_j $(j \in J)$ is a convex function, and g_i $(i \in I_{+0}^{0+})$, h_i $(i \in I_{0+}^{-})$, and $-h_i$ $(i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+})$ are ∂-quasi-convex at $\bar{x}$.*

(i) Then, $\bar{x}$ is a local Borwein proper efficient solution of problem MMPVC,

(ii) If $I_{0+}^{-} = I_{+0}^{0+} = \emptyset$, then $\bar{x}$ is a Borwein proper efficient solution of problem MMPVC.

Proof. (i) Since $I_{0+}^{-} \subseteq I_{0+}$, we have

$$g_i(\bar{x}) > 0,\ \ h_i(\bar{x}) = 0, \quad \forall\, i \in I_{0+}^{-}. \tag{22}$$

Since $I_{+0}^{0+} \subseteq I_{+0}$, we have

$$g_i(\bar{x}) = 0,\ \ h_i(\bar{x}) > 0, \quad \forall\, i \in I_{+0}^{0+}. \tag{23}$$

Since g_i $(i \in I_{0+}^{-})$ and h_i $(i \in I_{+0}^{0+})$ are continuous functions with (22) and (23), there exists a neighborhood U of $\bar{x}$ such that

$$g_i(x) > 0,\ h_i(x) = 0\,, \quad \forall\, x \in S \cap U,\ \forall\, i \in I_{0+}^{-}, \tag{24}$$

$$h_i(x) > 0,\ g_i(x) \le 0\,, \quad \forall\, x \in S \cap U,\ \forall\, i \in I_{+0}^{0+}. \tag{25}$$

Since problem MMPVC satisfies (6)–(9), there exist $\xi_j^f \in \partial f_j(\bar{x})$ $(j \in J)$, $\xi_i^h \in \partial h_i(\bar{x})$ $(i \in I_0)$, $\xi_i^g \in \partial g_i(\bar{x})$ $(i \in I_{+0})$ such that

$$\sum_{j \in J} \zeta_j^f \xi_j^f - \sum_{i \in I_0} \zeta_i^h \xi_i^h + \sum_{i \in I_{+0}} \zeta_i^g \xi_i^g = 0_{\mathbb{R}^n}. \tag{26}$$

Suppose to the contrary that $\bar{x}$ is not a local Borwein proper efficient solution. Then, there exist $v = (v_1, \ldots, v_p) \neq 0_p$ and a neighborhood V of $\bar{x}$ such that

$$v \in \mathrm{cl}(\mathrm{cone}(f(S \cap V) - f(\bar{x}))) \cap (-\mathbb{R}_+^p).$$

Thus, there exist $x_n \in S \cap U \cap V$ with $x_n \to \bar{x}$ and $\theta_n > 0$ such that

$$\lim_{n \to \infty} \theta_n (f_j(x_n) - f_j(\bar{x})) = v_j \le 0\,, \quad j = 1, 2, \ldots, p,$$

and at least one $v_j < 0$; without loss of generality we may assume that $v_1 < 0$. Since f_j is a continuous convex function, we have

$$\langle \xi_j^f, x_n - \bar{x} \rangle \leq f_j(x_n) - f_j(\bar{x}),$$

resulting in

$$\limsup_{n\to\infty} \langle \xi_1^f, \theta_n(x_n - \bar{x}) \rangle \leq \lim_{n\to\infty} \theta_n(f_1(x_n) - f_1(\bar{x})) = v_1 < 0,$$
$$\limsup_{n\to\infty} \langle \xi_j^f, \theta_n(x_n - \bar{x}) \rangle \leq \lim_{n\to\infty} \theta_n(f_j(x_n) - f_j(\bar{x})) = v_j \leq 0\,, \quad j = 2, \ldots, p,$$

and so

$$\limsup_{n\to\infty} \sum_{j\in J} \zeta_j^f \langle \xi_j^f, \theta_n(x_n - \bar{x}) \rangle \leq \sum_{j\in J} \zeta_j^f v_j < 0.$$

By (26), we have

$$-\sum_{i\in I_0} \zeta_i^h \xi_i^h + \sum_{i\in I_{+0}} \zeta_i^g \xi_i^g = -\sum_{j\in J} \zeta_j^f \xi_j^f,$$

implying

$$\begin{aligned}
&\liminf_{n\to\infty} \left(\sum_{i\in I_0} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n - \bar{x}) \rangle + \sum_{i\in I_{+0}} \zeta_i^g \langle \xi_i^g, \theta_n(x_n - \bar{x}) \rangle \right) \\
&= \liminf_{n\to\infty} \left(-\sum_{j\in J} \zeta_j^f \langle \xi_j^f, \theta_n(x_n - \bar{x}) \rangle \right) = -\limsup_{n\to\infty} \left(\sum_{j\in J} \zeta_j^f \langle \xi_j^f, \theta_n(x_n - \bar{x}) \rangle \right) \\
&\geq -\sum_{j\in J} \zeta_j^f v_j > 0.
\end{aligned} \tag{27}$$

Since g_i $(i \in I_{+0}^{0+})$ and h_i $(i \in I_{0+}^{-})$ are ∂-quasiconvex at $\bar{x}$, it follows from (22)–(25) that

$$g_i(x_n) \leq 0 = g_i(\bar{x}) \Rightarrow \langle \xi_i^g, x_n - \bar{x} \rangle \leq 0\,, \quad \forall\, i \in I_{+0}^{0+},$$

$$h_i(x_n) = 0 \leq h_i(\bar{x}) \Rightarrow \langle \xi_i^h, x_n - \bar{x} \rangle \leq 0\,, \quad \forall\, i \in I_{0+}^{-}.$$

On the other hand, by the definition of index sets, combining $x_n \in S$, we have

$$-h_i(x_n) \leq 0 = -h_i(\bar{x}), \quad \forall\, i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}. \tag{28}$$

Since $-\xi_i^h \in \partial(-h_i)(\bar{x})$ $(i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+})$, the ∂-quasi convexity of $-h_i$ and (28) imply that

$$\langle -\xi_i^h, x_n - \bar{x} \rangle \leq 0\,, \quad \forall\, i \in I_{0+}^{+} \cup I_{00}^{+} \cup I_{0-}^{+}.$$

The above inequality, Equations (20) and (21) imply that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i\in I_{0+}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n - \bar{x}) \rangle \\
&\quad \leq \limsup_{n\to\infty} \sum_{i\in I_{0+}^{+}} \underbrace{\zeta_i^h}_{>0} \underbrace{\langle -\xi_i^h, \theta_n(x_n - \bar{x}) \rangle}_{\leq 0} \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{0+}^{-}} -\underbrace{\zeta_i^h}_{<0} \underbrace{\langle \xi_i^h, \theta_n(x_n - \bar{x}) \rangle}_{\leq 0} \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{0+}^{0}} -\underbrace{\zeta_i^h}_{=0} \langle \xi_i^h, \theta_n(x_n - \bar{x}) \rangle \leq 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i\in I_{00}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle \\
&\quad \le \limsup_{n\to\infty} \sum_{i\in I_{00}^+} \underbrace{\zeta_i^h}_{>0} \underbrace{\langle -\xi_i^h, \theta_n(x_n-\bar{x})\rangle}_{\le 0} \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{00}^0} -\underbrace{\zeta_i^H}_{=0} \langle \xi_i^H, \theta_n(x_n-\bar{x})\rangle \le 0,
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i\in I_{0-}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle \\
&\quad \le \limsup_{n\to\infty} \sum_{i\in I_{0-}^+} \underbrace{\zeta_i^h}_{>0} \underbrace{\langle -\xi_i^h, \theta_n(x_n-\bar{x})\rangle}_{\le 0} \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{0-}^0} -\underbrace{\zeta_i^h}_{=0} \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle \le 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i\in I_{+0}} \zeta_i^g \langle \xi_i^g, \theta_n(x_n-\bar{x})\rangle \\
&\quad \le \limsup_{n\to\infty} \sum_{i\in I_{+0}^{0+}} \underbrace{\zeta_i^g}_{>0} \underbrace{\langle \xi_i^g, \theta_n(x_n-\bar{x})\rangle}_{\le 0} \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{+0}^{00}} \underbrace{\zeta_i^g}_{=0} \langle \xi_i^g, \theta_n(x_n-\bar{x})\rangle \le 0.
\end{aligned} \tag{32}$$

Adding (29)–(32) and noting $I_0 = I_{0+} \cup I_{00} \cup I_{0-}$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\sum_{i\in I_0} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle + \sum_{i\in I_{+0}} \zeta_i^g \langle \xi_i^g, \theta_n(x_n-\bar{x})\rangle \right) \\
&\quad \le \limsup_{n\to\infty} \sum_{i\in I_{0+}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle + \limsup_{n\to\infty} \sum_{i\in I_{00}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle \\
&\quad + \limsup_{n\to\infty} \sum_{i\in I_{0-}} -\zeta_i^h \langle \xi_i^h, \theta_n(x_n-\bar{x})\rangle + \limsup_{n\to\infty} \sum_{i\in I_{+0}} \zeta_i^g \langle \xi_i^g, \theta_n(x_n-\bar{x})\rangle \le 0,
\end{aligned}$$

which contradicts (27). Therefore, $\bar{x}$ is a local Borwein proper efficient solution of problem MMPVC.

(ii). Now, assume that $I_{0+}^- = I_{+0}^{0+} = \emptyset$. We begin our proof from "since problem MMPVC satisfies (6)–(9)" in the proof (i) and remove the neighborhoods U and V and $x_n \to \bar{x}$ from it. We immediately obtain that $\bar{x}$ is a global Borwein proper efficient solution of problem MMPVC. □

Remark 5. *In Theorem 10 of [18], Sadeghieh et al. established the following result. Suppose that f_j $(j \in J)$ is ∂-pseudoconvex at $\bar{x}$, and other conditions are the same as Theorem 2. Then, Theorem 2 holds for Pareto efficient solutions.*

In the following, we give an example to illustrate Theorem 2.

Example 4. *Let f_j $(j = 1, 2)$, h_i, g_i $(i = 1, 2) : \mathbb{R}^2 \to \mathbb{R}$. Consider the following problem MMPVC3:*

$$\begin{aligned}
&\min \ (|x_1| - x_2 \,,\, x_1^2 - x_2) \\
&\text{s.t. } \ x_2 \le 0, \\
&x_1 + |x_2| \ge 0, \\
&x_2(x_1 + |x_2|) \ge 0.
\end{aligned}$$

where

$$\begin{aligned} &f_1(x_1,x_2) = |x_1| - x_2, \quad f_2(x_1,x_2) = x_1^2 - x_2, \\ &h_1(x_1,x_2) = -x_2, \quad h_2(x_1,x_2) = x_1 + |x_2|, \\ &g_1(x_1,x_2) = -1, \quad g_2(x_1,x_2) = -x_2. \end{aligned}$$

Let $\bar{x} := (0,0)$. *We can calculate that*

$$\begin{aligned} &I_{+0} = \emptyset, \quad I_{+-} = \emptyset, \quad I_{0-} = \{1\}, \quad I_{00} = \{2\}, \quad I_{0+} = \emptyset, \quad I_{+} = \emptyset, \quad I_0 = \{1,2\}, \\ &\partial f_1(\bar{x}) = \{(d_1,-1) \mid |d_1| \le 1\}, \quad \partial f_2(\bar{x}) = \{(0,-1)\}, \\ &\partial h_1(\bar{x}) = \{(0,-1)\}\ , \quad \partial h_2(\bar{x}) = \{(1,d_2) \mid |d_2| \le 1\}\ . \end{aligned}$$

Let $\zeta_1^f = \zeta_2^f = \frac{1}{2}$, $\zeta_1^h = 1$, $\zeta_2^h = 0$, $\zeta_1^g = \zeta_2^g = 0$. *Then, we can calculate that*

$$I_{0+}^{-} = \emptyset, \quad I_{+0}^{0+} = \emptyset, \quad I_{0+}^{+} = \emptyset, \quad I_{00}^{+} = \emptyset, \quad I_{0-}^{+} = \{1\},$$

and

$$0_2 \in \frac{1}{2}\partial f_1(\bar{x}) + \frac{1}{2}\partial f_2(\bar{x}) - \partial h_1(\bar{x}) - 0\partial h_2(\bar{x}) + 0\,\partial g_1(\bar{x}) + 0\,\partial g_2(\bar{x}),$$

and $\zeta_1^f + \zeta_2^f = 1$, *which imply that (6)–(9) hold. Obviously,* $-h_1$ *is* ∂*-quasi-convex at* $\bar{x}$. *Since* $I_{0+}^{-} = I_{+0}^{0+} = \emptyset$, *by Theorem 2,* $\bar{x}$ *a global Borwein proper efficient solution of problem MMPVC3.*

4. Conclusions

In this paper, motivated by [18], we establish optimality conditions for Borwein proper efficient solutions of nonsmooth multiobjective problems with vanishing constraints by using the property of locally Lipschitz functions and the Clarke subdifferential. These results are extensions of the corresponding results of [18,19]. Since each Borwein proper efficient solution is a Pareto efficient solution, our results will bring potential applications in enhancing the accuracy of machine design. In the future, we will consider optimality conditions for other proper efficient solutions of nonsmooth vector optimization problems with vanishing constraints.

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Abbreviations

The following abbreviations are used in this manuscript:

MMPVC	Multiobjective mathematical programming with vanishing constraints
MPVC	Mathematical programming with vanishing constraints
PADQ	Proper Abadie data qualification
EADQ	Extended Abadie data qualification

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